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COEXISTING ATTRACTORS AND BUBBLING ROUTE TO CHAOS IN MODIFIED COUPLED DUFFING OSCILLATORS

BERC DERUNI[1], AVADIS S. HACINLIYAN[1,2], ENGIN KANDIRAN[3], ALI C. KELES[2], SMAIL KAOUACHE[4], M.-S. ABDELOUAHAB[4], N.-E. HAMRI[4]

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[1]DEPARTMENT OF PHYSICS, UNIVERSITY OF YEDITEPE, TURKEY.

[2]DEPARTMENT OF INFORMATION SYSTEMS AND TECHNOLOGIES, UNIVERSITY OF YEDITEPE, TURKEY.

[3]DEPARTMENT OF SOFTWARE DEVELOPMENT, UNIVERSITY OF YEDITEPE, TURKEY.

[4]LABORATORY OF MATHEMATICS AND THEIR INTERACTIONS, UNIVERSITY CENTER OF ABDELHAFID BOUSSOUF, MILA 43000, ALGERIA.
berc890@gmail.com
ahacinliyan@yeditepe.edu.tr
engin.kandiran@yeditepe.edu.tr
cihan.keles@yeditepe.edu.tr
s.kaouache@centr-univ-mila.dz
medsalah3@yahoo.fr
n.hamri@centre-univ-mila.dz

ABSTRACT. In this article dynamical behavior of coupled Duffing oscillators is analyzed under a small modification. The oscillators have cubic damping instead of linear one. Although single duffing oscillator has complex dynamics, coupled duffing systems possess a much more complex structure. The dynamical behavior of the system is investigated both numerically and analytically. Numerical results indicate that the system has double scroll attractor with suitable parameter values. On the other hand, bifurcation diagrams illustrate rich behavior of the system, and it is seen that, system enters into chaos with different routes. Beside classical bifurcations, bubbling route to chaos is observed for suitable parameter settings. On the other hand, Multistability of the system is indicated with the coexisting attractors, such that under same parameter setting the system shows different periodic and chaotic attractors. Moreover, chaotic synchronization of coupled oscillators is illustrated in final section.

Key words and phrases: Coupled duffing oscillator; Double scroll attractor; Chaos; Coexisting attractors; Multistability; Chaos synchronization.

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1. INTRODUCTION

Dynamics of coupled non-linear oscillators attracted much attention in recent years. Most important reason is that, they have wide range of application for modelling dynamical systems not only in physics, but also in various branches of science such as chemistry, biology, physiology [1, 2, 3] etc. Despite their apparent simplicity, dynamical behavior of coupled oscillators is very complicated such that analytical methods are not enough to describe all the features of the system. One must use numerical methods beside analytical methods, which is crucial for a better understanding of dynamical features.

In the present paper, dynamical behavior of coupled Duffing oscillators is investigated. Coupled Duffing oscillators are modified by introducing cubic damping instead of a linear damping. This modification makes substantial changes in dynamics of oscillators. In the numerical phase portraits, it is observed that the system possess chaotic double scroll attractor similar to the case in the famous Chua circuit [4]. One parameter bifurcation diagrams show the complexity of this system such that the system enters chaos through different routes [5]. Another interesting feature that appears in those bifurcation diagrams is period doubling bubbles [6, 7, 8]. Those bubbles make the system enter chaotic state by successive period doubling bifurcations.

On the other hand the multi stable characteristic of the system is seen by coexisting attractors [9, 10]. For a given set of parameters, the state of the system changes along with the initial conditions. Different periodic and chaotic attractors are simultaneously produced in the system for suitable parameter settings.

The final part is devoted to synchronization of those coupled oscillators. Several approaches of chaos synchronization have been developed, such as complete synchronization [11], generalized synchronization [12], inverse matrix projective synchronization [13] and modified projective synchronization [14]. Of particular interest is the duffing oscillators. For this aim, periodic driving force is incorporated for each oscillator with same amplitude and frequency. Before complete synchronization is achieved, intermittent loss of synchronization is shown.

The rest of the paper is organized as follows: the proposed system is described and analyzed in Section 2. In Section 3 bifurcations and the routes to chaos are investigated. In Section 4 coexisting attractors and multi stability of the system is shown. In Section 5 Chaos synchronization is implemented for coupled Duffing oscillators. Finally, conclusions are stated in Section 6.

2. DESCRIPTION OF THE MODEL AND LINEAR STABILITY ANALYSIS

The classical Duffing Oscillator is governed by the following equation of motion,

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \frac{dV(x)}{dx} = 0 \tag{2.1}$$

where δ is damping parameter and the potential of the system is

$$V(x) = \alpha\frac{x^2}{2} + \beta\frac{x^4}{4} \tag{2.2}$$

Depending on the parameters α and β one distinguishes three different cases of Duffing oscillator, namely single well $(\alpha > 0, \beta > 0)$, double well$(\alpha < 0, \beta > 0)$ and double hump $(\alpha > 0, \beta < 0)$. Each of them exhibits non-linear phenomena with different chaotic and periodic motions.

When two or more system 2.1 interact with specific coupling, the dynamics gets much more complex and attractive. There have been different types of coupled oscillators each describing interesting features. However most often used ones can be listed as coupled Duffing oscillators [15, 16, 17, 18], coupled Van der Pol oscillators[19, 20, 21, 22], coupled Van der Pol Duffing oscillators [23, 24, 25]. For the present paper two coupled duffing oscillators are analyzed including a cubic damping in both oscillators and considering a linear bidirectional coupling between them. The potential for the system is given by following

$$V(x,y) = -\frac{x^2}{2} + \frac{x^4}{16} - \frac{y^2}{2} + \frac{y^4}{16} + \frac{k}{2}(x-y)^2 \tag{2.3}$$

Instead of considering linear damping as in 2.1 cubic nonlinear damping (Van der Pol) type is incorporated and the following set of equations are derived

$$\begin{aligned} \frac{d^2x}{dt^2} &= x - \frac{1}{4}x^3 + k(y-x) - \delta x^2 \frac{dx}{dt} \\ \frac{d^2y}{dt^2} &= y - \frac{1}{4}y^3 + k(x-y) - \delta' y^2 \frac{dy}{dt} \end{aligned} \tag{2.4}$$

Before going to details of system 2.4, it is convenient to transform it to first-order differential equations. One can write the system in the following form

$$\begin{aligned} \frac{dx}{dt} &= p_x \\ \frac{dp_x}{dt} &= x - \frac{1}{4}x^3 + k(y-x) - \delta x^2 p_x \\ \frac{dy}{dt} &= p_y \\ \frac{dp_y}{dt} &= y - \frac{1}{4}y^3 + k(x-y) - \delta' y^2 p_y \end{aligned} \tag{2.5}$$

The equilibrium points of the system are determined by setting the right hand side of system (2.5) to zero. In general case the system possesses 9 equilibrium points, but only 3 of them are independent of parameters. The other 6 depend on coupling parameter k and for small interval of k they stay real, otherwise they are complex. In the present study the stability analysis is carried out for the first three equilibrium points. These points are $E_1 = (0,0,0,0)$ and $E_{2,3} = (\pm 2, 0, \pm 2, 0)$. It is important to note that the system is invariant under the transformation $(x, p_x, y, p_y) \leftrightarrow (-x, -p_x, -y, -p_y)$. This implies that the stability nature of $2^{nd}$and 3^{rd} equilibrium points must be the same.
Stability of equilibrium points can be determined with the investigation of eigenvalues of its Jacobian matrix. For the system (2.5) the Jacobian matrix is given by

$$M_J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1-k-\frac{3}{4}x^2 - 2\delta x p_x & -\delta x^2 & k & 0 \\ 0 & 0 & 0 & 1 \\ k & 0 & 1-k-\frac{3}{4}y^2 - 2\delta' y p_y & -\delta' y^2 \end{bmatrix} \tag{2.6}$$

and the eigenvalues are roots of the characteristic equation $\det(M_J - \lambda I_d) = 0$. For the first equilibrium point E_1 the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = \sqrt{-2k+1}$, $\lambda_4 = -\sqrt{-2k+1}$. Since there is at least one positive real eigenvalue, the equilibrium point E_1 is unstable saddle. Since the behavior of $2^{nd}$and 3^{rd} equilibrium points are the same, only stability of E_3 is carried out. For the equilibrium point E_3, the eigenvalues satisfy the characteristic equation of the form

$$\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0, \tag{2.7}$$

where $c_i (i = 0,1,2,3)$ are defined as

$$\begin{aligned} c_0 &= 4(k+1), \quad c_1 = 4(\delta k + \delta')(2+k) \\ c_2 &= 2(2+k+8\delta\delta'), \quad c_3 = 4(\delta + \delta') \end{aligned} \tag{2.8}$$

The roots of equation (2.7) are obtained with Newton-Raphson algorithm for changing k between $0 < k < 4$ while keeping damping terms constant $\delta = -0.109$ and $\delta' = 1.10$. The results as the spectrum of eigenvalues are shown in Figure 1.

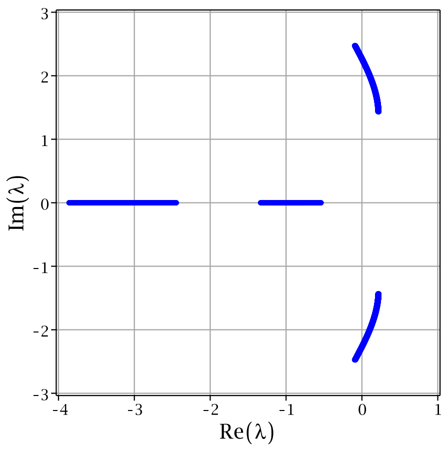
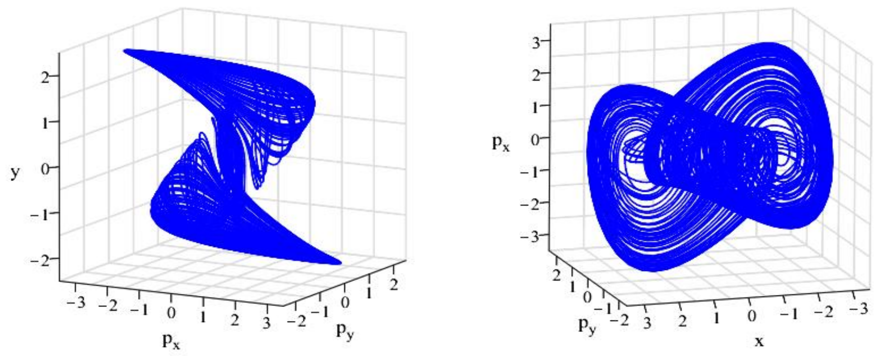


Figure 1: Eigenvalues of solutions of the characteristic equation (2.7) in complex plane (Re(λ), Im*(λ)).*

Since M_J is a real matrix, complex eigenvalues appear in complex conjugate pairs responsible for the observed symmetry on real axis. Four branches of eigenvalues appearing in Figure 1 indicate that the equilibrium point E_3 is not stable furthermore the complex eigenvalues are crossing the imaginary axis, which is an indication of Hopf bifurcation for varying k.

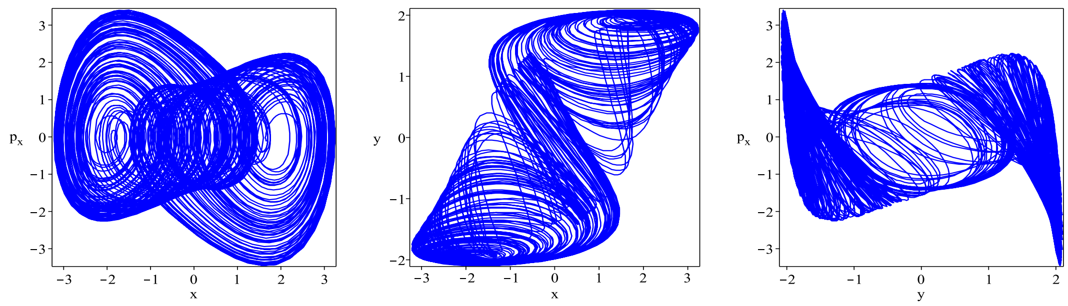
3. BIFURCATIONS AND ONSET OF CHAOS

With numerical investigation it is observed that this system possess double scroll attractor shown in Figure 2 and Figure 3 similar to the Chua attractor, and similar mechanisms take place such that two unstable fixed points give rise to attractors of the same type under appropriate parameter settings and two attractors eventually merge giving rise to a double scroll attractor.
Dynamical behavior of the system is illustrated with the aid of bifurcation diagrams and Lyapunov exponents in order to determine routes to chaos. For this aim, equations 2.5 are solved numerically with a given initial condition and local maximum of x is plotted against bifurcation parameter. On the other hand Lyapunov exponents are calculated with the Wolf algorithm [26]. In order to show bifurcation structure, two of the parameters are held constant and 3^{rd} parameter is allowed to change, and it is seen that system supports variety of local and global bifurcations which is shown in Figure 4a, especially period doubling cascades and symmetry breaking bifurcations takes place in large fraction of area.



(a) (b)

Figure 2: Double scroll attractor appearing in the phase space of, ***(a)*** (p_x, p_y, y) ***(b)*** (x, p_x, p_y) *for the parameters of* $k = 1.13$, $\delta = -0.109$, $\delta' = 1.15$ *with the initial conditions of* $(0.5, 0.0, 0.5, 0.0)$

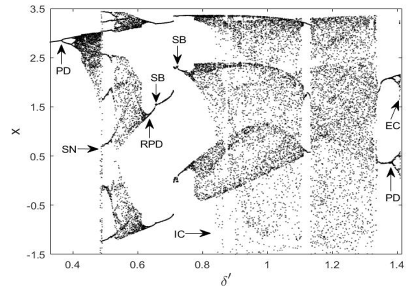


(a) (b) (c)

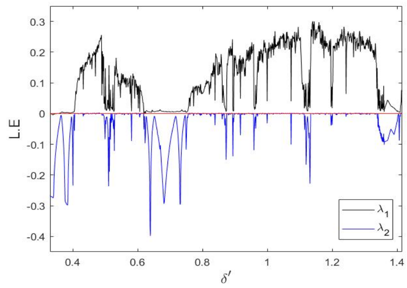
Figure 3: 2D views of double scroll attractor in ***(a)*** $(x - p_x)$ *space,* ***(b)*** $(x - y)$*space,* ***(c)*** $(y - p_y)$ *space*

Figure 4 is obtained for $k = 0.921$, $\delta = -0.109$, while δ' is increased from 0.2 up to 1.5. A period-1 oscillation take place after supercritical Hopf bifurcation which is not depicted in figure then by a period doubling bifurcation chaos starts. For δ' near 0.50 attractor collides with saddle and periodic windows appear. System reenters chaotic state near δ' 0.53 but then reverse period doubling bifurcationδ' near 0.62 happens and period-3 attractor seems to appear. After $\delta' = 0.73$, chaotic motions are triggered again by period doubling bifurcation. The system also suffers intermittent route to chaos for $\delta' = 0.86$ where internal crisis take place. One last period doubling bifurcation happens δ' close to 1.4 but system can not stay in chaos too much since it is immediately followed by exterior crisis and chaos suddenly disappears yielding unbounded motion.

Figure 4b represents largest and second Lyapunov exponents. In chaotic regions largest Lyapunov exponents gets greater and in periodic regions they are close to zero. For points where period doubling bifurcation occurs they tend to touch zero.

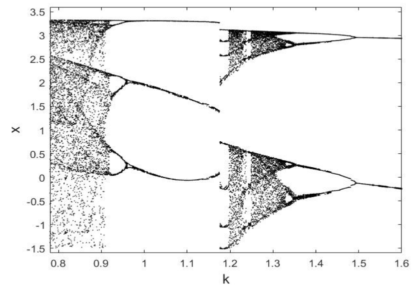


(a)

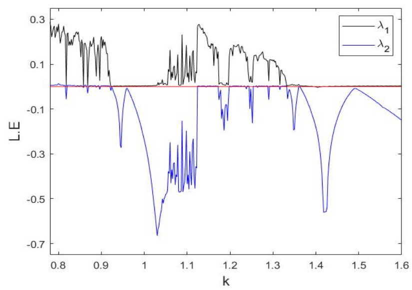


(b)

*Figure 4: **(a)** Bifurcation diagram for varying δ'. **(b)** First and second Lyapunov exponents for varying δ'.*



(a)



(b)

*Figure 5: **(a)**Bifurcation diagram for varying k, **(b)**First and second Lyapunov exponents for varying k*

In Figure 5a bifurcation diagram with respect to coupling parameter k is represented, and corresponding Lyapunov exponents are shown in Figure 5b. While largest Lyapunov exponents indicating chaotic regions, it is notable that second Lyapunov exponents give indication of hyperchaotic regime for k close to 0.8 with a maximum value of $\lambda_2 = 0.014$. The other parameters are set to $\delta = -0.104$, $\delta' = 0.967$ and diagrams are obtained for initial conditions of $(-0.2, 0.0, -0.2, 0.0)$.
Another interesting feature is that for suitable parameters system shows period doubling bubbles [27, 28] which are frequently encountered in many nonlinear systems. Figure 6 depicts this behavior where small chaotic regions exist inside bubbles for short interval of k, but system gets out of chaos with those structures.

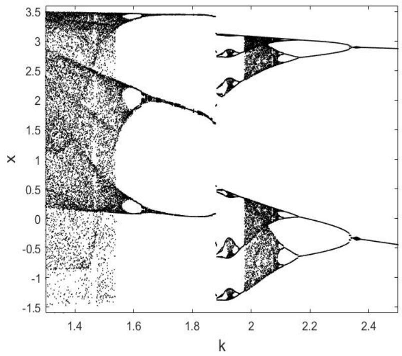


Figure 6: Bifurcation diagram with respect to k showing period doubling bubbles for $\delta = -0.154$, $\delta' = 1.1$.

In many nonlinear systems it is well established that the periodic orbits can be created with period doubling bifurcation and then destroyed with reverse period doubling bifurcation. This phenomenon is known as antimonotonicity [29, 30] and it has been reported in many dynamical systems. For this purpose, we investigate those bubble structures in bifurcation diagram. We observe that for small range of bifurcation parameter the system supports this phenomenon.
Figure 7c clearly shows how the system enters chaos within bubbles. In the bifurcation diagram with respect to parameter k successive period doubling bifurcation yields chaos, then reverse period doubling bifurcations take away the system from chaos as the parameter δ' slowly changed.

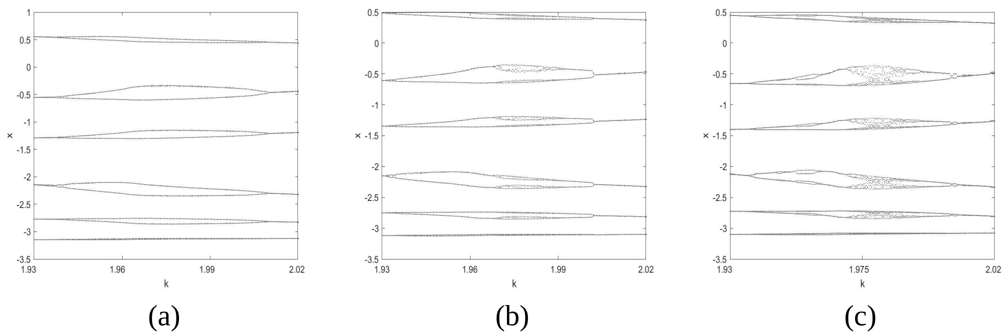


*Figure 7: Formation of chaos inside bubbles **(a)** $\delta' = 1.24$ **(b)** $\delta' = 1.10$ **(c)** $\delta' = 1.06$.*

4. COEXISTING ATTRACTORS AND MULTISTABILITY

The coexistence of attractors is one of the most striking features in non-linear dynamics. In recent years the existence of multiple attractors [31, 32, 33, 34] for the same parameter settings

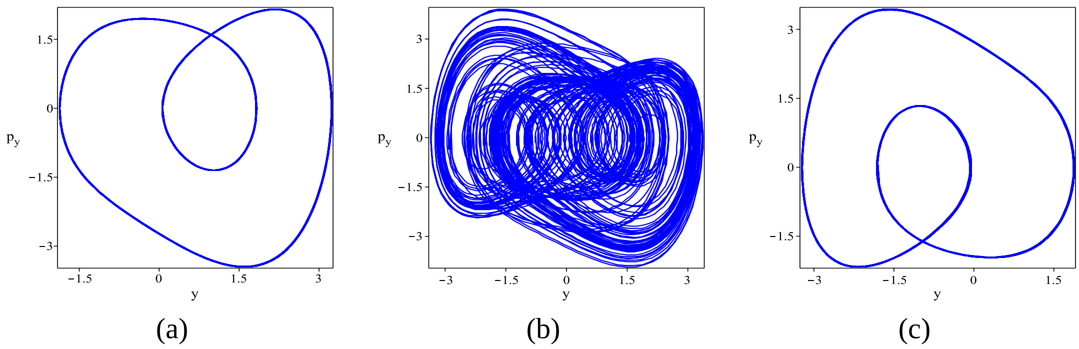


Figure 8: *Coexistence of attractors,* ***(a)*** *period-1 attractor with initial conditions of* $(-0.5, 0.5, 2.5, 0.0)$*,* ***(b)*** *chaotic attractor with initial conditions of* $(-0.5, 1.0, 2.5, 0.0)$*,* ***(c)*** *period-1 attractor with initial conditions of* $(-0.5, 0.5, -2.5, 0.0)$*. The parameters are* $k = 1.0$*,* $\delta = 1.5$*,* $\delta' = -0.109$*.*

has been studied extensively and such a phenomenon has been observed in various systems. In this respect we analyze the system numerically with same parameter settings but different initial conditions. It was observed that the system is sensitive to initial conditions such that multi stable characteristic [32, 35, 36] of the system is revealed with the coexistence of different attractors. Some of the numerical results are shown below

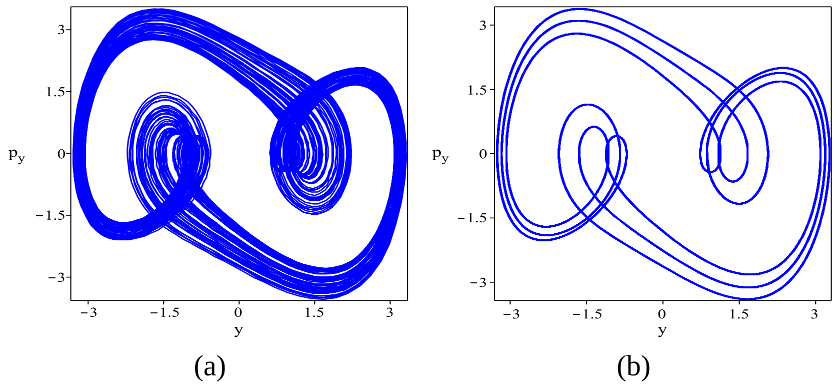


Figure 9: Coexistence of attractors, ***(a)*** *chaotic attractor with initial conditions of* $(-0.5, 0.5, 1.5, 2.5)$*,* ***(b)*** *period-3 attractor with initial conditions of* $(0.5, 0.5, 1.5, 2.5)$*. The parameters are* $k = 0.87$*,* $\delta = 0.58$*,* $\delta' = -0.109$*.*

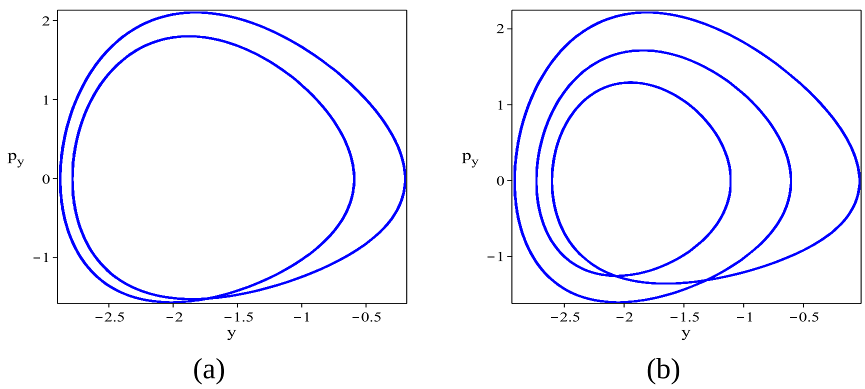


Figure 10: Coexistence of attractors, ***(a)*** *period-2 attractor with initial conditions of* $(0.5, -0.5, 1.5, 2.5)$*,* ***(b)*** *period-3 attractor with initial conditions of* $(0.5, 0.5, 1.5, 2.5)$*. The parameters are* $k = 0.87$*,* $\delta = 0.35$*,* $\delta' = -0.109$*.*

5. SYNCHRONIZATION AND CHAOS CONTROL

Since its introduction by Pecora and Carrol [37] in 1990 chaos synchronization has been attracted much interest recently. The motivation behind this research field is its huge potential for application such as communication systems, time series analysis, brain modelling, cardiac rhythm activity, earthquake dynamics etc.
In this section, chaotic synchronization of two coupled Duffing oscillators are analysed numerically and it is shown that there is a threshold value of coupling between oscillators after which complete synchronization takes place.
In order to achieve our goal, we introduce periodic driving forces onto system 2.4 such that the equation of motions reads

$$\begin{aligned} \frac{d^2x}{dt^2} &= x - \frac{1}{4}x^3 + k(y-x) - \delta x^2 \frac{dx}{dt} + g\sin(wt) \\ \frac{d^2y}{dt^2} &= y - \frac{1}{4}y^3 + k(x-y) - \delta' y^2 \frac{dy}{dt} + g\sin(wt) \end{aligned} \tag{5.1}$$

Damping parameters are equalized $\delta = \delta'$ so that in the no coupling $k = 0$ case two identical chaotic oscillators are obtained for a suitable parameter setting. On the other hand, driving forces and frequencies are set to $g = 0.702$, and $w = 1.15$ respectively. Chaotic systems are very sensitive to initial conditions and small deviation from the initial conditions leads to unpredictable results, however when coupling is turned on two systems starts to synchronize.

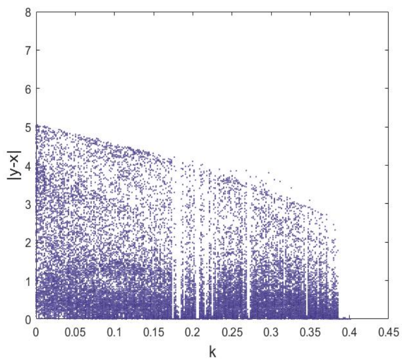
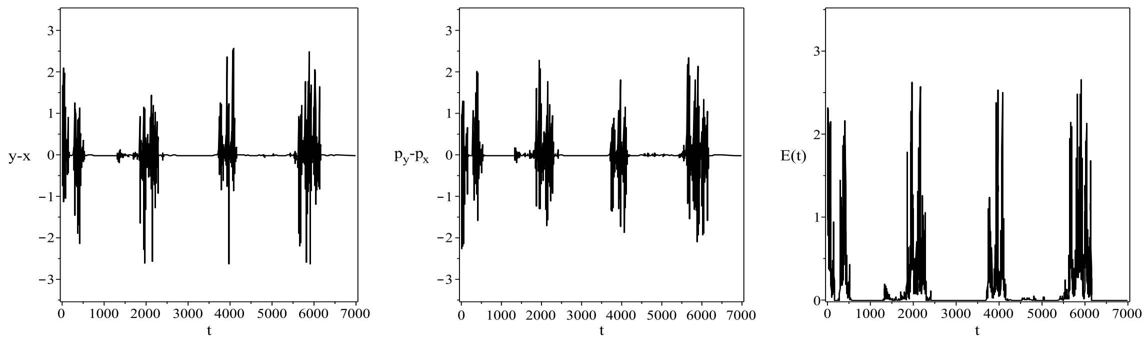


Figure 11: Bifurcation of $|e(t)|$ along k .

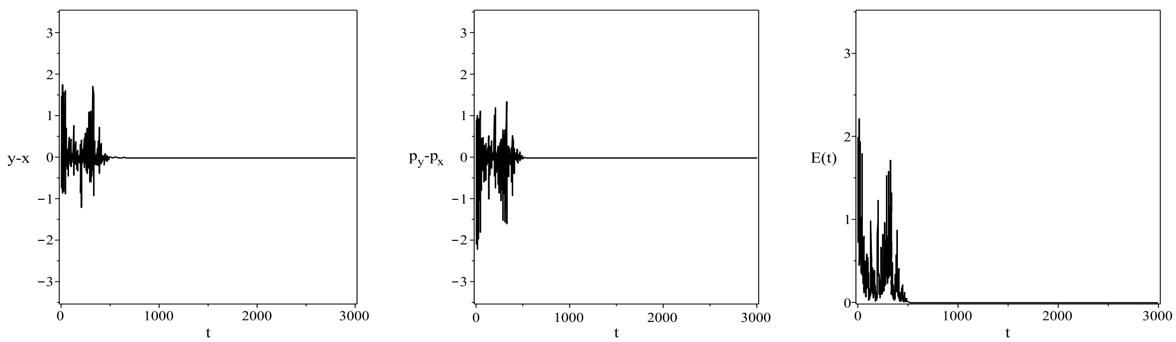
In order to validate the synchronization of two oscillators the fluctuation of the quantity $e(t) = y(t) - x(t)$ which is called error dynamics, can be tracked. Before synchronization state, fluctuations of $e(t)$ are chaotic. When synchronization starts the fluctuations tend to zero, and the dynamics is restricted on $y = x$ subspace also known as synchronization manifold [38, 39]. Equations for oscillators (9) are solved numerically for initial conditions of $(-0.2, -0.2, 0.4, 0.3)$. Our numerical simulations show a synchronization state for near $k = 0.40$. To this aim, the bifurcation diagram is shown in Figure 11 where absolute value of the error function [40] $e(t)$ is plotted against coupling parameter k.
For $k = 0.37$ the fluctuations for error dynamics are shown in Figure 12. Synchronization can not achieved due to intermittent losses [41].



(a) (b) (c)

Figure 12: Intermittent losses of synchronization for $k = 0.37$*. Fluctuation of errors* ***(a)***$y - x$*,* ***(b)*** $p_y - p_x$*,* ***(c)***$E = \sqrt{(y-x)^2 + (p_y - p_x)^2}$

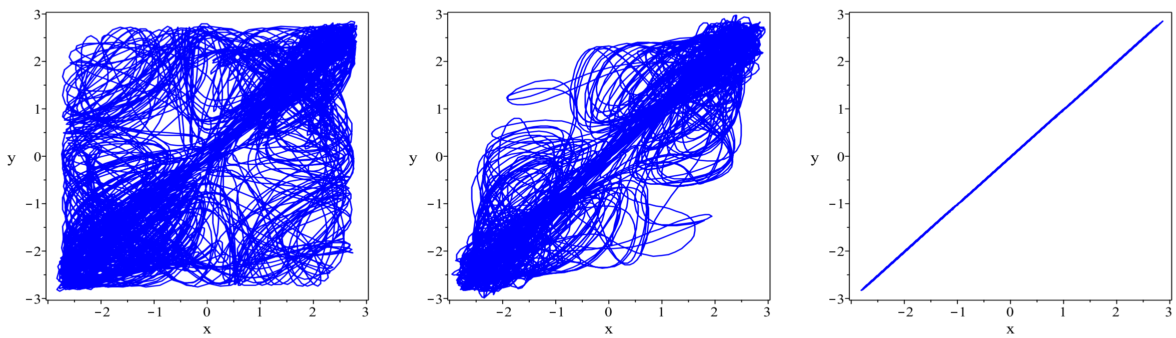
For $k = 0.40$ two oscillators are in complete synchronization which is shown in Figure 13.



(a) (b) (c)

Figure 13: Synchronization for $k = 0.40$*. Fluctuation of errors* ***(a)***$y - x$ ***(b)*** $p_y - p_x$ ***(c)***$E = \sqrt{(y-x)^2 + (p_y - p_x)^2}$

After a transient of $t = 500$ complete synchronization achieved for coupled oscillators. On the other hand the changes in phase portraits are indicated for varying k in Figure 14. It can also be seen from phase portraits above that synchronization is achieved for increasing k, as k approaches 0.40 the motions are getting more and more bounded in $x = y$ plane.



(a) (b) (c)

Figure 14: Phase portraits for ***(a)*** *nonsynchronous state* $k = 0.05$*,* ***(b)*** *nonsynchronous state* $k = 0.30$ ***(c)****Synchronous state* $k = 0.40$

6. CONCLUSION

In this paper, the dynamics of coupled double well duffing oscillators is investigated with addition of nonlinear damping. The characteristics of the system is analyzed both analytically and numerically. By using linear stability analysis, the behavior of 3 equilibrium points are determined and it is shown that two symmetric unstable equilibrium points give rise to Hopf bifurcation and the possibility of double scroll attractor.

Using nonlinear dynamical tools such as bifurcation diagrams, phase portraits and Lyapunov exponents, the complicated structure of the system is investigated. It is shown that the system possesses double scroll attractor as in the case of Chua circuit instead of classical double well attractor of duffing system. Variety of local and global bifurcations such as Hopf bifurcation, period doubling, symmetry breaking, reverse period doubling and crises were seen. The formation of bubbles are shown and bubbling route to chaos is identified in the vicinity of bifurcation diagrams. Although 2^{nd} Lyapunov exponents are seem to be constant, the system gives an indication of hyperchaotic behavior for small interval of coupling parameter. Furthermore, multi stability of the system is presented with the coexisting chaotic and periodic solutions of the system under the same parameter settings.

The system is open to detecting many other interesting features and new routes to chaos. The other equilibrium points may also give rise to attractors with similar mechanisms and they need more general analysis. Hyperchaotic behavior in the system can be extended with a further investigation of parameters and initial conditions. On the other hand, approximating methods can be applied also to obtain analytical expressions for bifurcating solutions, which can further enlighten the complex behavior of the system.

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