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HERMITE-HADAMARD TYPE INEQUALITIES FOR MN-CONVEX FUNCTIONS

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ABSTRACT. The present work endeavours to briefly present some of the fundamental results connected to the Hermite-Hadamard inequality for special classes of convex functions such as *AG, AH, GA, GG, GH, HA, HG* and *HH*-convex functions in which the author have been involved during the last five years. For simplicity, we call these classes of functions such as *MN*-convex functions, where *M* and *N* stand for any of the Arithmetic (*A*), Geometric (*G*) or Harmonic (*H*) weighted means of positive real numbers. The survey is intended for use by both researchers in various fields of Approximation Theory and Mathematical Inequalities, domains which have grown exponentially in the last decade, as well as by postgraduate students and scientists applying inequalities in their specific areas.

Key words and phrases: Hermite-Hadamard type inequalities, Convex functions, Special convexity, Integral inequalities.

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Preface

As the celebrated Danish mathematician J. L. W. V. Jensen anticipated in 1906:

"*Il me semble que la notion de fonction convexe est à peu prés aussi fondamentale que celles-ci: fonction positive, fonction croissante. Si je ne tromp pas en ceci, la notion devra trouver sa place dans les expositions élémentaires de la théorie des fonctions réelles*".
the concept of convex functions has indeed found an important place in *Modern Mathematics* as can be seen in a large number of research articles and books devoted to the field these days.

In this context, the Hermite-Hadamard inequality, which, we can say, is the first fundamental result for convex functions with a natural geometrical interpretation and many applications, has attracted and continues to attract much interest in elementary mathematics.

Many mathematicians have devoted their efforts to generalize, refine, counterpart and extend it for different classes of functions such as: quasi-convex functions, Godunova-Levin class of functions, log-convex and r-convex functions, p-functions, etc... or apply it for special means such as p-logarithmic means, identric mean, Stolarsky means, etc.

For a monograph devoted to this famous result and its ramifications as mentioned above, we recommend the freely available online book by S. S. Dragomir & C. E. M. Pearce entitled "*Selected Topics on Hermite-Hadamard Inequalities and Applications*", RGMIA Monographs, Victoria University, 2000. [`https://rgmia.org/monographs/hermite_hadamard.html`].

The present work endeavours to briefly present some of the fundamental results connected to the Hermite-Hadamard inequality for special classes of convex functions such as *AG, AH, GA, GG, GH, HA, HG* and *HH*-convex functions in which the author have been involved during the last five years. For simplicity, we call these classes of functions such as *MN*-convex functions, where *M* and *N* stand for any of the Arithmetic (*A*), Geometric (*G*) or Harmonic (*H*) weighted means of positive real numbers.

The survey is intended for use by both researchers in various fields of Approximation Theory and Mathematical Inequalities, domains which have grown exponentially in the last decade, as well as by postgraduate students and scientists applying inequalities in their specific areas.

For the sake of completeness, all the results presented are completely proved and the original references where they have been firstly obtained are mentioned.

CHAPTER 1

Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}, \tag{0.1}$$

where $a, b \in \mathbb{R}$, $a < b$. It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [**28**]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [**2**]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [**28**]. Since (0.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x, y \in X,\ x \ne y$. Define the segment

$$[x,y] := \{(1-t)x + ty,\ t \in [0,1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x,y) : [0,1] \to \mathbb{R},\ g(x,y)(t) := f[(1-t)x + ty],\ t \in [0,1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$.

For any convex function defined on a segment $[x, y] \subset X$, we have the *Hermite-Hadamard integral inequality* (see [**5**, p. 2], [**6**, p. 2])

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f[(1-t)x+ty]dt \le \frac{f(x)+f(y)}{2}, \tag{0.2}$$

which can be derived from the classical Hermite-Hadamard inequality (0.1) for the convex function $g(x, y) : [0, 1] \to \mathbb{R}$.

We recall some facts on the lateral derivatives of a convex function.

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_-(x) \le f'_+(x) \le f'_-(y) \le f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \ge f(a) + (x-a)\varphi(a) \text{ for any } x, a \in I.$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \le \varphi(x) \le f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

The following reverses of the Hermite-Hadamard inequality for functions of a real variable hold.

LEMMA 0.1 (Dragomir, 2002 [**5**] and [**6**]). *Let* $f:[a,b]\to\mathbb{R}$ *be a convex function on* $[a,b]$. *Then*

$$0\leq\frac{1}{8}\left[f_{+}\left(\frac{a+b}{2}\right)-f_{-}\left(\frac{a+b}{2}\right)\right](b-a)\leq\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\leq\frac{1}{8}\left[f_{-}(b)-f_{+}(a)\right](b-a)\tag{0.3}$$

and

$$0\leq\frac{1}{8}\left[f_{+}\left(\frac{a+b}{2}\right)-f_{-}\left(\frac{a+b}{2}\right)\right](b-a)\leq\frac{1}{b-a}\int_{a}^{b}f(x)\,dx-f\left(\frac{a+b}{2}\right)\leq\frac{1}{8}\left[f_{-}(b)-f_{+}(a)\right](b-a).\tag{0.4}$$

The constant $\frac{1}{8}$ *is best possible in all inequalities from (0.3) and (0.4).*

This result will be used several time in the following to derive similar inequalities for *MN*-convex functions.

CHAPTER 2

Inequalities for AG-Convex Functions

1. Preliminary Facts on Log-convex Functions

A function $f : I \to [0, \infty)$ is said to be *AG-convex* or *log-convex* or *multiplicatively convex* if $\log f$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality:

$$f(tx + (1-t)y) \le [f(x)]^t [f(y)]^{1-t}. \tag{1.1}$$

We note that if f and g are convex and g is increasing, then $g \circ f$ is convex; moreover, since $f = \exp(\log f)$, it follows that a log-convex function is convex, but the converse may not necessarily be true. This follows directly from (1.1) because, by the *arithmetic-geometric mean inequality*, we have

$$[f(x)]^t [f(y)]^{1-t} \le tf(x) + (1-t) f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Let us recall the *Hermite-Hadamard inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}, \tag{1.2}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval $I,\ a, b \in I$ and $a < b$.

Note that if we apply the above inequality for the log-convex functions $f : I \to (0, \infty)$, we have that

$$\ln\left[f\left(\frac{a+b}{2}\right)\right] \le \frac{1}{b-a}\int_a^b \ln f(x)\,dx \le \frac{\ln f(a) + \ln f(b)}{2}, \tag{1.3}$$

from which we get

$$f\left(\frac{a+b}{2}\right) \le \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \le \sqrt{f(a)\,f(b)}, \tag{1.4}$$

which is an inequality of Hermite-Hadamard's type for log-convex functions.

By using simple properties of log-convex functions Dragomir and Mond proved in 1998 the following result:

THEOREM 1.1 (Dragomir-Mond, 1998 [**22**]). *Let $f : I \to [0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequality:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \sqrt{f(x)\,f(a+b-x)}dx \le \sqrt{f(a)\,f(b)}. \tag{1.5}$$

The inequality between the first and second term in (1.5) may be improved as follows [**22**]. A different upper bound for the middle term in (1.5) can be also provided.

THEOREM 1.2 (Dragomir-Mond, 1998 [**22**]). *Let $f : I \to (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequalities:*

$$\begin{aligned} (1.6)\quad f\left(\frac{a+b}{2}\right) &\leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \\ &\leq \frac{1}{b-a}\int_a^b \sqrt{f(x)\,f(a+b-x)}dx \leq \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\leq L(f(a), f(b)), \end{aligned}$$

where $L(p, q)$ is the logarithmic mean of the strictly positive real numbers $p, q,$ i.e.,

$$L(p,q) := \frac{p-q}{\ln p - \ln q} \text{ if } p \neq q \text{ and } L(p,p) := p.$$

The last inequality in (1.6) was obtained in a different context in [**24**].

As shown in [**35**], the following result also holds:

THEOREM 1.3 (Sulaiman, 2011 [**35**]). *Let $f : I \to (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequalities:*

$$(1.7)\qquad f\left(\frac{a+b}{2}\right) \leq \left(\frac{1}{b-a}\int_a^b \sqrt{f(x)}dx\right)^2 \leq \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The following result improving the classical first Hermite-Hadamard inequality for differentiable log-convex functions also hold [**4**]:

THEOREM 1.4 (Dragomir, 2001 [**4**]). *Let $f : I \to (0, \infty)$ be a differentiable log-convex function on the interval of real numbers $\mathring{I}$ (the interior of I) and $a, b \in \mathring{I}$ with $a < b$. Then the following inequalities hold:*

$$\begin{aligned} (1.8)\quad & \frac{\frac{1}{b-a}\int_a^b f(x)\,dx}{f\left(\frac{a+b}{2}\right)} \\ &\geq L\left(\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right], \exp\left[-\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right]\right) \geq 1. \end{aligned}$$

The second Hermite-Hadamard inequality can be improved as follows [**4**].

THEOREM 1.5 (Dragomir, 2001 [**4**]). *Let $f : I \to \mathbb{R}$ be as in Theorem 1.4. Then we have the inequality:*

$$\begin{aligned} (1.9)\quad \frac{\frac{f(a)+f(b)}{2}}{\frac{1}{b-a}\int_a^b f(x)\,dx} &\geq 1 + \log\left[\frac{\int_a^b f(x)\,dx}{\int_a^b f(x)\exp\left[\frac{f'(x)}{f(x)}\left(\frac{a+b}{2} - x\right)\right]dx}\right] \\ &\geq 1 + \log\left[\frac{\frac{1}{b-a}\int_a^b f(x)\,dx}{f\left(\frac{a+b}{2}\right)}\right] \geq 1. \end{aligned}$$

Motivated by the above results, we present in this chapter some new inequalities for log-convex functions, some of them improving earlier results.

2. NEW INEQUALITIES FOR LOG-CONVEX FUNCTIONS

The following refinement of the Hermite-Hadamard inequality holds:

LEMMA 2.1 (Dragomir, 1994 [**3**]). *Let $h:[a,b]\to\mathbb{R}$ be a convex function and $a=x_0<x_1<...<x_{n-1}<x_n=b$ an arbitrary division of $[a,b]$ with $n\geq 2$. Then*

$$h\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\sum_{i=0}^{n-1}h\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\leq\frac{1}{b-a}\int_a^b h(x)\,dx\leq\frac{1}{b-a}\sum_{i=0}^{n-1}\frac{h(x_i)+h(x_{i+1})}{2}(x_{i+1}-x_i)\leq\frac{h(a)+h(b)}{2}. \tag{2.1}$$

The inequality (2.1) was obtained in 1994 as a particular case of a more general result, see [**3**] and also mentioned in [**23**, p. 22]. For a direct proof, see the recent paper [**7**].

THEOREM 2.2 (Dragomir, 2015 [**8**]). *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$ and $a=x_0<x_1<...<x_{n-1}<x_n=b$ an arbitrary division of $[a,b]$ with $n\geq 1$. Then*

$$f\left(\frac{a+b}{2}\right)\leq\prod_{i=1}^{n-1}\left[f\left(\frac{x_i+x_{i+1}}{2}\right)\right]^{\frac{x_{i+1}-x_i}{b-a}}\leq\exp\left(\frac{1}{b-a}\int_a^b\ln f(x)\,dx\right)\leq\prod_{i=1}^{n-1}\left[\sqrt{f(x_i)f(x_{i+1})}\right]^{\frac{x_{i+1}-x_i}{b-a}}\leq\sqrt{f(a)f(b)}. \tag{2.2}$$

PROOF. If we write the inequality (2.1) for the function $h=\ln f$ then we get

$$\ln f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\ln f\left(\frac{x_i+x_{i+1}}{2}\right)\leq\frac{1}{b-a}\int_a^b\ln f(x)\,dx\leq\frac{1}{b-a}\sum_{i=0}^{n-1}\frac{\ln f(x_i)+\ln f(x_{i+1})}{2}(x_{i+1}-x_i)\leq\frac{\ln f(a)+\ln f(b)}{2}.$$

This inequality is equivalent to the desired result (2.2). ∎

COROLLARY 2.3. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$ and $x\in[a,b]$, then*

$$f\left(\frac{a+b}{2}\right)\leq\left[f\left(\frac{a+x}{2}\right)\right]^{\frac{x-a}{b-a}}\left[f\left(\frac{x+b}{2}\right)\right]^{\frac{b-x}{b-a}}\leq\exp\left(\frac{1}{b-a}\int_a^b\ln f(x)\,dx\right)\leq\left[\sqrt{f(a)}\right]^{\frac{x-a}{b-a}}\sqrt{f(x)}\left[\sqrt{f(b)}\right]^{\frac{b-x}{b-a}}\leq\sqrt{f(a)f(b)} \tag{2.3}$$

and, equivalently

$$\ln f\left(\frac{a+b}{2}\right) \leq \frac{x-a}{b-a}\ln f\left(\frac{a+x}{2}\right)+\frac{b-x}{b-a}\ln f\left(\frac{x+b}{2}\right) \tag{2.4}$$
$$\leq \frac{1}{b-a}\int_a^b \ln f(x)\,dx$$
$$\leq \frac{1}{2}\left[\ln f(x)+\frac{(x-a)\ln f(a)+(b-x)\ln f(b)}{b-a}\right]$$
$$\leq \frac{\ln f(a)+\ln f(b)}{2}.$$

REMARK 2.1. If we take in (2.4) $x=\frac{a+b}{2}$, then we get

$$\ln f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[\ln f\left(\frac{3a+b}{4}\right)+\ln f\left(\frac{a+3b}{4}\right)\right] \tag{2.5}$$
$$\leq \frac{1}{b-a}\int_a^b \ln f(x)\,dx$$
$$\leq \frac{1}{2}\left[\ln f\left(\frac{a+b}{2}\right)+\frac{\ln f(a)+\ln f(b)}{2}\right] \leq \frac{\ln f(a)+\ln f(b)}{2}.$$

From the second inequality in (2.5) we get

$$0 \leq \frac{1}{b-a}\int_a^b \ln f(x)\,dx-\ln f\left(\frac{a+b}{2}\right)$$
$$\leq \frac{\ln f(a)+\ln f(b)}{2}-\frac{1}{b-a}\int_a^b \ln f(x)\,dx,$$

which shows that the integral term in (1.3) is closer to the left side than to the right side of that inequality.

In the case of log-convex functions we have:

THEOREM 2.4 (Dragomir, 2015 [**8**]). *Let* $f:[a,b]\to(0,\infty)$ *be a log-convex function on* $[a,b]$. *Then*

$$1 \leq \exp\left(\frac{1}{8}\left[\frac{f_+\left(\frac{a+b}{2}\right)-f_-\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\right](b-a)\right) \tag{2.6}$$
$$\leq \frac{\sqrt{f(a)\,f(b)}}{\exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right)}$$
$$\leq \exp\left(\frac{1}{8}\left[\frac{f_-(b)}{f(b)}-\frac{f_+(a)}{f(a)}\right](b-a)\right)$$

and

$$1 \leq \exp\left(\frac{1}{8}\left[\frac{f_+\left(\frac{a+b}{2}\right)-f_-\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\right](b-a)\right) \leq \frac{\exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right)}{f\left(\frac{a+b}{2}\right)} \leq \exp\left(\frac{1}{8}\left[\frac{f_-(b)}{f(b)}-\frac{f_+(a)}{f(a)}\right](b-a)\right). \tag{2.7}$$

PROOF. If we write the inequality (0.3) for the convex function $h = \ln f$

$$0 \leq \frac{1}{8}\left[\frac{f_+\left(\frac{a+b}{2}\right)-f_-\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\right](b-a) \leq \frac{\ln f(a)+\ln f(b)}{2}-\frac{1}{b-a}\int_a^b \ln f(x)\,dx \leq \frac{1}{8}\left[\frac{f_-(b)}{f(b)}-\frac{f_+(a)}{f(a)}\right](b-a)$$

that is equivalent to the desired result (2.6). The inequality (2.7) follows from (0.4). ∎

We also have the following result:

THEOREM 2.5 (Dragomir, 2015 [**8**]). *Let* $f : [a,b] \to (0,\infty)$ *be a log-convex function on* $[a,b]$ *and* $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ *an arbitrary division of* $[a,b]$ *with* $n \geq 1$. *Then*

$$\exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \leq \frac{1}{b-a}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \sqrt{f(x)\,f(x_i+x_{i+1}-x)}dx \leq \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.8}$$

PROOF. Observe that we have

$$\exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] = \exp\left[\frac{1}{b-a}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \ln f(x)\,dx\right] = \exp\left[\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{b-a}\left(\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \ln f(x)\,dx\right)\right]. \tag{2.9}$$

Since $\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{b-a} = 1$, then by Jensen's inequality for the convex function $\exp$ we have

$$\exp\left[\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{b-a}\left(\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \ln f(x)\,dx\right)\right] \leq \sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{b-a}\exp\left(\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \ln f(x)\,dx\right). \tag{2.10}$$

Utilising the inequality (1.6) on each of the intervals $[x_i, x_{i+1}]$ for $i \in \{0, ..., n-1\}$ we have

$$\exp\left[\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \ln f(x)\,dx\right] \leq \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \sqrt{f(x)\,f(x_i+x_{i+1}-x)}dx \leq \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)\,dx, \tag{2.11}$$

for any $i \in \{0, ..., n-1\}$.

If we multiply the inequality (2.11) by $\frac{x_{i+1}-x_i}{b-a}$ and sum over i from 0 to $n-1$ then we get

$$\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{b-a}\exp\left(\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \ln f(x)\,dx\right) \leq \frac{1}{b-a}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \sqrt{f(x)\,f(x_i+x_{i+1}-x)}dx \leq \frac{1}{b-a}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)\,dx = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.12}$$

Making use of (2.9), (2.10) and (2.12) we get the desired result (2.8). ∎

COROLLARY 2.6. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$ and $y\in[a,b]$, then*

$$\exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \leq \frac{1}{b-a}\left[\int_a^y \sqrt{f(x)\,f(a+y-x)}dx + \int_y^b \sqrt{f(x)\,f(b+y-x)}dx\right] \leq \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.13}$$

The following result also holds:

THEOREM 2.7 (Dragomir, 2015 [**8**]). *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$. Then for any $p>0$ we have the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \leq \left(\frac{1}{b-a}\int_a^b f^p(x)\,f^p(a+b-x)\,dx\right)^{\frac{1}{2p}} \leq \left(\frac{1}{b-a}\int_a^b f^{2p}(x)\,dx\right)^{\frac{1}{2p}} \leq \begin{cases} [L_{2p-1}(f(a),f(b))]^{1-\frac{1}{2p}}\,[L(f(a),f(b))]^{\frac{1}{2p}}, & p\neq\frac{1}{2}; \\ L(f(a),f(b)), & p=\frac{1}{2}. \end{cases} \tag{2.14}$$

If $p \in \left(0, \frac{1}{2}\right)$, *then we have*

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \leq \left(\frac{1}{b-a}\int_a^b f^p(x)\,f^p(a+b-x)\,dx\right)^{\frac{1}{2p}} \leq \left(\frac{1}{b-a}\int_a^b f^{2p}(x)\,dx\right)^{\frac{1}{2p}} \leq \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.15}$$

PROOF. If f is a log-convex function on $[a,b]$ then f^{2p} is log-convex on $[a,b]$ for $p>0$ and by (1.6) we have

$$f^{2p}\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f^{2p}(x)\,dx\right] \leq \frac{1}{b-a}\int_a^b f^p(x)\,f^p(a+b-x)\,dx \leq \frac{1}{b-a}\int_a^b f^{2p}(x)\,dx \leq L\left(f^{2p}(a), f^{2p}(b)\right). \tag{2.16}$$

Taking the power $\frac{1}{2p}$ in (2.16) we get

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \leq \left(\frac{1}{b-a}\int_a^b f^p(x)\,f^p(a+b-x)\,dx\right)^{\frac{1}{2p}} \leq \left(\frac{1}{b-a}\int_a^b f^{2p}(x)\,dx\right)^{\frac{1}{2p}} \leq \left[L\left(f^{2p}(a), f^{2p}(b)\right)\right]^{\frac{1}{2p}}. \tag{2.17}$$

Observe that, for $p \neq \frac{1}{2}$,

$$\left[L\left(f^{2p}(a), f^{2p}(b)\right)\right]^{\frac{1}{2p}} = \left[\frac{f^{2p}(a) - f^{2p}(b)}{\ln f^{2p}(a) - \ln f^{2p}(b)}\right]^{\frac{1}{2p}} = \left[L_{2p-1}(f(a), f(b))\right]^{1-\frac{1}{2p}}\left[L(f(a), f(b))\right]^{\frac{1}{2p}}$$

and by (2.17) we get the desired result (2.14).

The last inequality in (2.15) follows by the following integral inequality for power $q \in (0,1)$, namely

$$\frac{1}{b-a}\int_a^b f^q(x)\,dx \leq \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^q,$$

that follows by Jensen's inequality for concave functions. ∎

REMARK 2.2. If we take in (2.14) $p = 1,$ then we get

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \leq \left(\frac{1}{b-a}\int_a^b f(x) f(a+b-x)\,dx\right)^{\frac{1}{2}} \leq \left(\frac{1}{b-a}\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}} \leq [A(f(a), f(b))]^{\frac{1}{2}}[L(f(a), f(b))]^{\frac{1}{2}}. \tag{2.18}$$

If we take $p = \frac{1}{4}$ in (2.15), then we get

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \leq \left(\frac{1}{b-a}\int_a^b \sqrt[4]{f(x) f(a+b-x)\,dx}\right)^2 \leq \left(\frac{1}{b-a}\int_a^b \sqrt{f(x)}dx\right)^2 \leq \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.19}$$

This improves the inequality (1.7).

3. RELATED INEQUALITIES FOR LOG-CONVEX FUNCTIONS

In this section we establish some related results for log-convex functions.

THEOREM 3.1 (Dragomir, 2015 [**8**]). *Let $f : [a, b] \to (0, \infty)$ be a log-convex function on $[a, b]$. Then for any $x \in [a, b]$ we have*

$$f(b)(b-x) + f(a)(x-a) - \int_a^b f(y)\,dy \geq \int_a^b f(y) \ln f(y)\,dy - \ln f(x)\int_a^b f(y)\,dy. \tag{3.1}$$

In particular,

$$\frac{f(b)+f(a)}{2} - \frac{1}{b-a}\int_a^b f(y)\,dy \geq \frac{1}{b-a}\int_a^b f(y)\ln f(y)\,dy - \ln f\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b f(y)\,dy, \tag{3.2}$$

$$\frac{f(b)\sqrt{b} + f(a)\sqrt{a}}{\sqrt{b}+\sqrt{a}} - \frac{1}{b-a}\int_a^b f(y)\,dy \geq \frac{1}{b-a}\int_a^b f(y)\ln f(y)\,dy - \ln f\left(\sqrt{ab}\right)\frac{1}{b-a}\int_a^b f(y)\,dy \tag{3.3}$$

and

$$\frac{f(b)\,b + f(a)\,a}{a+b} - \frac{1}{b-a}\int_a^b f(y)\,dy \geq \frac{1}{b-a}\int_a^b f(y)\ln f(y)\,dy - \ln f\left(\frac{2ab}{a+b}\right)\frac{1}{b-a}\int_a^b f(y)\,dy. \tag{3.4}$$

PROOF. Since the function $\ln f$ is convex on $[a,b]$, then by the gradient inequality we have

$$\ln f(x) - \ln f(y) \geq \frac{f'_+(y)}{f(y)}(x-y) \tag{3.5}$$

for any $x \in [a,b]$ and $y \in (a,b)$.

If we multiply (3.5) by $f(y) > 0$ and integrate on $[a,b]$ over y we get

$$\ln f(x) \int_a^b f(y)\,dy - \int_a^b f(y) \ln f(y)\,dy \geq f(b)(x-b) + f(a)(a-x) + \int_a^b f(y)\,dy,$$

which is equivalent to (3.1).

The inequality (3.2) follows by (3.1) on taking $x = \frac{a+b}{2}$.

If we take in (3.1) $x = \sqrt{ab}$, then we get

$$f(b)\sqrt{b}\left(\sqrt{b}-\sqrt{a}\right) + f(a)\sqrt{a}\left(\sqrt{b}-\sqrt{a}\right) - \int_a^b f(y)\,dy \geq \int_a^b f(y)\ln f(y)\,dy - \ln f\left(\sqrt{ab}\right)\int_a^b f(y)\,dy,$$

which is equivalent to (3.3).

If we take in (3.1) $x = \frac{2ab}{a+b}$, then we get

$$f(b)\,b\left(\frac{b-a}{a+b}\right) + f(a)\,a\left(\frac{b-a}{a+b}\right) - \int_a^b f(y)\,dy \geq \int_a^b f(y)\ln f(y)\,dy - \ln f\left(\frac{2ab}{a+b}\right)\int_a^b f(y)\,dy,$$

which is equivalent to (3.4). ∎

COROLLARY 3.2. *Let* $f : [a,b] \to (0,\infty)$ *be a log-convex function on* $[a,b]$. *Then*

$$\frac{f(b)+f(a)}{2} - \frac{1}{b-a}\int_a^b f(y)\,dy \geq \int_a^b f(y)\ln f(y)\,dy - \int_a^b f(y)\,dy \frac{1}{b-a}\int_a^b \ln f(y)\,dy \geq 0. \tag{3.6}$$

PROOF. If we take the integral mean over x in (3.1), then we get

$$\frac{1}{b-a}\int_a^b [f(b)(b-x) + f(a)(x-a)]\,dx - \int_a^b f(y)\,dy \geq \int_a^b f(y)\ln f(y)\,dy - \int_a^b f(y)\,dy \frac{1}{b-a}\int_a^b \ln f(x)$$

and since

$$\frac{1}{b-a}\int_a^b [f(b)(b-x) + f(a)(x-a)]\,dx = \frac{f(b)+f(a)}{2} - \frac{1}{b-a}\int_a^b f(y)\,dy$$

then the first inequality in (3.6) is proved.

Since $\ln$ is an increasing function on $(0,\infty)$, then we have

$$(f(x) - f(y))(\ln f(x) - \ln f(y)) \geq 0$$

for any $x, y \in [a,b]$, showing that the functions f and $\ln f$ are synchronous on $[a,b]$.

By making use of the Čebyšev integral inequality for synchronous functions $g, h : [a, b] \to \mathbb{R}$, namely

$$\frac{1}{b-a}\int_a^b g(x) h(x) dx \geq \frac{1}{b-a}\int_a^b g(x) dx \frac{1}{b-a}\int_a^b h(x) dx,$$

then we have

$$\frac{1}{b-a}\int_a^b f(x) \ln f(x) dx \geq \frac{1}{b-a}\int_a^b f(x) dx \frac{1}{b-a}\int_a^b \ln f(x) dx,$$

which proves the last part of (3.6). ∎

The inequality (3.6) improves the well know result for convex functions

$$\frac{f(b)+f(a)}{2} \geq \frac{1}{b-a}\int_a^b f(y) dy.$$

We have:

COROLLARY 3.3. *Let* $f : [a, b] \to (0, \infty)$ *be a log-convex function on* $[a, b]$. *If* $f(a) \neq f(b)$ *and*

$$\alpha_f := \frac{\int_a^b f'(y) y dy}{\int_a^b f'(y) dy} = \frac{bf(b) - af(a) - \int_a^b f(y) dy}{f(b) - f(a)} \in [a, b], \tag{3.7}$$

then

$$\ln f(\alpha_f) \geq \frac{\int_a^b f(y) \ln f(y) dy}{\int_a^b f(y) dy}. \tag{3.8}$$

PROOF. Follows from (3.1) by observing that

$$f(b)(b - \alpha_f) + f(a)(\alpha_f - a) = \int_a^b f(y) dy.$$

∎

REMARK 3.1. We observe that if $f : [a, b] \to (0, \infty)$ is nondecreasing with $f(a) \neq f(b)$ the condition (3.7) is satisfied.

We also have:

COROLLARY 3.4. *Let* $f : [a, b] \to (0, \infty)$ *be a log-convex function on* $[a, b]$. *Then*

$$\begin{aligned} &f(b)\left(b - \frac{\int_a^b y f(y) dy}{\int_a^b f(y) dy}\right) + f(a)\left(\frac{\int_a^b y f(y) dy}{\int_a^b f(y) dy} - a\right) - \int_a^b f(y) dy \\ &\geq \int_a^b f(y) \ln f(y) dy - \int_a^b f(y) dy \ln f\left(\frac{\int_a^b y f(y) dy}{\int_a^b f(y) dy}\right) \geq 0. \end{aligned} \tag{3.9}$$

PROOF. The first inequality follows by (3.1) on taking

$$x = \frac{\int_a^b y f(y) dy}{\int_a^b f(y) dy} \in [a, b]$$

since $f(y) > 0$ for any $y \in [a, b]$.

By Jensen's inequality for the convex function $\ln f$ and the positive weight f we have

$$\frac{\int_a^b f(y) \ln f(y) dy}{\int_a^b f(y) dy} \geq f\left(\frac{\int_a^b f(y) y dy}{\int_a^b f(y) dy}\right),$$

which proves the second inequality in (3.9). ∎

4. FURTHER RESULTS FOR LOG-CONVEX FUNCTIONS

The following result holds.

THEOREM 4.1 (Dragomir, 2015 [**9**]). *Let $f : [a,b] \to (0,\infty)$ be a log-convex function. Then for every $t \in [0,1]$ we have*

$$\int_a^b f(x)\,dx \geq \int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx \geq \begin{cases} \frac{1}{1-2t}\displaystyle\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du, \text{ if } t\neq \frac{1}{2} \\ (b-a)\,f\left(\frac{a+b}{2}\right), \text{ if } t=\frac{1}{2}. \end{cases} \tag{4.1}$$

PROOF. The cases $t = 0, \frac{1}{2}, 1$ are obvious. Assume that $t \in (0,1) \setminus \left\{\frac{1}{2}\right\}$. By the log-convexity of f we have

$$[f(x)]^{1-t}[f(a+b-x)]^t \geq f((1-2t)x + t(a+b)) \tag{4.2}$$

for any $x \in [a,b]$.

Integrating the inequality (4.2) over x on $[a,b]$ we have

$$\int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx \geq \int_a^b f((1-2t)x + t(a+b))\,dx.$$

Since $t \neq \frac{1}{2}$, then $u := (1-2t)x + t(a+b)$ is a change of variable with $du = (1-2t)\,dx$. For $x = a$ we get $u = (1-t)a + tb$ and for $x = b$ we get $u = ta + (1-t)b$. Therefore

$$\int_a^b f((1-2t)x + t(a+b))\,dx = \frac{1}{1-2t}\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du$$

and the second inequality in (4.1) is proved.

By the Hölder integral inequality for $p = \frac{1}{1-t}$, $q = \frac{1}{t}$ we have

$$\begin{aligned}\int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx &\leq \left(\int_a^b \left([f(x)]^{1-t}\right)^{\frac{1}{1-t}}dx\right)^{1-t}\left(\int_a^b \left([f(a+b-x)]^{t}\right)^{\frac{1}{t}}dx\right)^{t} \\ &= \int_a^b f(x)\,dx,\end{aligned}$$

which proves the first inequality in (4.1). ∎

COROLLARY 4.2. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function. Then for every $t\in[0,1]\setminus\left\{\frac{1}{2}\right\}$ we have for $p>0$ that*

$$\begin{aligned}
&\left(\frac{1}{b-a}\int_a^b f^{2p}(x)\,dx\right)^{\frac{1}{2p}} \\
&\geq\left(\frac{1}{b-a}\int_a^b \left[f^{2p}(x)\right]^{1-t}\left[f^{2p}(a+b-x)\right]^{t}dx\right)^{\frac{1}{2p}} \\
&\geq\left(\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f^{2p}(u)\,du\right)^{\frac{1}{2p}} \\
&\geq\left(\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f^{p}(u)\,f^{p}(a+b-u)\,dx\right)^{\frac{1}{2p}} \\
&\geq\exp\left[\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b}\ln f(u)\,du\right]\geq f\left(\frac{a+b}{2}\right).
\end{aligned}\tag{4.3}$$

PROOF. Follows from Theorem 4.1 applied for the log-convex function f^{2p} with $p>0$ and by Theorem 1.4 applied for the interval $[(1-t)a+tb,ta+(1-t)b]$ when $t\in\left(0,\frac{1}{2}\right)$or $[ta+(1-t)b,(1-t)a+tb]))$ when $t\in\left(\frac{1}{2},1\right)$. ∎

If we take $p=1$ in (4.3), then we get

$$\begin{aligned}
\left(\frac{1}{b-a}\int_a^b f^{2}(x)\,dx\right)^{\frac{1}{2}} &\geq\left(\frac{1}{b-a}\int_a^b \left[f^{2}(x)\right]^{1-t}\left[f^{2}(a+b-x)\right]^{t}dx\right)^{\frac{1}{2}} \\
&\geq\left(\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f^{2}(u)\,du\right)^{\frac{1}{2}}
\end{aligned}\tag{4.4}$$

$$\begin{aligned}
&\geq\left(\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,f(a+b-u)\,dx\right)^{\frac{1}{2}} \\
&\geq\exp\left[\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b}\ln f(u)\,du\right]\geq f\left(\frac{a+b}{2}\right).
\end{aligned}$$

If we take $p = \frac{1}{2}$ in (4.3), then we get

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)\,dx \\
&\geq \frac{1}{b-a}\int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx \\
&\geq \frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du \\
&\geq \frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} \sqrt{f(u)\,f(a+b-u)}dx \\
&\geq \exp\left[\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} \ln f(u)\,du\right] \geq f\left(\frac{a+b}{2}\right).
\end{aligned}
\tag{4.5}
$$

COROLLARY 4.3. *Let $f : [a,b] \to (0,\infty)$ be a log-convex function. Then for every $t \in [0,1] \setminus \left\{\frac{1}{2}\right\}$ we have for $p \in \left(0, \frac{1}{2}\right)$ that*

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)\,dx &\geq \frac{1}{b-a}\int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx \\
&\geq \frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du \\
&\geq \left(\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f^{2p}(u)\,du\right)^{\frac{1}{2p}} \\
&\geq \left(\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f^{p}(u)\,f^{p}(a+b-u)\,du\right)^{\frac{1}{2p}} \\
&\geq \exp\left[\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} \ln f(u)\,du\right] \geq f\left(\frac{a+b}{2}\right).
\end{aligned}
\tag{4.6}
$$

Follows by Theorem 4.1 and Theorem 1.4 for $p \in \left(0, \frac{1}{2}\right)$.

If we take $p=\frac{1}{4}$ in (4.6), then we get

$$\begin{aligned}(4.7)\qquad \frac{1}{b-a}\int_a^b f(x)\,dx &\geq \frac{1}{b-a}\int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx \\ &\geq \frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du \\ &\geq \left(\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b}\sqrt{f(u)}du\right)^2 \\ &\geq \left(\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b}\sqrt[4]{f(u)f(a+b-u)}du\right)^2 \\ &\geq \exp\left[\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b}\ln f(u)\,du\right] \geq f\left(\frac{a+b}{2}\right).\end{aligned}$$

5. WEIGHTED INEQUALITIES

We have the following generalized weighted version of the inequality (1.5).

THEOREM 5.1 (Dragomir, 2015 [**9**]). *Let $f:[a,b]\to(0,\infty)$ be a log-convex function. If $w:[a,b]\to[0,\infty)$ is integrable and $\int_a^b w(x)\,dx>0$, then*

$$(5.1)\qquad f\left(\frac{a+b}{2}\right)\leq\left(\frac{\int_a^b w(x)f^p(x)f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}}\leq\sqrt{f(a)f(b)}$$

for any $p>0$.

In particular, we have

$$(5.2)\qquad f\left(\frac{a+b}{2}\right)\leq\left(\frac{\int_a^b w(x)f(x)f(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2}}\leq\sqrt{f(a)f(b)}.$$

PROOF. We know that, see [**22**] or [**23**, p. 198], if g is log-convex, then

$$(5.3)\qquad g\left(\frac{a+b}{2}\right)\leq\sqrt{g(x)g(a+b-x)}\leq\sqrt{g(a)g(b)}$$

for any $x\in[a,b]$.

For any $p>0$ the function f^{2p} is log-convex and by (5.3) we have

$$(5.4)\qquad f^{2p}\left(\frac{a+b}{2}\right)\leq f^p(x)f^p(a+b-x)\leq f^p(a)f^p(b)$$

for any $x\in[a,b]$.

If we multiply (5.4) by $w(x)\geq 0$ and integrate, then we get

$$(5.5)\qquad f^{2p}\left(\frac{a+b}{2}\right)\leq\frac{\int_a^b w(x)f^p(x)f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\leq f^p(a)f^p(b).$$

Taking the power $\frac{1}{2p}$ in (5.5) we obtain the desired result (5.1). ∎

We also have the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b w(x)\sqrt{f(x)f(a+b-x)}dx}{\int_a^b w(x)\,dx} \leq \sqrt{f(a)f(b)}, \tag{5.6}$$

that is a weighted version of (1.5).

If we take $p = \frac{1}{4}$ in (5.1), then we get

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b w(x)\sqrt[4]{f(x)f(a+b-x)}dx}{\int_a^b w(x)\,dx}\right)^2 \leq \sqrt{f(a)f(b)}. \tag{5.7}$$

Using Jensen's inequality for the power $p \geq 1$ ($p \in (0,1)$), namely

$$\left(\frac{\int_a^b w(x)\,g(x)\,dx}{\int_a^b w(x)\,dx}\right)^p \leq (\geq) \frac{\int_a^b w(x)\,g^p(x)\,dx}{\int_a^b w(x)\,dx},$$

we can state the following more precise result:

COROLLARY 5.2. *Let $f : [a,b] \to (0,\infty)$ be a log-convex function and $w : [a,b] \to [0,\infty)$ be integrable and $\int_a^b w(x)\,dx > 0$.*

If $p \geq 1$, then

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b w(x)\,f(x)\,f(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2}} \tag{5.8}$$
$$\leq \left(\frac{\int_a^b w(x)\,f^p(x)\,f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}} \leq \sqrt{f(a)f(b)}.$$

If $p \in (0,1)$, then

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b w(x)\,f^p(x)\,f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}} \tag{5.9}$$
$$\leq \left(\frac{\int_a^b w(x)\,f(x)\,f(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2}} \leq \sqrt{f(a)f(b)}.$$

REMARK 5.1. Let $f : [a,b] \to (0,\infty)$ be a log-convex function. We observe that if we take in (5.1) $w(x) = f^{-p}(a+b-x)$, $p > 0$, then we get

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b f^p(x)\,dx}{\int_a^b f^{-p}(x)\,dx}\right)^{\frac{1}{2p}} \leq \sqrt{f(a)f(b)} \tag{5.10}$$

for any $p > 0$.

In particular, we have the inequalities

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b f(x)\,dx}{\int_a^b \frac{1}{f(x)}dx}\right)^{\frac{1}{2}} \leq \sqrt{f(a)f(b)}, \tag{5.11}$$

and

$$f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b \sqrt{f(x)}dx}{\int_a^b \frac{1}{\sqrt{f(x)}}dx} \leq \sqrt{f(a)f(b)}. \tag{5.12}$$

THEOREM 5.3 (Dragomir, 2015 [**9**]). *Let* $f:[a,b]\to(0,\infty)$ *be a log-convex function. If* $w:[a,b]\to[0,\infty)$ *is integrable and* $\int_a^b w(x)\,dx>0$, *then*

$$f\left(\frac{\int_a^b w(x)\,xdx}{\int_a^b w(x)\,dx}\right)\le\exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right)\le[f(b)]^{\frac{1}{b-a}\left(\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}-a\right)}[f(a)]^{\frac{1}{b-a}\left(b-\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}\right)}\le\frac{1}{b-a}\left[\left(\frac{\int_a^b xw(x)\,dx}{\int_a^b w(x)\,dx}-a\right)f(b)+\left(b-\frac{\int_a^b xw(x)\,dx}{\int_a^b w(x)\,dx}\right)f(a)\right].\tag{5.13}$$

PROOF. Since $\ln f$ is convex, then by Jensen's inequality we have

$$\ln f\left(\frac{\int_a^b w(x)\,xdx}{\int_a^b w(x)\,dx}\right)\le\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}.\tag{5.14}$$

Taking the exponential in (5.14) we get the first inequality in (5.13).

Since $\ln f$ is convex, then

$$\ln f(x)=\ln f\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\le\frac{x-a}{b-a}\ln f(b)+\frac{b-x}{b-a}\ln f(a)\tag{5.15}$$

for any $x\in[a,b]$.

By taking the weighted integral mean in (5.15) we obtain

$$\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\le\frac{1}{b-a}\left[\left(\frac{\int_a^b xw(x)\,dx}{\int_a^b w(x)\,dx}-a\right)\ln f(b)+\left(b-\frac{\int_a^b xw(x)\,dx}{\int_a^b w(x)\,dx}\right)\ln f(a)\right]=\ln\left([f(b)]^{\frac{1}{b-a}\left(\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}-a\right)}[f(a)]^{\frac{1}{b-a}\left(b-\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}\right)}\right).\tag{5.16}$$

By taking the exponential in (5.16), we get the second inequality in (5.13).

The last part of (5.13) follows by the weighted geometric mean-arithmetic mean inequality. ∎

REMARK 5.2. If we take $w(x)=1$, $x\in[a,b]$ in the first two inequalities (5.13), we recapture (1.4).

We also have the alternative result:

THEOREM 5.4 (Dragomir, 2015 [**9**]). *Let* $f:[a,b]\to(0,\infty)$ *be a log-convex function. If* $w:[a,b]\to[0,\infty)$ *is integrable and* $\int_a^b w(x)\,dx>0$, *then*

$$\begin{aligned}(5.17)\qquad & f\left(\frac{\int_a^b w(x)\,xdx}{\int_a^b w(x)\,dx}\right)\\ &\le \exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right)\\ &\le \frac{\int_a^b w(x)\,f(x)\,dx}{\int_a^b w(x)\,dx}\le\left(\frac{[f(a)]^b}{[f(b)]^a}\right)^{\frac{1}{b-a}}\frac{\int_a^b w(x)\left(\frac{f(b)}{f(a)}\right)^{\frac{x}{b-a}}dx}{\int_a^b w(x)\,dx}\\ &\le \frac{1}{b-a}\left[\left(\frac{\int_a^b xw(x)\,dx}{\int_a^b w(x)\,dx}-a\right)f(b)+\left(b-\frac{\int_a^b xw(x)\,dx}{\int_a^b w(x)\,dx}\right)f(a)\right].\end{aligned}$$

PROOF. Using Jensen's inequality for the exponential function we have

$$\exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right)\le\frac{\int_a^b w(x)\,f(x)\,dx}{\int_a^b w(x)\,dx}$$

and the second inequality in (5.17) is proved.

From (5.15) and the arithmetic mean - geometric mean inequality we have

$$\begin{aligned}(5.18)\qquad f(x)&\le[f(b)]^{\frac{x-a}{b-a}}[f(a)]^{\frac{b-x}{b-a}}=\left(\frac{[f(a)]^b}{[f(b)]^a}\right)^{\frac{1}{b-a}}\left(\frac{f(b)}{f(a)}\right)^{\frac{x}{b-a}}\\ &\le\frac{x-a}{b-a}f(b)+\frac{b-x}{b-a}f(a)\end{aligned}$$

for any $x\in[a,b]$.

By taking the weighted integral mean in (5.18) we get

$$\begin{aligned}\frac{\int_a^b w(x)\,f(x)\,dx}{\int_a^b w(x)\,dx}&\le\left(\frac{[f(a)]^b}{[f(b)]^a}\right)^{\frac{1}{b-a}}\frac{\int_a^b w(x)\left(\frac{f(b)}{f(a)}\right)^{\frac{x}{b-a}}dx}{\int_a^b w(x)\,dx}\\ &\le\frac{1}{b-a}\left[\left(\frac{\int_a^b xw(x)\,dx}{\int_a^b w(x)\,dx}-a\right)f(b)+\left(b-\frac{\int_a^b xw(x)\,dx}{\int_a^b w(x)\,dx}\right)f(a)\right]\end{aligned}$$

and the last part of (5.17) is proved. ∎

REMARK 5.3. If we take $w(x)=1$, $x\in[a,b]$ in (5.17), then we have

$$\begin{aligned}(5.19)\qquad f\left(\frac{a+b}{2}\right)&\le\exp\left(\frac{1}{b-a}\int_a^b\ln f(x)\,dx\right)\\ &\le\frac{1}{b-a}\int_a^b f(x)\,dx\le L(f(a),f(b))\le\frac{f(a)+f(b)}{2}.\end{aligned}$$

6. INEQUALITIES FOR SYMMETRIC WEIGHTS

We say that the weight $w:[a,b]\to[0,\infty)$ is *symmetric* on $[a,b]$ if

$$w(a+b-x)=w(x)\ \text{ for all } x\in[a,b].$$

It is well known that if $f:[a,b]\to\mathbb{R}$ is convex and $w:[a,b]\to[0,\infty)$ is integrable and symmetric on $[a,b]$, then the *Fejér inequality* holds

$$f\left(\frac{a+b}{2}\right)\leq\frac{\int_a^b w(x)f(x)\,dx}{\int_a^b w(x)\,dx}\leq\frac{f(a)+f(b)}{2}. \tag{6.1}$$

If $f:[a,b]\to(0,\infty)$ is a log-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ is integrable and symmetric on $[a,b]$, then by (6.1) we have

$$\ln f\left(\frac{a+b}{2}\right)\leq\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\leq\frac{\ln f(a)+\ln f(b)}{2},$$

which is equivalent to

$$f\left(\frac{a+b}{2}\right)\leq\exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right)\leq\sqrt{f(a)f(b)}. \tag{6.2}$$

THEOREM 6.1 (Dragomir, 2015 [**9**]). *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ be integrable and symmetric on $[a,b]$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right) \\ &\leq \frac{\int_a^b w(x)\sqrt{f(x)f(a+b-x)}dx}{\int_a^b w(x)\,dx}\leq\frac{\int_a^b w(x)f(x)\,dx}{\int_a^b w(x)\,dx}. \end{aligned} \tag{6.3}$$

PROOF. By Jensen's integral inequality for the exponential we have

$$\begin{aligned} &\exp\left(\frac{\int_a^b w(x)\ln\sqrt{f(x)f(a+b-x)}dx}{\int_a^b w(x)\,dx}\right) \\ &\leq\frac{\int_a^b w(x)\exp\left(\ln\sqrt{f(x)f(a+b-x)}\right)dx}{\int_a^b w(x)\,dx} \\ &=\frac{\int_a^b w(x)\sqrt{f(x)f(a+b-x)}dx}{\int_a^b w(x)\,dx}. \end{aligned} \tag{6.4}$$

Observe, by the symmetry of w, that

$$\begin{aligned} &\int_a^b w(x)\ln\sqrt{f(x)f(a+b-x)}dx \\ &=\frac{1}{2}\left[\int_a^b w(x)\ln f(x)\,dx+\int_a^b w(x)\ln f(a+b-x)\,dx\right] \\ &=\frac{1}{2}\left[\int_a^b w(x)\ln f(x)\,dx+\int_a^b w(a+b-x)\ln f(a+b-x)\,dx\right] \\ &=\int_a^b w(x)\ln f(x)\,dx \end{aligned}$$

since, obviously

$$\int_a^b w(a+b-x)\ln f(a+b-x)\,dx=\int_a^b w(x)\ln f(x)\,dx.$$

By (6.4) we then get the second inequality in (6.3).

By Cauchy-Buniakovski-Schwarz integral inequality we also have

$$\int_a^b w(x)\sqrt{f(x)f(a+b-x)}dx \le \sqrt{\int_a^b w(x)f(x)\,dx}\sqrt{\int_a^b w(x)f((a+b-x))\,dx} = \int_a^b w(x)f(x)\,dx,$$

which proves the third inequality in (6.3). ∎

The above inequality (6.2) may be generalized as follows by replacing f with f^{2p} for $p > 0.$

COROLLARY 6.2. *Let* $f : [a,b] \to (0,\infty)$ *be a log-convex function on* $[a,b]$ *and* $w : [a,b] \to [0,\infty)$ *be integrable and symmetric on* $[a,b]$ *. Then for any* $p > 0$ *we have*

$$f\left(\frac{a+b}{2}\right) \le \exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right) \le \left(\frac{\int_a^b w(x)f^p(x)f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}} \le \left(\frac{\int_a^b w(x)f^{2p}(x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}}. \tag{6.5}$$

REMARK 6.1. We observe that for $p \ge 1$ we have

$$f\left(\frac{a+b}{2}\right) \le \exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right) \le \left(\frac{\int_a^b w(x)f(x)f(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2}} \le \left(\frac{\int_a^b w(x)f^p(x)f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}} \le \left(\frac{\int_a^b w(x)f^{2p}(x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}} \tag{6.6}$$

and for $p \in (0,1)$

$$f\left(\frac{a+b}{2}\right) \leq \exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right) \tag{6.7}$$
$$\leq \left(\frac{\int_a^b w(x) f^p(x) f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}}$$
$$\leq \left(\frac{\int_a^b w(x) f(x) f(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2}}$$
$$\leq \left(\frac{\int_a^b w(x) f^2(x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2}}.$$

Finally, we have:

THEOREM 6.3. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ be integrable and symmetric on $[a,b]$. Then for any $p>0$ we have*

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b f^p(x)\,w(x)\,dx}{\int_a^b \frac{w(x)dx}{f^p(x)}}\right)^{\frac{1}{2p}} \leq \sqrt{f(a)\,f(b)}. \tag{6.8}$$

PROOF. From (5.4) we have

$$f^{2p}\left(\frac{a+b}{2}\right)\frac{1}{f^p(a+b-x)} \leq f^p(x) \leq f^p(a)\,f^p(b)\,\frac{1}{f^p(a+b-x)} \tag{6.9}$$

for any $x\in[a,b]$.

If we multiply by $w(x)\geq 0$ and integrate on $[a,b]$, then we get

$$f^{2p}\left(\frac{a+b}{2}\right)\int_a^b \frac{w(x)\,dx}{f^p(a+b-x)} \leq \int_a^b f^p(x)\,w(x)\,dx$$
$$\leq f^p(a)\,f^p(b)\int_a^b \frac{w(x)\,dx}{f^p(a+b-x)}.$$

Since, by symmetry of w we have

$$\int_a^b \frac{w(x)\,dx}{f^p(a+b-x)} = \int_a^b \frac{w(a+b-x)\,dx}{f^p(a+b-x)} = \int_a^b \frac{w(x)\,dx}{f^p(x)},$$

which implies that

$$f^{2p}\left(\frac{a+b}{2}\right)\int_a^b \frac{w(x)\,dx}{f^p(x)} \leq \int_a^b f^p(x)\,w(x)\,dx \leq f^p(a)\,f^p(b)\int_a^b \frac{w(x)\,dx}{f^p(x)}.$$

and the inequality (6.8) is proved. ∎

CHAPTER 3

Inequalities for AH-Convex Functions

1. Preliminary Facts on AH-convex Functions

Let X be a linear space and C a convex subset in X. A function $f : C \to \mathbb{R}\setminus\{0\}$ is called *AH-convex (concave)* on the convex set C if the following inequality holds

$$f((1-\lambda)x+\lambda y) \leq (\geq) \frac{f(x)f(y)}{(1-\lambda)f(y)+\lambda f(x)} \tag{AH}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

An important case which provides many examples is that one in which the function is assumed to be positive for any $x \in C$. In that situation the inequality (AH) is equivalent to

$$(1-\lambda)\frac{1}{f(x)} + \lambda\frac{1}{f(y)} \leq (\geq) \frac{1}{f((1-\lambda)x+\lambda y)}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

Therefore we can state the following fact:

CRITERION 1. *Let X be a linear space and C a convex subset in X. The function $f : C \to (0,\infty)$ is AH-convex (concave) on C if and only if $\frac{1}{f}$ is concave (convex) on C in the usual sense.*

If we apply the Hermite-Hadamard inequality (0.2) for the function $\frac{1}{f}$ then we state the following result:

PROPOSITION 1.1 (Dragomir, 2014 [**10**]). *Let X be a linear space and C a convex subset in X. If the function $f : C \to (0,\infty)$ is AH-convex (concave) on C, then*

$$\frac{f(x)+f(y)}{2f(x)f(y)} \leq (\geq) \int_0^1 \frac{d\lambda}{f((1-\lambda)x+\lambda y)} \leq (\geq) \frac{1}{f\left(\frac{x+y}{2}\right)} \tag{1.1}$$

for any $x, y \in C$.

Motivated by the above results, in this chapter we present some Hermite-Hadamard type inequalities for *AH*-convex (concave) functions, first in the general setting of linear spaces and then in the particular case of functions of a real variable.

2. Some Hermite-Hadamard Type Inequalities

The following result holds:

THEOREM 2.1 (Dragomir, 2014 [**10**]). *Let X be a linear space and C a convex subset in X. If the function $f : C \to (0,\infty)$ is AH-convex (concave) on C, then for any $x, y \in C$ we have*

$$\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda \leq (\geq) \frac{G^2(f(x),f(y))}{L(f(x),f(y))}. \tag{2.1}$$

where $L(a,b)$ is the Logarithmic mean of positive numbers a, b and the geometric mean is $G(a,b)=\sqrt{ab}$.

PROOF. Let $x, y \in C$ with $x \neq y$. If $f : C \to (0, \infty)$ is *AH*-convex (concave) on C, then $\frac{1}{f}$ is concave (convex) on C. This implies that the function

$$\varphi_{x,y} : [0,1] \to (0,\infty), \varphi_{x,y}(t) = \frac{1}{f((1-\lambda)x + \lambda y)}$$

is concave (convex) on $[0,1]$ and therefore continuous on $(0,1)$ with $\varphi_{x,y}(0) = \frac{1}{f(x)}$ and $\varphi_{x,y}(1) = \frac{1}{f(y)}$. The function $[0,1] \ni t \mapsto f((1-t)x + ty)$ is continuous on $(0,1)$ and since $f(x), f(y) > 0$ are finite, then the Lebesgue integral $\int_0^1 f((1-t)x + ty)\, dt$ exists and by (AH) we have

$$\int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda \leq (\geq) f(x) f(y) \int_0^1 \frac{d\lambda}{(1-\lambda) f(y) + \lambda f(x)}. \tag{2.2}$$

If $f(y) = f(x)$, then

$$\int_0^1 \frac{d\lambda}{(1-\lambda) f(y) + \lambda f(x)} = \frac{1}{f(y)}.$$

If $f(y) \neq f(x)$, then by changing the variable $u = \lambda(f(x) - f(y)) + f(y)$ we have

$$\int_0^1 \frac{d\lambda}{(1-\lambda) f(y) + \lambda f(x)} = \frac{\ln f(x) - \ln f(y)}{f(x) - f(y)} = \frac{1}{L(f(x), f(y))}.$$

By the use of (2.2) we get the desired result (2.1). ∎

REMARK 2.1. Using the following well known inequalities

$$H(a,b) \leq G(a,b) \leq L(a,b)$$

we have

$$\int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda \leq \frac{G^2(f(x), f(y))}{L(f(x), f(y))} \leq G(f(x), f(y)) \tag{2.3}$$

for any $x, y \in C$, provided that $f : C \to (0, \infty)$ is *AH*-convex.

If $f : C \to (0, \infty)$ is *AH*-concave, then

$$\int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda \geq \frac{G^2(f(x), f(y))}{L(f(x), f(y))} \geq \frac{G(f(x), f(y))}{L(f(x), f(y))} H(f(x), f(y)) \tag{2.4}$$

for any $x, y \in C$.

THEOREM 2.2 (Dragomir, 2014 [**10**]). *Let X be a linear space and C a convex subset in X. If the function $f : C \to (0, \infty)$ is AH-convex (concave) on C, then for any $x, y \in C$ we have*

$$f\left(\frac{x+y}{2}\right) \leq (\geq) \frac{\int_0^1 f((1-\lambda)x + \lambda y) f(\lambda x + (1-\lambda) y)\, d\lambda}{\int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda}. \tag{2.5}$$

PROOF. By the definition of *AH*-convexity (concavity) we have

$$f\left(\frac{u+v}{2}\right) \leq (\geq) \frac{2 f(u) f(v)}{f(u) + f(v)} \tag{2.6}$$

for any $u, v \in C$.

Let $x, y \in C$ and $\lambda \in [0,1]$. If we take in (2.6) $u = (1-\lambda)x + \lambda y$ and $v = \lambda x + (1-\lambda) y$, then we get

$$\begin{aligned} &\frac{1}{2} f\left(\frac{x+y}{2}\right) [f((1-\lambda)x + \lambda y) + f(\lambda x + (1-\lambda) y)] \\ &\leq (\geq) f((1-\lambda)x + \lambda y) f(\lambda x + (1-\lambda) y). \end{aligned} \tag{2.7}$$

Integrating the inequality on $[0,1]$ over $\lambda \in [0,1]$ and taking into account that

$$\int_0^1 f((1-\lambda)x + \lambda y)\,d\lambda = \int_0^1 f(\lambda x + (1-\lambda)y)\,d\lambda$$

we deduce from (2.7) the desired result (2.5). ∎

REMARK 2.2. By the Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\begin{aligned}(2.8)\qquad &\int_0^1 f((1-\lambda)x + \lambda y) f(\lambda x + (1-\lambda)y)\,d\lambda \\ &\le \left[\int_0^1 f^2((1-\lambda)x + \lambda y)\,d\lambda \int_0^1 f^2(\lambda x + (1-\lambda)y)\,d\lambda\right]^{1/2} \\ &= \int_0^1 f^2((1-\lambda)x + \lambda y)\,d\lambda\end{aligned}$$

for any $x, y \in C$.

If the function $f : C \to (0,\infty)$ is *AH*-convex on C, then we have

$$\begin{aligned}(2.9)\qquad f\left(\frac{x+y}{2}\right) &\le \frac{\int_0^1 f((1-\lambda)x + \lambda y) f(\lambda x + (1-\lambda)y)\,d\lambda}{\int_0^1 f((1-\lambda)x + \lambda y)\,d\lambda} \\ &\le \frac{\int_0^1 f^2((1-\lambda)x + \lambda y)\,d\lambda}{\int_0^1 f((1-\lambda)x + \lambda y)\,d\lambda}.\end{aligned}$$

If the function $\psi_{x,y}(t) = f((1-t)x + ty)$, for some given $x, y \in C$ with $x \ne y$, is monotonic nondecreasing on $[0,1]$, then $\chi_{x,y}(t) = f(tx + (1-t)y)$ is monotonic nonincreasing on $[0,1]$ and by Čebyšev's inequality for monotonic opposite functions we have

$$\begin{aligned}&\int_0^1 f((1-\lambda)x + \lambda y) f(\lambda x + (1-\lambda)y)\,d\lambda \\ &\le \int_0^1 f((1-\lambda)x + \lambda y)\,d\lambda \int_0^1 f(\lambda x + (1-\lambda)y)\,d\lambda \\ &= \left(\int_0^1 f((1-\lambda)x + \lambda y)\,d\lambda\right)^2.\end{aligned}$$

So, for some given $x, y \in C$ with $x \ne y$, if $\psi_{x,y}(t) = f((1-t)x + ty)$ is monotonic nondecreasing (nonincreasing) on $[0,1]$ and if the function $f : C \to (0,\infty)$ is *AH*-convex on C, then we have

$$\begin{aligned}(2.10)\qquad f\left(\frac{x+y}{2}\right) &\le \frac{\int_0^1 f((1-\lambda)x + \lambda y) f(\lambda x + (1-\lambda)y)\,d\lambda}{\int_0^1 f((1-\lambda)x + \lambda y)\,d\lambda} \\ &\le \int_0^1 f((1-\lambda)x + \lambda y)\,d\lambda.\end{aligned}$$

3. MORE RESULTS FOR SCALAR CASE

If the function f is defined on an interval I and $a, b \in I$ with $a < b$, then

$$\int_0^1 f((1-\lambda)x + \lambda y)\,d\lambda = \frac{1}{b-a}\int_a^b f(t)\,dt$$

and the inequalities (1.1), (2.1) and (2.5) can be written as [**10**]

$$\frac{f(a)+f(b)}{2f(a)f(b)} \leq (\geq) \frac{1}{b-a}\int_a^b \frac{1}{f(t)}dt \leq (\geq) \frac{1}{f\left(\frac{a+b}{2}\right)}, \tag{3.1}$$

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq (\geq) \frac{G^2(f(a),f(b))}{L(f(a),f(b))}, \tag{3.2}$$

and

$$f\left(\frac{a+b}{2}\right) \leq (\geq) \frac{\int_a^b f(t)\,f(a+b-t)\,dt}{\int_a^b f(t)\,dt}, \tag{3.3}$$

respectively, where $f : I \to (0,\infty)$ is assumed to be *AH*-convex (concave) on I.

The following proposition holds:

PROPOSITION 3.1 (Dragomir, 2014 [**10**]). *Let $f : I \to (0,\infty)$ be AH-convex (concave) on I. Let $x, y \in \mathring{I}$, the interior of I, then there exists $\varphi(y) \in \left[f'_-(y), f'_+(y)\right]$ such that*

$$\frac{f(y)}{f(x)} - 1 \leq (\geq) \frac{\varphi(y)}{f(y)}(y-x) \tag{3.4}$$

holds.

PROOF. Let $x, y \in \mathring{I}$. Since the function $\frac{1}{f}$ is concave (convex) then the lateral derivatives $f'_-(y), f'_+(y)$ exist for $y \in \mathring{I}$ and $\left(\frac{1}{f}\right)'_{-(+)}(y) = -\frac{f'_{-(+)}(y)}{f^2(y)}$.

Since $\frac{1}{f}$ is concave (convex) then we have the *gradient inequality*

$$\frac{1}{f(y)} - \frac{1}{f(x)} \geq (\leq) \lambda(y)(y-x) = -\lambda(y)(x-y)$$

with $\lambda(y) \in \left[-\frac{f'_+(y)}{f^2(y)}, -\frac{f'_-(y)}{f^2(y)}\right]$, which is equivalent to

$$\frac{1}{f(y)} - \frac{1}{f(x)} \geq (\leq) \frac{\varphi(y)}{f^2(y)}(x-y) \tag{3.5}$$

with $\varphi(y) \in \left[f'_-(y), f'_+(y)\right]$.

The inequality (3.5) is equivalent to (3.4). ∎

COROLLARY 3.2. *Let $f : I \to (0,\infty)$ be AH-convex (concave) on I. If f is differentiable on $\mathring{I}$ then for any $x, y \in \mathring{I}$, we have*

$$\frac{f(y)}{f(x)} - 1 \leq (\geq) \frac{f'(y)}{f(y)}(y-x). \tag{3.6}$$

The following result also holds:

THEOREM 3.3 (Dragomir, 2014 [**10**]). *Let $f : I \to (0,\infty)$ be AH-convex (concave) on I. If $a, b \in I$ with $a < b$, then we have the inequality*

$$\frac{1}{b-a}\int_a^b f^2(t)\,dt \leq (\geq) \left[\frac{b-s}{b-a}f(b) + \frac{s-a}{b-a}f(a)\right] f(s) \tag{3.7}$$

for any $s \in [a,b]$.

In particular, we have

$$\frac{1}{b-a}\int_a^b f^2(t)\,dt \leq (\geq) f\left(\frac{a+b}{2}\right)\frac{f(a)+f(b)}{2} \tag{3.8}$$

and

$$\frac{1}{b-a}\int_a^b f^2(t)\,dt \le (\ge) f(a) f(b). \tag{3.9}$$

PROOF. If the function $f : I \to (0,\infty)$ is *AH*-convex (concave) on I, then the function f is differentiable almost everywhere on I and we have the inequality

$$\frac{f(t)}{f(s)} - 1 \le (\ge) \frac{f'(t)}{f(t)}(t-s) \tag{3.10}$$

for every $s \in [a,b]$ and almost every $t \in [a,b]$.

Multiplying (3.10) by $f(t) > 0$ and integrating over $t \in [a,b]$ we have

$$\frac{1}{f(s)}\int_a^b f^2(t)\,dt - \int_a^b f(t)\,dt \le (\ge) \int_a^b f'(t)(t-s)\,dt. \tag{3.11}$$

Integrating by parts we have

$$\int_a^b f'(t)(t-s)\,dt = f(b)(b-s) + f(a)(s-a) - \int_a^b f(t)\,dt$$

and by (3.11) we get the desired result (3.7).

We observe that (3.8) follows by (3.7) for $s = \frac{a+b}{2}$ while (3.9) follows by (3.7) for either $s = a$ or $s = b$. ∎

REMARK 3.1. By the Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)^2 \le \frac{1}{b-a}\int_a^b f^2(t)\,dt$$

and if we assume that $f : I \to (0,\infty)$ is *AH*-convex on I, then we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \left(\frac{1}{b-a}\int_a^b f^2(t)\,dt\right)^{1/2} \le \sqrt{f\left(\frac{a+b}{2}\right)\frac{f(a)+f(b)}{2}} \tag{3.12}$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \left(\frac{1}{b-a}\int_a^b f^2(t)\,dt\right)^{1/2} \le \sqrt{f(a) f(b)}. \tag{3.13}$$

The following result also holds:

THEOREM 3.4 (Dragomir, 2014 [**10**]). *Let $f : I \to (0,\infty)$ be AH-convex (concave) on I. If $a, b \in I$ with $a < b$, then we have the inequality*

$$\int_a^b \ln f(t)\,dt + \frac{1}{f(s)}\int_a^b f(t)\,dt \le (\ge)\, b-a+(b-s)\ln f(b) + (s-a)\ln f(a) \tag{3.14}$$

for any $s \in [a,b]$.

In particular, we have

$$\frac{1}{b-a}\int_a^b \ln f(t)\,dt + \frac{1}{f\left(\frac{a+b}{2}\right)}\frac{1}{b-a}\int_a^b f(t)\,dt \le (\ge)\, 1 + \ln\sqrt{f(b)\,f(a)} \tag{3.15}$$

and

$$\frac{1}{b-a}\int_a^b \ln f(t)\,dt + \left[\frac{f(b)+f(a)}{2f(a)\,f(b)}\right]\frac{1}{b-a}\int_a^b f(t)\,dt \le (\ge)\, 1 + \ln\sqrt{f(b)\,f(a)}. \tag{3.16}$$

PROOF. Integrating the inequality (3.10) over $t \in [a,b]$ we have

$$\frac{1}{f(s)}\int_a^b f(t)\,dt - (b-a) \le (\ge) \int_a^b \frac{f'(t)}{f(t)}(t-s)\,dt. \tag{3.17}$$

Observe that

$$\int_a^b \frac{f'(t)}{f(t)}(t-s)\,dt = (b-s)\ln f(b) + (s-a)\ln f(a) - \int_a^b \ln f(t)\,dt$$

and by (3.17) we get

$$\frac{1}{f(s)}\int_a^b f(t)\,dt - (b-a) \le (\ge)\, (b-s)\ln f(b) + (s-a)\ln f(a) - \int_a^b \ln f(t)\,dt,$$

which is equivalent to

$$\int_a^b \ln f(t)\,dt + \frac{1}{f(s)}\int_a^b f(t)\,dt \le (\ge)\, b-a + (b-s)\ln f(b) + (s-a)\ln f(a)$$

for any $s \in [a,b]$.

If we take in (3.14) $s = \frac{a+b}{2}$ then we get the desired result (3.15).

If we take in (3.14) $s = a$ and $s = b$ we get

$$\int_a^b \ln f(t)\,dt + \frac{1}{f(a)}\int_a^b f(t)\,dt \le (\ge)\, b-a + (b-a)\ln f(b)$$

and

$$\int_a^b \ln f(t)\,dt + \frac{1}{f(b)}\int_a^b f(t)\,dt \le (\ge)\, b-a + (b-a)\ln f(a),$$

which by addition produces

$$\int_a^b \ln f(t)\,dt + \left[\frac{f(b)+f(a)}{2f(a)\,f(b)}\right]\int_a^b f(t)\,dt \le (\ge)\, b-a + (b-a)\ln\sqrt{f(b)\,f(a)},$$

which is equivalent to (3.16). ∎

REMARK 3.2. We observe that

$$(b-s)\ln f(b)+(s-a)\ln f(a)=0$$

iff

$$s=\frac{b\ln f(b)-a\ln f(a)}{\ln f(b)-\ln f(a)}=\frac{L(f(a),f(b))}{L\left([f(a)]^{a},[f(b)]^{b}\right)}.$$

If

$$s=\frac{L(f(a),f(b))}{L\left([f(a)]^{a},[f(b)]^{b}\right)}\in I$$

then from (3.14) we have

$$\frac{1}{b-a}\int_{a}^{b}\ln f(t)\,dt+\frac{\frac{1}{b-a}\int_{a}^{b}f(t)\,dt}{f\left(\frac{L(f(a),f(b))}{L\left([f(a)]^{a},[f(b)]^{b}\right)}\right)}\leq(\geq)\,1. \tag{3.18}$$

CHAPTER 4

Inequalities for GA-Convex Functions

1. Preliminary Facts and Results on GA-Convex Functions

Let $I\subset(0,\infty)$ be an interval; a real-valued function $f:I\rightarrow\mathbb{R}$ is said to be *GA-convex* (concave) on I if

$$f\left(x^{1-\lambda}y^{\lambda}\right)\leq(\geq)\left(1-\lambda\right)f\left(x\right)+\lambda f\left(y\right) \tag{1.1}$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Since the condition (1.1) can be written as

$$f\circ\exp\left(\left(1-\lambda\right)\ln x+\lambda\ln y\right)\leq(\geq)\left(1-\lambda\right)f\circ\exp\left(\ln x\right)+\lambda f\circ\exp\left(\ln y\right), \tag{1.2}$$

then we observe that $f:I\rightarrow\mathbb{R}$ is *GA-convex* (concave) on I if and only if $f\circ\exp$ is convex (concave) on $\ln I:=\{\ln z,z\in I\}$. If $I=[a,b]$ then $\ln I=[\ln a,\ln b]$.

It is known that the function $f\left(x\right)=\ln\left(1+x\right)$ is *GA*-convex on $(0,\infty)$ [**1**].

For real and positive values of x, the *Euler gamma* function Γ and its *logarithmic derivative* ψ, the so-called *digamma function*, are defined by

$$\Gamma\left(x\right):=\int_{0}^{\infty}t^{x-1}e^{-t}dt\text{ and }\psi\left(x\right):=\frac{\Gamma'\left(x\right)}{\Gamma\left(x\right)}.$$

It has been shown in [**38**] that the function $f:(0,\infty)\rightarrow\mathbb{R}$ defined by

$$f\left(x\right)=\psi\left(x\right)+\frac{1}{2x}$$

is *GA*-concave on $(0,\infty)$ while the function $g:(0,\infty)\rightarrow\mathbb{R}$ defined by

$$g\left(x\right)=\psi\left(x\right)+\frac{1}{2x}+\frac{1}{12x^{2}}$$

is *GA*-convex on $(0,\infty)$.

If $[a,b]\subset(0,\infty)$ and the function $g:[\ln a,\ln b]\rightarrow\mathbb{R}$ is convex (concave) on $[\ln a,\ln b]$, then the function $f:[a,b]\rightarrow\mathbb{R}$, $f\left(t\right)=g\left(\ln t\right)$ is GA-convex (concave) on $[a,b]$.

Indeed, if $x,y\in[a,b]$ and $\lambda\in[0,1]$, then

$$\begin{aligned}f\left(x^{1-\lambda}y^{\lambda}\right)&=g\left(\ln\left(x^{1-\lambda}y^{\lambda}\right)\right)=g\left[\left(1-\lambda\right)\ln x+\lambda\ln y\right]\\&\leq(\geq)\left(1-\lambda\right)g\left(\ln x\right)+\lambda g\left(\ln y\right)=\left(1-\lambda\right)f\left(x\right)+\lambda f\left(y\right)\end{aligned}$$

showing that f is GA-convex (concave) on $[a,b]$.

In [**38**] the authors obtained the following Hermite-Hadamard type inequality.

THEOREM 1.1 (Zhang et al., 2010 [**38**]). *If* $b>a>0$ *and* $f:[a,b]\rightarrow\mathbb{R}$ *is a differentiable GA-convex (concave) function on* $[a,b]$, *then*

$$\begin{aligned}f\left(I\left(a,b\right)\right)&\leq(\geq)\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\\&\leq(\geq)\frac{b-L\left(a,b\right)}{b-a}f\left(b\right)+\frac{L\left(a,b\right)-a}{b-a}f\left(a\right).\end{aligned} \tag{1.3}$$

The differentiability of the function is not necessary in Theorem 1.1 for the first inequality (1.3) to hold. A proof of this fact is proved below after some short preliminaries. The second inequality in (1.3) has been proved in [**38**] without differentiability assumption.

Now, since $f \circ \exp$ is convex on $[\ln a, \ln b]$ it follows that f has finite lateral derivatives on $(\ln a, \ln b)$ and by gradient inequality for convex functions we have

$$f \circ \exp(x) - f \circ \exp(y) \geq (x - y) \varphi(\exp y) \exp y \tag{1.4}$$

where $\varphi(\exp y) \in \left[f'_{-}(\exp y), f'_{+}(\exp y)\right]$ for any $x, y \in (\ln a, \ln b)$.

If $s, t \in (a, b)$ and we take in (1.4) $x = \ln t, y = \ln s$, then we get

$$f(t) - f(s) \geq (\ln t - \ln s) \varphi(s) s \tag{1.5}$$

where $\varphi(s) \in \left[f'_{-}(s), f'_{+}(s)\right]$.

Now, if we take the integral mean on $[a, b]$ in the inequality (1.5) we get

$$\frac{1}{b-a} \int_a^b f(t) \, dt - f(s) \geq \left(\frac{1}{b-a} \int_a^b \ln t dt - \ln s\right) \varphi(s) s$$

and since

$$\frac{1}{b-a} \int_a^b \ln t dt = \ln I(a, b)$$

then we get

$$\frac{1}{b-a} \int_a^b f(t) \, dt \geq f(s) + (\ln I(a, b) - \ln s) \varphi(s) s \tag{1.6}$$

for any $s \in (a, b)$ and $\varphi(s) \in \left[f'_{-}(s), f'_{+}(s)\right]$. This is an inequality of interest in itself.

Now, if we take in (1.6) $s = I(a, b) \in (a, b)$ then we get the first inequality in (1.3) for GA-convex functions.

If f is differentiable and GA-convex on (a, b), then we have from (1.6) the inequality

$$\frac{1}{b-a} \int_a^b f(t) \, dt \geq f(s) + (\ln I(a, b) - \ln s) f'(s) s \tag{1.7}$$

for any $s \in (a, b)$.

If we take in (1.7) $s = \frac{a+b}{2} = A(a, b)$, then we get

$$\frac{1}{b-a} \int_a^b f(t) \, dt \geq f(A(a, b)) - f'(A(a, b)) A(a, b) \ln\left(\frac{A(a, b)}{I(a, b)}\right). \tag{1.8}$$

If we assume that $f'(A(a, b)) \leq 0$, then, since $I(a, b) \leq A(a, b)$, we get

$$\frac{1}{b-a} \int_a^b f(t) \, dt \geq f(A(a, b)) \tag{1.9}$$

provided that f is differentiable and GA-convex on (a, b).

Also, if we take in (1.7) $s = L(a, b)$, then we get

$$\frac{1}{b-a} \int_a^b f(t) \, dt \geq f(L(a, b)) + f'(L(a, b)) L(a, b) \ln\left(\frac{I(a, b)}{L(a, b)}\right). \tag{1.10}$$

If we assume that $f'(L(a, b)) \geq 0$, then we get from (1.10) that

$$\frac{1}{b-a} \int_a^b f(t) \, dt \geq f(L(a, b)) \tag{1.11}$$

provided that f is differentiable and GA-convex on (a, b).

Now, if we take in (1.7) $s=\sqrt{ab}=G(a,b)$, then we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f(G(a,b)) + f'(G(a,b))\,G(a,b)\ln\left(\frac{I(a,b)}{G(a,b)}\right). \tag{1.12}$$

Since

$$\ln\left(\frac{I(a,b)}{G(a,b)}\right) = \ln I(a,b) - \ln G(a,b) = \frac{A(a,b)-L(a,b)}{L(a,b)},$$

then (1.12) is equivalent to

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f(G(a,b)) + f'(G(a,b))\,G(a,b)\,\frac{A(a,b)-L(a,b)}{L(a,b)}. \tag{1.13}$$

If $f'(G(a,b)) \geq 0$, then we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f(G(a,b)) \tag{1.14}$$

provided that f is differentiable and GA-convex on (a,b).

Motivated by the above results we present in this chapter other inequalities of Hermite-Hadamard type for GA-convex functions.

2. SOME NEW INEQUALITIES FOR GA-CONVEX FUNCTIONS

We start with the following result that provide in the right side of (1.3) a bound in terms of the identric mean.

THEOREM 2.1 (Dragomir, 2015 [**11**]). *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex (concave) function on $[a,b]$. Then we have*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)\,dt &\leq (\geq) \frac{(\ln b - \ln I(a,b))\,f(a) + (\ln I(a,b) - \ln a)\,f(b)}{\ln b - \ln a} \\ &= \frac{b-L(a,b)}{b-a}f(b) + \frac{L(a,b)-a}{b-a}f(a).\end{aligned} \tag{2.1}$$

PROOF. Since is a *GA*-convex (concave) function on $[a,b]$ then $f\circ\exp$ is convex (concave) and we have

$$\begin{aligned} f(t) &= f\circ\exp(\ln t) \\ &= f\circ\exp\left(\frac{(\ln b-\ln t)\ln a + (\ln t - \ln a)\ln b}{\ln b - \ln a}\right) \\ &\leq (\geq) \frac{(\ln b - \ln t)\,f\circ\exp(\ln a) + (\ln t - \ln a)\,f\circ\exp(\ln b)}{\ln b - \ln a} \\ &= \frac{(\ln b - \ln t)\,f(a) + (\ln t - \ln a)\,f(b)}{\ln b - \ln a}\end{aligned} \tag{2.2}$$

for any $t\in[a,b]$.

This inequality is of interest in itself as well.

If we take the integral mean in (2.2) we get

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\leq (\geq) \frac{\left(\ln b - \frac{1}{b-a}\int_a^b \ln t dt\right) f(a) + \left(\frac{1}{b-a}\int_a^b \ln t dt - \ln a\right) f(b)}{\ln b - \ln a}\end{aligned}$$

and since

$$\frac{1}{b-a}\int_a^b \ln t dt = \ln I(a,b),$$

then we obtain the desired result (2.1).

Now, we observe that

$$\frac{\ln b - \ln I(a,b)}{\ln b - \ln a} = \frac{\ln b - \frac{b\ln b - a\ln a}{b-a} + 1}{\ln b - \ln a} = \frac{L(a,b) - a}{b-a}$$

and, similarly

$$\frac{\ln I(a,b) - \ln a}{\ln b - \ln a} = \frac{b - L(a,b)}{b-a},$$

which proves the last part of (2.1). ∎

If $f : I \subset (0,\infty) \to \mathbb{R}$ is a *GA*-convex (concave) on I then we have the inequality

(2.3) $$f(\sqrt{xy}) \le (\ge) \frac{f(x) + f(y)}{2}$$

for any $x, y \in I$.

The following refinement of (2.3), which is an inequality of Hermite-Hadamard type, holds (see [**32**] for an extension for *GA h*-convex functions). For the sake of completeness we give here a short proof.

LEMMA 2.2 (Noor et al., 2014 [**32**]). *Let* $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be a GA-convex (concave) function on* $[a,b]$. *Then we have*

(2.4) $$f\left(\sqrt{ab}\right) \le (\ge) \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t} dt \le (\ge) \frac{f(a) + f(b)}{2}.$$

PROOF. By the definition of *GA*-convex (concave) functions on $[a,b]$ we have

(2.5) $$f\left(a^{1-\lambda}b^{\lambda}\right) \le (\ge) (1-\lambda) f(a) + \lambda f(b)$$

for any $\lambda \in [0,1]$.

Integrating the inequality (2.5) on $[0,1]$ we get

(2.6) $$\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right) d\lambda \le (\ge) f(a)\int_0^1 (1-\lambda) d\lambda + f(b)\int_0^1 \lambda d\lambda.$$

Since $\int_0^1 (1-\lambda) d\lambda = \int_0^1 \lambda d\lambda = \frac{1}{2}$ and, by changing the variable $t = a^{1-\lambda}b^{\lambda}$, $\lambda \in [0,1]$, we have

$$\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right) d\lambda = \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t} dt$$

then by (2.6) we get the second inequality in (2.4).

By the inequality (2.3) we have

(2.7) $$f\left(\sqrt{ab}\right) = f\left(\sqrt{a^{1-\lambda}b^{\lambda}a^{\lambda}b^{1-\lambda}}\right) \le (\ge) \frac{1}{2}\left[f\left(a^{1-\lambda}b^{\lambda}\right) + f\left(a^{\lambda}b^{1-\lambda}\right)\right]$$

for any $\lambda \in [0,1]$.

Integrating the inequality (2.7) $[0,1]$ we get

(2.8) $$f\left(\sqrt{ab}\right) \le (\ge) \frac{1}{2}\left[\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right) d\lambda + \int_0^1 f\left(a^{\lambda}b^{1-\lambda}\right) d\lambda\right].$$

Since

$$\int_0^1 f\left(a^{\lambda}b^{1-\lambda}\right) d\lambda = \int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right) d\lambda = \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t} dt,$$

then by (2.8) we get the first inequality in (2.4). ∎

REMARK 2.1. The inequality (2.4) can be also written for any $d > c > 0$ with $c, d \in I$ as

$$f\left(\sqrt{cd}\right) \leq (\geq) \int_0^1 f\left(c^{1-\lambda} d^{\lambda}\right) d\lambda \leq (\geq) \frac{f(c) + f(d)}{2}, \tag{2.9}$$

provided *GA*-convex (concave) function on I.

We have the following representation result:

LEMMA 2.3 (Dragomir, 2015 [**11**]). *Let* $g : [x, y] \subset \mathbb{R} \to \mathbb{C}$ *be a Lebesgue integrable function on* $[x, y]$ *. Then for any* $\lambda \in [0, 1]$ *we have the representation*

$$\int_0^1 g[(1-t)x + ty]\,dt = (1-\lambda)\int_0^1 g[(1-t)((1-\lambda)x + \lambda y) + ty]\,dt + \lambda \int_0^1 g[(1-t)x + t((1-\lambda)x + \lambda y)]\,dt. \tag{2.10}$$

PROOF. For $\lambda = 0$ and $\lambda = 1$ the equality (2.10) is obvious. Let $\lambda \in (0, 1)$. Observe that

$$\int_0^1 g[(1-t)(\lambda y + (1-\lambda)x) + ty]\,dt = \int_0^1 g[((1-t)\lambda + t)y + (1-t)(1-\lambda)x]\,dt$$

and

$$\int_0^1 g[t(\lambda y + (1-\lambda)x) + (1-t)x]\,dt = \int_0^1 g[t\lambda y + (1-\lambda t)x]\,dt.$$

If we make the change of variable $u := (1-t)\lambda + t$ then we have $1 - u = (1-t)(1-\lambda)$ and $du = (1-\lambda)\,dt$. Then

$$\int_0^1 g[((1-t)\lambda + t)y + (1-t)(1-\lambda)x]\,dt = \frac{1}{1-\lambda}\int_\lambda^1 g[uy + (1-u)x]\,du.$$

If we make the change of variable $u := \lambda t$ then we have $du = \lambda dt$ and

$$\int_0^1 g[t\lambda y + (1-\lambda t)x]\,dt = \frac{1}{\lambda}\int_0^\lambda g[uy + (1-u)x]\,du.$$

Therefore

$$(1-\lambda)\int_0^1 g[(1-t)(\lambda y + (1-\lambda)x) + ty]\,dt + \lambda\int_0^1 g[t(\lambda y + (1-\lambda)x) + (1-t)x]\,dt = \int_0^1 g[uy + (1-u)x]\,du$$

and the identity (2.10) is proved. ∎

COROLLARY 2.4. *Let* $f : [a, b] \subset (0, \infty) \to \mathbb{C}$ *be a Lebesgue integrable function on* $[a, b]$ *. Then for any* $\lambda \in [0, 1]$ *we have the representation*

$$\int_0^1 f\left(a^{1-s}b^s\right) ds = (1-\lambda)\int_0^1 f\left(\left[a^{(1-\lambda)}b^{\lambda}\right]^{1-s} b^s\right) ds + \lambda\int_0^1 f\left(a^{1-s}\left[a^{(1-\lambda)}b^{\lambda}\right]^s\right) ds. \tag{2.11}$$

PROOF. Using (2.10) we have

$$\int_0^1 f\left(a^{1-s}b^s\right)ds = \int_0^1 f\circ\exp\left((1-s)\ln a + s\ln b\right)ds$$
$$= (1-\lambda)\int_0^1 f\circ\exp\left[(1-s)\left((1-\lambda)\ln a+\lambda\ln b\right)+s\ln b\right]ds$$
$$+\lambda\int_0^1 f\circ\exp\left[(1-s)\ln a+s\left((1-\lambda)\ln a+\lambda\ln b\right)\right]ds$$
$$=(1-\lambda)\int_0^1 f\left(\left[a^{(1-\lambda)}b^{\lambda}\right]^{1-s}b^s\right)ds+\lambda\int_0^1 f\left(a^{1-s}\left[a^{(1-\lambda)}b^{\lambda}\right]^s\right)ds$$
$$=(1-\lambda)\int_0^1 f\left(\left[a^{(1-\lambda)}b^{\lambda}\right]^{1-s}b^s\right)ds+\lambda\int_0^1 f\left(a^{1-s}\left[a^{(1-\lambda)}b^{\lambda}\right]^s\right)ds$$

and the identity (2.11) is proved. ∎

We are able now to provide a refinement of (2.4) as follows:

THEOREM 2.5 (Dragomir, 2015 [**11**]). *Let* $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ *be a GA-convex (concave) function on* $[a,b]$. *Then for any* $\lambda\in[0,1]$ *we have*

$$f\left(\sqrt{ab}\right)\le(\ge)(1-\lambda)f\left(a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}\right)+\lambda f\left(a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\right) \tag{2.12}$$
$$\le(\ge)\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{t}dt$$
$$\le(\ge)\frac{1}{2}\left[f\left(a^{1-\lambda}b^{\lambda}\right)+(1-\lambda)f(b)+\lambda f(a)\right]$$
$$\le(\ge)\frac{f(a)+f(b)}{2}.$$

PROOF. We prove the inequalities only for the *GA*-convex case. Using the inequality (2.9) we have

$$f\left(\sqrt{a^{1-\lambda}b^{\lambda}b}\right)\le\int_0^1 f\left(\left[a^{1-\lambda}b^{\lambda}\right]^{1-s}b^s\right)ds\le\frac{f\left(a^{1-\lambda}b^{\lambda}\right)+f(b)}{2},$$

that is equivalent to

$$f\left(a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}\right)\le\int_0^1 f\left(\left[a^{1-\lambda}b^{\lambda}\right]^{1-s}b^s\right)ds\le\frac{f\left(a^{1-\lambda}b^{\lambda}\right)+f(b)}{2}, \tag{2.13}$$

for any $\lambda\in[0,1]$.

We also have

$$f\left(\sqrt{aa^{1-\lambda}b^{\lambda}}\right)\le\int_0^1 f\left(a^{1-s}\left[a^{1-\lambda}b^{\lambda}\right]^s\right)ds\le\frac{f(a)+f\left(a^{1-\lambda}b^{\lambda}\right)}{2}$$

that is equivalent to

$$f\left(a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\right)\le\int_0^1 f\left(a^{1-s}\left[a^{1-\lambda}b^{\lambda}\right]^s\right)ds\le\frac{f(a)+f\left(a^{1-\lambda}b^{\lambda}\right)}{2} \tag{2.14}$$

for any $\lambda\in[0,1]$.

If we multiply (2.13) by $1-\lambda$ and (2.14) by λ and add the obtained inequalities we get, by the identity (2.11), that

$$\begin{aligned}(1-\lambda) f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right)+\lambda f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) &\leq \int_0^1 f\left(a^{1-s} b^s\right) ds \\ &\leq(1-\lambda) \frac{f\left(a^{1-\lambda} b^\lambda\right)+f(b)}{2}+\lambda \frac{f(a)+f\left(a^{1-\lambda} b^\lambda\right)}{2} \\ &=\frac{1}{2}\left[f\left(a^{1-\lambda} b^\lambda\right)+(1-\lambda) f(b)+\lambda f(a)\right]\end{aligned}$$

for any $\lambda \in[0,1]$, which proves the second and third inequality in (2.12).

By the *GA*-convexity we have

$$\begin{aligned}(1-\lambda) f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right)+\lambda f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) &\geq f\left[\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right)^{1-\lambda}\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right)^\lambda\right] \\ &=f\left(a^{\frac{1}{2}} b^{\frac{1}{2}}\right),\end{aligned}$$

which proves the first inequality in (2.12).

By the *GA*-convexity we also have

$$\begin{aligned}&\frac{1}{2}\left[f\left(a^{1-\lambda} b^\lambda\right)+(1-\lambda) f(b)+\lambda f(a)\right] \\ &\qquad \leq \frac{1}{2}[(1-\lambda) f(a)+\lambda f(b)+(1-\lambda) f(b)+\lambda f(a)]=\frac{f(a)+f(b)}{2},\end{aligned}$$

which proves the last inequality in (2.12). ∎

COROLLARY 2.6. *With the assumptions of Theorem 2.5 we have*

$$\begin{aligned}f(\sqrt{ab}) &\leq(\geq) \frac{1}{2}\left[f\left(a^{\frac{1}{4}} b^{\frac{3}{4}}\right)+f\left(a^{\frac{3}{4}} b^{\frac{1}{4}}\right)\right] \\ &\leq(\geq) \frac{1}{\ln b-\ln a} \int_a^b \frac{f(t)}{t} dt \\ &\leq(\geq) \frac{1}{2}\left[f(\sqrt{ab})+\frac{f(b)+f(a)}{2}\right] \leq(\geq) \frac{f(a)+f(b)}{2}.\end{aligned} \tag{2.15}$$

3. RELATED INEQUALITIES

The following result also holds:

THEOREM 3.1 (Dragomir, 2015 [**11**]). *Let $f:[a,b] \subset(0,\infty) \rightarrow \mathbb{R}$ be a GA-convex (concave) function on $[a,b]$. Then for any $t \in[a,b]$ we have*

$$\begin{aligned}&\frac{1}{\ln b-\ln a} \int_a^b \frac{f(s)}{s} ds \\ &\leq(\geq) \frac{1}{2}\left[f(t)+\frac{f(b)(\ln b-\ln t)+f(a)(\ln t-\ln a)}{\ln b-\ln a}\right] \\ &\leq(\geq) \frac{f(a)+f(b)}{2}.\end{aligned} \tag{3.1}$$

PROOF. We give a proof only for the *GA*-convex case. From the inequality (1.5) we have that

$$f(t)-f(s) \geq(\ln t-\ln s) f_+'(s) s \tag{3.2}$$

for any $s \in (a, b)$ and $t \in [a, b]$.

We divide by $s > 0$ and integrate on $[a, b]$ over s to get

$$f(t) \int_a^b \frac{1}{s} ds - \int_a^b \frac{f(s)}{s} ds \geq \left(\int_a^b f'_+(s) \, ds \right) \ln t - \int_a^b f'_+(s) \ln s ds \tag{3.3}$$

for any $t \in [a, b]$.

However

$$\int_a^b \frac{1}{s} ds = \ln b - \ln a, \quad \int_a^b f'_+(s) \, ds = f(b) - f(a)$$

and

$$\int_a^b f'_+(s) \ln s ds = f(b) \ln b - f(a) \ln a - \int_a^b \frac{f(s)}{s} ds.$$

Therefore, by (3.3) we get

$$f(t)(\ln b - \ln a) - \int_a^b \frac{f(s)}{s} ds$$
$$\geq (f(b) - f(a)) \ln t - f(b) \ln b + f(a) \ln a + \int_a^b \frac{f(s)}{s} ds$$

namely

$$f(t)(\ln b - \ln a) - (f(b) - f(a)) \ln t + f(b) \ln b - f(a) \ln a \geq 2 \int_a^b \frac{f(s)}{s} ds,$$

which can be written as

$$f(t)(\ln b - \ln a) + f(b)(\ln b - \ln t) + f(a)(\ln t - \ln a) \geq 2 \int_a^b \frac{f(s)}{s} ds$$

and the first inequality in (3.1) is proved.

Using (2.2) we have

$$f(t) + \frac{f(b)(\ln b - \ln t) + f(a)(\ln t - \ln a)}{\ln b - \ln a}$$
$$\leq \frac{(\ln b - \ln t) f(a) + (\ln t - \ln a) f(b)}{\ln b - \ln a}$$
$$+ \frac{f(b)(\ln b - \ln t) + f(a)(\ln t - \ln a)}{\ln b - \ln a} = f(a) + f(b)$$

for any $t \in [a, b]$ that proves the last part of (3.1). ∎

By taking the integral mean in the inequality (3.1) we have:

COROLLARY 3.2. *With the assumptions in Theorem 3.1 we have*

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \leq (\geq) \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f(t) \, dt \right.$$
$$\left. + \frac{f(b)(\ln b - \ln I(a,b)) + f(a)(\ln I(a,b) - \ln a)}{\ln b - \ln a} \right]$$
$$\leq (\geq) \frac{f(a) + f(b)}{2}. \tag{3.4}$$

Since a simple calculation reveals (see the proof of Theorem 2.1) that

$$\frac{f(b)(\ln b-\ln I(a,b))+f(a)(\ln I(a,b)-\ln a)}{\ln b-\ln a} = \frac{L(a,b)-a}{b-a}f(b)+\frac{b-L(a,b)}{b-a}f(a),$$

then the inequality (3.4) is equivalent to

$$\frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds \le (\ge) \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{L(a,b)-a}{b-a}f(b)+\frac{b-L(a,b)}{b-a}f(a)\right] \le (\ge) \frac{f(a)+f(b)}{2}. \tag{3.5}$$

REMARK 3.1. Taking specific values for $t\in[a,b]$ in (3.1) we get the following results

$$\frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds \le (\ge) \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)\left(\ln b-\ln\frac{a+b}{2}\right)+f(a)\left(\ln\frac{a+b}{2}-\ln a\right)}{\ln b-\ln a}\right] \le (\ge) \frac{f(a)+f(b)}{2}, \tag{3.6}$$

$$\frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds \le (\ge) \frac{1}{2}\left[f\left(\sqrt{ab}\right)+\frac{f(a)+f(b)}{2}\right] \le (\ge) \frac{f(a)+f(b)}{2} \tag{3.7}$$

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds &\le (\ge) \frac{1}{2}\left[f(I(a,b))+\frac{f(b)(\ln b-\ln I(a,b))+f(a)(\ln I(a,b)-\ln a)}{\ln b-\ln a}\right]\\ &= \frac{1}{2}\left[f(I(a,b))+\frac{L(a,b)-a}{b-a}f(b)+\frac{b-L(a,b)}{b-a}f(a)\right]\\ &\le (\ge) \frac{f(a)+f(b)}{2},\end{aligned} \tag{3.8}$$

and

$$\frac{1}{\ln b-\ln a}\int_a^b \frac{f(s)}{s}ds \le (\ge) \frac{1}{2}\left[f(L(a,b))+\frac{f(b)(\ln b-\ln L(a,b))+f(a)(\ln L(a,b)-\ln a)}{\ln b-\ln a}\right] \le (\ge) \frac{f(a)+f(b)}{2}. \tag{3.9}$$

Now, observe that

$$f(b)(\ln b-\ln t)+f(a)(\ln t-\ln a)=0$$

iff

$$\ln t=\frac{f(b)\ln b-f(a)\ln a}{f(b)-f(a)}=\ln\left(\frac{b^{f(b)}}{a^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}},$$

which is equivalent to

$$t=\left(\frac{b^{f(b)}}{a^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}}.$$

Therefore, if

$$t=\left(\frac{b^{f(b)}}{a^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}}\in[a,b]$$

then by (3.1) we get

$$\frac{1}{\ln b-\ln a}\int_a^b\frac{f(s)}{s}ds\le(\ge)\frac{1}{2}f\left(\left(\frac{b^{f(b)}}{a^{f(a)}}\right)^{\frac{1}{f(b)-f(a)}}\right)\le(\ge)\frac{f(a)+f(b)}{2}. \tag{3.10}$$

The following result also holds

THEOREM 3.3 (Dragomir, 2015 [**11**]). *Let* $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ *be a GA-convex (concave) function on* $[a,b]$. *Then for any* $t\in[a,b]$ *we have*

$$\frac{1}{2}\left[f(t)+\frac{f(b)b(\ln b-\ln t)+af(a)(\ln t-\ln a)}{b-a}\right]-\frac{1}{b-a}\int_a^b f(s)\,ds \tag{3.11}$$
$$\ge(\le)\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(s)\ln s ds-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\ln t\right].$$

PROOF. We give a proof only for the *GA*-convex case. Integrate over s in the inequality (3.2) to get

$$f(t)(b-a)-\int_a^b f(s)\,ds\ge\ln t\int_a^b f'_+(s)\,sds-\int_a^b f'_+(s)\,s\ln sds \tag{3.12}$$

for any $t\in[a,b]$.

Observe that, integrating by parts in the Lebesgue integral, we have

$$\int_a^b f'_+(s)\,sds=bf(b)-af(a)-\int_a^b f(s)\,ds$$

and

$$\int_a^b f'_+(s)\,s\ln sds=f(b)\,b\ln b-f(a)\,a\ln a-\int_a^b f(s)\ln sds-\int_a^b f(s)\,ds.$$

Using the inequality (3.12) we get

$$f(t)(b-a)-\int_a^b f(s)\,ds$$
$$\geq \ln t\left(bf(b)-af(a)-\int_a^b f(s)\,ds\right)$$
$$-f(b)\,b\ln b+f(a)\,a\ln a+\int_a^b f(s)\ln s ds+\int_a^b f(s)\,ds$$
$$=bf(b)\ln t-af(a)\ln t-\ln t\int_a^b f(s)\,ds$$
$$-f(b)\,b\ln b+f(a)\,a\ln a+\int_a^b f(s)\ln s ds+\int_a^b f(s)\,ds$$

that is equivalent to

$$f(t)(b-a)+f(b)\,b(\ln b-\ln t)+af(a)(\ln t-\ln a)-2\int_a^b f(s)\,ds$$
$$\geq \int_a^b f(s)\ln s ds-\ln t\int_a^b f(s)\,ds,$$

for any $t\in[a,b]$ and the inequality (3.11) is proved. ∎

COROLLARY 3.4. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex function on $[a,b]$. Then*

$$\frac{bf(b)(\ln b-\ln I(a,b))+af(a)(\ln I(a,b)-\ln a)}{b-a}-\frac{1}{b-a}\int_a^b f(s)\,ds \tag{3.13}$$
$$\geq \frac{1}{b-a}\int_a^b f(s)\ln s ds-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\ln I(a,b).$$

Moreover, if f is monotonic nondecreasing, then

$$\frac{bf(b)(\ln b-\ln I(a,b))+af(a)(\ln I(a,b)-\ln a)}{b-a}-\frac{1}{b-a}\int_a^b f(s)\,ds \tag{3.14}$$
$$\geq \frac{1}{b-a}\int_a^b f(s)\ln s ds-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\ln I(a,b)\geq 0.$$

PROOF. Integrating over t on $[a,b]$ and dividing by $b-a$ in (3.11) we get (3.13).

Now, since f is monotonic nondecreasing on $[a,b]$, then by Čebyšev inequality for synchronous functions, we have

$$\frac{1}{b-a}\int_a^b f(s)\ln s ds\geq\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\frac{1}{b-a}\int_a^b \ln t dt$$

that proves (3.14). ∎

COROLLARY 3.5. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex function on $[a,b]$. Then*

$$\frac{1}{2}\left[f\left(\exp\left(\mu_f\right)\right)+\frac{f(b)\,b\left(\ln b-\mu_f\right)+af(a)\left(\mu_f-\ln a\right)}{b-a}\right] \tag{3.15}$$
$$\geq \frac{1}{b-a}\int_a^b f(s)\,ds,$$

provided that

$$\mu_f := \frac{\int_a^b f(s)\ln s ds}{\int_a^b f(s)\,ds} \in [\ln a, \ln b].$$

REMARK 3.2. If we take $t = \sqrt{ab}$ in (3.11), then we get

$$\begin{aligned}(3.16)\quad &\frac{1}{2}\left[f\left(\sqrt{ab}\right) + \frac{f(b)\,b + af(a)}{2L(a,b)}\right] - \frac{1}{b-a}\int_a^b f(s)\,ds \\ &\geq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(s)\ln s ds - \left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\ln\sqrt{ab}\right].\end{aligned}$$

If we take $t = I(a,b)$ in (3.11), then we get

$$\begin{aligned}(3.17)\quad &\frac{1}{2}\left[f(I(a,b)) + \frac{f(b)\,b(\ln b - \ln I(a,b)) + af(a)(\ln I(a,b) - \ln a)}{b-a}\right] \\ &- \frac{1}{b-a}\int_a^b f(s)\,ds \\ &\geq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(s)\ln s ds - \left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)\ln I(a,b)\right].\end{aligned}$$

Finally, we have

THEOREM 3.6 (Dragomir, 2015 [**11**]). *Let* $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be a GA-convex (concave) function on* $[a,b]$. *Then we have*

$$\begin{aligned}(3.18)\quad &\frac{1}{8}\left[f'_+\left(\sqrt{ab}\right) - f'_-\left(\sqrt{ab}\right)\right]\sqrt{ab}(\ln b - \ln a) \\ &\leq (\geq)\,\frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(s)}{s}ds \\ &\leq (\geq)\,\frac{1}{8}\left[f'_-(b)\,b - f'_+(a)\,a\right](\ln b - \ln a)\end{aligned}$$

and

$$\begin{aligned}(3.19)\quad &\frac{1}{8}\left[f'_+\left(\sqrt{ab}\right) - f'_-\left(\sqrt{ab}\right)\right]\sqrt{ab}(\ln b - \ln a) \\ &\leq (\geq)\,\frac{1}{\ln b - \ln a}\int_a^b \frac{f(s)}{s}ds - f\left(\sqrt{ab}\right) \\ &\leq (\geq)\,\frac{1}{8}\left[f'_-(b)\,b - f'_+(a)\,a\right](\ln b - \ln a).\end{aligned}$$

PROOF. Consider the function $h : [\ln a, \ln b] \to \mathbb{R}$ defined by $h(t) = f \circ \exp(t)$. Since f is a *GA*-convex (concave) function on $[a,b]$, then we have the lateral derivatives

$$h'_{\pm}(t) = \left(f'_{\pm} \circ \exp(t)\right)\exp t,\ t \in [\ln a, \ln b].$$

If we apply Lemma 0.1 we deduce the desired result. ∎

REMARK 3.3. If the function $f : I \subset (0,\infty) \to \mathbb{R}$ is differentiable and a *GA*-convex function on $[a,b] \subset \mathring{I}$ then we have the following inequalities

$$\begin{aligned}(3.20)\quad 0 &\leq \frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(s)}{s}ds \\ &\leq \frac{1}{8}\left[f'(b)\,b - f'(a)\,a\right](\ln b - \ln a)\end{aligned}$$

and

$$0 \le \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds - f\left(\sqrt{ab}\right) \le \frac{1}{8} \left[f'(b) b - f'(a) a\right] (\ln b - \ln a) . \tag{3.21}$$

4. SOME REFINEMENTS

In 1994, [**3**] (see also [**23**, p. 22]) we proved the following refinement of Hermite-Hadamard inequality.

LEMMA 4.1. *Let* $g : [c,d] \to \mathbb{R}$ *be a convex function on* $[c,d]$. *Then for any division* $c = x_0 < x_1 < ... < x_{n-1} < x_n = d$ *with* $n \ge 1$ *we have the inequalities*

$$\begin{aligned} g\left(\frac{c+d}{2}\right) &\le \frac{1}{d-c} \sum_{i=0}^{n-1} (x_{i+1} - x_i) g\left(\frac{x_{i+1} + x_i}{2}\right) \\ &\le \frac{1}{d-c} \int_c^d g(x) dx \le \frac{1}{d-c} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \frac{g(x_i) + g(x_{i+1})}{2} \\ &\le \frac{1}{2} [g(c) + g(d)] . \end{aligned} \tag{4.1}$$

We have:

THEOREM 4.2 (Dragomir, 2015 [**12**]). *Let* $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be a GA-convex function on* $[a,b]$. *Then for any division* $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$ *with* $n \ge 1$ *we have the inequalities*

$$\begin{aligned} f\left(\sqrt{ab}\right) &\le \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) f\left(\sqrt{t_i t_{i+1}}\right) \\ &\le \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\ &\le \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) \frac{f(t_i) + f(t_{i+1})}{2} \le \frac{1}{2} [f(a) + f(b)] . \end{aligned} \tag{4.2}$$

PROOF. If $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is a *GA*-convex function on $[a,b]$, then $f \circ \exp$ is convex on $[\ln a, \ln b]$. Let $c = \ln a$, $d = \ln b$ and the division $x_i = \ln t_i$, $i \in \{0, ..., n-1\}$ of the interval $[c,d]$. If we write the inequality (4.1) for $g = f \circ \exp$ on the interval $[c,d]$ and for the division

$$\ln a = \ln t_0 < \ln t_1 < ... < \ln t_{n-1} < \ln t_n = \ln b$$

we have

$$(f\circ\exp)\left(\frac{\ln a+\ln b}{2}\right) \tag{4.3}$$
$$\le \frac{1}{\ln b-\ln a}\sum_{i=0}^{n-1}(\ln t_{i+1}-\ln t_i)(f\circ\exp)\left(\frac{\ln t_{i+1}+\ln t_i}{2}\right)$$
$$\le \frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}(f\circ\exp)(x)\,dx$$
$$\le \frac{1}{\ln b-\ln a}\sum_{i=0}^{n-1}(\ln t_{i+1}-\ln t_i)\frac{(f\circ\exp)(\ln t_i)+(f\circ\exp)(\ln t_{i+1})}{2}$$
$$\le \frac{1}{2}\left[(f\circ\exp)(\ln a)+(f\circ\exp)(\ln b)\right],$$

that is equivalent to

$$f\left(\sqrt{ab}\right)\le \frac{1}{\ln b-\ln a}\sum_{i=0}^{n-1}(\ln t_{i+1}-\ln t_i)f\left(\sqrt{t_i t_{i+1}}\right) \tag{4.4}$$
$$\le \frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}(f\circ\exp)(x)\,dx$$
$$\le \frac{1}{\ln b-\ln a}\sum_{i=0}^{n-1}(\ln t_{i+1}-\ln t_i)\frac{f(t_i)+f(t_{i+1})}{2}\le\frac{1}{2}\left[f(a)+f(b)\right].$$

By using the change of variable $\exp(x)=t$, we have $x=\ln t$, $dx=\frac{dt}{t}$ and $\int_{\ln a}^{\ln b}(f\circ\exp)(x)\,dx=\int_a^b\frac{f(t)}{t}dt$ and by (4.4) we get the desired result (4.2). ∎

COROLLARY 4.3. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GA-convex function on $[a,b]$. If $a\le t\le b$, then we have*

$$f\left(\sqrt{ab}\right) \tag{4.5}$$
$$\le \frac{1}{\ln b-\ln a}\left[(\ln t-\ln a)f\left(\sqrt{at}\right)+(\ln b-\ln t)f\left(\sqrt{tb}\right)\right]$$
$$\le \frac{1}{\ln b-\ln a}\int_a^b\frac{f(s)}{s}ds$$
$$\le \frac{1}{2}\left[f(t)+\frac{(\ln t-\ln a)f(a)+(\ln b-\ln t)f(b)}{\ln b-\ln a}\right]$$
$$\le \frac{1}{2}\left[f(a)+f(b)\right].$$

REMARK 4.1. Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a *GA*-convex function on $[a,b]$. For a division $0=\lambda_0<\lambda_1<...<\lambda_{n-1}<\lambda_n=1$ with $n\ge1$ of the interval $[0,1]$, consider the division

$$t_i:=a\left(\frac{b}{a}\right)^{\lambda_i}=a^{1-\lambda_i}b^{\lambda_i},\ i\in\{0,...,n-1\} \tag{4.6}$$

of the interval $[a,b]$.

Observe that

$$\ln t_{i+1} - \ln t_i = \ln \left[a \left(\frac{b}{a} \right)^{\lambda_{i+1}} \right] - \ln \left[a \left(\frac{b}{a} \right)^{\lambda_i} \right] = (\ln b - \ln a)(\lambda_{i+1} - \lambda_i)$$

and

$$\sqrt{t_i t_{i+1}} = \sqrt{a^{1-\lambda_i} b^{\lambda_i} a^{1-\lambda_{i+1}} b^{\lambda_{i+1}}} = a^{1-\frac{\lambda_i+\lambda_{i+1}}{2}} b^{\frac{\lambda_i+\lambda_{i+1}}{2}}$$

for any $i \in \{0, ..., n-1\}$.

If we write the inequality (4.2) for the division (4.6) we get

$$\begin{aligned} f\left(\sqrt{ab}\right) &\leq \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) f\left(a^{1-\frac{\lambda_i+\lambda_{i+1}}{2}} b^{\frac{\lambda_i+\lambda_{i+1}}{2}}\right) \\ &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(s)}{s} ds \\ &\leq \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) \frac{f\left(a^{1-\lambda_i} b^{\lambda_i}\right) + f\left(a^{1-\lambda_{i+1}} b^{\lambda_{i+1}}\right)}{2} \leq \frac{1}{2}[f(a) + f(b)], \end{aligned} \tag{4.7}$$

for any division $0 = \lambda_0 < \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 1$ with $n \geq 1$ of the interval $[0, 1]$.

If we write the inequality (4.7) for $0 < \lambda < 1$, then we get

$$\begin{aligned} f\left(\sqrt{ab}\right) &\leq (1-\lambda) f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) + \lambda f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \\ &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \\ &\leq \frac{1}{2}\left[f\left(a^{1-\lambda} b^{\lambda}\right) + (1-\lambda) f(b) + \lambda f(a)\right] \leq \frac{f(a) + f(b)}{2}. \end{aligned} \tag{4.8}$$

The inequality (4.8) was obtained in [**11**].

In the following chapter we establish some weighted Hermite-Hadamard type inequalities for *GA*-convex functions.

5. WEIGHTED INEQUALITIES

We have the following weighted inequality:

THEOREM 5.1 (Dragomir, 2015 [**12**]). *Let $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ be a GA-convex function on $[a, b]$ and $w : [a, b] \to [0, \infty)$ an integrable function on $[a, b]$ with $\int_a^b w(t)\, dt > 0$,then*

$$\begin{aligned} & f\left(\exp\left(\frac{\int_a^b w(t) \ln t dt}{\int_a^b w(t)\, dt}\right)\right) \\ & \leq \frac{\int_a^b w(t) f(t)\, dt}{\int_a^b w(t)\, dt} \\ & \leq \frac{\left(\ln b - \frac{\int_a^b w(t) \ln t dt}{\int_a^b w(t) dt}\right) f(a) + \left(\frac{\int_a^b w(t) \ln t dt}{\int_a^b w(t) dt} - \ln a\right) f(b)}{\ln b - \ln a}. \end{aligned} \tag{5.1}$$

PROOF. Observe that for $t \in [a, b]$ we have

$$\ln t = \frac{(\ln b - \ln t) \ln a + (\ln t - \ln a) \ln b}{\ln b - \ln a}.$$

By the convexity of $f\circ\exp$ we have

$$f(t)=(f\circ\exp)(\ln t)=(f\circ\exp)\left(\frac{(\ln b-\ln t)\ln a+(\ln t-\ln a)\ln b}{\ln b-\ln a}\right)\tag{5.2}$$
$$\leq\frac{\ln b-\ln t}{\ln b-\ln a}f(a)+\frac{\ln t-\ln a}{\ln b-\ln a}f(b)$$

for any $t\in[a,b]$.

If we multiply (5.2) by $w(t)\geq 0$, $t\in[a,b]$ and integrate over t on $[a,b]$ then we get

$$\int_a^b f(t)w(t)\,dt$$
$$\leq\frac{\ln b\int_a^b w(t)\,dt-\int_a^b w(t)\ln t dt}{\ln b-\ln a}f(a)+\frac{\int_a^b w(t)\ln t dt-\ln a\int_a^b w(t)\,dt}{\ln b-\ln a}f(b)$$

and the second inequality in (5.1) is proved.

By Jensen's inequality we have

$$\frac{\int_a^b w(t)f(t)\,dt}{\int_a^b w(t)\,dt}=\frac{\int_a^b w(t)(f\circ\exp)(\ln t)\,dt}{\int_a^b w(t)\,dt}\geq(f\circ\exp)\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)\,dt}\right)$$

and the first part of (5.1) is proved. ∎

COROLLARY 5.2. *Let* $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ *be a GA-convex function on* $[a,b]$, *then*

$$f\left(\sqrt{ab}\right)\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{t}dt\leq\frac{f(a)+f(b)}{2}.\tag{5.3}$$

PROOF. If we take $w(t)=\frac{1}{t}$ in (5.1), then we have

$$f\left(\exp\left(\frac{\int_a^b\frac{\ln t}{t}dt}{\int_a^b\frac{1}{t}dt}\right)\right)\tag{5.4}$$
$$\leq\frac{\int_a^b\frac{f(t)}{t}dt}{\int_a^b\frac{1}{t}dt}$$
$$\leq\frac{\left(\ln b-\frac{\int_a^b\frac{\ln t}{t}dt}{\int_a^b\frac{1}{t}dt}\right)f(a)+\left(\frac{\int_a^b\frac{\ln t}{t}dt}{\int_a^b\frac{1}{t}dt}-\ln a\right)f(b)}{\ln b-\ln a}.$$

Since

$$\int_a^b\frac{\ln t}{t}dt=\frac{1}{2}\left[(\ln b)^2-(\ln a)^2\right],\quad\int_a^b\frac{1}{t}dt=\ln b-\ln a$$

then we get from (5.4)

$$f\left(\exp\left(\frac{\ln b+\ln a}{2}\right)\right)\leq\frac{\int_a^b\frac{f(t)}{t}dt}{\ln b-\ln a}$$
$$\leq\frac{\left(\ln b-\frac{\ln b+\ln a}{2}\right)f(a)+\left(\frac{\ln b+\ln a}{2}-\ln a\right)f(b)}{\ln b-\ln a}$$

and the inequality (5.3) is proved. ∎

COROLLARY 5.3 (Dragomir, 2015 [**12**]). *Let* $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be a GA-convex function on* $[a,b]$ *, then for any* $p \in \mathbb{R}$ *with* $p \neq 0, -1$ *we have*

$$\begin{aligned} L_p^p(a,b)\, f\left(\left[I\left(a^{p+1},b^{p+1}\right)\right]^{\frac{1}{p+1}}\right) &\leq \frac{1}{b-a}\int_a^b t^p f(t)\,dt \\ &\leq \frac{\left(L\left(a^{p+1},b^{p+1}\right)-a^{p+1}\right)f(a)+\left(b^{p+1}-L\left(a^{p+1},b^{p+1}\right)\right)f(b)}{(p+1)(b-a)}. \end{aligned} \tag{5.5}$$

If $p=0,$ *then we have*

$$f(I(a,b)) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{(L(a,b)-a)f(a)+(b-L(a,b))f(b)}{b-a}. \tag{5.6}$$

PROOF. If we take $w(t)=t^p$, with $p\neq 0,-1$ in (5.1), then we have

$$\begin{aligned} f\left(\exp\left(\frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt}\right)\right) &\leq \frac{\int_a^b t^p f(t)\,dt}{\int_a^b t^p dt} \\ &\leq \frac{\left(\ln b-\frac{\int_a^b t^p\ln t dt}{\int_a^b t^p dt}\right)f(a)+\left(\frac{\int_a^b t^p\ln t dt}{\int_a^b t^p dt}-\ln a\right)f(b)}{\ln b-\ln a}. \end{aligned} \tag{5.7}$$

We have

$$\int_a^b t^p dt = \frac{1}{p+1}\left(b^{p+1}-a^{p+1}\right)$$

and

$$\int_a^b t^p \ln t dt = \frac{1}{p+1}\left(b^{p+1}\ln b - a^{p+1}\ln a - \frac{1}{p+1}\left(b^{p+1}-a^{p+1}\right)\right).$$

Then

$$\frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt} = \ln\left[I\left(a^{p+1},b^{p+1}\right)\right]^{\frac{1}{p+1}},$$

$$\ln b - \frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt} = \frac{1}{p+1}\frac{L\left(a^{p+1},b^{p+1}\right)-a^{p+1}}{L\left(a^{p+1},b^{p+1}\right)}$$

and, similarly,

$$\frac{\int_a^b t^p \ln t dt}{\int_a^b t^p dt} - \ln a = \frac{1}{p+1}\frac{b^{p+1}-L\left(a^{p+1},b^{p+1}\right)}{L\left(a^{p+1},b^{p+1}\right)}.$$

From (5.7) we then have

$$\begin{aligned} f\left(\exp\left(\ln\left[I\left(a^{p+1},b^{p+1}\right)\right]^{\frac{1}{p+1}}\right)\right) &\leq \frac{\frac{1}{b-a}\int_a^b t^p f(t)\,dt}{\frac{1}{b-a}\int_a^b t^p dt} \\ &\leq \frac{\left(\frac{1}{p+1}\frac{L\left(a^{p+1},b^{p+1}\right)-a^{p+1}}{L\left(a^{p+1},b^{p+1}\right)}\right)f(a)+\left(\frac{1}{p+1}\frac{b^{p+1}-L\left(a^{p+1},b^{p+1}\right)}{L\left(a^{p+1},b^{p+1}\right)}\right)f(b)}{\ln b-\ln a}, \end{aligned}$$

i.e.

$$f\left(\left[I\left(a^{p+1},b^{p+1}\right)\right]^{\frac{1}{p+1}}\right)\leq\frac{\frac{1}{b-a}\int_a^b t^p f(t)\,dt}{L_p^p(a,b)}$$
$$\leq\frac{\left(\frac{1}{p+1}\frac{L\left(a^{p+1},b^{p+1}\right)-a^{p+1}}{L(a^{p+1},b^{p+1})}\right)f(a)+\left(\frac{1}{p+1}\frac{b^{p+1}-L\left(a^{p+1},b^{p+1}\right)}{L(a^{p+1},b^{p+1})}\right)f(b)}{\ln b-\ln a}$$
$$=\frac{\frac{L\left(a^{p+1},b^{p+1}\right)-a^{p+1}}{L(a^{p+1},b^{p+1})}f(a)+\frac{b^{p+1}-L\left(a^{p+1},b^{p+1}\right)}{L(a^{p+1},b^{p+1})}f(b)}{\ln b^{p+1}-\ln a^{p+1}}.$$

By multiplying this inequality with $L_p^p(a,b)$ we get

$$L_p^p(a,b)\,f\left(\left[I\left(a^{p+1},b^{p+1}\right)\right]^{\frac{1}{p+1}}\right)\leq\frac{1}{b-a}\int_a^b t^p f(t)\,dt$$
$$\leq\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\frac{\frac{L\left(a^{p+1},b^{p+1}\right)-a^{p+1}}{L(a^{p+1},b^{p+1})}f(a)+\frac{b^{p+1}-L\left(a^{p+1},b^{p+1}\right)}{L(a^{p+1},b^{p+1})}f(b)}{\ln b^{p+1}-\ln a^{p+1}}$$
$$=\frac{\left(L\left(a^{p+1},b^{p+1}\right)-a^{p+1}\right)f(a)+\left(b^{p+1}-L\left(a^{p+1},b^{p+1}\right)\right)f(b)}{(p+1)(b-a)}.$$

If we perform the calculations in the above inequalities for $p=0$ we get the desired inequality (5.6). We omit the details. ∎

REMARK 5.1. If we take $p=1$ in (5.5), then we get

$$f\left(\sqrt{I\left(a^2,b^2\right)}\right)\leq\frac{1}{b-a}\int_a^b tf(t)\,dt\tag{5.8}$$
$$\leq\frac{\left(A(a,b)\,L(a,b)-a^2\right)f(a)+\left(b^2-A(a,b)\,L(a,b)\right)f(b)}{2(b-a)}.$$

CHAPTER 5

Inequalities for GG-Convex Functions

1. Some Preliminary Facts for GG-convex Functions

The function $f : I \subset (0, \infty) \to (0, \infty)$ is called *GG-convex* on the interval I of real umbers $\mathbb{R}$ if [**1**]

$$f\left(x^{1-\lambda} y^{\lambda}\right) \leq [f(x)]^{1-\lambda} [f(y)]^{\lambda} \tag{1.1}$$

for any $x, y \in I$ and $\lambda \in [0, 1]$. If the inequality is reversed in (1.1) then the function is called *GG-concave*.

This concept was introduced in 1928 by P. Montel [**30**], however, the roots of the research in this area can be traced long before him [**31**].

It is easy to see that [**31**], the function $f : I \subset (0, \infty) \to (0, \infty)$ is *GG-convex* if and only if the the function $g : \ln I \to \mathbb{R}$, $g = \ln \circ f \circ \exp$ is convex on $\ln I$.

It is known that [**31**] every real analytic function $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with non-negative coefficients c_n is a *GG*-convex function on $(0, r)$, where r is the radius of convergence for f. Therefore functions like $\exp$, $\sinh$, $\cosh$ are *GG*-convex on $\mathbb{R}$, $\tan$, $\sec$, $\csc$, $\frac{1}{x} - \cot x$ are *GG*-convex on $\left(0, \frac{\pi}{2}\right)$ and $\frac{1}{1-x}$, $\ln \frac{1}{1-x}$ or $\frac{1+x}{1-x}$ are *GG*-convex on $(0, 1)$. Also, Γ function is a strictly *GG*-convex function on $[1, \infty)$.

It is also known that [**31**], if a function f is *GG*-convex, then so is $x^{\alpha} f^{\beta}(x)$ for all $\alpha \in \mathbb{R}$ and all $\beta > 0$. If f is continuous, and one of the functions $f(x)^x$ and $f(e^{1/\log x})$ is *GG*-convex, then so is the other.

The following result is due to P. Montel [**30**]:

PROPOSITION 1.1. *Let $f : [0, a] \to [0, \infty)$ be a continuous GG-convex on $(0, a)$, then $F(x) := \int_0^x f(t)\, dt$ is continuous GG-convex on $(0, a)$.*

Therefore, as pointed out in [**31**], the *Lobacevski's function*

$$L(x) := -\int_0^x \ln(\cos t)\, dt$$

is *GG*-convex on $(0, \pi/2)$ and the integral sine

$$\operatorname{Si}(x) := \int_0^x \frac{\sin t}{t} dt$$

is *GG*-concave on $(0, \pi/2)$.

The following characterizations hold [**31**].

PROPOSITION 1.2. *Let $f : I \subset (0, \infty) \to (0, \infty)$ be a differentiable function on the interior $\mathring{I}$ of I. The following statements are equivalent:*

(i) The function f is GG-convex on I;

(ii) The function $\frac{xf'(x)}{f(x)}$ is nondecreasing on $\mathring{I}$;

(iii) We have the inequality

$$\frac{f(x)}{f(y)} \geq \left(\frac{x}{y}\right)^{\frac{yf'(y)}{f(y)}} \tag{1.2}$$

for any $x \in I$ *and* $y \in \mathring{I}$;

and

PROPOSITION 1.3. *Let* $f : I \subset (0, \infty) \to (0, \infty)$ *be a twice differentiable function on the interior* $\mathring{I}$ *of* I*. The following statements are equivalent:*
(i) The function f *is GG-convex on* I;
(ii) We have the inequality

$$x \left[f(x) f''(x) - (f'(x))^2 \right] + f(x) f'(x) \geq 0 \tag{1.3}$$

for any $x \in \mathring{I}$.

In 2010, Zhang and Zheng [**37**] proved the following inequality for a *GG*-convex function f on $[a, b]$:

$$\frac{1}{\ln b - \ln a} \int_a^b f(t)\, dt \leq L(af(a), bf(b)). \tag{1.4}$$

In 2011, Mitroi and Spiridon [**29**] established amongst other the following double inequality:

$$f(I(a,b)) \leq \exp\left(\frac{1}{b-a} \int_a^b \ln f(t)\, dt \right) \leq [f(b)]^{\frac{b-L(a,b)}{b-a}} [f(a)]^{\frac{L(a,b)-a}{b-a}}, \tag{1.5}$$

where $f : [a, b] \subset (0, \infty) \to (0, \infty)$ is *GG*-convex and $I(a, b)$ is the *identric mean*.

In 2013, Işcan [**25**] also proved the following result:

$$f\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{1}{t} \sqrt{f(t) f\left(\frac{ab}{t}\right)} dt \tag{1.6}$$

$$\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \leq L(f(a), f(b))$$

provided that $f : [a, b] \subset (0, \infty) \to (0, \infty)$ is *GG*-convex.

The function $g : I \subset (0, \infty) \to \mathbb{R}$ is called *GA*-convex if [**1**]

$$g\left(x^{1-\lambda} y^{\lambda}\right) \leq (1 - \lambda) g(x) + \lambda g(y) \tag{1.7}$$

for any $x, y \in I$ and $\lambda \in [0, 1]$.

One can observe that the function $f : I \subset (0, \infty) \to (0, \infty)$ is *GG*-convex if and only if $g := \ln f$ is *GA*-convex on I.

In order to prove in a different way (1.5) by using *GA*-convexity we use the following result:

LEMMA 1.4. *If* $b > a > 0$ *and* $g : [a, b] \to \mathbb{R}$ *is a GA-convex function on* $[a, b]$, *then*

$$g(I(a,b)) \leq \frac{1}{b-a} \int_a^b g(t)\, dt \leq \frac{b - L(a,b)}{b-a} g(b) + \frac{L(a,b) - a}{b-a} g(a). \tag{1.8}$$

This result was proved in the case of differentiable functions in [**38**] and in general case in [**11**].

Since f is *GG*-convex, then $g := \ln f$ is *GA*-convex and by (1.8) we have

$$\ln f(I(a,b)) \leq \frac{1}{b-a} \int_a^b \ln f(t)\, dt$$

$$\leq \frac{b - L(a,b)}{b-a} \ln f(b) + \frac{L(a,b) - a}{b-a} \ln f(a)$$

that is equivalent to

$$\ln f\left(I\left(a,b\right)\right)\leq\frac{1}{b-a}\int_{a}^{b}\ln f\left(t\right)dt\leq\ln\left(\left[f\left(b\right)\right]^{\frac{b-L(a,b)}{b-a}}\left[f\left(a\right)\right]^{\frac{L(a,b)-a}{b-a}}\right)$$

and by taking the exponential, with the desired inequality (1.5).

Motivated by the above results we establish in this paper some new inequalities of Hermite-Hadamard type for *GG*-convex functions. Applications for special means are also provided.

2. INEQUALITIES FOR GG-CONVEX FUNCTIONS

We have the following generalization of Zhang and Zheng result (1.4):

THEOREM 2.1 (Dragomir, 2015 [**13**]). *Let* $f:[a,b]\subset(0,\infty)\to(0,\infty)$ *be a GG-convex function on* $[a,b]$. *Then for every* $p>0$ *and* $q\in\mathbb{R}$ *we have the inequality:*

$$\frac{1}{\ln b-\ln a}\int_{a}^{b}t^{q-1}f^{p}\left(t\right)dt\leq L\left(a^{q}f^{p}\left(a\right),b^{q}f^{p}\left(b\right)\right). \tag{2.1}$$

For $p=1$ we then have (see (1.6))

$$\frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{f\left(t\right)}{t}dt\leq L\left(f\left(a\right),f\left(b\right)\right) \tag{2.2}$$

for $q=0$, and (see (1.4))

$$\frac{1}{\ln b-\ln a}\int_{a}^{b}f\left(t\right)dt\leq L\left(af\left(a\right),bf\left(b\right)\right) \tag{2.3}$$

for $q=1$.

From a different perspective we have:

THEOREM 2.2 (Dragomir, 2015 [**13**]). *Let* $f:[a,b]\subset(0,\infty)\to(0,\infty)$ *be a GG-convex function on* $[a,b]$. *If* $p>0$ *and*

(i) $q\in\mathbb{R}\setminus\{0,1\}$, *then we have the inequalities*

$$\begin{aligned}f\left(\sqrt{ab}\right)\left(L_{q-1}^{q-1}\left(a,b\right)\right)^{\frac{1}{2p}}&\leq\left(\frac{1}{b-a}\int_{a}^{b}f^{p}\left(t\right)f^{p}\left(\frac{ab}{t}\right)t^{q-1}dt\right)^{\frac{1}{2p}}\\&\leq\sqrt{f\left(a\right)f\left(b\right)}\left(L_{q-1}^{q-1}\left(a,b\right)\right)^{\frac{1}{2p}}.\end{aligned} \tag{2.4}$$

(ii) We have the inequalities

$$f\left(\sqrt{ab}\right)\leq\left(\frac{1}{b-a}\int_{a}^{b}f^{p}\left(t\right)f^{p}\left(\frac{ab}{t}\right)dt\right)^{\frac{1}{2p}}\leq\sqrt{f\left(a\right)f\left(b\right)}. \tag{2.5}$$

(iii) We have the inequalities

$$f\left(\sqrt{ab}\right)\leq\left(\frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{f^{p}\left(t\right)f^{p}\left(\frac{ab}{t}\right)}{t}dt\right)^{\frac{1}{2p}}\leq\sqrt{f\left(a\right)f\left(b\right)}. \tag{2.6}$$

If we take $p=\frac{1}{2}$ above, then we get

$$\begin{aligned}f\left(\sqrt{ab}\right)L_{q-1}^{q-1}\left(a,b\right)&\leq\frac{1}{b-a}\int_{a}^{b}t^{q-1}\sqrt{f\left(t\right)f\left(\frac{ab}{t}\right)}dt\\&\leq L_{q-1}^{q-1}\left(a,b\right)\sqrt{f\left(a\right)f\left(b\right)}\end{aligned} \tag{2.7}$$

for $q \in \mathbb{R}\setminus\{0,1\}$,

$$f\left(\sqrt{ab}\right) \le \frac{1}{b-a}\int_a^b \sqrt{f(t)\, f\left(\frac{ab}{t}\right)}dt \le \sqrt{f(a)\, f(b)} \tag{2.8}$$

and

$$f\left(\sqrt{ab}\right) \le \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t}\sqrt{f(t)\, f\left(\frac{ab}{t}\right)}dt \le \sqrt{f(a)\, f(b)}. \tag{2.9}$$

The inequality (2.9) was stated for *GG*-convex and monotonically decreasing functions in [**26**].

If we take $p = 1$ above, then we also have

$$\begin{aligned} f\left(\sqrt{ab}\right)\sqrt{L_{q-1}^{q-1}(a,b)} &\le \sqrt{\frac{1}{b-a}\int_a^b f(t)\, f\left(\frac{ab}{t}\right) t^{q-1}dt} \\ &\le \sqrt{f(a)\, f(b)}\sqrt{L_{q-1}^{q-1}(a,b)}, \end{aligned} \tag{2.10}$$

$$f\left(\sqrt{ab}\right) \le \sqrt{\frac{1}{b-a}\int_a^b f(t)\, f\left(\frac{ab}{t}\right) dt} \le \sqrt{f(a)\, f(b)} \tag{2.11}$$

for $q \in \mathbb{R}\setminus\{0,1\}$, and

$$f\left(\sqrt{ab}\right) \le \sqrt{\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)\, f\left(\frac{ab}{t}\right)}{t}dt} \le \sqrt{f(a)\, f(b)}. \tag{2.12}$$

The inequality (2.11) was stated for *GG*-convex and monotonically decreasing functions in [**36**].

The following result holds:

THEOREM 2.3 (Dragomir, 2015 [**13**]). *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$. Then for any $\lambda \in [0,1]$ we have*

$$\begin{aligned} f\left(\sqrt{ab}\right) &\le \left[f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right)\right]^{1-\lambda}\left[f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right)\right]^{\lambda} \\ &\le \exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(t)}{t}dt\right) \\ &\le \sqrt{f\left(a^{1-\lambda}b^{\lambda}\right)[f(b)]^{1-\lambda}[f(a)]^{\lambda}} \le \sqrt{f(a)\, f(b)}. \end{aligned} \tag{2.13}$$

If we take in (2.13) $\lambda = \frac{1}{2}$, then we get

$$\begin{aligned} f\left(\sqrt{ab}\right) &\le \sqrt{f\left(\sqrt[4]{ab^3}\right) f\left(\sqrt[4]{a^3 b}\right)} \\ &\le \exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(t)}{t}dt\right) \\ &\le \sqrt{f\left(\sqrt{ab}\right)}\sqrt[4]{f(b)\, f(a)} \le \sqrt{f(a)\, f(b)}. \end{aligned} \tag{2.14}$$

We also have:

THEOREM 2.4 (Dragomir, 2015 [**13**]). *Let* $f:[a,b]\subset(0,\infty)\to(0,\infty)$ *be a GG-convex function on* $[a,b]$. *Then we have the inequalities*

$$1\le\frac{\sqrt{f(a)f(b)}}{\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right)}\le\left(\frac{b}{a}\right)^{\frac{1}{8}\left[\frac{f'_-(b)b}{f(b)}-\frac{f'_+(a)a}{f(a)}\right]} \tag{2.15}$$

and

$$1\le\frac{\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right)}{f\left(\sqrt{ab}\right)}\le\left(\frac{b}{a}\right)^{\frac{1}{8}\left[\frac{f'_-(b)b}{f(b)}-\frac{f'_+(a)a}{f(a)}\right]}. \tag{2.16}$$

We also have:

THEOREM 2.5 (Dragomir, 2015 [**13**]). *Let* $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ *be a GG-convex function on* $[a,b]$. *Then for any* $t\in[a,b]$ *we have*

$$\begin{aligned}&\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right)\\&\le\sqrt{f(t)}\sqrt{f(b)^{\frac{\ln b-\ln t}{\ln b-\ln a}}f(a)^{\frac{\ln t-\ln a}{\ln b-\ln a}}}\le\sqrt{f(a)f(b)}.\end{aligned} \tag{2.17}$$

If we take in (2.17) $t=G(a,b)=\sqrt{ab}$, then we get the second part of (2.14), namely

$$\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(t)}{t}dt\right)\le\sqrt{f\left(\sqrt{ab}\right)}\sqrt[4]{f(b)f(a)}\le\sqrt{f(a)f(b)}.$$

If we take $t=I(a,b)$ in (2.17) we also get

$$\begin{aligned}&\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right)\\&\le\sqrt{f(I(a,b))}\sqrt{f(b)^{\frac{\ln b-\ln I(a,b)}{\ln b-\ln a}}f(a)^{\frac{\ln I(a,b)-\ln a}{\ln b-\ln a}}}\le\sqrt{f(a)f(b)}.\end{aligned} \tag{2.18}$$

Observe that

$$\frac{\ln b-\ln I(a,b)}{\ln b-\ln a}=\frac{\ln b-\frac{b\ln b-a\ln a}{b-a}+1}{\ln b-\ln a}=\frac{L(a,b)-a}{b-a}$$

and, similarly,

$$\frac{\ln I(a,b)-\ln a}{\ln b-\ln a}=\frac{b-L(a,b)}{b-a}.$$

Therefore the inequality (2.18) is equivalent to:

$$\begin{aligned}&\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right)\\&\le\sqrt{f(I(a,b))}\sqrt{f(b)^{\frac{L(a,b)-a}{b-a}}f(a)^{\frac{b-L(a,b)}{b-a}}}\le\sqrt{f(a)f(b)}.\end{aligned} \tag{2.19}$$

We also have

$$\begin{aligned}&\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right)\\&\le\exp\left(\frac{1}{2}\cdot\frac{1}{b-a}\int_a^b\ln f(t)\,dt\right)\sqrt{f(b)^{\frac{L(a,b)-a}{b-a}}f(a)^{\frac{b-L(a,b)}{b-a}}}\le\sqrt{f(a)f(b)}.\end{aligned} \tag{2.20}$$

Since : $[a,b] \subset (0,\infty) \rightarrow (0,\infty)$ is a *GG*-convex function on $[a,b]$, then for any $p>0$ we have

$$f^{p}\left(a^{1-\lambda}b^{\lambda}\right) \leq \left[f^{p}(a)\right]^{1-\lambda}\left[f^{p}(b)\right]^{\lambda} \tag{2.21}$$

for all $\lambda \in [0,1]$.

Now if we multiply (2.21) by $\left(a^{1-\lambda}b^{\lambda}\right)^{q}>0$ for $q\in\mathbb{R}$ we get

$$\begin{aligned} f^{p}\left(a^{1-\lambda}b^{\lambda}\right)\left(a^{1-\lambda}b^{\lambda}\right)^{q} &\leq \left[f^{p}(a)\right]^{1-\lambda}\left[f^{p}(b)\right]^{\lambda}\left(a^{1-\lambda}b^{\lambda}\right)^{q} \\ &= \left[a^{q}f^{p}(a)\right]^{1-\lambda}\left[b^{q}f^{p}(b)\right]^{\lambda} \end{aligned} \tag{2.22}$$

for all $\lambda \in [0,1]$.

If we integrate the inequality (2.22) over $\lambda \in [0,1]$, then we get

$$\int_0^1 f^{p}\left(a^{1-\lambda}b^{\lambda}\right)\left(a^{1-\lambda}b^{\lambda}\right)^{q} d\lambda \leq \int_0^1 \left[a^{q}f^{p}(a)\right]^{1-\lambda}\left[b^{q}f^{p}(b)\right]^{\lambda} d\lambda. \tag{2.23}$$

Observe that

$$\int_0^1 \left[a^{q}f^{p}(a)\right]^{1-\lambda}\left[b^{q}f^{p}(b)\right]^{\lambda} d\lambda = L\left(a^{q}f^{p}(a), b^{q}f^{p}(b)\right)$$

and by the change of variable $a^{1-\lambda}b^{\lambda}=t$ we have

$$\int_0^1 f^{p}\left(a^{1-\lambda}b^{\lambda}\right)\left(a^{1-\lambda}b^{\lambda}\right)^{q} d\lambda = \frac{1}{\ln b-\ln a}\int_a^b t^{q-1}f^{p}(t)\,dt,$$

which, together with (2.23) produces the desired result (2.1)

This proves Theorem 2.1.

To prove Theorem 2.2, we observe that if f is *GG*-convex, then we have, as in [**25**]:

$$f\left(\sqrt{ab}\right) \leq \sqrt{f\left(a^{1-\lambda}b^{\lambda}\right) f\left(a^{\lambda}b^{1-\lambda}\right)} \leq \sqrt{f(a)\,f(b)} \tag{2.24}$$

for all $\lambda \in [0,1]$.

If we take the power $2p>0$ in (2.24) we get

$$f^{2p}\left(\sqrt{ab}\right) \leq f^{p}\left(a^{1-\lambda}b^{\lambda}\right) f^{p}\left(a^{\lambda}b^{1-\lambda}\right) \leq f^{p}(a)\,f^{p}(b) \tag{2.25}$$

for all $\lambda \in [0,1]$.

By multiplying with $\left(a^{1-\lambda}b^{\lambda}\right)^{q}>0$ for $q\in\mathbb{R}\setminus\{0\}$ and integrating over λ on $[0,1]$ we get

$$\begin{aligned} f^{2p}\left(\sqrt{ab}\right)\int_0^1 \left(a^{1-\lambda}b^{\lambda}\right)^{q} d\lambda &\leq \int_0^1 f^{p}\left(a^{1-\lambda}b^{\lambda}\right) f^{p}\left(a^{\lambda}b^{1-\lambda}\right)\left(a^{1-\lambda}b^{\lambda}\right)^{q} d\lambda \\ &\leq f^{p}(a)\,f^{p}(b)\int_0^1 \left(a^{1-\lambda}b^{\lambda}\right)^{q} d\lambda. \end{aligned} \tag{2.26}$$

Since

$$\int_0^1 \left(a^{1-\lambda}b^{\lambda}\right)^{q} d\lambda = \int_0^1 \left(a^{q}\right)^{1-\lambda}\left(b^{q}\right)^{\lambda} d\lambda = L\left(a^{q},b^{q}\right) = L_{q-1}^{q-1}(a,b)\,L(a,b)$$

for $q\neq 1$ and for $q=1$, $\int_0^1 a^{1-\lambda}b^{\lambda}d\lambda = L(a,b)$, while

$$\int_0^1 f^{p}\left(a^{1-\lambda}b^{\lambda}\right) f^{p}\left(a^{\lambda}b^{1-\lambda}\right)\left(a^{1-\lambda}b^{\lambda}\right)^{q} d\lambda = \frac{1}{\ln b-\ln a}\int_a^b f^{p}(t)\,f^{p}\left(\frac{ab}{t}\right) t^{q-1}dt,$$

then from (2.26) we have for $q \neq 0, 1$ that

$$f^{2p}\left(\sqrt{ab}\right) L_{q-1}^{q-1}(a,b) L(a,b) \leq \frac{1}{\ln b - \ln a} \int_a^b f^p(t) f^p\left(\frac{ab}{t}\right) t^{q-1} dt \leq f^p(a) f^p(b) L_{q-1}^{q-1}(a,b) L(a,b). \tag{2.27}$$

For $q = 1$ we also have

$$f^{2p}\left(\sqrt{ab}\right) \leq \frac{1}{b-a} \int_a^b f^p(t) f^p\left(\frac{ab}{t}\right) dt \leq f^p(a) f^p(b). \tag{2.28}$$

If $q = 0$, then by (2.25) we have

$$f^{2p}\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f^p(t) f^p\left(\frac{ab}{t}\right)}{t} dt \leq f^p(a) f^p(b) \tag{2.29}$$

and Theorem 2.2 is completely proved.

In order to prove Theorem 2.3 we use the following result for *GA*-convex functions [**11**].

LEMMA 2.6. *Let $g : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a GA-convex function on $[a,b]$. Then for any $\lambda \in [0,1]$ we have*

$$\begin{aligned} g\left(\sqrt{ab}\right) &\leq (1-\lambda) g\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right) + \lambda g\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right) \\ &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \\ &\leq \frac{1}{2}\left[g\left(a^{1-\lambda} b^{\lambda}\right) + (1-\lambda) g(b) + \lambda g(a)\right] \leq \frac{g(a) + g(b)}{2}. \end{aligned} \tag{2.30}$$

If f is a *GG*-convex function, then by writing the inequality (2.30) for $g = \ln f$ we have

$$\begin{aligned} \ln f\left(\sqrt{ab}\right) &\leq \ln\left(\left[f\left(a^{\frac{1-\lambda}{2}} b^{\frac{\lambda+1}{2}}\right)\right]^{1-\lambda} \left[f\left(a^{\frac{2-\lambda}{2}} b^{\frac{\lambda}{2}}\right)\right]^{\lambda}\right) \\ &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{\ln f(t)}{t} dt \\ &\leq \ln \sqrt{f\left(a^{1-\lambda} b^{\lambda}\right) [f(b)]^{1-\lambda} [f(a)]^{\lambda}} \leq \ln \sqrt{f(a) f(b)}. \end{aligned} \tag{2.31}$$

By taking the exponential in (2.31) we get the desired result (2.13).

We have the following reverse inequalities [**11**].

LEMMA 2.7. *Let $g : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a GA-convex function on $[a,b]$. Then we have*

$$\begin{aligned} 0 &\leq \frac{g(a) + g(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{g(s)}{s} ds \\ &\leq \frac{1}{8}\left[g'_-(b) b - g'_+(a) a\right] (\ln b - \ln a) \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{g(s)}{s} ds - g\left(\sqrt{ab}\right) \\ &\leq \frac{1}{8}\left[g'_-(b) b - g'_+(a) a\right] (\ln b - \ln a). \end{aligned} \tag{2.33}$$

If we take in (2.32) $g = \ln f$, then we get

$$\frac{\ln f(a) + \ln f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s}ds \leq \frac{1}{8}\left[\frac{f'_-(b)\,b}{f(b)} - \frac{f'_+(a)\,a}{f(a)}\right](\ln b - \ln a),$$

to

$$\ln\sqrt{f(a)f(b)} \leq \ln\left(\exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s}ds\right)\left(\frac{b}{a}\right)^{\frac{1}{8}\left[\frac{f'_-(b)b}{f(b)} - \frac{f'_+(a)a}{f(a)}\right]}\right). \tag{2.34}$$

Taking the exponential in (2.34) we get

$$\sqrt{f(a)f(b)} \leq \exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s}ds\right)\left(\frac{b}{a}\right)^{\frac{1}{8}\left[\frac{f'_-(b)b}{f(b)} - \frac{f'_+(a)a}{f(a)}\right]}$$

and the inequality (2.15).

The inequality (2.16) follows by (2.33) in a similar way and the details are omitted.

Further, we use the following result for *GA*-convex functions [**11**].

LEMMA 2.8. *Let* $g : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be a GA-convex function on* $[a,b]$. *Then for any* $t \in [a,b]$ *we have*

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{g(s)}{s}ds \leq (\geq)\,\frac{1}{2}\left[g(t) + \frac{g(b)(\ln b - \ln t) + g(a)(\ln t - \ln a)}{\ln b - \ln a}\right] \leq (\geq)\,\frac{g(a) + g(b)}{2}. \tag{2.35}$$

If f is *GG*-convex, then if we write (2.35) for $g = \ln f$, we have

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s}ds \leq \ln\left(\sqrt{f(t)}\sqrt{f(b)^{\frac{\ln b - \ln t}{\ln b - \ln a}} f(a)^{\frac{\ln t - \ln a}{\ln b - \ln a}}}\right) \leq \ln\sqrt{f(a)f(b)} \tag{2.36}$$

and by taking the exponential in (2.36) we get the desired result (2.17).

If we take the integral mean in (2.35) we get

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{g(s)}{s}ds \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b g(t)\,dt + \frac{g(b)\left(\ln b - \frac{1}{b-a}\int_a^b \ln t dt\right) + g(a)\left(\frac{1}{b-a}\int_a^b \ln t dt - \ln a\right)}{\ln b - \ln a}\right] \leq \frac{g(a) + g(b)}{2}, \tag{2.37}$$

namely

$$\begin{aligned}(2.38)\quad &\frac{1}{\ln b-\ln a}\int_a^b \frac{g(s)}{s}ds\\ &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b g(t)\,dt+\frac{g(b)(\ln b-\ln I(a,b))+g(a)(\ln I(a,b)-\ln a)}{\ln b-\ln a}\right]\\ &\leq \frac{g(a)+g(b)}{2}.\end{aligned}$$

Since

$$\begin{aligned}&\frac{g(b)(\ln b-\ln I(a,b))+g(a)(\ln I(a,b)-\ln a)}{\ln b-\ln a}\\ &=\frac{L(a,b)-a}{b-a}g(b)+\frac{b-L(a,b)}{b-a}g(a),\end{aligned}$$

then by (2.38) we have

$$\begin{aligned}&\frac{1}{\ln b-\ln a}\int_a^b \frac{g(s)}{s}ds\\ &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b g(t)\,dt+\frac{L(a,b)-a}{b-a}g(b)+\frac{b-L(a,b)}{b-a}g(a)\right]\\ &\leq \frac{g(a)+g(b)}{2}.\end{aligned}$$

Writing this inequality for $\ln f$ we have

$$\begin{aligned}&\frac{1}{\ln b-\ln a}\int_a^b \frac{\ln f(s)}{s}ds\\ &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \ln f(t)\,dt+\frac{L(a,b)-a}{b-a}\ln f(b)+\frac{b-L(a,b)}{b-a}\ln f(a)\right]\\ &\leq \frac{\ln f(a)+\ln f(b)}{2}\end{aligned}$$

and the inequality (2.20) is thus proved.

3. SEVERAL REFINEMENTS

We have the following result for an arbitrary division:

THEOREM 3.1 (Dragomir, 2015 [**14**]). *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GG-convex function on $[a,b]$. For a division $0=\lambda_0<\lambda_1<...<\lambda_{n-1}<\lambda_n=1$ with $n\geq 1$ of the interval $[0,1]$, we have*

$$\begin{aligned}(3.1)\quad f\left(\sqrt{ab}\right) &\leq \prod_{i=0}^{n-1}\left[f\left(a^{1-\frac{\lambda_i+\lambda_{i+1}}{2}}b^{\frac{\lambda_i+\lambda_{i+1}}{2}}\right)\right]^{\lambda_{i+1}-\lambda_i}\\ &\leq \exp\left(\frac{1}{\ln b-\ln a}\int_a^b \frac{\ln f(s)}{s}ds\right)\\ &\leq \prod_{i=0}^{n-1}\left[f\left(a^{1-\lambda_i}b^{\lambda_i}\right)f\left(a^{1-\lambda_{i+1}}b^{\lambda_{i+1}}\right)\right]^{\frac{\lambda_{i+1}-\lambda_i}{2}}\leq\sqrt{f(a)f(b)}.\end{aligned}$$

For a division of the interval $[a,b]$ we also have:

THEOREM 3.2 (Dragomir, 2015 [**14**]). *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GG-convex function on $[a,b]$. Then for any division $a=t_0<t_1<...<t_{n-1}<t_n=b$ with $n\geq 1$ we have the inequalities*

$$f\left(\sqrt{ab}\right)\leq\left(\prod_{i=0}^{n-1}\left[f\left(\sqrt{t_it_{i+1}}\right)\right]^{\ln\left(\frac{t_{i+1}}{t_i}\right)}\right)^{\frac{1}{\ln\left(\frac{b}{a}\right)}} \tag{3.2}$$
$$\leq\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right)$$
$$\leq\prod_{i=0}^{n-1}\left(\left[\sqrt{f(t_i)f(t_{i+1})}\right]^{\ln\left(\frac{t_{i+1}}{t_i}\right)}\right)^{\frac{1}{\ln\left(\frac{b}{a}\right)}}\leq\sqrt{f(a)f(b)}.$$

If we write the inequality (3.2) for $a=t_0<t_1=t<t_2=b$, then we get

$$f\left(\sqrt{ab}\right)\leq\left[f\left(\sqrt{at}\right)\right]^{\frac{\ln t-\ln a}{\ln b-\ln a}}\left[f\left(\sqrt{tb}\right)\right]^{\frac{\ln b-\ln t}{\ln b-\ln a}} \tag{3.3}$$
$$\leq\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right)$$
$$\leq\sqrt{f(t)}\sqrt{[f(a)]^{\frac{\ln t-\ln a}{\ln b-\ln a}}[f(b)]^{\frac{\ln b-\ln t}{\ln b-\ln a}}}\leq\sqrt{f(a)f(b)}$$

for any $t\in[a,b]$.

From a different perspective we have:

THEOREM 3.3 (Dragomir, 2015 [**14**]). *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a GG-convex function on $[a,b]$. Then for any division $a=t_0<t_1<...<t_{n-1}<t_n=b$ with $n\geq 1$ we have the inequalities*

$$f\left(\sqrt{ab}\right)\leq\exp\left[\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right] \tag{3.4}$$
$$\leq\frac{1}{\ln b-\ln a}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\frac{1}{s}\sqrt{f(s)f\left(\frac{t_it_{i+1}}{s}\right)}ds$$
$$\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(s)}{s}ds.$$

If we take $n=1$ in (3.4), then we have

$$f\left(\sqrt{ab}\right)\leq\exp\left[\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right] \tag{3.5}$$
$$\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{s}\sqrt{f(s)f\left(\frac{ab}{s}\right)}ds$$
$$\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(s)}{s}ds.$$

The first inequality in (3.5) provides a refinement of (1.6).

If $a \le t \le b$, then by (3.5) written for $n = 2$ we get

$$f\left(\sqrt{ab}\right) \le \exp\left[\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(s)}{s} ds\right] \le \frac{1}{\ln b - \ln a}\left[\int_a^t \frac{1}{s}\sqrt{f(s) f\left(\frac{at}{s}\right)} ds + \int_t^b \frac{1}{s}\sqrt{f(s) f\left(\frac{tb}{s}\right)} ds\right] \le \frac{1}{\ln b - \ln a}\int_a^b \frac{f(s)}{s} ds. \tag{3.6}$$

We have the following results for differentiable functions:

THEOREM 3.4 (Dragomir, 2015 [**14**]). *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a differentiable GG-convex function on $[a,b]$. Then we have the inequalities*

$$\frac{\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t} dt}{f\left(\sqrt{ab}\right)} \ge L\left(\left(\frac{b}{a}\right)^{\frac{f'(\sqrt{ab})\sqrt{ab}}{2f(\sqrt{ab})}}, \left(\frac{b}{a}\right)^{-\frac{f'(\sqrt{ab})\sqrt{ab}}{2f(\sqrt{ab})}}\right) \ge 1 \tag{3.7}$$

and

$$\frac{\sqrt{f(a) f(b)}}{\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t} dt} \ge 1 + \log\left[\frac{\int_a^b \frac{f(t)}{t} dt}{\int_a^b \frac{f(t)}{t}\left(\frac{\sqrt{ab}}{t}\right)^{\frac{f'(t)t}{f(t)}} dt}\right] \ge 1 + \log\left[\frac{\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t} dt}{f\left(\sqrt{ab}\right)}\right] \ge 1 \tag{3.8}$$

where $L(\cdot,\cdot)$ is the logarithmic mean.

Now we can state the following result for weighted integrals:

THEOREM 3.5 (Dragomir, 2015 [**14**]). *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$ and $w : [a,b] \to [0,\infty)$ an integrable function on $[a,b]$, then*

$$f\left(\exp\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t) dt}\right)\right) \le \exp\left(\frac{\int_a^b w(t)\ln f(t) dt}{\int_a^b w(t) dt}\right) \le [f(a)]^{\frac{\ln b - \frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t) dt}}{\ln b - \ln a}} [f(b)]^{\frac{\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t) dt} - \ln a}{\ln b - \ln a}}. \tag{3.9}$$

In particular we have for any *GG*-convex function $f : [a,b] \subset (0,\infty) \to (0,\infty)$ that

$$\left[f\left(\left[I\left(a^{p+1}, b^{p+1}\right)\right]^{\frac{1}{p+1}}\right)\right]^{L_p^p(a,b)} \le \exp\left(\frac{1}{b-a}\int_a^b t^p \ln f(t) dt\right) \le [f(a)]^{\frac{L\left(a^{p+1},b^{p+1}\right) - a^{p+1}}{(p+1)(b-a)}} [f(b)]^{\frac{b^{p+1} - L\left(a^{p+1},b^{p+1}\right)}{(p+1)(b-a)}}, \tag{3.10}$$

for any $p \in \mathbb{R}$ with $p \neq 0, -1$.

If $p = 0$, namely we take $w(t) = 1$ in (3.9), then we get (1.5).

In the paper [**12**] we obtained the following result for a division of the interval $[0,1]$:

LEMMA 3.6. *Let* $g : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be a GA-convex function on* $[a,b]$. *For any division* $0 = \lambda_0 < \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 1$ *with* $n \geq 1$ *of the interval* $[0,1]$, *we have*

$$g\left(\sqrt{ab}\right) \leq \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) g\left(a^{1-\frac{\lambda_i+\lambda_{i+1}}{2}} b^{\frac{\lambda_i+\lambda_{i+1}}{2}}\right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{g(s)}{s} ds \leq \sum_{i=0}^{n-1} (\lambda_{i+1} - \lambda_i) \frac{g\left(a^{1-\lambda_i} b^{\lambda_i}\right) + g\left(a^{1-\lambda_{i+1}} b^{\lambda_{i+1}}\right)}{2} \leq \frac{1}{2} [g(a) + g(b)]. \tag{3.11}$$

If we write the inequality (3.11) for the *GA*-convex function $g = \ln f$, where f is *GG*-convex, then we have

$$\ln f\left(\sqrt{ab}\right) \leq \ln \left[\prod_{i=0}^{n-1} f\left(a^{1-\frac{\lambda_i+\lambda_{i+1}}{2}} b^{\frac{\lambda_i+\lambda_{i+1}}{2}}\right)\right]^{(\lambda_{i+1}-\lambda_i)} \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{\ln f(s)}{s} ds \leq \ln \prod_{i=0}^{n-1} \left[f\left(a^{1-\lambda_i} b^{\lambda_i}\right) f\left(a^{1-\lambda_{i+1}} b^{\lambda_{i+1}}\right)\right]^{\frac{\lambda_{i+1}-\lambda_i}{2}} \leq \ln \sqrt{f(a) f(b)}. \tag{3.12}$$

By taking the exponential in (3.12) we get the desired result (3.11).

In [**12**] we also proved the following result for a division of the interval $[a,b]$:

LEMMA 3.7. *Let* $g : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be a GA-convex function on* $[a,b]$. *Then for any division* $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$ *with* $n \geq 1$ *we have the inequalities*

$$g\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) g\left(\sqrt{t_i t_{i+1}}\right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{g(s)}{s} ds \leq \frac{1}{\ln b - \ln a} \sum_{i=0}^{n-1} (\ln t_{i+1} - \ln t_i) \frac{g(t_i) + g(t_{i+1})}{2} \leq \frac{1}{2} [f(a) + f(b)]. \tag{3.13}$$

If we write the inequality for the *GA*-convex function $g = \ln f$, where f is *GG*-convex, then we have

$$\ln f\left(\sqrt{ab}\right) \leq \ln \left(\prod_{i=0}^{n-1} \left[f\left(\sqrt{t_i t_{i+1}}\right)\right]^{\ln\left(\frac{t_{i+1}}{t_i}\right)}\right)^{\frac{1}{\ln\left(\frac{b}{a}\right)}} \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{\ln f(s)}{s} ds \leq \ln \prod_{i=0}^{n-1} \left(\left[f(t_i) f(t_{i+1})\right]^{\frac{1}{2}\ln\left(\frac{t_{i+1}}{t_i}\right)}\right)^{\frac{1}{\ln\left(\frac{b}{a}\right)}} \leq \ln \sqrt{f(a) f(b)}. \tag{3.14}$$

By taking the exponential in (3.14) we get the desired result (3.2).

In paper [**8**] we have established the following result for log-convex functions:

LEMMA 3.8. *Let* $g : [c, d] \to (0, \infty)$ *be a log-convex function on* $[c, d]$ *and* $c = y_0 < y_1 < \ldots < y_{n-1} < y_n = d$ *an arbitrary division of* $[c, d]$ *with* $n \geq 1$*. Then*

$$\exp\left[\frac{1}{d-c}\int_c^d \ln g(y)\,dy\right] \leq \frac{1}{d-c}\sum_{i=0}^{n-1}\int_{y_i}^{y_{i+1}} \sqrt{g(y)\,g(y_i+y_{i+1}-y)}dy \leq \frac{1}{d-c}\int_c^d g(y)\,dy. \tag{3.15}$$

If $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ is a *GG*-convex function on $[a, b]$ and $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ with $n \geq 1$ is a division of $[a, b]$ then by taking $g := f \circ \exp$, $y_i := \ln t_i$, $i \in \{0, \ldots, n-1\}$, $c = \ln a$ and $d = \ln b$ we have that g is log-convex on $[c, d]$ and $y_i := \ln t_i$, $i \in \{0, \ldots, n-1\}$ is a division of $[c, d]$.

By writing the inequality (3.15) for these choices, we have

$$\exp\left[\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \ln f(e^y)\,dy\right] \leq \frac{1}{\ln b - \ln a}\sum_{i=0}^{n-1}\int_{\ln t_i}^{\ln t_{i+1}} \sqrt{f(e^y)\,f(e^{\ln t_i+\ln t_{i+1}-y})}dy \leq \frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} f(e^y)\,dydy, \tag{3.16}$$

that is equivalent to

$$\exp\left[\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \ln f(e^y)\,dy\right] \leq \frac{1}{\ln b - \ln a}\sum_{i=0}^{n-1}\int_{\ln t_i}^{\ln t_{i+1}} \sqrt{f(e^y)\,f\left(\frac{t_i t_{i+1}}{e^y}\right)}dy \leq \frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} f(e^y)\,dy. \tag{3.17}$$

If we make the change of variable $e^y = t$ in (3.17), then we deduce the desired result (3.4).

In [**4**] we proved the following result for log-convex functions:

LEMMA 3.9. *Let* $g : I \to (0, \infty)$ *be a differentiable log-convex function on the interval of real numbers* $\mathring{I}$ *(the interior of* I*) and* $c, d \in \mathring{I}$ *with* $c < d$*. Then the following inequalities hold:*

$$\frac{\frac{1}{d-c}\int_c^d g(x)\,dx}{g\left(\frac{c+d}{2}\right)} \geq L\left(\exp\left[\frac{g'\left(\frac{c+d}{2}\right)}{g\left(\frac{c+d}{2}\right)}\left(\frac{d-c}{2}\right)\right], \exp\left[-\frac{g'\left(\frac{c+d}{2}\right)}{g\left(\frac{c+d}{2}\right)}\left(\frac{d-c}{2}\right)\right]\right) \geq 1. \tag{3.18}$$

If $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is a *GG*-convex function on $[a,b]$, then by taking $g:=f\circ\exp$, $c=\ln a$ and $d=\ln b$ we have that g is log-convex on $[c,d]$ we have

$$\text{(3.19)}\quad \frac{\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b} f\left(e^{x}\right)dx}{f\circ\exp\left(\frac{\ln a+\ln b}{2}\right)} \geq L\left(\exp\left[\frac{f'\circ\exp\left(\frac{\ln a+\ln b}{2}\right)\exp\left(\frac{\ln a+\ln b}{2}\right)}{f\circ\exp\left(\frac{\ln a+\ln b}{2}\right)}\left(\frac{\ln b-\ln a}{2}\right)\right],\right.$$
$$\left.\exp\left[-\frac{f'\circ\exp\left(\frac{\ln a+\ln b}{2}\right)\exp\left(\frac{\ln a+\ln b}{2}\right)}{f\circ\exp\left(\frac{\ln a+\ln b}{2}\right)}\left(\frac{\ln b-\ln a}{2}\right)\right]\right)\geq 1,$$

that is equivalent to

$$\text{(3.20)}\quad \frac{\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b} f\left(e^{x}\right)dx}{f\left(\sqrt{ab}\right)} \geq L\left(\exp\left[\frac{f'\left(\sqrt{ab}\right)\sqrt{ab}}{f\left(\sqrt{ab}\right)}\left(\frac{\ln b-\ln a}{2}\right)\right],\right.$$
$$\left.\exp\left[-\frac{f'\left(\sqrt{ab}\right)\sqrt{ab}}{f\left(\sqrt{ab}\right)}\left(\frac{\ln b-\ln a}{2}\right)\right]\right)\geq 1,$$

and the inequality (3.7) is proved.

The inequality (3.8) follows by the following result [**4**] that holds for the differentiable log-convex function $g:[c,d]\to\mathbb{R}$

$$\text{(3.21)}\quad \frac{\frac{g(c)+g(d)}{2}}{\frac{1}{d-c}\int_c^d g(x)\,dx}\geq 1+\log\left[\frac{\int_c^d g(x)\,dx}{\int_c^d g(x)\exp\left[\frac{g'(x)}{g(x)}\left(\frac{c+d}{2}-x\right)\right]dx}\right]$$
$$\geq 1+\log\left[\frac{\frac{1}{d-c}\int_c^d g(x)\,dx}{g\left(\frac{c+d}{2}\right)}\right]\geq 1.$$

We omit the details.

We have the following result for *GA*-convex functions [**12**]:

LEMMA 3.10. *Let* $g:[a,b]\subset(0,\infty)\to\mathbb{R}$ *be a GA-convex function on* $[a,b]$ *and* $w:[a,b]\to[0,\infty)$ *an integrable function on* $[a,b]$, *then*

$$\text{(3.22)}\quad g\left(\exp\left(\frac{\int_a^b w(t)\ln t\,dt}{\int_a^b w(t)\,dt}\right)\right)\leq\frac{\int_a^b w(t)\,g(t)\,dt}{\int_a^b w(t)\,dt}$$
$$\leq\frac{\left(\ln b-\frac{\int_a^b w(t)\ln t\,dt}{\int_a^b w(t)dt}\right)g(a)+\left(\frac{\int_a^b w(t)\ln t\,dt}{\int_a^b w(t)dt}-\ln a\right)g(b)}{\ln b-\ln a}.$$

If we take $g = \ln f$, then we get from (3.22) that

$$\begin{aligned}(3.23)\quad \ln f\left(\exp\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)\,dt}\right)\right) &\le \frac{\int_a^b w(t)\ln f(t)\,dt}{\int_a^b w(t)\,dt}\\ &\le \frac{\left(\ln b - \frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}\right)\ln f(a) + \left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt} - \ln a\right)\ln f(b)}{\ln b - \ln a}\\ &= \ln\left([f(a)]^{\frac{\ln b - \frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}}{\ln b - \ln a}}[f(b)]^{\frac{\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt} - \ln a}{\ln b - \ln a}}\right)\end{aligned}$$

and by taking the exponential in (3.23) we get the desired result (3.9).

In [**12**] we obtained the following particular inequalities for $g : [a,b] \subset (0,\infty) \to \mathbb{R}$, a *GA*-convex function

$$\begin{aligned}(3.24)\quad L_p^p(a,b)\,g\left(\left[I\left(a^{p+1},b^{p+1}\right)\right]^{\frac{1}{p+1}}\right) &\le \frac{1}{b-a}\int_a^b t^p g(t)\,dt\\ &\le \frac{\left(L\left(a^{p+1},b^{p+1}\right) - a^{p+1}\right)g(a) + \left(b^{p+1} - L\left(a^{p+1},b^{p+1}\right)\right)g(b)}{(p+1)(b-a)},\end{aligned}$$

for any $p \in \mathbb{R}$ with $p \ne 0, -1$.

If $p = 0$, namely we take $w(t) = 1$ in (3.22), then we have

$$(3.25)\qquad g(I(a,b)) \le \frac{1}{b-a}\int_a^b g(t)\,dt \le \frac{(L(a,b) - a)\,g(a) + (b - L(a,b))\,f(b)}{b-a}.$$

If $p = -1$, namely we take $w(t) = \frac{1}{t}$ in (3.22), then we have

$$(3.26)\qquad f\left(\sqrt{ab}\right) \le \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt \le \frac{f(a) + f(b)}{2}.$$

If $p = 1$ (3.24), then we have

$$\begin{aligned}(3.27)\quad f\left(\sqrt{I\left(a^2,b^2\right)}\right) &\le \frac{1}{b-a}\int_a^b t f(t)\,dt\\ &\le \frac{\left(A(a,b)\,L(a,b) - a^2\right)f(a) + \left(b^2 - A(a,b)\,L(a,b)\right)f(b)}{2(b-a)}.\end{aligned}$$

4. POWER INEQUALITIES

We start with the following inequality for powers of *GG*-convex functions:

THEOREM 4.1 (Dragomir, 2015 [**15**]). *Let* $f : [a,b] \subset (0,\infty) \to (0,\infty)$ *be a GG-convex function on* $[a,b]$.

(i) If $p \in \left(0, \frac{1}{2}\right]$ *, then we have*

$$f\left(\sqrt{ab}\right) \le \exp\left[\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(t)}{t}dt\right] \le \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t} f^p(t) f^p\left(\frac{\sqrt{ab}}{t}\right)dt\right)^{\frac{1}{2p}} \le \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t} f^{2p}(t)\,dt\right)^{\frac{1}{2p}} \le \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt \le L(f(a), f(b)). \tag{4.1}$$

(ii) For $p > 0$*, but* $p \neq \frac{1}{2}$*, we also have*

$$f\left(\sqrt{ab}\right) \le \exp\left[\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(t)}{t}dt\right] \le \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t} f^p(t) f^p\left(\frac{\sqrt{ab}}{t}\right)dt\right)^{\frac{1}{2p}} \le \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t} f^{2p}(t)\,dt\right)^{\frac{1}{2p}} \le \left[L_{2p-1}(f(a), f(b))\right]^{1-\frac{1}{2p}}\left[L(f(a), f(b))\right]^{\frac{1}{2p}}. \tag{4.2}$$

If we take $p = \frac{1}{4}$ in (4.1), then we get

$$f\left(\sqrt{ab}\right) \le \exp\left[\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(t)}{t}dt\right] \le \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t}\sqrt[4]{f(t) f\left(\frac{\sqrt{ab}}{t}\right)}dt\right)^2 \le \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t}\sqrt{f(t)}dt\right)^2 \le \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt \le L(f(a), f(b)). \tag{4.3}$$

The case $p = \frac{1}{2}$ produces

$$f\left(\sqrt{ab}\right) \le \exp\left[\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(t)}{t}dt\right] \le \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t}\sqrt{f(t) f\left(\frac{\sqrt{ab}}{t}\right)}dt \le \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt \le L(f(a), f(b)). \tag{4.4}$$

Also, if we take $p = 1$ in (4.2), then we get

$$f\left(\sqrt{ab}\right) \le \exp\left[\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(t)}{t}dt\right] \le \sqrt{\frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t} f(t) f\left(\frac{\sqrt{ab}}{t}\right) dt} \le \sqrt{\frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t} f^2(t)\, dt} \le \sqrt{A(f(a), f(b))}\sqrt{L(f(a), f(b))}. \tag{4.5}$$

We also have:

THEOREM 4.2 (Dragomir, 2015 [**15**]). *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$. Then for every $s \in [a,b]$ we have the inequality*

$$(\ln b - \ln s) f(b) + (\ln s - \ln a) f(a) - \int_a^b \frac{f(t)}{t}dt \ge \int_a^b \frac{f(t)\ln f(t)}{t}dt - \ln f(s)\int_a^b \frac{f(t)}{t}dt. \tag{4.6}$$

If we take in (4.6) $s = G(a,b) = \sqrt{ab}$, then we get

$$(\ln b - \ln a)\frac{f(b) + f(a)}{2} - \int_a^b \frac{f(t)}{t}dt \ge \int_a^b \frac{f(t)\ln f(t)}{t}dt - \ln f(G(a,b))\int_a^b \frac{f(t)}{t}dt. \tag{4.7}$$

Also, if we take $s = I(a,b)$, then we get from (4.6) that

$$(\ln b - \ln I(a,b)) f(b) + (\ln I(a,b) - \ln a) f(a) - \int_a^b \frac{f(t)}{t}dt \ge \int_a^b \frac{f(t)\ln f(t)}{t}dt - \ln f(I(a,b))\int_a^b \frac{f(t)}{t}dt. \tag{4.8}$$

Since simple calculations show that

$$\ln b - \ln I(a,b) = \frac{L(a,b) - a}{L(a,b)},\ \ \ln I(a,b) - \ln a = \frac{b - L(a,b)}{L(a,b)},$$

and then the inequality (4.8) can be written as

$$\frac{L(a,b) - a}{L(a,b)} f(b) + \frac{b - L(a,b)}{L(a,b)} f(a) - \int_a^b \frac{f(t)}{t}dt \ge \int_a^b \frac{f(t)\ln f(t)}{t}dy - \ln f(I(a,b))\int_a^b \frac{f(t)}{t}dt. \tag{4.9}$$

Moreover, if we take the integral mean in (4.6), then we get

$$(\ln b - \ln I(a,b)) f(b) + (\ln I(a,b) - \ln a) f(a) - \int_a^b \frac{f(t)}{t}dt \ge \int_a^b \frac{f(t)\ln f(t)}{t}dt - \frac{1}{b-a}\int_a^b \ln f(s)\, ds \int_a^b \frac{f(t)}{t}dt. \tag{4.10}$$

This can be also written as

$$\begin{aligned}(4.11)\quad &\frac{L(a,b)-a}{L(a,b)}f(b)+\frac{b-L(a,b)}{L(a,b)}f(a)-\int_a^b\frac{f(t)}{t}dt\\ &\geq\int_a^b\frac{f(t)\ln f(t)}{t}dt-\frac{1}{b-a}\int_a^b\ln f(s)\,ds\int_a^b\frac{f(t)}{t}dt.\end{aligned}$$

From a different perspective we have:

THEOREM 4.3 (Dragomir, 2015 [**15**]). *Let* $f:[a,b]\subset(0,\infty)\to(0,\infty)$ *be a GG-convex function on* $[a,b]$. *Then we have the inequality*

$$\begin{aligned}(4.12)\quad &\frac{f(b)+f(a)}{2}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{t}dt\\ &\geq\int_a^b\frac{f(t)}{t}\ln f(t)\,dt-\int_a^b\frac{f(t)}{t}dt\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(t)}{t}dt\geq 0.\end{aligned}$$

Also, we can state the following result as well:

THEOREM 4.4 (Dragomir, 2015 [**15**]). *Let* $f:[a,b]\subset(0,\infty)\to(0,\infty)$ *be a GG-convex function on* $[a,b]$. *Then we have the inequality*

$$\begin{aligned}(4.13)\quad &f(b)\left(\ln b-\frac{\int_a^b\frac{f(t)\ln t}{t}dt}{\int_a^b\frac{f(t)}{t}dt}\right)+f(a)\left(\frac{\int_a^b\frac{f(t)\ln t}{t}dt}{\int_a^b\frac{f(t)}{t}dt}-\ln a\right)-\int_a^b\frac{f(t)}{t}dt\\ &\geq\int_a^b\frac{f(t)}{t}\ln f(t)\,dt-\int_a^b\frac{f(t)}{t}\ln f\left(\exp\left(\frac{\int_a^b\frac{f(t)\ln t}{t}dt}{\int_a^b\frac{f(t)}{t}dt}\right)\right)\geq 0.\end{aligned}$$

Finally, we have

THEOREM 4.5 (Dragomir, 2015 [**15**]). *Let* $f:[a,b]\subset(0,\infty)\to(0,\infty)$ *be a GG-convex function on* $[a,b]$. *Then we have the inequality*

$$\begin{aligned}(4.14)\quad &\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{f^{2p}(t)}{t}dt\right)^{\frac{1}{2p}}\\ &\geq\left(\frac{1}{\ln b-\ln a}\int_a^b\left[f^{2p}(t)\right]^{1-\alpha}\left[f^{2p}\left(\frac{ab}{t}\right)\right]^{\alpha}dt\right)^{\frac{1}{2p}}\\ &\geq\left(\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}}\frac{f^{2p}(t)}{t}dt\right)^{\frac{1}{2p}}\\ &\geq\left(\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}}\frac{f^{p}(t)f^{p}\left(\frac{ab}{t}\right)}{t}dt\right)^{\frac{1}{2p}}\\ &\geq\exp\left[\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}}\frac{\ln f(t)}{t}dt\right]\geq f\left(\sqrt{ab}\right),\end{aligned}$$

for every $\alpha\in[0,1]\setminus\left\{\frac{1}{2}\right\}$ *and* $p>0$.

Finally, by taking $p=1$ in (4.14) we get

$$\begin{aligned}
&\sqrt{\frac{1}{\ln b-\ln a}\int_a^b \frac{f^2(t)}{t}dt} \\
&\geq \sqrt{\frac{1}{\ln b-\ln a}\int_a^b \left[f^2(t)\right]^{1-\alpha}\left[f^2\left(\frac{ab}{t}\right)\right]^{\alpha} dt} \\
&\geq \sqrt{\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{f^2(t)}{t}dt} \\
&\geq \sqrt{\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{f(t)f\left(\frac{ab}{t}\right)}{t}dt} \\
&\geq \exp\left[\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{\ln f(t)}{t}dt\right] \geq f\left(\sqrt{ab}\right).
\end{aligned} \tag{4.15}$$

In the recent paper [**8**] we established the following inequalities for a log-convex function $g:[c,d]\to(0,\infty)$:

If $p\in\left(0,\frac{1}{2}\right]$, then we have

$$\begin{aligned}
g\left(\frac{c+d}{2}\right) &\leq \exp\left[\frac{1}{d-c}\int_c^d \ln g(x)\,dx\right] \\
&\leq \left(\frac{1}{d-c}\int_c^d g^p(x)\,g^p(c+d-x)\,dx\right)^{\frac{1}{2p}} \\
&\leq \left(\frac{1}{d-c}\int_c^d g^{2p}(x)\,dx\right)^{\frac{1}{2p}} \leq \frac{1}{d-c}\int_c^d g(x)\,dx \\
&\leq L(g(c),g(d)).
\end{aligned} \tag{4.16}$$

For $p>0$, but $p\neq\frac{1}{2}$, we also have

$$\begin{aligned}
g\left(\frac{c+d}{2}\right) &\leq \exp\left[\frac{1}{d-c}\int_c^d \ln g(x)\,dx\right] \\
&\leq \left(\frac{1}{d-c}\int_c^d g^p(x)\,g^p(c+d-x)\,dx\right)^{\frac{1}{2p}} \\
&\leq \left(\frac{1}{d-c}\int_c^d g^{2p}(x)\,dx\right)^{\frac{1}{2p}} \\
&\leq \left[L_{2p-1}(g(c),g(d))\right]^{1-\frac{1}{2p}}\left[L(g(c),g(d))\right]^{\frac{1}{2p}}.
\end{aligned} \tag{4.17}$$

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is a *GG*-convex function on $[a,b]$, then by taking

$g := f \circ \exp$; $c = \ln a$ and $d = \ln b$ we have that g is log-convex on $[c, d]$ and by (4.16) and (4.17) we get

$$\begin{aligned} &f \circ \exp\left(\frac{\ln a + \ln b}{2}\right) \\ &\leq \exp\left[\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \ln\left[f \circ \exp(x)\right] dx\right] \\ &\leq \left(\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \left[f \circ \exp(x)\right]^p \left[f \circ \exp(\ln a + \ln b - x)\right]^p dx\right)^{\frac{1}{2p}} \\ &\leq \left(\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \left[f \circ \exp(x)\right]^{2p} dx\right)^{\frac{1}{2p}} \\ &\leq \frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} f \circ \exp(x)\, dx \leq L\left(f \circ \exp(\ln a), f \circ \exp(d)\right) \end{aligned} \tag{4.18}$$

for $p \in \left(0, \frac{1}{2}\right]$ and

$$\begin{aligned} &f \circ \exp\left(\frac{\ln a + \ln b}{2}\right) \\ &\leq \exp\left[\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \ln\left[f \circ \exp(x)\right] dx\right] \\ &\leq \left(\frac{1}{\ln b - \ln a}\left[f \circ \exp(x)\right]^p \left[f \circ \exp(\ln a + \ln b - x)\right]^p dx\right)^{\frac{1}{2p}} \\ &\leq \left(\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \left[f \circ \exp(x)\right]^{2p} dx\right)^{\frac{1}{2p}} \\ &\leq \left[L_{2p-1}\left(f \circ \exp(\ln a), f \circ \exp(\ln b)\right)\right]^{1-\frac{1}{2p}} \\ &\times \left[L\left(f \circ \exp(\ln a), f \circ \exp(\ln b)\right)\right]^{\frac{1}{2p}} \end{aligned} \tag{4.19}$$

for $p > 0$, but $p \neq \frac{1}{2}$.

The inequalities (4.18) and (4.19) can be equivalently written as

$$\begin{aligned} f\left(\sqrt{ab}\right) &\leq \exp\left[\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \ln\left[f \circ \exp(x)\right] dx\right] \\ &\leq \left(\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \left[f \circ \exp(x)\right]^p \left[f\left(\frac{\sqrt{ab}}{\exp(x)}\right)\right]^p dx\right)^{\frac{1}{2p}} \\ &\leq \left(\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \left[f \circ \exp(x)\right]^{2p} dx\right)^{\frac{1}{2p}} \\ &\leq \frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} f \circ \exp(x)\, dx \leq L\left(f(a), f(b)\right) \end{aligned} \tag{4.20}$$

and

$$f\left(\sqrt{ab}\right) \leq \exp\left[\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \ln\left[f \circ \exp(x)\right] dx\right] \tag{4.21}$$
$$\leq \left(\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \left[f \circ \exp(x)\right]^p \left[f\left(\frac{\sqrt{ab}}{\exp(x)}\right)\right]^p dx\right)^{\frac{1}{2p}}$$
$$\leq \left(\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \left[f \circ \exp(x)\right]^{2p} dx\right)^{\frac{1}{2p}}$$
$$\leq \left[L_{2p-1}\left(f(a), f(b)\right)\right]^{1-\frac{1}{2p}} \left[L\left(f(a), f(b)\right)\right]^{\frac{1}{2p}}.$$

Now, by making the change of variable $\exp(x) = t$ in the integrals from (4.20) and (4.21) we obtain the desired results (4.1) and (4.2).

In [**8**] we also obtained the following result:

Let $g : [c, d] \to (0, \infty)$ be a log-convex function on $[c, d]$. Then for any $x \in [c, d]$ we have

$$g(d)(d - x) + g(c)(x - c) - \int_c^d g(y) dy \geq \int_c^d g(y) \ln g(y) dy - \ln g(x) \int_c^d g(y) dy. \tag{4.22}$$

A simple proof of this fact is as follows.

Since the function $\ln g$ is convex on $[c, d]$, then by the gradient inequality we have

$$\ln g(x) - \ln g(y) \geq \frac{g'_+(y)}{g(y)}(x - y) \tag{4.23}$$

for any $x \in [c, d]$ and $y \in (c, d)$.

If we multiply (4.23) by $g(y) > 0$ and integrate on $[c, d]$ over y we get

$$\ln g(x) \int_c^d g(y) dy - \int_c^d g(y) \ln g(y) dy \geq g(d)(x - d) + g(c)(c - x) + \int_c^d g(y) dy,$$

which is equivalent to (4.22).

If $f : [a, b] \subset (0, \infty) \to (0, \infty)$ is a *GG*-convex function on $[a, b]$, then by taking

$g := f \circ \exp$; $c = \ln a$, $d = \ln b$ and $x = \ln s$, $s \in [a, b]$, we have that g is log-convex on $[c, d]$ and by (4.22) we get

$$(\ln b - \ln s) f(b) + (\ln s - \ln a) f(a) - \int_{\ln a}^{\ln b} f \circ \exp(y) dy$$
$$\geq \int_{\ln a}^{\ln b} f \circ \exp(y) \ln\left(f \circ \exp(y)\right) dy - \ln f(s) \int_{\ln a}^{\ln b} f \circ \exp(y) dy,$$

that holds for each $s \in [a, b]$.

Now, if in this last inequality we make the change of variable $\exp(y) = t$, then we obtain the desired result (4.6).

We have the following result for log-convex functions that improves the trapezoid inequality for the convex function $h:[c,d]\to\mathbb{R}$

$$\frac{h(d)+h(c)}{2}-\frac{1}{d-c}\int_c^d h(y)\,dy\geq 0.$$

We have [**8**] :

LEMMA 4.6. *Let* $g:[c,d]\to(0,\infty)$ *be a log-convex function on* $[c,d]$*. Then*

$$\begin{aligned}(4.24)\quad &\frac{g(d)+g(c)}{2}-\frac{1}{d-c}\int_c^d g(y)\,dy\\ &\geq \int_c^d g(y)\ln g(y)\,dy-\int_c^d g(y)\,dy\frac{1}{d-c}\int_c^d \ln g(y)\,dy\geq 0.\end{aligned}$$

PROOF. If we take the integral mean over x in (4.22), then we get

$$\begin{aligned}&\frac{1}{d-c}\int_c^d [g(d)(d-x)+g(c)(x-c)]\,dx-\int_c^d g(y)\,dy\\ &\geq \int_c^d g(y)\ln g(y)\,dy-\int_c^d g(y)\,dy\frac{1}{d-c}\int_c^d \ln g(x)\,dx\end{aligned}$$

and since

$$\frac{1}{d-c}\int_c^d [g(d)(d-x)+g(c)(x-c)]\,dx=\frac{g(d)+g(c)}{2}-\frac{1}{d-c}\int_c^d g(y)\,dy$$

then the first inequality in (4.24) is proved.

Since $\ln$ is an increasing function on $(0,\infty)$, then we have

$$(g(x)-g(y))(\ln g(x)-\ln g(y))\geq 0$$

for any $x,y\in[c,d]$, showing that the functions g and $\ln g$ are synchronous on $[c,d]$.

By making use of the Čebyšev integral inequality for synchronous functions $g,h:[c,d]\to\mathbb{R}$, namely

$$\frac{1}{d-c}\int_c^d g(x)h(x)\,dx\geq\frac{1}{d-c}\int_c^d g(x)\,dx\frac{1}{d-c}\int_c^d h(x)\,dx,$$

then we have

$$\frac{1}{d-c}\int_c^d g(x)\ln g(x)\,dx\geq\frac{1}{d-c}\int_c^d g(x)\,dx\frac{1}{d-c}\int_c^d \ln g(x)\,dx,$$

which proves the last part of (4.24). ∎

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is a *GG*-convex function on $[a,b]$, then by taking $g:=f\circ\exp$; $c=\ln a$, and $d=\ln b$, we have that g is log-convex on $[c,d]$ and by (4.24) we get

$$\begin{aligned}(4.25)\quad &\frac{f(b)+f(a)}{2}-\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy\\ &\geq\int_{\ln a}^{\ln b} f\circ\exp(y)\ln(f\circ\exp(y))\,dy\\ &\quad-\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\ln(f\circ\exp(y))\,dy\geq 0.\end{aligned}$$

By changing the variable $\exp(y)=t$ in (4.25) we deduce the desired inequality (4.12).

We have [**8**]:

LEMMA 4.7. *Let* $g:[c,d]\to(0,\infty)$ *be a log-convex function on* $[c,d]$. *Then*

$$g(d)\left(d-\frac{\int_c^d yg(y)\,dy}{\int_c^d g(y)\,dy}\right)+g(c)\left(\frac{\int_c^d yg(y)\,dy}{\int_c^d g(y)\,dy}-c\right)-\int_c^d g(y)\,dy \geq \int_c^d g(y)\ln g(y)\,dy-\int_c^d g(y)\,dy\ln g\left(\frac{\int_c^d yg(y)\,dy}{\int_c^d g(y)\,dy}\right)\geq 0. \tag{4.26}$$

PROOF. The first inequality follows by (4.22) on taking

$$x=\frac{\int_c^d yg(y)\,dy}{\int_c^d g(y)\,dy}\in[c,d]$$

since $g(y)>0$ for any $y\in[c,d]$.

By Jensen's inequality for the convex function $\ln g$ and the positive weight g we have

$$\frac{\int_c^d g(y)\ln g(y)\,dy}{\int_c^d g(y)\,dy}\geq g\left(\frac{\int_c^d g(y)\,ydy}{\int_c^d g(y)\,dy}\right),$$

which proves the second inequality in (4.26). ∎

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is a *GG*-convex function on $[a,b]$, then by taking $g:=f\circ\exp$; $c=\ln a$, and $d=\ln b$, we have that g is log-convex on $[c,d]$ and by (4.26) we have

$$f(b)\left(\ln b-\frac{\int_{\ln a}^{\ln b} yf\circ\exp(y)\,dy}{\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy}\right)+f(c)\left(\frac{\int_{\ln a}^{\ln b} yf\circ\exp(y)\,dy}{\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy}-\ln a\right) -\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy \geq \int_{\ln a}^{\ln b} f\circ\exp(y)\ln(f\circ\exp(y))\,dy -\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy\times\ln(f\circ\exp)\left(\frac{\int_{\ln a}^{\ln b} y(f\circ\exp(y))\,dy}{\int_{\ln a}^{\ln b} f\circ\exp(y)\,dy}\right)\geq 0. \tag{4.27}$$

By changing the variable $\exp(y)=t$ in (4.27) we get (4.13).

In [**8**] we also proved the following result:

Let $g:[c,d]\to(0,\infty)$ be a log-convex function. Then for every $\alpha\in[0,1]\setminus\left\{\frac{1}{2}\right\}$ we have for $p>0$ that

$$\left(\frac{1}{d-c}\int_c^d g^{2p}(x)\,dx\right)^{\frac{1}{2p}} \tag{4.28}$$
$$\geq\left(\frac{1}{d-c}\int_c^d\left[g^{2p}(x)\right]^{1-\alpha}\left[g^{2p}(c+d-x)\right]^{\alpha}dx\right)^{\frac{1}{2p}}$$
$$\geq\left(\frac{1}{(1-2\alpha)(d-c)}\int_{(1-\alpha)c+\alpha d}^{\alpha c+(1-\alpha)d}g^{2p}(u)\,du\right)^{\frac{1}{2p}}$$
$$\geq\left(\frac{1}{(1-2\alpha)(d-c)}\int_{(1-\alpha)c+\alpha d}^{\alpha c+(1-\alpha)d}g^{p}(u)\,g^{p}(c+d-u)\,dx\right)^{\frac{1}{2p}}$$
$$\geq\exp\left[\frac{1}{(1-2\alpha)(d-c)}\int_{(1-\alpha)c+\alpha d}^{\alpha c+(1-\alpha)d}\ln g(u)\,du\right]\geq g\left(\frac{c+d}{2}\right).$$

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is a *GG*-convex function on $[a,b]$, then by taking $g:=f\circ\exp$; $c=\ln a$, and $d=\ln b$, we have that g is log-convex on $[c,d]$ and by (4.28) we have

$$\left(\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}f^{2p}\circ\exp(x)\,dx\right)^{\frac{1}{2p}}$$
$$\geq\left(\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\left[f^{2p}\circ\exp(x)\right]^{1-\alpha}\left[f^{2p}\left(\frac{ab}{\exp(x)}\right)\right]^{\alpha}dx\right)^{\frac{1}{2p}}$$
$$\geq\left(\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{\ln(a^{1-\alpha}b^{\alpha})}^{\ln\left(a^{\alpha}b^{1-\alpha}\right)}f^{2p}\circ\exp(x)\,dx\right)^{\frac{1}{2p}}$$
$$\geq\left(\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{\ln(a^{1-\alpha}b^{\alpha})}^{\ln\left(a^{\alpha}b^{1-\alpha}\right)}f^{p}\circ\exp(x)\,f^{p}\left(\frac{ab}{\exp(x)}\right)dx\right)^{\frac{1}{2p}}$$
$$\geq\exp\left[\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{\ln(a^{1-\alpha}b^{\alpha})}^{\ln\left(a^{\alpha}b^{1-\alpha}\right)}\ln f\circ\exp(x)\,dx\right]\geq f\left(\sqrt{ab}\right),$$

which, by changing the variable $t=\exp x$, is equivalent to (4.14).

5. INEQUALITIES FOR GENERAL WEIGHTS

We have:

THEOREM 5.1. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$ and $w : [a,b] \to [0,\infty)$ an integrable function on $[a,b]$, then*

$$f\left(\sqrt{ab}\right) \leq \left(\frac{\int_a^b w(t) f^p(t) f^p\left(\frac{ab}{t}\right) dt}{\int_a^b w(t)\, dt}\right)^{\frac{1}{2p}} \leq \sqrt{f(a) f(b)} \tag{5.1}$$

for any $p > 0$.

PROOF. From the definition of *GG*-convex functions we have

$$f\left(x^{1-\lambda} y^{\lambda}\right) \leq [f(x)]^{1-\lambda} [f(y)]^{\lambda} \tag{5.2}$$

and

$$f\left(x^{\lambda} y^{1-\lambda}\right) \leq [f(x)]^{\lambda} [f(y)]^{1-\lambda} \tag{5.3}$$

for any $x, y \in [a,b]$ and $\lambda \in [0,1]$.

By multiplication of (5.2) with (5.3) we get

$$f\left(x^{1-\lambda} y^{\lambda}\right) f\left(x^{\lambda} y^{1-\lambda}\right) \leq f(x) f(y)$$

for any $x, y \in [a,b]$ and $\lambda \in [0,1]$.

Therefore

$$f\left(a^{1-\lambda} b^{\lambda}\right) f\left(a^{\lambda} b^{1-\lambda}\right) \leq f(a) f(b) \tag{5.4}$$

for any $\lambda \in [0,1]$.

From (5.2) we also have

$$f\left(\sqrt{xy}\right) \leq \sqrt{f(x) f(y)} \tag{5.5}$$

for any $x, y \in [a,b]$.

By taking $x = a^{1-\lambda} b^{\lambda}$, $y = a^{\lambda} b^{1-\lambda}$ in (5.5) and then squaring we get

$$f^2\left(\sqrt{ab}\right) \leq f\left(a^{1-\lambda} b^{\lambda}\right) f\left(a^{\lambda} b^{1-\lambda}\right). \tag{5.6}$$

Since for any $t \in [a,b]$ there is a unique $\lambda \in [0,1]$ such that $t = a^{1-\lambda} b^{\lambda}$, we obtain from (5.4) and (5.6) that

$$f^2\left(\sqrt{ab}\right) \leq f(t) f\left(\frac{ab}{t}\right) \leq f(a) f(b) \tag{5.7}$$

for any $t \in [a,b]$.

If we take the power $p > 0$ in (5.7), multiply by $w(t) \geq 0$ for $t \in [a,b]$ and integrate, we get

$$\begin{aligned} f^{2p}\left(\sqrt{ab}\right) \int_a^b w(t)\, dt &\leq \int_a^b w(t) f^p(t) f^p\left(\frac{ab}{t}\right) dt \\ &\leq f^p(a) f^p(b) \int_a^b w(t)\, dt \end{aligned} \tag{5.8}$$

that is equivalent to

$$f^{2p}\left(\sqrt{ab}\right) \leq \frac{\int_a^b w(t) f^p(t) f^p\left(\frac{ab}{t}\right) dt}{\int_a^b w(t)\, dt} \leq f^p(a) f^p(b) \tag{5.9}$$

and by taking the power $\frac{1}{2p}$ we get the desired result (5.1). ∎

We observe that for $p=1$ we get the inequality

$$f\left(\sqrt{ab}\right)\le\left(\frac{\int_a^b w(t)\,f(t)\,f\left(\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2}}\le\sqrt{f(a)\,f(b)}, \tag{5.10}$$

while from $p=\frac{1}{2}$ we get

$$f\left(\sqrt{ab}\right)\le\frac{\int_a^b w(t)\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}dt}{\int_a^b w(t)\,dt}\le\sqrt{f(a)\,f(b)}. \tag{5.11}$$

If we take $p=\frac{1}{4}$ in (5.1), then we get

$$f\left(\sqrt{ab}\right)\le\left(\frac{\int_a^b w(t)\sqrt[4]{f(t)\,f\left(\frac{ab}{t}\right)}dt}{\int_a^b w(t)\,dt}\right)^{2}\le\sqrt{f(a)\,f(b)}. \tag{5.12}$$

Using *Jensen's inequality* for the power $p\ge 1$ $(p\in(0,1))$, namely

$$\left(\frac{\int_a^b w(x)\,g(x)\,dx}{\int_a^b w(x)\,dx}\right)^{p}\le(\ge)\,\frac{\int_a^b w(x)\,g^p(x)\,dx}{\int_a^b w(x)\,dx},$$

we can state the following more precise result:

COROLLARY 5.2. *Let* $f:[a,b]\subset(0,\infty)\to(0,\infty)$ *be a GG-convex function on* $[a,b]$ *and* $w:[a,b]\to[0,\infty)$ *an integrable function on* $[a,b]$.
(i) If $p\ge 1$, *then*

$$f\left(\sqrt{ab}\right)\le\left(\frac{\int_a^b w(t)\,f(t)\,f\left(\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2}} \le\left(\frac{\int_a^b w(t)\,f^p(t)\,f^p\left(\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2p}}\le\sqrt{f(a)\,f(b)}. \tag{5.13}$$

(ii) If $p\in(0,1)$, *then*

$$f\left(\sqrt{ab}\right)\le\left(\frac{\int_a^b w(t)\,f^p(t)\,f^p\left(\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2p}} \le\left(\frac{\int_a^b w(t)\,f(t)\,f\left(\frac{ab}{t}\right)dt}{\int_a^b w(t)\,dt}\right)^{\frac{1}{2}}\le\sqrt{f(a)\,f(b)}. \tag{5.14}$$

If we take in Corollary 5.2 $w(t)=1$, $t\in[a,b]$, then for any $f:[a,b]\subset(0,\infty)\to(0,\infty)$ a *GG*-convex function, we have for $p\ge 1$ that

$$f\left(\sqrt{ab}\right)\le\left(\frac{1}{b-a}\int_a^b f(t)\,f\left(\frac{ab}{t}\right)dt\right)^{\frac{1}{2}} \le\left(\frac{1}{b-a}\int_a^b f^p(t)\,f^p\left(\frac{ab}{t}\right)dt\right)^{\frac{1}{2p}}\le\sqrt{f(a)\,f(b)} \tag{5.15}$$

and for $p \in (0,1)$, that

$$f\left(\sqrt{ab}\right) \le \left(\frac{1}{b-a}\int_a^b f^p(t) f^p\left(\frac{ab}{t}\right)dt\right)^{\frac{1}{2p}} \le \left(\frac{1}{b-a}\int_a^b f(t) f\left(\frac{ab}{t}\right)dt\right)^{\frac{1}{2}} \le \sqrt{f(a)f(b)}. \tag{5.16}$$

If we take in Corollary 5.2 $w(t) = \frac{1}{t}$, $t \in [a,b]$, then for any $f : [a,b] \subset (0,\infty) \to (0,\infty)$ a *GG*-convex function, we have for $p \ge 1$ that

$$f\left(\sqrt{ab}\right) \le \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t) f\left(\frac{ab}{t}\right)}{t}dt\right)^{\frac{1}{2}} \le \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{f^p(t) f^p\left(\frac{ab}{t}\right)}{t}dt\right)^{\frac{1}{2p}} \le \sqrt{f(a)f(b)} \tag{5.17}$$

and for $p \in (0,1)$, that

$$f\left(\sqrt{ab}\right) \le \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{f^p(t) f^p\left(\frac{ab}{t}\right)}{t}dt\right)^{\frac{1}{2p}} \le \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t) f\left(\frac{ab}{t}\right)}{t}dt\right)^{\frac{1}{2}} \le \sqrt{f(a)f(b)}. \tag{5.18}$$

If we take $p = \frac{1}{2}$ in the first inequality in (5.18), then we get

$$f\left(\sqrt{ab}\right) \le \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t}\sqrt{f(t) f\left(\frac{ab}{t}\right)}dt \tag{5.19}$$

that has been obtained by Işcan in [**25**].

THEOREM 5.3. *Let* $f : [a,b] \subset (0,\infty) \to (0,\infty)$ *be a GG-convex function on* $[a,b]$ *and* $w : [a,b] \to [0,\infty)$ *an integrable function on* $[a,b]$, *then*

$$\begin{aligned} f\left(\exp\left(\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}\right)\right) &\le \exp\left(\frac{\int_a^b w(t)\ln f(t)dt}{\int_a^b w(t)dt}\right) \\ &\le \frac{\int_a^b w(t) f(t)dt}{\int_a^b w(t)dt} \le \left(\frac{[f(a)]^{\ln b}}{[f(b)]^{\ln a}}\right)^{\frac{1}{\ln b - \ln a}} \frac{\int_a^b w(t)\left(\frac{f(b)}{f(a)}\right)^{\frac{\ln t}{\ln b - \ln a}}dt}{\int_a^b w(t)dt} \\ &\le \left(\frac{\ln b - \frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt}}{\ln b - \ln a}\right) f(a) + \left(\frac{\frac{\int_a^b w(t)\ln t dt}{\int_a^b w(t)dt} - \ln a}{\ln b - \ln a}\right) f(b). \end{aligned} \tag{5.20}$$

PROOF. If we use Jensen's inequality for the exponential function and nonnegative weight w, we have

$$\exp\left(\frac{\int_a^b w(t)\ln f(t)dt}{\int_a^b w(t)dt}\right) \le \frac{\int_a^b w(t) f(t)dt}{\int_a^b w(t)dt},$$

and the second inequality in (5.20) is proved.

Let $t = a^{1-\lambda}b^{\lambda} \in [a,b]$ with $\lambda \in [0,1]$, then $\lambda = \frac{\ln t - \ln a}{\ln b - \ln a}$. By the *GG*-convexity of f we have

$$f(t) = f\left(a^{1-\lambda}b^{\lambda}\right) \leq [f(a)]^{1-\lambda}[f(b)]^{\lambda} = \left(\frac{[f(a)]^{\ln b}}{[f(b)]^{\ln a}}\right)^{\frac{1}{\ln b - \ln a}} \left(\frac{f(b)}{f(a)}\right)^{\frac{\ln t}{\ln b - \ln a}} \leq \frac{\ln b - \ln t}{\ln b - \ln a} f(a) + \frac{\ln t - \ln a}{\ln b - \ln a} f(b) \tag{5.21}$$

for any $t \in [a,b]$.

Now, if we take the weighted integral mean in (5.21), then we get the last part of (5.20). ∎

By choosing $w(t) = 1$, $t \in [a,b]$ in (5.20), we deduce

$$f\left(\exp\left(\frac{1}{b-a}\int_a^b \ln t dt\right)\right) \leq \exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \left(\frac{[f(a)]^{\ln b}}{[f(b)]^{\ln a}}\right)^{\frac{1}{\ln b - \ln a}} \frac{\int_a^b \left(\frac{f(b)}{f(a)}\right)^{\frac{\ln t}{\ln b - \ln a}} dt}{b-a} \leq \left(\frac{\ln b - \frac{1}{b-a}\int_a^b \ln t dt}{\ln b - \ln a}\right) f(a) + \left(\frac{\frac{1}{b-a}\int_a^b \ln t dt - \ln a}{\ln b - \ln a}\right) f(b),$$

and since $\frac{1}{b-a}\int_a^b \ln t dt = \ln I(a,b)$, hence

$$f(I(a,b)) \leq \exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \left(\frac{[f(a)]^{\ln b}}{[f(b)]^{\ln a}}\right)^{\frac{1}{\ln b - \ln a}} \frac{\int_a^b \left(\frac{f(b)}{f(a)}\right)^{\frac{\ln t}{\ln b - \ln a}} dt}{b-a} \leq \frac{\ln b - \ln I(a,b)}{\ln b - \ln a} f(a) + \frac{\ln I(a,b) - \ln a}{\ln b - \ln a} f(b). \tag{5.22}$$

If we take $w(t) = \frac{1}{t}$, $t \in [a,b]$ in (5.20), then we get

$$f\left(\exp\left(\frac{\int_a^b \frac{1}{t}\ln t dt}{\int_a^b \frac{1}{t} dt}\right)\right) \leq \exp\left(\frac{\int_a^b \frac{1}{t}\ln f(t)\,dt}{\int_a^b \frac{1}{t} dt}\right) \leq \frac{\int_a^b \frac{1}{t} f(t)\,dt}{\int_a^b \frac{1}{t} dt} \leq \left(\frac{[f(a)]^{\ln b}}{[f(b)]^{\ln a}}\right)^{\frac{1}{\ln b - \ln a}} \frac{\int_a^b \frac{1}{t}\left(\frac{f(b)}{f(a)}\right)^{\frac{\ln t}{\ln b - \ln a}} dt}{\int_a^b w(t)\,dt} \leq \left(\frac{\ln b - \frac{\int_a^b \frac{1}{t}\ln t dt}{\int_a^b \frac{1}{t} dt}}{\ln b - \ln a}\right) f(a) + \left(\frac{\frac{\int_a^b \frac{1}{t}\ln t dt}{\int_a^b \frac{1}{t} dt} - \ln a}{\ln b - \ln a}\right) f(b). \tag{5.23}$$

This is equivalent, after suitable calculations, to

$$f\left(\sqrt{ab}\right)\leq\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}\ln f\left(t\right)dt\right)\leq\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}f\left(t\right)dt\leq L\left(f\left(a\right),f\left(b\right)\right). \tag{5.24}$$

The third inequality in (5.24) has been obtained in a different way by Işcan in [**25**].

6. OTHER WEIGHTED INEQUALITIES

We have:

THEOREM 6.1. *Let* $f:[a,b]\subset(0,\infty)\to(0,\infty)$ *be a GG-convex function on* $[a,b]$ *and* $w:[a,b]\to[0,\infty)$ *an integrable function on* $[a,b]$ *and such that* $w\left(\frac{ab}{t}\right)=w\left(t\right)$ *for any* $t\in[a,b]$ *. Then we have the inequalities*

$$f\left(\sqrt{ab}\right)\leq\exp\left(\frac{1}{2}\cdot\frac{\int_a^b\left(1+\frac{ab}{t^2}\right)w\left(t\right)\ln f\left(t\right)dt}{\int_a^b w\left(t\right)dt}\right)\leq\sqrt{f\left(a\right)f\left(b\right)}. \tag{6.1}$$

PROOF. By taking the log in (5.7) we get

$$2\ln f\left(\sqrt{ab}\right)\leq\ln f\left(t\right)+\ln f\left(\frac{ab}{t}\right)\leq\ln f\left(a\right)+\ln f\left(b\right) \tag{6.2}$$

for any $t\in[a,b]$.

If we multiply (6.2) by $w\left(t\right)\geq0$ with $t\in[a,b]$ and use the fact that $w\left(\frac{ab}{t}\right)=w\left(t\right)$ for any $t\in[a,b]$, then we get

$$2w\left(t\right)\ln f\left(\sqrt{ab}\right)\leq w\left(t\right)\ln f\left(t\right)+w\left(\frac{ab}{t}\right)\ln f\left(\frac{ab}{t}\right)\leq w\left(t\right)\left[\ln f\left(a\right)+\ln f\left(b\right)\right] \tag{6.3}$$

for any $t\in[a,b]$.

If we integrate the inequality (6.3) on $[a,b]$ we get

$$\begin{aligned}2\ln f\left(\sqrt{ab}\right)\int_a^b w\left(t\right)dt&\leq\int_a^b w\left(t\right)\ln f\left(t\right)dt+\int_a^b w\left(\frac{ab}{t}\right)\ln f\left(\frac{ab}{t}\right)dt\\&\leq\left[\ln f\left(a\right)+\ln f\left(b\right)\right]\int_a^b w\left(t\right)dt.\end{aligned} \tag{6.4}$$

By changing the variable $u=\frac{ab}{t}$, we have $dt=-\frac{ab}{u^2}du$ and

$$\int_a^b w\left(\frac{ab}{t}\right)\ln f\left(\frac{ab}{t}\right)dt=\int_a^b w\left(t\right)\ln f\left(t\right)\frac{ab}{t^2}dt$$

and by (6.4) we get

$$\begin{aligned}2\ln f\left(\sqrt{ab}\right)\int_a^b w\left(t\right)dt&\leq\int_a^b w\left(t\right)\ln f\left(t\right)dt+\int_a^b w\left(t\right)\ln f\left(t\right)\frac{ab}{t^2}dt\\&\leq\left[\ln f\left(a\right)+\ln f\left(b\right)\right]\int_a^b w\left(t\right)dt,\end{aligned}$$

which is equivalent to the desired result (6.1). ∎

If we take in (6.1) $w(t) = 1$, $t \in [a,b]$, then we get

$$f\left(\sqrt{ab}\right) \le \exp\left(\frac{1}{2} \cdot \frac{\int_a^b \left(1 + \frac{ab}{t^2}\right) \ln f(t)\, dt}{b-a}\right) \le \sqrt{f(a) f(b)}. \tag{6.5}$$

Another example of weight w that satisfies the condition $w\left(\frac{ab}{t}\right) = w(t)$ for any $t \in [a,b]$ is $w(t) = \left|\ln\left(\frac{\sqrt{ab}}{t}\right)\right|$, with $t \in [a,b] \subset (0,\infty)$.

The following result also holds:

THEOREM 6.2. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$ and $w : [a,b] \to [0,\infty)$ an integrable function on $[a,b]$ and such that $w\left(\frac{ab}{t}\right) = w(t)$ for any $t \in [a,b]$. Then we have the inequalities*

$$f\left(\sqrt{ab}\right) \le \exp\left(\frac{\int_a^b \frac{1}{t} w(t) \ln f(t)\, dt}{\int_a^b \frac{1}{t} w(t)\, dt}\right) \le \sqrt{f(a) f(b)}. \tag{6.6}$$

PROOF. From (6.2) for $t = a^{1-\lambda} b^{\lambda}$ with $\lambda \in [0,1]$, we have

$$2 \ln f\left(\sqrt{ab}\right) \le \ln f\left(a^{1-\lambda} b^{\lambda}\right) + \ln f\left(a^{\lambda} b^{1-\lambda}\right) \le \ln f(a) + \ln f(b) \tag{6.7}$$

for any $\lambda \in [0,1]$.

Since $w\left(\frac{ab}{t}\right) = w(t)$ for any $t \in [a,b]$, then $w\left(a^{1-\lambda} b^{\lambda}\right) = w\left(a^{\lambda} b^{1-\lambda}\right)$ for any $\lambda \in [0,1]$ and by (6.7) we have

$$\begin{aligned} 2 \ln f\left(\sqrt{ab}\right) w\left(a^{1-\lambda} b^{\lambda}\right) &\le w\left(a^{1-\lambda} b^{\lambda}\right) \ln f\left(a^{1-\lambda} b^{\lambda}\right) + w\left(a^{\lambda} b^{1-\lambda}\right) \ln f\left(a^{\lambda} b^{1-\lambda}\right) \\ &\le w\left(a^{1-\lambda} b^{\lambda}\right) \left[\ln f(a) + \ln f(b)\right] \end{aligned} \tag{6.8}$$

for any $\lambda \in [0,1]$.

Integrating the inequality over $\lambda \in [0,1]$ we have

$$\begin{aligned} 2 \ln f\left(\sqrt{ab}\right) \int_0^1 w\left(a^{1-\lambda} b^{\lambda}\right) d\lambda &\le \int_0^1 w\left(a^{1-\lambda} b^{\lambda}\right) \ln f\left(a^{1-\lambda} b^{\lambda}\right) d\lambda + \int_0^1 w\left(a^{\lambda} b^{1-\lambda}\right) \ln f\left(a^{\lambda} b^{1-\lambda}\right) d\lambda \\ &\le \left[\ln f(a) + \ln f(b)\right] \int_0^1 w\left(a^{1-\lambda} b^{\lambda}\right) d\lambda \end{aligned} \tag{6.9}$$

and since

$$\int_0^1 w\left(a^{1-\lambda} b^{\lambda}\right) \ln f\left(a^{1-\lambda} b^{\lambda}\right) d\lambda = \int_0^1 w\left(a^{\lambda} b^{1-\lambda}\right) \ln f\left(a^{\lambda} b^{1-\lambda}\right) d\lambda,$$

hence by (6.9) we get

$$\ln f\left(\sqrt{ab}\right) \le \frac{\int_0^1 w\left(a^{1-\lambda} b^{\lambda}\right) \ln f\left(a^{1-\lambda} b^{\lambda}\right) d\lambda}{\int_0^1 w\left(a^{1-\lambda} b^{\lambda}\right) d\lambda} \le \ln\left(\sqrt{f(a) f(b)}\right). \tag{6.10}$$

By changing the variable $a^{1-\lambda} b^{\lambda} = t$, then $(1-\lambda)\ln a + \lambda \ln b = \ln t$ which gives that

$$\lambda = \frac{\ln t - \ln a}{\ln b - \ln a}.$$

Therefore $d\lambda = \frac{1}{t}dt$,

$$\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)\ln f\left(a^{1-\lambda}b^{\lambda}\right)d\lambda = \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t}w(t)\ln f(t)\,dt$$

and

$$\int_0^1 w\left(a^{1-\lambda}b^{\lambda}\right)d\lambda = \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t}w(t)\,dt$$

and by (6.10) we get the desired result (6.6). ∎

If we take in (6.6)

$$w(t) = \left|\ln\left(\frac{\sqrt{ab}}{t}\right)\right| = \left|\ln t - \frac{\ln a + \ln b}{2}\right|$$

and since

$$\int_a^b \frac{1}{t}w(t)\,dt = \int_a^b \frac{1}{t}\left|\ln t - \frac{\ln a + \ln b}{2}\right|dt = \frac{1}{4}(\ln b - \ln a)^2$$

then we get

$$\frac{1}{4}(\ln b - \ln a)^2 \ln f\left(\sqrt{ab}\right) \le \int_a^b \frac{1}{t}\left|\ln\left(\frac{\sqrt{ab}}{t}\right)\right|\ln f(t)\,dt \le \frac{1}{4}(\ln b - \ln a)^2 \ln\left(\sqrt{f(a)f(b)}\right). \tag{6.11}$$

We also have:

THEOREM 6.3. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$ and $w : [a,b] \to [0,\infty)$ an integrable function on $[a,b]$ and such that $w\left(\frac{ab}{t}\right) = w(t)$ for any $t \in [a,b]$. Then we have the inequalities*

$$f\left(\sqrt{ab}\right) \le \exp\left(\frac{1}{2}\cdot\frac{\int_a^b\left(1+\frac{ab}{t^2}\right)w(t)\ln f(t)\,dt}{\int_a^b w(t)\,dt}\right) \le \frac{\int_a^b w(t)\sqrt{f(t)f\left(\frac{ab}{t}\right)}dt}{\int_a^b w(t)\,dt} \le \frac{\sqrt{ab\int_a^b \frac{w(t)f(t)}{t^2}dt\int_a^b w(t)f(t)\,dt}}{\int_a^b w(t)\,dt} \le \frac{1}{2}\frac{\int_a^b\left(1+\frac{ab}{t^2}\right)w(t)f(t)\,dt}{\int_a^b w(t)\,dt}. \tag{6.12}$$

PROOF. As in the proof of Theorem 6.1 we have

$$\frac{1}{2}\int_a^b\left(1+\frac{ab}{t^2}\right)w(t)\ln f(t)\,dt = \int_a^b w(t)\ln\sqrt{f(t)f\left(\frac{ab}{t}\right)}dt.$$

Then by Jensen's inequality for the exponential and the weight w we have

$$\exp\left(\frac{1}{2}\cdot\frac{\int_a^b\left(1+\frac{ab}{t^2}\right)w(t)\ln f(t)\,dt}{\int_a^b w(t)\,dt}\right)$$
$$=\exp\left(\frac{\int_a^b w(t)\ln\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}dt}{\int_a^b w(t)\,dt}\right)$$
$$\le\frac{\int_a^b w(t)\exp\left(\ln\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}\right)dt}{\int_a^b w(t)\,dt}=\frac{\int_a^b w(t)\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}dt}{\int_a^b w(t)\,dt}$$

that proves the second part of (6.12).

By Cauchy-Bunyakowsky-Schwarz inequality and the property of w we have

$$\int_a^b w(t)\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}dt\le\sqrt{\int_a^b w(t)\,f(t)\,dt\int_a^b w(t)\,f\left(\frac{ab}{t}\right)dt}$$
$$=\sqrt{ab\int_a^b\frac{w(t)\,f(t)}{t^2}dt\int_a^b w(t)\,f(t)\,dt},$$

which proves the third inequality in (6.12).

By the geometric mean - arithmetic mean inequality we also have

$$\sqrt{ab\int_a^b\frac{w(t)\,f(t)}{t^2}dt\int_a^b w(t)\,f(t)\,dt}\le\frac{1}{2}\int_a^b\left(1+\frac{ab}{t^2}\right)w(t)\,f(t)\,dt$$

that proves the last part of (6.12). ∎

If we take $w(t)=1$, $t\in[a,b]$ in (6.12), then we get

$$f\left(\sqrt{ab}\right)\le\exp\left(\frac{1}{2(b-a)}\int_a^b\left(1+\frac{ab}{t^2}\right)\ln f(t)\,dt\right)\tag{6.13}$$
$$\le\frac{1}{b-a}\int_a^b\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}dt\le\frac{1}{b-a}\sqrt{ab\int_a^b\frac{f(t)}{t^2}dt\int_a^b f(t)\,dt}$$
$$\le\frac{1}{2(b-a)}\int_a^b\left(1+\frac{ab}{t^2}\right)f(t)\,dt.$$

We observe that, if in the first inequality in (5.16) we take $p=\frac{1}{2}$, then we have

$$f\left(\sqrt{ab}\right)\le\frac{1}{b-a}\int_a^b\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}dt.\tag{6.14}$$

Therefore the first part of (6.13) is a refinement of (6.14).

CHAPTER 6

Inequalities for GH-Convex Functions

1. Preliminary Facts for GH-convex Functions

Let X be a linear space and C a convex subset in X. A function $f : C \to \mathbb{R}\setminus\{0\}$ is called *AH-convex (concave)* on the convex set C if the following inequality holds

$$f((1-\lambda)x+\lambda y) \le (\ge) \frac{f(x)f(y)}{(1-\lambda)f(y)+\lambda f(x)} \tag{AH}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

An important case which provides many examples is that one in which the function is assumed to be positive for any $x \in C$. In that situation the inequality (AH) is equivalent to

$$(1-\lambda)\frac{1}{f(x)} + \lambda\frac{1}{f(y)} \le (\ge) \frac{1}{f((1-\lambda)x+\lambda y)}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

Therefore we can state the following fact:

Criterion 2. *Let X be a linear space and C a convex subset in X. The function $f : C \to (0,\infty)$ is AH-convex (concave) on C if and only if $\frac{1}{f}$ is concave (convex) on C in the usual sense.*

If we apply the Hermite-Hadamard inequality for the function $\frac{1}{f}$ then we state the following result:

Proposition 1.1 (Dragomir, 2015 [**17**]). *Let X be a linear space and C a convex subset in X. If the function $f : C \to (0,\infty)$ is AH-convex (concave) on C, then*

$$\frac{f(x)+f(y)}{2f(x)f(y)} \le (\ge) \int_0^1 \frac{d\lambda}{f((1-\lambda)x+\lambda y)} \le (\ge) \frac{1}{f\left(\frac{x+y}{2}\right)} \tag{1.1}$$

for any $x, y \in C$.

Following [**1**], we can introduce the concept of *GH-convex (concave)* function $f : I \subset (0,\infty) \to \mathbb{R}$ on an interval of positive numbers I as satisfying the condition

$$f\left(x^{1-\lambda}y^{\lambda}\right) \le (\ge) \frac{1}{(1-\lambda)\frac{1}{f(x)} + \lambda\frac{1}{f(y)}} = \frac{f(x)f(y)}{(1-\lambda)f(y)+\lambda f(x)}. \tag{1.2}$$

Since

$$f\left(x^{1-\lambda}y^{\lambda}\right) = f \circ \exp\left[(1-\lambda)\ln x + \lambda \ln y\right]$$

and

$$\frac{f(x)f(y)}{(1-\lambda)f(y)+\lambda f(x)} = \frac{f\circ\exp(\ln x)\, f\circ\exp(\ln y)}{(1-\lambda)f\circ\exp(y)+\lambda f\circ\exp(x)}$$

then $f : I \subset (0,\infty) \to \mathbb{R}$ is *GH*-convex (concave) on I if and only if $f \circ \exp$ is *AH-convex (concave)* on $\ln I := \{x \mid x = \ln t,\ t \in I\}$.

Motivated by the above results, in this paper we establish some Hermite-Hadamard type inequalities for GH-convex (concave) functions. Some examples for special means are provided as well.

2. INEQUALITIES FOR GH-CONVEX FUNCTIONS

As a direct consequence of Hermite-Hadamard inequality we have:

THEOREM 2.1 (Dragomir, 2015 [**17**]). *Let* $f : [a,b] \subset (0,\infty) \to (0,\infty)$ *be GH-convex (concave) on* $[a,b]$. *Then*

$$\frac{f(a)+f(b)}{2f(a)f(b)} \leq (\geq) \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{tf(t)}dt \leq (\geq) \frac{1}{f\left(\sqrt{ab}\right)}. \tag{2.1}$$

From a different perspective we have:

THEOREM 2.2 (Dragomir, 2015 [**17**]). *Let* $f : [a,b] \subset (0,\infty) \to (0,\infty)$ *be GH-convex (concave) on* $[a,b]$. *Then*

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt \leq (\geq) \frac{G^2(f(a),f(b))}{L(f(a),f(b))}. \tag{2.2}$$

Using the following well known inequality $G(a,b) \leq L(a,b)$ we have a simpler upper bound

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt \leq \frac{G^2(f(a),f(b))}{L(f(a),f(b))} \leq G(f(a),f(b)) \tag{2.3}$$

provided that $f : [a,b] \subset (0,\infty) \to (0,\infty)$ is *GH*-convex on $[a,b]$.

We have also the complementary result:

THEOREM 2.3 (Dragomir, 2015 [**17**]). *Let* $f : [a,b] \subset (0,\infty) \to (0,\infty)$ *be GH-convex (concave) on* $[a,b]$. *Then*

$$f\left(\sqrt{ab}\right) \leq (\geq) \frac{\int_a^b \frac{1}{t}f(t)f\left(\frac{ab}{t}\right)dt}{\int_a^b \frac{1}{t}f(t)dt}. \tag{2.4}$$

We observe that by Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\int_a^b \frac{1}{t}f(t)f\left(\frac{ab}{t}\right)dt \leq \left(\int_a^b \frac{1}{t^2}f^2(t)dt\right)^{1/2}\left(\int_a^b f^2\left(\frac{ab}{t}\right)dt\right)^{1/2}. \tag{2.5}$$

If we change the variable $\frac{ab}{t} = s$, then $dt = -\frac{ab}{s^2}ds$ and we have

$$\int_a^b f^2\left(\frac{ab}{t}\right)dt = ab\int_a^b \frac{1}{s^2}f^2(s)ds.$$

From (2.5) we get

$$\begin{aligned}\int_a^b \frac{1}{t}f(t)f\left(\frac{ab}{t}\right)dt &\leq \left(\int_a^b \frac{1}{t^2}f^2(t)dt\right)^{1/2}\left(ab\int_a^b \frac{1}{s^2}f^2(s)ds\right)^{1/2} \\ &= \sqrt{ab}\int_a^b \frac{1}{t^2}f^2(t)dt.\end{aligned}$$

Now, if $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is *GH*-convex, then from (2.4) we have

$$f\left(\sqrt{ab}\right)\le\frac{\int_a^b\frac{1}{t}f(t)\,f\left(\frac{ab}{t}\right)dt}{\int_a^b\frac{1}{t}f(t)\,dt}\le\sqrt{ab}\,\frac{\int_a^b\frac{1}{t^2}f^2(t)\,dt}{\int_a^b\frac{1}{t}f(t)\,dt}. \tag{2.6}$$

If the function $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is monotonic either nonincreasing or nondecreasing, then the functions $f(\cdot)$ and $f\left(\frac{ab}{\cdot}\right)$ have opposite monotonicities. By the Čebyšev weighted integral inequality for asynchronous functions g and h and the positive weight $w\ge 0$, namely

$$\int_a^b w(t)\,dt\int_a^b w(t)\,g(t)\,h(t)\,dt\le\int_a^b w(t)\,g(t)\,dt\int_a^b w(t)\,h(t)\,dt,$$

we have

$$\int_a^b\frac{1}{t}dt\int_a^b\frac{1}{t}f(t)\,f\left(\frac{ab}{t}\right)dt\le\int_a^b\frac{1}{t}f(t)\,dt\int_a^b\frac{1}{t}f\left(\frac{ab}{t}\right)dt,$$

i.e.,

$$\int_a^b\frac{1}{t}f(t)\,f\left(\frac{ab}{t}\right)dt\le\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}f(t)\,dt\int_a^b\frac{1}{t}f\left(\frac{ab}{t}\right)dt.$$

So, if $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is *GH*-convex and monotonic on $[a,b]$, then from (2.4) we have

$$f\left(\sqrt{ab}\right)\le\frac{\int_a^b\frac{1}{t}f(t)\,f\left(\frac{ab}{t}\right)dt}{\int_a^b\frac{1}{t}f(t)\,dt}\le\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}f\left(\frac{ab}{t}\right)dt \tag{2.7}$$

or, equivalently

$$f\left(\sqrt{ab}\right)\le\frac{\int_a^b\frac{1}{t}f(t)\,f\left(\frac{ab}{t}\right)dt}{\int_a^b\frac{1}{t}f(t)\,dt}\le\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}f(t)\,dt. \tag{2.8}$$

THEOREM 2.4 (Dragomir, 2015 [**17**]). *Let $f:I\subset(0,\infty)\to(0,\infty)$ be GH-convex (concave) on I. If $x,y\in\mathring{I}$, the interior of I, then there exists $\varphi(y)\in\left[f_-'(y),f_+'(y)\right]$ such that*

$$\frac{f(y)}{f(x)}-1\le(\ge)\,\frac{\varphi(y)\,y}{f(y)}(\ln y-\ln x). \tag{2.9}$$

In particular, we have:

COROLLARY 2.5. *Let $f:I\subset(0,\infty)\to(0,\infty)$ be GH-convex (concave) on I and differentiable on $\mathring{I}$. If $x,y\in\mathring{I}$, then*

$$\frac{f(y)}{f(x)}-1\le(\ge)\,\frac{f'(y)\,y}{f(y)}(\ln y-\ln x). \tag{2.10}$$

We also have:

THEOREM 2.6 (Dragomir, 2015 [**17**]). *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be GH-convex (concave) on $[a,b]$. Then*

$$\int_a^b\frac{1}{s}f^2(s)\,ds\le(\ge)\,[(\ln b-\ln u)\,f(b)+(\ln u-\ln a)\,f(a)]\,f(u), \tag{2.11}$$

for any $u\in[a,b]$.

If we take in (2.11) $u = G(a,b) = \sqrt{ab}$, then we get

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{1}{s} f^2(s)\,ds \le (\ge)\, A(f(a), f(b))\, f(G(a,b)). \tag{2.12}$$

If we take in (2.11) either $u = a$ or $u = b$, then we have

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{1}{s} f^2(s)\,ds \le (\ge)\, f(b)\, f(a). \tag{2.13}$$

Also, by taking in (2.11) $u = I(a,b)$, the *identric mean*, then we get

$$\int_a^b \frac{1}{s} f^2(s)\,ds \le (\ge) \left[(\ln b - \ln I(a,b)) f(b) + (\ln I(a,b) - \ln a) f(a)\right] f(I(a,b)). \tag{2.14}$$

Since simple calculations show that

$$\ln b - \ln I(a,b) = \frac{L(a,b) - a}{L(a,b)},\ \ \ln I(a,b) - \ln a = \frac{b - L(a,b)}{L(a,b)},$$

and then the inequality (2.14) is equivalent to

$$\int_a^b \frac{1}{s} f^2(s)\,ds \le (\ge)\, f(I(a,b)) \left[\frac{L(a,b) - a}{L(a,b)} f(b) + \frac{b - L(a,b)}{L(a,b)} f(a)\right]. \tag{2.15}$$

Since $f : [a,b] \subset (0,\infty) \to (0,\infty)$ is *GH*-convex (concave) on $[a,b]$, hence $f \circ \exp$ is *AH*-convex (concave) on $[\ln a, \ln b]$. By the inequality (1.1) for $f \circ \exp$ and $\ln a, \ln b$ we have

$$\frac{f\circ\exp(\ln a) + f\circ\exp(\ln b)}{2 f\circ\exp(\ln a)\, f\circ\exp(\ln b)} \le (\ge) \frac{1}{f\circ\exp\left(\frac{\ln a + \ln b}{2}\right)} \tag{2.16}$$

that is equivalent to

$$\frac{f(a) + f(b)}{2 f(a) f(b)} \le (\ge) \int_0^1 \frac{d\lambda}{f(a^{1-\lambda} b^\lambda)} \le (\ge) \frac{1}{f\left(\sqrt{ab}\right)}. \tag{2.17}$$

If we change the variable $t = a^{1-\lambda} b^\lambda$, then $(1-\lambda)\ln a + \lambda \ln b = \ln t$, which gives $\lambda = \frac{\ln t - \ln a}{\ln b - \ln a}$ and $d\lambda = \frac{1}{(\ln b - \ln a)t} dt$. We have then

$$\int_0^1 \frac{d\lambda}{f(a^{1-\lambda} b^\lambda)} = \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{t f(t)} dt$$

and by (2.17) we obtain the desired result (2.1).

From the definition of *GH*-convex (concave) functions on $[a,b]$ and by integration we get

$$\int_0^1 f\left(a^{1-\lambda} b^\lambda\right) d\lambda \le (\ge)\, f(a) f(b) \int_0^1 \frac{d\lambda}{(1-\lambda) f(a) + \lambda f(b)}. \tag{2.18}$$

If $f(a) = f(b)$, then the integral

$$\int_0^1 \frac{d\lambda}{(1-\lambda) f(a) + \lambda f(b)} \tag{2.19}$$

reduces to $\frac{1}{f(a)}$.

If $f(a) \ne f(b)$, then by changing the variable $u = (1-\lambda) f(a) + \lambda f(b)$ in (2.19) we have

$$\int_0^1 \frac{d\lambda}{(1-\lambda) f(a) + \lambda f(b)} = \frac{1}{f(b) - f(a)} \int_{f(a)}^{f(b)} \frac{du}{u} = \frac{1}{L(f(a), f(b))}.$$

Also, as above, if we change the variable $t = a^{1-\lambda}b^{\lambda}$, then

$$\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right) d\lambda = \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt.$$

Replacing these values in (2.18), we get the desired result (2.2).

If we take in the definition of *GH*-convex functions $\lambda = \frac{1}{2}$, then we get

$$f\left(\sqrt{xy}\right) \le (\ge) \frac{2f(x)f(y)}{f(y)+f(x)}. \tag{2.20}$$

If we replace in (2.20), $x = a^{1-\lambda}b^{\lambda}$ and $y = a^{\lambda}b^{1-\lambda}$, then we get

$$f\left(\sqrt{ab}\right)\left[f\left(a^{1-\lambda}b^{\lambda}\right) + f\left(a^{\lambda}b^{1-\lambda}\right)\right] \le (\ge)\, 2f\left(a^{1-\lambda}b^{\lambda}\right) f\left(a^{\lambda}b^{1-\lambda}\right). \tag{2.21}$$

By integrating this inequality over λ on $[0,1]$ we obtain

$$f\left(\sqrt{ab}\right)\left[\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right) d\lambda + \int_0^1 f\left(a^{\lambda}b^{1-\lambda}\right) d\lambda\right] \le (\ge)\, 2\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right) f\left(a^{\lambda}b^{1-\lambda}\right) d\lambda. \tag{2.22}$$

Observe that

$$\int_0^1 f\left(a^{\lambda}b^{1-\lambda}\right) d\lambda = \int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right) d\lambda = \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}dt$$

and

$$\int_0^1 f\left(a^{1-\lambda}b^{\lambda}\right) f\left(a^{\lambda}b^{1-\lambda}\right) d\lambda = \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)\, f\left(\frac{ab}{t}\right)}{t}dt.$$

Making use of (2.22) we deduce the desired result (2.4).

The following lemma is of interest in itself:

LEMMA 2.7 (Dragomir, 2015 [**17**]). *Let $f : I \subset \mathbb{R} \to (0,\infty)$ be AH-convex (concave) on I. If $x, y \in \mathring{I}$, the interior of I, then there exists $\varphi(y) \in \left[f'_-(y), f'_+(y)\right]$ such that*

$$\frac{f(y)}{f(x)} - 1 \le (\ge) \frac{\varphi(y)}{f(y)}(y - x) \tag{2.23}$$

holds.

PROOF. Let $x, y \in \mathring{I}$. Since the function $\frac{1}{f}$ is concave (convex) then the lateral derivatives $f'_-(y), f'_+(y)$ exists for $y \in \mathring{I}$ and $\left(\frac{1}{f}\right)'_{-(+)}(y) = -\frac{f'_{-(+)}(y)}{f^2(y)}$.

Since $\frac{1}{f}$ is concave (convex) then we have the *gradient inequality*

$$\frac{1}{f(y)} - \frac{1}{f(x)} \ge (\le)\, \lambda(y)(y - x) = -\lambda(y)(x - y)$$

with $\lambda(y) \in \left[-\frac{f'_+(y)}{f^2(y)}, -\frac{f'_-(y)}{f^2(y)}\right]$, which is equivalent to

$$\frac{1}{f(y)} - \frac{1}{f(x)} \ge (\le) \frac{\varphi(y)}{f^2(y)}(x - y) \tag{2.24}$$

with $\varphi(y) \in \left[f'_-(y), f'_+(y)\right]$.

The inequality (2.24) can be also written as

$$\frac{f(y)}{f(x)} - 1 \le (\ge) \frac{\varphi(y)}{f(y)}(y-x)$$

and the inequality (2.23) is proved. ∎

Now, since $f : I \subset (0,\infty) \to (0,\infty)$ is GH-convex (concave) on I, then the function $f \circ \exp$ is AH-convex (concave) on $\ln I$.

Let $u, v \in \ln \mathring{I}$, then by (2.23) we have

$$\frac{f(e^v)}{f(e^u)} - 1 \le (\ge) \frac{\varphi(e^v)e^v}{f(e^v)}(v-u) \tag{2.25}$$

with $\varphi(e^v) \in \left[f'_-(e^v), f'_+(e^v)\right]$.

If $x, y \in \mathring{I}$ and we take $u = \ln x$, $v = \ln y$ in (2.25) then we get

$$\frac{f(y)}{f(x)} - 1 \le (\ge) \frac{\varphi(y)y}{f(y)}(\ln y - \ln x)$$

with $\varphi(y) \in \left[f'_-(y), f'_+(y)\right]$.

This proves Theorem 2.4.

The following lemma is of interest in itself.

LEMMA 2.8 (Dragomir, 2015 [**17**]). *Let $g : [c,d] \subset (0,\infty) \to (0,\infty)$ be AH-convex (concave) on $[c,d]$, then we have the inequality*

$$\frac{1}{d-c}\int_c^d g^2(t)\,dt \le (\ge) \left[\frac{d-s}{d-c}g(d) + \frac{s-c}{d-c}g(c)\right]g(s) \tag{2.26}$$

for any $s \in [c,d]$.

PROOF. If the function $g : [c,d] \subset (0,\infty) \to (0,\infty)$ is AH-convex (concave) on $[c,d]$, then the function g is differentiable almost everywhere on $[c,d]$ and we have the inequality

$$\frac{g(t)}{g(s)} - 1 \le (\ge) \frac{g'(t)}{g(t)}(t-s) \tag{2.27}$$

for every $s \in [c,d]$ and almost every $t \in [c,d]$.

Multiplying (2.27) by $g(t) > 0$ and integrating over $t \in [c,d]$ we have

$$\frac{1}{g(s)}\int_c^d g^2(t)\,dt - \int_c^d g(t)\,dt \le (\ge) \int_c^d g'(t)(t-s)\,dt. \tag{2.28}$$

Integrating by parts we also have

$$\int_c^d g'(t)(t-s)\,dt = g(d)(d-s) + g(c)(s-c) - \int_c^d g(t)\,dt$$

and by (2.28) we get the desired result (2.26). ∎

Now, since $f : [a,b] \subset (0,\infty) \to (0,\infty)$ is GH-convex (concave) on I, then the function $g = f \circ \exp$ is AH-convex (concave) on $[c,d] = [\ln a, \ln b]$.

From (2.26) we then have for $s = \ln u$, $u \in [a,b]$ that

$$\begin{aligned}\frac{1}{\ln b - \ln a}&\int_{\ln a}^{\ln b} f^2 \circ \exp(t)\,dt \\ &\le (\ge) \left[\frac{\ln b - \ln u}{\ln b - \ln a}f \circ \exp(\ln b) + \frac{\ln u - \ln a}{\ln b - \ln a}f \circ \exp(\ln a)\right] f \circ \exp(\ln u).\end{aligned}$$

This is equivalent to

$$\text{(2.29)} \quad \frac{1}{\ln b - \ln a} \int_{\ln a}^{\ln b} f^2 \circ \exp(t)\, dt \\ \leq (\geq) \left[\frac{\ln b - \ln u}{\ln b - \ln a} f(b) + \frac{\ln u - \ln a}{\ln b - \ln a} f(a) \right] f(u),$$

for any $u \in [a, b]$.

If we make the change of variable $s = \exp(t)$, then $t = \ln s$, $dt = \frac{ds}{s}$ and by (2.29) we get the desired inequality (2.11).

CHAPTER 7

Inequalities for HA-Convex Functions

1. Some Preliminary Facts for HA-Convex Functions

Following [**1**] we say that the function $f:I\subset R\setminus\{0\}\to\mathbb{R}$ is *HA-convex* or *harmonically convex* if

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq(1-t)f(x)+tf(y) \tag{1.1}$$

for all $x,y\in I$ and $t\in[0,1]$. If the inequality in (1.1) is reversed, then f is said to be *HA-concave* or *harmonically concave*.

In order to avoid any confusion with the class of *AH-convex* functions, namely the functions satisfying the condition

$$f((1-t)x+ty)\leq\frac{f(x)f(y)}{(1-t)f(y)+tf(x)} \tag{1.2}$$

we will call the class of functions satisfying (1.1) as *HA*-convex functions.

If $I\subset(0,\infty)$ and f is convex and nondecreasing function then f is *HA*-convex and if f is *HA*-convex and nonincreasing function then f is convex.

If $[a,b]\subset I\subset(0,\infty)$ and if we consider the function $g:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$, defined by $g(t)=f\left(\frac{1}{t}\right)$, then f is *HA*-convex on $[a,b]$ if and only if g is convex in the usual sense on $\left[\frac{1}{b},\frac{1}{a}\right]$.

If we write the Hermite-Hadamard inequality for the convex function $g(t)=f\left(\frac{1}{t}\right)$ on the closed interval $\left[\frac{1}{b},\frac{1}{a}\right]$ we have

$$f\left(\frac{1}{\frac{\frac{1}{a}+\frac{1}{b}}{2}}\right)\leq\frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}}f\left(\frac{1}{t}\right)dt\leq\frac{f\left(\frac{1}{\frac{1}{b}}\right)+f\left(\frac{1}{\frac{1}{a}}\right)}{2}$$

that is equivalent to

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}}f\left(\frac{1}{t}\right)dt\leq\frac{f(b)+f(a)}{2} \tag{1.3}$$

Using the change of variable $s=\frac{1}{t}$, then

$$\int_{\frac{1}{b}}^{\frac{1}{a}}f\left(\frac{1}{t}\right)dt=\int_a^b\frac{f(s)}{s^2}ds$$

and by (1.3) we get

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_a^b\frac{f(s)}{s^2}ds\leq\frac{f(b)+f(a)}{2}. \tag{1.4}$$

The inequality (1.4) has been obtained in a different manner in [**27**].

Motivated by the above results, we establish in this paper some new inequalities of Hermite-Hadamard type for *HA*-convex functions.

2. A REFINEMENT

We have the following representation result.

LEMMA 2.1 (Dragomir, 2015 [**18**]). *Let $g : [x,y] \subset \mathbb{R} \to \mathbb{C}$ be a Lebesgue integrable function on $[x,y]$. Then for any $\lambda \in [0,1]$ we have the representation*

$$\int_0^1 g[(1-t)x+ty]\,dt = (1-\lambda)\int_0^1 g[(1-t)((1-\lambda)x+\lambda y)+ty]\,dt + \lambda\int_0^1 g[(1-t)x+t((1-\lambda)x+\lambda y)]\,dt. \tag{2.1}$$

PROOF. For $\lambda=0$ and $\lambda=1$ the equality (2.1) is obvious.

Let $\lambda\in(0,1)$. Observe that

$$\int_0^1 g[(1-t)(\lambda y+(1-\lambda)x)+ty]\,dt = \int_0^1 g[((1-t)\lambda+t)y+(1-t)(1-\lambda)x]\,dt$$

and

$$\int_0^1 g[t(\lambda y+(1-\lambda)x)+(1-t)x]\,dt = \int_0^1 g[t\lambda y+(1-\lambda t)x]\,dt.$$

If we make the change of variable $u:=(1-t)\lambda+t$ then we have $1-u=(1-t)(1-\lambda)$ and $du=(1-\lambda)du$. Then

$$\int_0^1 g[((1-t)\lambda+t)y+(1-t)(1-\lambda)x]\,dt = \frac{1}{1-\lambda}\int_\lambda^1 g[uy+(1-u)x]\,du.$$

If we make the change of variable $u:=\lambda t$ then we have $du=\lambda dt$ and

$$\int_0^1 g[t\lambda y+(1-\lambda t)x]\,dt = \frac{1}{\lambda}\int_0^\lambda g[uy+(1-u)x]\,du.$$

Therefore

$$(1-\lambda)\int_0^1 g[(1-t)(\lambda y+(1-\lambda)x)+ty]\,dt + \lambda\int_0^1 g[t(\lambda y+(1-\lambda)x)+(1-t)x]\,dt = \int_0^1 g[uy+(1-u)x]\,du$$

and the identity (2.1) is proved. ∎

COROLLARY 2.2 (Dragomir, 2015 [**18**]). *Let $f : [a,b] \subset (0,\infty) \to \mathbb{C}$ be a Lebesgue integrable function on $[a,b]$ and $\lambda\in[0,1]$, then we have the representation*

$$\int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)dt = (1-\lambda)\int_0^1 f\left(\frac{ab}{(1-t)((1-\lambda)a+\lambda b)+tb}\right)dt + \lambda\int_0^1 f\left(\frac{ab}{(1-t)a+t((1-\lambda)a+\lambda b)}\right)dt. \tag{2.2}$$

PROOF. Consider the function $g:\left[\frac{1}{b},\frac{1}{a}\right]$, $g(s)=f\left(\frac{1}{s}\right)$, $s\in\left[\frac{1}{b},\frac{1}{a}\right]$.

We have by (2.1) for g and $x = \frac{1}{b}$, $y = \frac{1}{a}$ that

$$\begin{aligned}
(2.3) \quad & \int_0^1 f\left(\frac{ab}{(1-t)\,a+tb}\right) dt \\
&= \int_0^1 f\left(\frac{1}{(1-t)\,\frac{1}{b}+t\frac{1}{a}}\right) dt = \int_0^1 g\left((1-t)\,\frac{1}{b}+t\frac{1}{a}\right) dt \\
&= (1-\lambda)\int_0^1 g\left[(1-t)\left((1-\lambda)\,\frac{1}{b}+\lambda\frac{1}{a}\right)+t\frac{1}{a}\right] dt \\
&+ \lambda\int_0^1 g\left[(1-t)\,\frac{1}{b}+t\left((1-\lambda)\,\frac{1}{b}+\lambda\frac{1}{a}\right)\right] dt
\end{aligned}$$

$$\begin{aligned}
&= (1-\lambda)\int_0^1 f\left(\frac{ab}{(1-t)\left((1-\lambda)\,a+\lambda b\right)+tb}\right) dt \\
&+ \lambda\int_0^1 f\left(\frac{ab}{(1-t)\,a+t\left((1-\lambda)\,a+\lambda b\right)}\right) dt.
\end{aligned}$$

∎

The following result holds.

THEOREM 2.3 (Dragomir, 2015 [**18**]). *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then for any $\lambda \in [0,1]$ we have the inequalities*

$$\begin{aligned}
(2.4) \quad f\left(\frac{2ab}{a+b}\right) &\le (1-\lambda)\,f\left(\frac{2ab}{(1-\lambda)\,a+(\lambda+1)\,b}\right)+\lambda f\left(\frac{2ab}{(2-\lambda)\,a+\lambda b}\right) \\
&\le \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt \\
&\le \frac{1}{2}\left[f\left(\frac{ab}{(1-\lambda)\,a+\lambda b}\right)+(1-\lambda)\,f(a)+\lambda f(b)\right] \\
&\le \frac{f(a)+f(b)}{2}.
\end{aligned}$$

PROOF. Consider the function $g : \left[\frac{1}{b},\frac{1}{a}\right]$, $g(s) = f\left(\frac{1}{s}\right)$, $s \in \left[\frac{1}{b},\frac{1}{a}\right]$.

Since g is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$, then by Hermite-Hadamard inequality for convex functions we have for $\lambda \in [0,1]$

$$\begin{aligned}
(2.5) \quad g\left(\frac{(1-\lambda)\,a+(\lambda+1)\,b}{2ab}\right) &= g\left(\frac{(1-\lambda)\,\frac{1}{b}+\lambda\frac{1}{a}+\frac{1}{a}}{2}\right) \\
&\le \int_0^1 g\left((1-t)\left((1-\lambda)\,\frac{1}{b}+\lambda\frac{1}{a}\right)+t\frac{1}{a}\right) dt \\
&\le \frac{g\left((1-\lambda)\,\frac{1}{b}+\lambda\frac{1}{a}\right)+g\left(\frac{1}{a}\right)}{2} = \frac{g\left(\frac{(1-\lambda)a+\lambda b}{ab}\right)+g\left(\frac{1}{a}\right)}{2}
\end{aligned}$$

and

$$g\left(\frac{(2-\lambda)a+\lambda b}{2ab}\right)=g\left(\frac{\frac{1}{b}+(1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}}{2}\right) \tag{2.6}$$
$$\leq\int_0^1 g\left((1-t)\frac{1}{b}+t\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)\right)dt$$
$$\leq\frac{g\left(\frac{1}{b}\right)+g\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)}{2}=\frac{g\left(\frac{1}{b}\right)+g\left(\frac{(1-\lambda)a+\lambda b}{ab}\right)}{2}.$$

If we multiply (2.5) by $(1-\lambda)$ and (2.6) by $\lambda,$ add the obtained inequalities and use the first part of the equality (2.3) we get

$$(1-\lambda)f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)+\lambda f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right) \tag{2.7}$$
$$\leq\int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)dt$$
$$\leq(1-\lambda)\frac{f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+f(a)}{2}+\lambda\frac{f(b)+f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)}{2}$$
$$=\frac{1}{2}\left[f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+(1-\lambda)f(a)+\lambda f(b)\right].$$

By the convexity of g we have

$$(1-\lambda)f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)+\lambda f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)$$
$$=(1-\lambda)g\left(\frac{(1-\lambda)a+(\lambda+1)b}{2ab}\right)+\lambda g\left(\frac{(2-\lambda)a+\lambda b}{2ab}\right)$$
$$\geq g\left(\frac{(1-\lambda)[(1-\lambda)a+(\lambda+1)b]}{2ab}+\frac{\lambda[(2-\lambda)a+\lambda b]}{2ab}\right)$$
$$=g\left(\frac{a+b}{2ab}\right)=f\left(\frac{2ab}{a+b}\right)$$

and

$$f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+(1-\lambda)f(a)+\lambda f(b)$$
$$=g\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)+(1-\lambda)f(a)+\lambda f(b)$$
$$\leq(1-\lambda)f(b)+\lambda f(a)+(1-\lambda)f(a)+\lambda f(b)$$
$$=f(a)+f(b)$$

and the desired inequality (2.4) is proved. ∎

COROLLARY 2.4. *With the assumptions of Theorem 2.3 we have*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{1}{2}\left[f\left(\frac{4ab}{a+3b}\right)+f\left(\frac{4ab}{3a+b}\right)\right]\leq\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt \tag{2.8}$$
$$\leq\frac{1}{2}\left[f\left(\frac{2ab}{a+b}\right)+\frac{f(a)+f(b)}{2}\right]\leq\frac{f(a)+f(b)}{2}.$$

3. INEQUALITIES FOR HA-CONVEX FUNCTIONS

We have:

THEOREM 3.1 (Dragomir, 2015 [**18**]). *Let* $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be an HA-convex function on the interval* $[a,b]$. *Then*

$$f(L(a,b)) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{(L(a,b)-a)\,bf(b)+(b-L(a,b))\,af(a)}{(b-a)\,L(a,b)}. \tag{3.1}$$

PROOF. Since $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is an *HA*-convex function on the interval $[a,b]$, then the function $g : \left[\frac{1}{b},\frac{1}{a}\right]$, $g(s) = f\left(\frac{1}{s}\right)$, is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$. Therefore f has partial derivatives in each point of (a,b) and by the gradient inequality for g we have for any $x,y \in (a,b)$ that

$$\begin{aligned} f(x)-f(y) = g\left(\frac{1}{x}\right)-g\left(\frac{1}{y}\right) &\ge g'_+\left(\frac{1}{y}\right)\left(-\frac{1}{y^2}\right)\left(\frac{1}{x}-\frac{1}{y}\right) \\ &= g'_+\left(\frac{1}{y}\right)\frac{x-y}{xy^3}. \end{aligned} \tag{3.2}$$

Since

$$g'_+(s) = f'_-\left(\frac{1}{s}\right)\left(-\frac{1}{s^2}\right),\ s \in \left(\frac{1}{b},\frac{1}{a}\right)$$

then

$$g'_+\left(\frac{1}{y}\right) = f'_-(y)\left(-y^2\right)$$

and by (3.2) we have

$$f(x)-f(y) \ge f'_-(y)\frac{x-y}{xy^3}\left(-y^2\right) = f'_-(y)\frac{y-x}{xy}.$$

Therefore we have

$$f(x)-f(y) \ge \left(\frac{1}{x}-\frac{1}{y}\right)f'_-(y) \tag{3.3}$$

for any $x,y \in (a,b)$.

If we take the integral mean over x in (3.3), then we have

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f(y) \ge \left(\frac{1}{L(a,b)}-\frac{1}{y}\right)f'_-(y) \tag{3.4}$$

for any $y \in (a,b)$.

Now, if we take $y = L(a,b)$ in (3.4), then we get the first inequality in (3.1).

Observe that for any $x \in [a,b]$ we have

$$\frac{1}{x} = \frac{\left(\frac{1}{a}-\frac{1}{x}\right)\frac{1}{b}+\left(\frac{1}{x}-\frac{1}{b}\right)\frac{1}{a}}{\frac{1}{a}-\frac{1}{b}}.$$

By the convexity of g on $\left[\frac{1}{b},\frac{1}{a}\right]$ we then have

$$\begin{aligned} f(x) = g\left(\frac{1}{x}\right) &= g\left(\frac{\left(\frac{1}{a}-\frac{1}{x}\right)\frac{1}{b}+\left(\frac{1}{x}-\frac{1}{b}\right)\frac{1}{a}}{\frac{1}{a}-\frac{1}{b}}\right) \\ &\le \frac{\left(\frac{1}{a}-\frac{1}{x}\right)f(b)+\left(\frac{1}{x}-\frac{1}{b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}} \end{aligned} \tag{3.5}$$

for any $x \in [a,b]$.

Taking the integral mean in (3.5) we get

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{\left(\frac{1}{a}-\frac{1}{b-a}\int_a^b \frac{1}{x}dx\right) f(b) + \left(\frac{1}{b-a}\int_a^b \frac{1}{x}dx - \frac{1}{b}\right) f(a)}{\frac{1}{a}-\frac{1}{b}}$$

$$= \frac{\frac{L(a,b)-a}{aL(a,b)} f(b) + \frac{b-L(a,b)}{L(a,b)b} f(a)}{\frac{b-a}{ab}}$$

and the second inequality in (3.1) is also proved. ∎

REMARK 3.1. If $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is a differentiable HA-convex function on the interval (a,b), then from (3.4) we have the following inequality

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f(y) \ge \left(\frac{1}{L(a,b)} - \frac{1}{y}\right) f'(y) \tag{3.6}$$

for any $y \in (a,b)$.

We have

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f(A(a,b)) \ge \frac{A(a,b)-L(a,b)}{L(a,b)A(a,b)} f'(A(a,b)) \tag{3.7}$$

and if $f'(A(a,b)) \ge 0$, then

$$\frac{1}{b-a}\int_a^b f(x)\,dx \ge f(A(a,b)). \tag{3.8}$$

We have

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f(I(a,b)) \ge \frac{I(a,b)-L(a,b)}{L(a,b)I(a,b)} f'(I(a,b)) \tag{3.9}$$

and if $f'(I(a,b)) \ge 0$, then

$$\frac{1}{b-a}\int_a^b f(x)\,dx \ge f(I(a,b)). \tag{3.10}$$

We have

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f(G(a,b)) \ge \frac{G(a,b)-L(a,b)}{L(a,b)G(a,b)} f'(G(a,b)) \tag{3.11}$$

and if $f'(G(a,b)) \le 0$, then

$$\frac{1}{b-a}\int_a^b f(x)\,dx \ge f(G(a,b)). \tag{3.12}$$

We have:

THEOREM 3.2 (Dragomir, 2015 [**18**]). *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)\frac{a+b}{2} \le \frac{1}{b-a}\int_a^b x f(x)\,dx \le \frac{bf(b)+af(a)}{2}. \tag{3.13}$$

PROOF. From the inequality (3.3), by multiplying with $x > 0$ we have

$$xf(x) - xf(y) \ge \left(1 - \frac{x}{y}\right) f'_-(y) \tag{3.14}$$

for any $x, y \in (a,b)$.

Taking the integral mean over $x \in [a,b]$ we have

$$\frac{1}{b-a}\int_a^b xf(x)\,dx - f(y)\frac{a+b}{2} \geq \left(1-\frac{1}{y}\frac{a+b}{2}\right) f'_-(y), \tag{3.15}$$

for any $y \in (a,b)$.

If we take in (3.15) $y = \frac{a+b}{2}$, then we get the first inequality in (3.13).

From the inequality (3.5) we also have

$$xf(x) \leq \frac{\left(\frac{x}{a}-1\right) f(b) + \left(1-\frac{x}{b}\right) f(a)}{\frac{1}{a}-\frac{1}{b}} \tag{3.16}$$

for any $x \in [a,b]$.

Taking the integral mean on (3.16) we get

$$\frac{1}{b-a}\int_a^b xf(x)\,dx \leq \frac{bf(b)+af(a)}{2} \tag{3.17}$$

and the second inequality in (3.13) is proved. ∎

REMARK 3.2. If $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is a differentiable HA-convex function on the interval (a,b), then from (3.15) we have

$$f(y)\frac{a+b}{2} - \frac{1}{b-a}\int_a^b xf(x)\,dx \leq \frac{A(a,b)-y}{y} f'(y), \tag{3.18}$$

for any $y \in (a,b)$.

If we take in (3.18) $y = I(a,b)$, then we get

$$f(I(a,b))\frac{a+b}{2} - \frac{1}{b-a}\int_a^b xf(x)\,dx \leq \frac{A(a,b)-I(a,b)}{I(a,b)} f'(I(a,b)). \tag{3.19}$$

If $f'(I(a,b)) \leq 0$, then

$$f(I(a,b))\frac{a+b}{2} \leq \frac{1}{b-a}\int_a^b xf(x)\,dx. \tag{3.20}$$

If we take in (3.18) $y = L(a,b)$, then we get

$$f(L(a,b))\frac{a+b}{2} - \frac{1}{b-a}\int_a^b xf(x)\,dx \leq \frac{A(a,b)-L(a,b)}{L(a,b)} f'(L(a,b)). \tag{3.21}$$

If $f'(L(a,b)) \leq 0$, then

$$f(L(a,b))\frac{a+b}{2} \leq \frac{1}{b-a}\int_a^b xf(x)\,dx. \tag{3.22}$$

If we take in (3.18) $y = G(a,b)$, then we get

$$f(G(a,b))\frac{a+b}{2} - \frac{1}{b-a}\int_a^b xf(x)\,dx \leq \frac{A(a,b)-G(a,b)}{G(a,b)} f'(G(a,b)). \tag{3.23}$$

If $f'(G(a,b)) \leq 0$, then

$$f(G(a,b))\frac{a+b}{2} \leq \frac{1}{b-a}\int_a^b xf(x)\,dx. \tag{3.24}$$

THEOREM 3.3 (Dragomir, 2015 [**18**]). *Let* $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ *be a HA-convex function on the interval* $[a,b]$. *Then*

$$\frac{1}{2}\left[f'_{+}\left(\frac{2ab}{a+b}\right)-f'_{-}\left(\frac{2ab}{a+b}\right)\right]\frac{ab}{(a+b)^{2}}(b-a)\leq\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_{a}^{b}\frac{f(t)}{t^{2}}dt\leq\frac{1}{8}\left[\frac{f'_{-}(b)b^{2}-f'_{+}(a)a^{2}}{ab}\right](b-a)\tag{3.25}$$

and

$$\frac{1}{2}\left[f'_{+}\left(\frac{2ab}{a+b}\right)-f'_{-}\left(\frac{2ab}{a+b}\right)\right]\frac{ab}{(a+b)^{2}}(b-a)\leq\frac{ab}{b-a}\int_{a}^{b}\frac{f(t)}{t^{2}}dt-f\left(\frac{2ab}{a+b}\right)\leq\frac{1}{8}\left[\frac{f'_{-}(b)b^{2}-f'_{+}(a)a^{2}}{ab}\right](b-a).\tag{3.26}$$

PROOF. Since $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is an *HA*-convex function on the interval $[a,b]$, then the function $g:\left[\frac{1}{b},\frac{1}{a}\right]$, $g(s)=f\left(\frac{1}{s}\right)$, is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$.

We know that

$$g'_{\pm}(s)=f'_{\mp}\left(\frac{1}{s}\right)\left(-\frac{1}{s^{2}}\right),\ s\in\left(\frac{1}{b},\frac{1}{a}\right).$$

If we apply Lemma 0.1 we get the desired result. ∎

COROLLARY 3.4. *If* $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ *is a differentiable HA-convex function on the interval* (a,b), *then*

$$0\leq\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_{a}^{b}\frac{f(t)}{t^{2}}dt\leq\frac{1}{8}\left[\frac{f'_{-}(b)b^{2}-f'_{+}(a)a^{2}}{ab}\right](b-a)\tag{3.27}$$

and

$$0\leq\frac{ab}{b-a}\int_{a}^{b}\frac{f(t)}{t^{2}}dt-f\left(\frac{2ab}{a+b}\right)\leq\frac{1}{8}\left[\frac{f'_{-}(b)b^{2}-f'_{+}(a)a^{2}}{ab}\right](b-a).\tag{3.28}$$

4. RELATED RESULTS

We have the following related result:

THEOREM 4.1 (Dragomir, 2015 [**18**]). *Let* $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ *be a HA-convex function on the interval* $[a,b]$. *Then*

$$f(x)-\frac{1}{b-a}\int_{a}^{b}f(y)dy\geq\frac{1}{b-a}\left[f(b)\left(\frac{1}{x}-\frac{1}{b}\right)+f(a)\left(\frac{1}{a}-\frac{1}{x}\right)\right]-\frac{1}{b-a}\int_{a}^{b}\frac{f(y)}{y^{2}}dy\tag{4.1}$$

for any $x \in [a, b]$ *.*

In particular, we have

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(y)\,dy \geq \frac{\frac{f(b)}{b}+\frac{f(a)}{a}}{a+b} - \frac{1}{b-a}\int_a^b \frac{f(y)}{y^2}dy \tag{4.2}$$

and

$$ab\left[f\left(\frac{2ab}{a+b}\right) - \frac{1}{b-a}\int_a^b f(y)\,dy\right] \geq \frac{f(b)+f(a)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(y)}{y^2}dy \geq 0 \tag{4.3}$$

PROOF. If we integrate over y in the inequality (3.3) we have

$$\begin{aligned}(b-a) f(x) - \int_a^b f(y)\,dy \\ &\geq \frac{1}{x}\int_a^b f'_-(y)\,dy - \int_a^b \frac{1}{y} f'_-(y)\,dy \\ &= \frac{1}{x}[f(b)-f(a)] - \left[\frac{f(b)}{b} - \frac{f(a)}{a} + \int_a^b \frac{f(y)}{y^2}dy\right]\end{aligned} \tag{4.4}$$

for any $x \in (a, b)$.

Observe that

$$\begin{aligned}\frac{1}{x}[f(b)-f(a)] - \frac{f(b)}{b} + \frac{f(a)}{a} - \int_a^b \frac{f(y)}{y^2}dy \\ &= f(b)\left(\frac{1}{x}-\frac{1}{b}\right) + f(a)\left(\frac{1}{a}-\frac{1}{x}\right) - \int_a^b \frac{f(y)}{y^2}dy\end{aligned}$$

and by (4.4) we have

$$(b-a) f(x) - \int_a^b f(y)\,dy \geq f(b)\left(\frac{1}{x}-\frac{1}{b}\right) + f(a)\left(\frac{1}{a}-\frac{1}{x}\right) - \int_a^b \frac{f(y)}{y^2}dy$$

for any $x \in (a, b)$, namely

$$(b-a) f(x) - \int_a^b f(y)\,dy \geq \frac{af(b)(b-x)+bf(a)(x-a)}{xba} - \int_a^b \frac{f(y)}{y^2}dy \tag{4.5}$$

and the inequality (4.1) is proved.

If we take $x = \frac{a+b}{2}$ in (4.5), then we get

$$\begin{aligned}f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(y)\,dy \\ &\geq \frac{af(b)\left(b-\frac{a+b}{2}\right)+bf(a)\left(\frac{a+b}{2}-a\right)}{\frac{a+b}{2}ba(b-a)} - \frac{1}{b-a}\int_a^b \frac{f(y)}{y^2}dy \\ &= \frac{\frac{f(b)}{b}+\frac{f(a)}{a}}{a+b} - \frac{1}{b-a}\int_a^b \frac{f(y)}{y^2}dy\end{aligned}$$

and the inequality (4.2) is proved.

If we take $x = \frac{2ab}{a+b}$ in (4.1), then we get

$$f\left(\frac{2ab}{a+b}\right) - \frac{1}{b-a}\int_a^b f(y)\,dy \geq \frac{1}{ab}\left(\frac{f(b)+f(a)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(y)}{y^2}dy\right)$$

and the first part of (4.3) is proved. The second part follows by (1.4). ∎

We also have:

THEOREM 4.2 (Dragomir, 2015 [**18**]). *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then*

$$\frac{a+b}{2}f(x) - \frac{1}{b-a}\int_a^b yf(y)\,dy \geq \frac{f(b)\frac{b-x}{x} + f(a)\frac{x-a}{x}}{b-a} - \frac{1}{x}\frac{1}{b-a}\int_a^b f(y)\,dy. \tag{4.6}$$

In particular, we have

$$\frac{2}{a+b}\left(\frac{1}{b-a}\int_a^b f(y)\,dy - \frac{f(b)+f(a)}{2}\right) \geq \frac{1}{b-a}\int_a^b yf(y)\,dy - \frac{a+b}{2}f\left(\frac{a+b}{2}\right) \geq 0 \tag{4.7}$$

and

$$\frac{a+b}{2}f\left(\frac{2ab}{a+b}\right) - \frac{1}{b-a}\int_a^b yf(y)\,dy \geq \frac{a+b}{2ab}\left(\frac{bf(b)+af(a)}{a+b} - \frac{1}{b-a}\int_a^b f(y)\,dy\right). \tag{4.8}$$

PROOF. From the inequality (3.3) we have we have

$$yf(x) - yf(y) \geq \left(\frac{y}{x} - 1\right) f'_-(y) \tag{4.9}$$

for any $x, y \in (a,b)$.

If we take the integral over y we have

$$f(x)\int_a^b y\,dy - \int_a^b yf(y)\,dy \geq \frac{1}{x}\int_a^b yf'_-(y)\,dy - \int_a^b f'_-(y)\,dy = f(b)\frac{b-x}{x} + f(a)\frac{x-a}{x} - \frac{1}{x}\int_a^b f(y)\,dy$$

that is equivalent to

$$\frac{1}{2}\left(b^2-a^2\right)f(x) - \int_a^b yf(y)\,dy \geq f(b)\frac{b-x}{x} + f(a)\frac{x-a}{x} - \frac{1}{x}\int_a^b f(y)\,dy \tag{4.10}$$

and , by dividing by $b-a$, we get the desired inequality (4.6).

Now, if we take $x=\frac{a+b}{2}$ in (4.10), then we get

$$\frac{1}{2}\left(b^2-a^2\right)f\left(\frac{a+b}{2}\right)-\int_a^b yf(y)\,dy \geq f(b)\frac{b-\frac{a+b}{2}}{\frac{a+b}{2}}+f(a)\frac{\frac{a+b}{2}-a}{\frac{a+b}{2}}-\frac{1}{\frac{a+b}{2}}\int_a^b f(y)\,dy,$$

which is equivalent to

$$(b-a)\left(\frac{a+b}{2}\right)f\left(\frac{a+b}{2}\right)-\int_a^b yf(y)\,dy \geq f(b)\frac{b-a}{a+b}+f(a)\frac{b-a}{a+b}-\frac{2}{a+b}\int_a^b f(y)\,dy.$$

Dividing by $b-a$ we have

$$\left(\frac{a+b}{2}\right)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b yf(y)\,dy \geq \frac{f(b)+f(a)}{a+b}-\frac{2}{a+b}\frac{1}{b-a}\int_a^b f(y)\,dy.$$

that is equivalent to first part of (4.7).

The last part of the inequality (4.7) follows from (3.13).

If we take in (4.6) $x=\frac{2ab}{a+b}$, then we get

$$\begin{aligned}\frac{a+b}{2}f\left(\frac{2ab}{a+b}\right)-\frac{1}{b-a}\int_a^b yf(y)\,dy &\geq \frac{f(b)\left(\frac{b(a+b)}{2ab}-1\right)+f(a)\left(1-\frac{a(a+b)}{2ab}\right)}{b-a}-\frac{a+b}{2ab}\frac{1}{b-a}\int_a^b f(y)\,dy\\ &=\frac{a+b}{2ab}\left(\frac{bf(b)+af(a)}{a+b}-\frac{1}{b-a}\int_a^b f(y)\,dy\right)\end{aligned}$$

and the inequality (4.8) is proved. ∎

5. FURTHER INEQUALITIES FOR HA-CONVEX FUNCTIONS

We start with the following characterization of *HA*-convex functions.

THEOREM 5.1 (Dragomir, 2015 [**19**]). *Let* $f,h:[a,b]\subset(0,\infty)\to\mathbb{R}$ *be so that* $h(t)=tf(t)$ *for* $t\in[a,b]$*. Then* f *is HA-convex on the interval* $[a,b]$ *if and only if* h *is convex on* $[a,b]$*.*

PROOF. Assume that f is *HA*-convex on the interval $[a,b]$. Then the function $g:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$, $g(t)=f\left(\frac{1}{t}\right)$ is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$. By replacing t with $\frac{1}{t}$ we have $f(t)=g\left(\frac{1}{t}\right)$.

If $\lambda\in[0,1]$ and $x,y\in[a,b]$ then, by the convexity of g on $\left[\frac{1}{b},\frac{1}{a}\right]$, we have

$$\begin{aligned}h((1-\lambda)x+\lambda y)&=[(1-\lambda)x+\lambda y]f((1-\lambda)x+\lambda y)\\ &\leq[(1-\lambda)x+\lambda y]\frac{(1-\lambda)xg\left(\frac{1}{x}\right)+\lambda yg\left(\frac{1}{y}\right)}{(1-\lambda)x+\lambda y}=(1-\lambda)h(x)+\lambda h(y),\end{aligned}$$

which shows that h is convex on $[a,b]$.

We have $f(t)=\frac{h(t)}{t}$ for $t\in[a,b]$. If $\lambda\in[0,1]$ and $x,y\in[a,b]$ then, by the convexity of h on $[a,b]$, we have

$$\begin{aligned} f\left(\frac{xy}{\lambda x+(1-\lambda)y}\right) &= \frac{h\left(\frac{xy}{\lambda x+(1-\lambda)y}\right)}{\frac{xy}{\lambda x+(1-\lambda)y}} \\ &= \frac{\lambda x+(1-\lambda)y}{xy}h\left(\frac{(1-\lambda)\frac{1}{x}x+\lambda\frac{1}{y}y}{(1-\lambda)\frac{1}{x}+\lambda\frac{1}{y}}\right) \\ &\le \frac{\lambda x+(1-\lambda)y}{xy}\frac{(1-\lambda)\frac{1}{x}h(x)+\lambda\frac{1}{y}h(y)}{(1-\lambda)\frac{1}{x}+\lambda\frac{1}{y}} \\ &= (1-\lambda)\frac{1}{x}h(x)+\lambda\frac{1}{y}h(y)=(1-\lambda)f(x)+\lambda f(y), \end{aligned}$$

which shows that f is *HA*-convex on the interval $[a,b]$. ∎

REMARK 5.1. If f is HA-convex on the interval $[a,b]$, then by Theorem 5.1 the function $h(t)=tf(t)$ is convex on $[a,b]$ and by Hermite-Hadamard inequality we get the inequality (3.13). This gives a direct proof of (3.13) and it is simpler than in [**18**].

In 1994, [**3**] (see also [**23**, p. 22]) we proved the following refinement of Hermite-Hadamard inequality. For a direct proof that is different from the one in [**3**], see the recent paper [**12**].

LEMMA 5.2. *Let $p:[c,d]\to\mathbb{R}$ be a convex function on $[c,d]$. Then for any division $c=y_0<y_1<...<y_{n-1}<y_n=d$ with $n\ge 1$ we have the inequalities*

$$\begin{aligned} p\left(\frac{c+d}{2}\right) &\le \frac{1}{d-c}\sum_{i=0}^{n-1}(y_{i+1}-y_i)\,p\left(\frac{y_{i+1}+y_i}{2}\right) \\ &\le \frac{1}{d-c}\int_c^d p(y)\,dy \le \frac{1}{d-c}\sum_{i=0}^{n-1}(y_{i+1}-y_i)\frac{p(y_i)+p(y_{i+1})}{2} \\ &\le \frac{1}{2}[p(c)+p(d)]. \end{aligned} \tag{5.1}$$

We can state the following result:

THEOREM 5.3 (Dragomir, 2015 [**19**]). *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then for any division $a=x_0<x_1<...<x_{n-1}<x_n=b$ with $n\ge 1$ we have the inequalities*

$$\begin{aligned} \frac{a+b}{2}f\left(\frac{a+b}{2}\right) &\le \frac{1}{2(b-a)}\sum_{i=0}^{n-1}\left(x_{i+1}^2-x_i^2\right)f\left(\frac{x_{i+1}+x_i}{2}\right) \\ &\le \frac{1}{b-a}\int_a^b xf(x)\,dx \\ &\le \frac{1}{b-a}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\frac{x_if(x_i)+x_{i+1}f(x_{i+1})}{2} \\ &\le \frac{1}{2}[af(a)+bf(b)]. \end{aligned} \tag{5.2}$$

Follows by Lemma 5.2 for the convex function $p(x)=xf(x)$, $x\in[a,b]$.

If we take $n=2$ and $x\in[a,b]$, then by (5.2) we have

$$\begin{aligned}(5.3)\quad &\frac{a+b}{2}f\left(\frac{a+b}{2}\right)\\ &\leq\frac{1}{2(b-a)}\left[\left(x^2-a^2\right)f\left(\frac{x+a}{2}\right)+\left(b^2-x^2\right)f\left(\frac{x+b}{2}\right)\right]\\ &\leq\frac{1}{b-a}\int_a^b tf(t)\,dt\\ &\leq\frac{1}{2(b-a)}\left[(b-a)xf(x)+(x-a)af(a)+(b-x)bf(b)\right]\\ &\leq\frac{1}{2}\left[af(a)+bf(b)\right].\end{aligned}$$

If in this inequality we choose $x=\frac{a+b}{2}$, then we get the inequality

$$\begin{aligned}(5.4)\quad &\frac{a+b}{2}f\left(\frac{a+b}{2}\right)\\ &\leq\frac{1}{2(b-a)}\left[\frac{b+3a}{4}f\left(\frac{b+3a}{4}\right)+\frac{a+3b}{4}f\left(\frac{a+3b}{4}\right)\right]\\ &\leq\frac{1}{b-a}\int_a^b tf(t)\,dt\\ &\leq\frac{1}{2}\left[\frac{a+b}{2}f\left(\frac{a+b}{2}\right)+\frac{af(a)+bf(b)}{2}\right]\leq\frac{1}{2}\left[af(a)+bf(b)\right].\end{aligned}$$

If we take in (5.3) $x=\frac{2ab}{a+b}$, then we get

$$\begin{aligned}(5.5)\quad &\frac{a+b}{2}f\left(\frac{a+b}{2}\right)\\ &\leq\frac{1}{4(a+b)^2}\left[a^2(a+3b)f\left(\frac{a(a+3b)}{2(a+b)}\right)+b^2(3a+b)f\left(\frac{b(3a+b)}{2(a+b)}\right)\right]\\ &\leq\frac{1}{b-a}\int_a^b tf(t)\,dt\\ &\leq\frac{1}{a+b}\left[abf\left(\frac{2ab}{a+b}\right)+\frac{a^2f(a)+b^2f(b)}{2}\right]\leq\frac{1}{2}\left[af(a)+bf(b)\right].\end{aligned}$$

We also have:

THEOREM 5.4 (Dragomir, 2015 [**19**]). *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then for any division $a=x_0<x_1<...<x_{n-1}<x_n=b$ with $n\geq 1$ we*

have the inequalities

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\sum_{j=0}^{n-1}\left(\frac{x_{j+1}-x_j}{x_{j+1}x_j}\right) f\left(\frac{2x_{j+1}x_j}{x_{j+1}+x_j}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{ab}{b-a}\sum_{i=0}^{n-1}\left(\frac{x_{j+1}-x_j}{x_{j+1}x_j}\right)\frac{f(x_j)+f(x_{j+1})}{2} \le \frac{f(b)+f(a)}{2}. \tag{5.6}$$

PROOF. Consider the convex function $p(x) = f\left(\frac{1}{x}\right)$ that is convex on the interval $\left[\frac{1}{b}, \frac{1}{a}\right]$. The division $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ with $n \ge 1$ produces the division $y_i = \frac{1}{x_{n-i}}$, $i \in \{0, ..., n\}$ of the interval $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Using the inequality (5.1) we get

$$f\left(\frac{1}{\frac{\frac{1}{b}+\frac{1}{a}}{2}}\right) \le \frac{1}{\frac{1}{a}-\frac{1}{b}}\sum_{i=0}^{n-1}\left(\frac{1}{x_{n-i-1}}-\frac{1}{x_{n-i}}\right) f\left(\frac{1}{\frac{\frac{1}{x_{n-i-1}}+\frac{1}{x_{n-i}}}{2}}\right) \le \frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right)dt \le \frac{1}{\frac{1}{a}-\frac{1}{b}}\sum_{i=0}^{n-1}\left(\frac{1}{x_{n-i-1}}-\frac{1}{x_{n-i}}\right)\frac{f\left(\frac{1}{\frac{1}{x_{n-i-1}}}\right)+f\left(\frac{1}{\frac{1}{x_{n-i}}}\right)}{2} \le \frac{1}{2}\left[f\left(\frac{1}{\frac{1}{b}}\right)+f\left(\frac{1}{\frac{1}{a}}\right)\right] \tag{5.7}$$

that is equivalent to

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\sum_{i=0}^{n-1}\left(\frac{x_{n-i}-x_{n-i-1}}{x_{n-i-1}x_{n-i}}\right) f\left(\frac{2x_{n-i-1}x_{n-i}}{x_{n-i}+x_{n-i-1}}\right) \le \frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right)dt \le \frac{ab}{b-a}\sum_{i=0}^{n-1}\left(\frac{x_{n-i}-x_{n-i-1}}{x_{n-i-1}x_{n-i}}\right)\frac{f(x_{n-i-1})+f(x_{n-i})}{2} \le \frac{1}{2}[f(b)+f(a)]. \tag{5.8}$$

By re-indexing the sums and taking into account that

$$\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right)dt = \int_a^b \frac{f(x)}{x^2}dx$$

we obtain the desired result (5.6). ∎

REMARK 5.2. If we take $n = 2$ and $x \in [a, b]$, then by (5.6) we have, after appropriate calculations, that

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{1}{x}\left[\frac{(x-a)\,bf\left(\frac{2ax}{a+x}\right) + (b-x)\,af\left(\frac{2xb}{x+b}\right)}{b-a}\right] \tag{5.9}$$
$$\leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx$$
$$\leq \frac{1}{2}\left[f(x) + \frac{(x-a)\,bf(a) + (b-x)\,af(b)}{x(b-a)}\right] \leq \frac{f(b)+f(a)}{2}.$$

If we take in (5.9) $x = \frac{2ab}{a+b} \in [a, b]$, then we get

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{1}{2}\left[f\left(\frac{4ab}{a+3b}\right) + f\left(\frac{4ab}{3a+b}\right)\right] \tag{5.10}$$
$$\leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx$$
$$\leq \frac{1}{2}\left[f\left(\frac{2ab}{a+b}\right) + \frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}.$$

If we take in (5.9) $x = \frac{a+b}{2} \in [a, b]$, then we get

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{bf\left(\frac{a(a+b)}{3a+b}\right) + af\left(\frac{b(a+b)}{a+3b}\right)}{a+b} \tag{5.11}$$
$$\leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx$$
$$\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{bf(a)+af(b)}{a+b}\right] \leq \frac{f(b)+f(a)}{2}.$$

6. RELATED RESULTS

LEMMA 6.1 (Dragomir, 2015 [**19**]). *Let $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ be an HA-convex function on the interval $[a, b]$. Then f has lateral derivatives in every point of (a, b) and*

$$f(t) - f(s) \geq sf'_{\pm}(s)\left(1 - \frac{s}{t}\right) \tag{6.1}$$

for any $s \in (a, b)$ and $t \in [a, b]$.

Also, we have

$$f(t) - f(a) \geq af'_{+}(a)\left(1 - \frac{a}{t}\right) \tag{6.2}$$

and

$$f(t) - f(b) \geq bf'_{-}(b)\left(1 - \frac{b}{t}\right) \tag{6.3}$$

for any $t \in [a, b]$ provided the lateral derivatives $f'_{+}(a)$ and $f'_{-}(b)$ are finite.

PROOF. If f is *HA*-convex function on the interval $[a, b]$, then the function $h(t) = tf(t)$ is convex on $[a, b]$, therefore the function f has lateral derivatives in each point of (a, b) and

$$h'_{\pm}(t) = f(t) + tf'_{\pm}(t)$$

for any $t \in (a,b)$. Also, if $f'_+(a)$ and $f'_-(b)$ are finite then

$$h'_+(a) = f(a) + af'_+(a) \text{ and } h'_-(b) = f(b) + bf'_-(b).$$

Writing the gradient inequality for the convex function h, namely

$$h(t) - h(s) \geq h'_\pm(s)(t-s)$$

for any $s \in (a,b)$ and $t \in [a,b]$, we have

$$tf(t) - sf(s) \geq \left[f(s) + sf'_\pm(s)\right](t-s) = f(s)(t-s) + sf'_\pm(s)(t-s)$$

that is equivalent to

$$tf(t) - tf(s) \geq sf'_\pm(s)(t-s)$$

for any $s \in (a,b)$ and $t \in [a,b]$.

Now, by dividing with $t > 0$ we get the desired result (6.1).

The rest follows by the corresponding properties of convex function h. ∎

The following result holds:

THEOREM 6.2 (Dragomir, 2015 [**19**]). *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then we have*

$$\begin{aligned}(6.4)\qquad &\frac{1}{16}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right]\left(b^2-a^2\right)\\ &\leq \frac{af(a)+bf(b)}{2} - \frac{1}{b-a}\int_a^b tf(t)\,dt\\ &\leq \frac{1}{8}\left[f(b)-f(a)\right](b-a) + \frac{1}{8}\left[bf'_-(b) - af'_+(a)\right](b-a)\end{aligned}$$

and

$$\begin{aligned}(6.5)\qquad &\frac{1}{16}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right]\left(b^2-a^2\right)\\ &\leq \frac{1}{b-a}\int_a^b tf(t)\,dt - \frac{a+b}{2}f\left(\frac{a+b}{2}\right)\\ &\leq \frac{1}{8}\left[f(b)-f(a)\right](b-a) + \frac{1}{8}\left[bf'_-(b) - af'_+(a)\right](b-a).\end{aligned}$$

PROOF. Follows by Lemma 0.1 for the convex function $h(t) = tf(t)$. ∎

COROLLARY 6.3. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a differentiable HA-convex function on the interval $[a,b]$. Then we have*

$$\begin{aligned}(6.6)\qquad 0 &\leq \frac{af(a)+bf(b)}{2} - \frac{1}{b-a}\int_a^b tf(t)\,dt\\ &\leq \frac{1}{8}\left[f(b)-f(a)\right](b-a) + \frac{1}{8}\left[bf'_-(b) - af'_+(a)\right](b-a)\end{aligned}$$

and

$$\begin{aligned}(6.7)\qquad 0 &\leq \frac{1}{b-a}\int_a^b tf(t)\,dt - \frac{a+b}{2}f\left(\frac{a+b}{2}\right)\\ &\leq \frac{1}{8}\left[f(b)-f(a)\right](b-a) + \frac{1}{8}\left[bf'_-(b) - af'_+(a)\right](b-a).\end{aligned}$$

We remark that from (6.6) we have

$$\frac{(3a+b)\,f(a)+(a+3b)\,f(b)}{8}-\frac{1}{8}\left[bf'_-(b)-af'_+(a)\right](b-a) \le \frac{1}{b-a}\int_a^b tf(t)\,dt \le \frac{af(a)+bf(b)}{2} \tag{6.8}$$

and from (6.7) we have

$$\frac{a+b}{2}f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b tf(t)\,dt \le \frac{a+b}{2}f\left(\frac{a+b}{2}\right)+\frac{1}{8}\left[f(b)-f(a)\right](b-a) + \frac{1}{8}\left[bf'_-(b)-af'_+(a)\right](b-a). \tag{6.9}$$

The following result also holds:

THEOREM 6.4 (Dragomir, 2015 [**19**]). *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an HA-convex function on the interval $[a,b]$.*

(i) If $bf(b)-af(a)\neq\int_a^b f(s)\,ds$ and

$$\alpha_f := \frac{\int_a^b s^2 f'(s)\,ds}{\int_a^b sf'(s)\,ds}=\frac{b^2 f(b)-a^2 f(a)-2\int_a^b sf(s)\,ds}{bf(b)-af(a)-\int_a^b f(s)\,ds}\in[a,b] \tag{6.10}$$

then

$$f(\alpha_f)\ge\frac{1}{b-a}\int_a^b f(s)\,ds. \tag{6.11}$$

(ii) If $f(b)\neq f(a)$ and

$$\beta_f=\frac{\int_a^b sf'(s)\,ds}{\int_a^b f'(s)\,ds}=\frac{bf(b)-af(a)-\int_a^b f(s)\,ds}{f(b)-f(a)}\in[a,b] \tag{6.12}$$

then

$$f(\beta_f)\ge\frac{1}{\ln b-\ln a}\int_a^b f(s)\,ds. \tag{6.13}$$

(iii) If $af(b)\neq bf(a)$ and

$$\gamma_f := \frac{(f(b)-f(a))\,ab}{af(b)-bf(a)}\in[a,b] \tag{6.14}$$

then

$$f(\gamma_f)\ge\frac{2ab}{b-a}\int_a^b\frac{f(s)}{s^2}ds. \tag{6.15}$$

PROOF. We know that if $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is an *HA*-convex function on the interval $[a,b]$ then the functions is differentiable except for at most countably many points. Then, from (6.1) we have

$$f(t)-f(s)\ge sf'(s)\left(1-\frac{s}{t}\right) \tag{6.16}$$

for any $t\in[a,b]$ and almost every $s\in(a,b)$.

(i) If we take the Lebesgue integral mean in (6.16), then we get

$$f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds \ge \frac{1}{b-a}\int_a^b sf'(s)\,ds-\frac{1}{t}\frac{1}{b-a}\int_a^b s^2 f'(s)\,ds \tag{6.17}$$

for any $t\in[a,b]$.

If we take $t=\alpha_f$ in (6.17) then we get the desired inequality (6.11).

(ii) If we divide the inequality (6.16) by s then we get

$$\frac{1}{s}f(t)-\frac{f(s)}{s}\ge f'(s)-\frac{1}{t}sf'(s) \tag{6.18}$$

for any $t\in[a,b]$ and almost every $s\in(a,b)$.

If we take the Lebesgue integral mean in (6.18), then we get

$$f(t)\frac{1}{b-a}\int_a^b \frac{1}{s}ds-\frac{1}{b-a}\int_a^b\frac{f(s)}{s}ds \ge \frac{1}{b-a}\int_a^b f'(s)\,ds-\frac{1}{t}\frac{1}{b-a}\int_a^b sf'(s)\,ds$$

that is equivalent to

$$\frac{f(t)}{L(a,b)}-\frac{1}{b-a}\int_a^b\frac{f(s)}{s}ds \ge \frac{f(b)-f(a)}{b-a}-\frac{1}{t}\frac{bf(b)-af(a)-\int_a^b f(s)\,ds}{b-a} \tag{6.19}$$

for any $t\in[a,b]$

If we take $t=\beta_f$ in (6.19) then we get the desired result (6.13).

(iii) If we divide the inequality (6.16) by s^2 then we get

$$\frac{1}{s^2}f(t)-\frac{f(s)}{s^2}\ge\frac{f'(s)}{s}-\frac{1}{t}f'(s) \tag{6.20}$$

for any $t\in[a,b]$ and almost every $s\in(a,b)$.

If we take the Lebesgue integral mean in (6.20), then we get

$$f(t)\frac{1}{b-a}\int_a^b\frac{1}{s^2}ds-\frac{1}{b-a}\int_a^b\frac{f(s)}{s^2}ds \ge \frac{1}{b-a}\int_a^b\frac{f'(s)}{s}ds-\frac{1}{t}\frac{1}{b-a}\int_a^b f'(s)\,ds,$$

which is equivalent to

$$f(t)\frac{1}{ab}-\frac{2}{b-a}\int_a^b\frac{f(s)}{s^2}ds\ge\frac{1}{b-a}\frac{af(b)-bf(a)}{ba}-\frac{1}{t}\frac{f(b)-f(a)}{b-a}.$$

∎

REMARK 6.1. We observe that a sufficient condition for (6.10) and (6.12) to hold is that f is increasing on $[a,b]$. If $f(a)<0<f(b)$, then the inequality (6.14) also holds.

We also have the following result:

THEOREM 6.5 (Dragomir, 2015 [**19**]). *Let* $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be an HA-convex function on the interval* $[a,b]$ *. Then we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{a+b-t}dt \leq \frac{af(a)+bf(b)}{a+b}. \tag{6.21}$$

PROOF. Since the function $h(t) = tf(t)$ is convex, then we have

$$\frac{x+y}{2}f\left(\frac{x+y}{2}\right) \leq \frac{xf(x)+yf(y)}{2}$$

for any $x, y \in [a,b]$.

If we divide this inequality by $xy > 0$ we get

$$\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right)f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\left(\frac{f(x)}{y}+\frac{f(y)}{x}\right), \tag{6.22}$$

for any $x, y \in [a,b]$.

If we replace x by $(1-t)a+tb$ and y by $ta+(1-t)b$ in (6.22), then we get

$$\begin{aligned} &\frac{1}{2}\left(\frac{1}{(1-t)a+tb}+\frac{1}{ta+(1-t)b}\right)f\left(\frac{a+b}{2}\right) \\ &\qquad \leq \frac{1}{2}\left(\frac{f((1-t)a+tb)}{ta+(1-t)b}+\frac{f(ta+(1-t)b)}{(1-t)a+tb}\right), \end{aligned} \tag{6.23}$$

for any $t \in [0,1]$.

Integrating (6.23) on $[0,1]$ over t we get

$$\begin{aligned} &\frac{1}{2}\left(\int_0^1 \frac{1}{(1-t)a+tb}dt+\int_0^1 \frac{1}{ta+(1-t)b}dt\right)f\left(\frac{a+b}{2}\right) \\ &\qquad \leq \frac{1}{2}\left(\int_0^1 \frac{f((1-t)a+tb)}{ta+(1-t)b}dt+\int_0^1 \frac{f(ta+(1-t)b)}{(1-t)a+tb}dt\right). \end{aligned} \tag{6.24}$$

Observe that, by the appropriate change of variable,

$$\int_0^1 \frac{1}{(1-t)a+tb}dt = \int_0^1 \frac{1}{ta+(1-t)b}dt = \frac{\ln b - \ln a}{b-a}$$

and

$$\int_0^1 \frac{f((1-t)a+tb)}{ta+(1-t)b}dt = \int_0^1 \frac{f(ta+(1-t)b)}{(1-t)a+tb} = \frac{1}{b-a}\int_a^b \frac{f(u)}{a+b-u}du$$

and by (6.24) we get the first inequality in (6.21).

From the convexity of h we also have

$$((1-t)a+tb)f((1-t)a+tb) \leq (1-t)af(a)+tbf(b)$$

and

$$(ta+(1-t)b)f(ta+(1-t)b) \leq taf(a)+(1-t)bf(b)$$

for any $t \in [0,1]$.

Add these inequalities to get

$$\begin{aligned} &((1-t)a+tb)f((1-t)a+tb)+(ta+(1-t)b)f(ta+(1-t)b) \\ &\leq af(a)+bf(b) \end{aligned}$$

for any $t \in [0,1]$.

If we divide this inequality by $((1-t)a+tb)(ta+(1-t)b)$, then we get

$$\frac{f((1-t)a+tb)}{ta+(1-t)b}+\frac{f(ta+(1-t)b)}{(1-t)a+tb}\leq\frac{af(a)+bf(b)}{((1-t)a+tb)(ta+(1-t)b)} \tag{6.25}$$

for any $t\in[0,1]$.

If we integrate the inequality (6.25) over t on $[0,1]$, then we obtain

$$\int_0^1\frac{f((1-t)a+tb)}{ta+(1-t)b}dt+\int_0^1\frac{f(ta+(1-t)b)}{(1-t)a+tb}dt \leq [af(a)+bf(b)]\int_0^1\frac{dt}{((1-t)a+tb)(ta+(1-t)b)}. \tag{6.26}$$

Since

$$\int_0^1\frac{dt}{((1-t)a+tb)(ta+(1-t)b)}=\frac{1}{b-a}\int_a^b\frac{du}{u(a+b-u)}$$

and

$$\frac{1}{u(a+b-u)}=\frac{1}{a+b}\left(\frac{1}{u}+\frac{1}{a+b-u}\right),$$

then

$$\int_a^b\frac{du}{u(a+b-u)}=\frac{2}{a+b}(\ln b-\ln a).$$

By (6.26) we then have

$$\frac{2}{b-a}\int_a^b\frac{f(u)}{a+b-u}du\leq 2\left[\frac{af(a)+bf(b)}{a+b}\right]\frac{\ln b-\ln a}{b-a},$$

which proves the second inequality in (6.21). ∎

CHAPTER 8

Inequalities for HG-Convex Functions

1. Some Preliminary Facts

Following [**1**] (see also [**27**]) we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \to (0, \infty)$ is *HG-convex* if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le [f(x)]^{1-t}[f(y)]^{t} \tag{1.1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.1) is reversed, then f is said to be *HG-concave*.

By the geometric-mean - arithmetic mean inequality we have that any *HG*-convex function is *HA*-convex. The converse is obviously not true.

We observe that $f : I \subset \mathbb{R} \setminus \{0\} \to (0, \infty)$ is *HG*-convex *if and only if* the function $\ln f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is *HA*-convex on I.

We have:

PROPOSITION 1.1 (Dragomir, 2015 [**20**]). *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ and define the associated functions $G_f : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ defined by $G_f(t) = \ln f\left(\frac{1}{t}\right)$ and $H_f : [a,b] \subset (0,\infty) \to \mathbb{R}$ defined by $H_f(t) = t \ln f(t)$. Then the following statements are equivalent:*

(i) The function f is HG-convex on $[a,b]$;

(ii) The function G_f is convex on $\left[\frac{1}{b}, \frac{1}{a}\right]$;

(iii) The function H_f is convex on $[a,b]$.

In this chapter we present some inequalities of Hermite-Hadamard type for *HG*-convex functions defined on positive intervals. Applications for special means are also provided.

2. Inequalities for HG-convex Functions

The following result holds.

THEOREM 2.1 (Dragomir, 2015 [**20**]). *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be an HG-convex function on the interval $[a,b]$. Then for any $\lambda \in [0,1]$ we have the inequalities*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\le \left[f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)\right]^{1-\lambda}\left[f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)\right]^{\lambda} \\ &\le \exp\left(\frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt\right) \\ &\le \sqrt{f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)[f(a)]^{1-\lambda}[f(b)]^{\lambda}} \le \sqrt{f(a)f(b)}. \end{aligned} \tag{2.1}$$

THEOREM 2.2. *Let $f : [a, b] \subset (0, \infty) \to (0, \infty)$ be an HG-convex function on the interval $[a, b]$. Then*

$$f(L(a,b)) \le \exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right) \le [f(b)]^{\frac{(L(a,b)-a)b}{(b-a)L(a,b)}}[f(a)]^{\frac{(b-L(a,b))a}{(b-a)L(a,b)}}. \tag{2.2}$$

If we write the classical Hermite-Hadamard inequality for the function H_f that is convex on $[a, b]$ when $f : [a, b] \subset (0, \infty) \to (0, \infty)$ is an *HG*-convex function on $[a, b]$ and perform the required calculations, we get:

THEOREM 2.3 (Dragomir, 2015 [**20**]). *Let $f : [a, b] \subset (0, \infty) \to (0, \infty)$ be an HG-convex function on the interval $[a, b]$. Then we have*

$$\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{a+b}{2}} \le \exp\left(\frac{1}{b-a}\int_a^b t\ln f(t)\,dt\right) \le \sqrt{[f(b)]^b[f(a)]^a}. \tag{2.3}$$

We have the reverse inequalities as well:

THEOREM 2.4 (Dragomir, 2015 [**20**]). *Let $f : [a, b] \subset (0, \infty) \to (0, \infty)$ be an HG-convex function on the interval $[a, b]$. Then we have*

$$1 \le \frac{\exp\left(\frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt\right)}{f\left(\frac{2ab}{a+b}\right)} \le \exp\left(\frac{1}{8}\left[\frac{f'_-(b)}{f(b)}b^2 - \frac{f'_+(a)\,a^2}{f(a)}\right]\left(\frac{b-a}{ab}\right)\right) \tag{2.4}$$

and

$$1 \le \frac{\sqrt{f(a)f(b)}}{\exp\left(\frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt\right)} \le \exp\left(\frac{1}{8}\left[\frac{f'_-(b)}{f(b)}b^2 - \frac{f'_+(a)\,a^2}{f(a)}\right]\left(\frac{b-a}{ab}\right)\right). \tag{2.5}$$

The following related result also holds:

THEOREM 2.5 (Dragomir, 2015 [**20**]). *Let $f : [a, b] \subset (0, \infty) \to (0, \infty)$ be an HG-convex function on the interval $[a, b]$. Then we have*

$$1 \le \frac{\sqrt{[f(a)]^a[f(b)]^b}}{\exp\left(\frac{1}{b-a}\int_a^b t\ln f(t)\,dt\right)} \le \left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)}\exp\left(\frac{1}{8}(b-a)\left(\frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)}\right)\right) \tag{2.6}$$

and

$$1 \le \frac{\exp\left(\frac{1}{b-a}\int_a^b t\ln f(t)\,dt\right)}{\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{a+b}{2}}} \le \left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)}\exp\left(\frac{1}{8}(b-a)\left(\frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)}\right)\right). \tag{2.7}$$

From a different perspective we have:

THEOREM 2.6 (Dragomir, 2015 [**20**]). *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be an HG-convex function on the interval $[a,b]$. Then*

$$\exp\left(\frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt\right) \le \sqrt{f(x)\,[f(b)]^{\frac{a(b-x)}{x(b-a)}}\,[f(a)]^{\frac{b(x-a)}{x(b-a)}}} \tag{2.8}$$

for any $x \in [a,b]$.

If we take in (2.8), $x = \frac{a+b}{2}$, then we get from (2.8) that

$$\exp\left(\frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt\right) \le \sqrt{f\left(\frac{a+b}{2}\right)[f(b)]^{\frac{a}{a+b}}\,[f(a)]^{\frac{b}{a+b}}}. \tag{2.9}$$

In [**18**], in order to improve Işcan's inequality [**27**] for *HA*-convex functions $g : [a,b] \subset (0,\infty) \to \mathbb{R}$,

$$g\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{g(t)}{t^2}dt \le \frac{g(a)+g(b)}{2}, \tag{2.10}$$

we obtained the following result:

$$\begin{aligned} g\left(\frac{2ab}{a+b}\right) &\le (1-\lambda)\,g\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right) + \lambda g\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right) \\ &\le \frac{ab}{b-a}\int_a^b \frac{g(t)}{t^2}dt \\ &\le \frac{1}{2}\left[g\left(\frac{ab}{(1-\lambda)a+\lambda b}\right) + (1-\lambda)\,g(a) + \lambda g(b)\right] \\ &\le \frac{g(a)+g(b)}{2}, \end{aligned} \tag{2.11}$$

where $\lambda \in [0,1]$.

Now, if $f : [a,b] \subset (0,\infty) \to (0,\infty)$ is an *HG*-convex function on the interval $[a,b]$, then $g := \ln f$ is *HA*-convex on $[a,b]$, and by (2.11) we get

$$\begin{aligned} \ln f\left(\frac{2ab}{a+b}\right) &\le (1-\lambda)\ln f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right) + \lambda \ln f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right) \\ &\le \frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt \\ &\le \frac{1}{2}\left[\ln f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right) + (1-\lambda)\ln f(a) + \lambda \ln f(b)\right] \\ &\le \frac{\ln f(a)+\ln f(b)}{2}, \end{aligned} \tag{2.12}$$

that is equivalent to

$$
\begin{aligned}
&\ln f\left(\frac{2ab}{a+b}\right) \\
&\qquad \leq \ln\left(\left[f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)\right]^{1-\lambda}\left[f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)\right]^{\lambda}\right) \\
&\qquad \leq \frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt \leq \ln\sqrt{f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)[f(a)]^{1-\lambda}[f(b)]^{\lambda}} \\
&\qquad \leq \ln\sqrt{f(a)f(b)},
\end{aligned}
$$

and by taking the exponential we get the desired result (2.1).

We have the following result for *HA*-convex functions [**18**]:

LEMMA 2.7. *Let* $g:[a,b]\subset(0,\infty)\to\mathbb{R}$ *be an HA-convex function on the interval* $[a,b]$. *Then*

$$
\begin{aligned}
(2.13)\quad g(L(a,b)) &\leq \frac{1}{b-a}\int_a^b g(x)\,dx \\
&\leq \frac{(L(a,b)-a)\,bg(b)+(b-L(a,b))\,ag(a)}{(b-a)L(a,b)}.
\end{aligned}
$$

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is an *HG*-convex function on the interval $[a,b]$, then $g:=\ln f$ is *HA*-convex on $[a,b]$, and by (2.13) we have

$$
\begin{aligned}
(2.14)\quad \ln f(L(a,b)) &\leq \frac{1}{b-a}\int_a^b \ln f(x)\,dx \\
&\leq \ln\left([f(b)]^{\frac{(L(a,b)-a)b}{(b-a)L(a,b)}}[f(a)]^{\frac{(b-L(a,b))a}{(b-a)L(a,b)}}\right).
\end{aligned}
$$

By taking the exponential in (2.14) we get the desired result (2.2).

If $\ell:[a,b]\subset(0,\infty)\to\mathbb{R}$ is an *HA*-convex function on the interval $[a,b]$, then the function $g:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$, $g(s)=\ell\left(\frac{1}{s}\right)$, is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$.

Now, by Lemma 0.1 we have

$$
\begin{aligned}
(2.15)\quad 0 &\leq \frac{g\left(\frac{1}{a}\right)+g\left(\frac{1}{b}\right)}{2}-\frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}} g(t)\,dt \\
&\leq \frac{1}{8}\left[g'_-\left(\frac{1}{a}\right)-g'_+\left(\frac{1}{b}\right)\right]\left(\frac{1}{a}-\frac{1}{b}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(2.16)\quad 0 &\leq \frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}} g(t)\,dt-g\left(\frac{\frac{1}{a}+\frac{1}{b}}{2}\right) \\
&\leq \frac{1}{8}\left[g'_-\left(\frac{1}{a}\right)-g'_+\left(\frac{1}{b}\right)\right]\left(\frac{1}{a}-\frac{1}{b}\right).
\end{aligned}
$$

We also have

$$
g'_{\pm}(s)=\ell'_{\mp}\left(\frac{1}{s}\right)\left(-\frac{1}{s^2}\right)
$$

and then

$$g'_{-}\left(\frac{1}{a}\right)=-\ell'_{+}\left(a\right)a^{2} \text{ and } g'_{+}\left(\frac{1}{b}\right)=-\ell'_{-}\left(b\right)b^{2}.$$

From (2.15) and (2.16) we have

$$(2.17)\quad 0\leq\frac{\ell\left(a\right)+\ell\left(b\right)}{2}-\frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}}\ell\left(\frac{1}{s}\right)ds \leq\frac{1}{8}\left[\ell'_{-}\left(b\right)b^{2}-\ell'_{+}\left(a\right)a^{2}\right]\left(\frac{b-a}{ab}\right)$$

and

$$(2.18)\quad 0\leq\frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}}\ell\left(\frac{1}{s}\right)ds-\ell\left(\frac{2ab}{a+b}\right) \leq\frac{1}{8}\left[\ell'_{-}\left(b\right)b^{2}-\ell'_{+}\left(a\right)a^{2}\right]\left(\frac{b-a}{ab}\right).$$

If we change the variable $\frac{1}{s}=u$, then $ds=-\frac{du}{u^{2}}$ and (2.17) and (2.18) can be written as

$$(2.19)\quad 0\leq\frac{\ell\left(a\right)+\ell\left(b\right)}{2}-\frac{ab}{b-a}\int_{a}^{b}\frac{\ell\left(t\right)}{t^{2}}dt \leq\frac{1}{8}\left[\ell'_{-}\left(b\right)b^{2}-\ell'_{+}\left(a\right)a^{2}\right]\left(\frac{b-a}{ab}\right)$$

and

$$(2.20)\quad 0\leq\frac{ab}{b-a}\int_{a}^{b}\frac{\ell\left(t\right)}{t^{2}}dt-\ell\left(\frac{2ab}{a+b}\right) \leq\frac{1}{8}\left[\ell'_{-}\left(b\right)b^{2}-\ell'_{+}\left(a\right)a^{2}\right]\left(\frac{b-a}{ab}\right).$$

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is an *HG*-convex function on the interval $[a,b]$, then $\ell:=\ln f$ is *HA*-convex on $[a,b]$, and by (2.19) and (2.20) we have

$$(2.21)\quad 0\leq\ln\sqrt{f\left(a\right)f\left(b\right)}-\frac{ab}{b-a}\int_{a}^{b}\frac{\ln f\left(t\right)}{t^{2}}dt \leq\frac{1}{8}\left[\frac{f'_{-}\left(b\right)}{f\left(b\right)}b^{2}-\frac{f'_{+}\left(a\right)a^{2}}{f\left(a\right)}\right]\left(\frac{b-a}{ab}\right)$$

and

$$(2.22)\quad 0\leq\frac{ab}{b-a}\int_{a}^{b}\frac{\ln f\left(t\right)}{t^{2}}dt-\ln f\left(\frac{2ab}{a+b}\right) \leq\frac{1}{8}\left[\frac{f'_{-}\left(b\right)}{f\left(b\right)}b^{2}-\frac{f'_{+}\left(a\right)a^{2}}{f\left(a\right)}\right]\left(\frac{b-a}{ab}\right),$$

and the Theorem 2.4 is proved.

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is an *HG*-convex function on the interval $[a,b]$, then H_f is convex on $[a,b]$ and by Lemma 0.1 we have after appropriate calculations

$$0\le\ln\sqrt{[f(a)]^a[f(b)]^b}-\frac{1}{b-a}\int_a^b t\ln f(t)\,dt \le\ln\left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)}+\frac{1}{8}(b-a)\left(\frac{bf'_-(b)}{f(b)}-\frac{af'_+(a)}{f(a)}\right)$$

and

$$0\le\frac{1}{b-a}\int_a^b t\ln f(t)\,dt-\ln\left(\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{a+b}{2}}\right) \le\ln\left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)}+\frac{1}{8}(b-a)\left(\frac{bf'_-(b)}{f(b)}-\frac{af'_+(a)}{f(a)}\right).$$

These inequalities are equivalent to

$$0\le\ln\left(\frac{\sqrt{[f(a)]^a[f(b)]^b}}{\exp\left(\frac{1}{b-a}\int_a^b t\ln f(t)\,dt\right)}\right) \le\ln\left[\left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)}\exp\left(\frac{1}{8}(b-a)\left(\frac{bf'_-(b)}{f(b)}-\frac{af'_+(a)}{f(a)}\right)\right)\right]$$

and

$$0\le\ln\left(\frac{\exp\left(\frac{1}{b-a}\int_a^b t\ln f(t)\,dt\right)}{\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{a+b}{2}}}\right) \le\ln\left[\left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)}\exp\left(\frac{1}{8}(b-a)\left(\frac{bf'_-(b)}{f(b)}-\frac{af'_+(a)}{f(a)}\right)\right)\right]$$

and by taking the exponential we get the desired results (2.6) and (2.7).

The following lemma is of interest in itself:

LEMMA 2.8 (Dragomir, 2015 [**20**]). *Let $g:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then*

$$\frac{1}{2x}\left(\frac{g(b)\,a(b-x)+g(a)\,b(x-a)}{b-a}+xg(x)\right)\ge\frac{ab}{b-a}\int_a^b\frac{g(y)}{y^2}dy \tag{2.23}$$

for any $x\in[a,b]$.

PROOF. Since $h(t)=tg(t)$ for $t\in[a,b]$ is convex, then by the gradient inequality for convex functions we have

$$xg(x)-yg(y)\ge\left(g(y)+yg'_-(y)\right)(x-y)$$

for any $x,y\in(a,b)$.

This is equivalent to

$$xg(x)-xg(y)\ge yg'_-(y)(x-y) \tag{2.24}$$

for any $x,y\in(a,b)$.

From (2.24) we have, by division with $xy^2 > 0$, that

$$\frac{1}{y^2}g(x) - \frac{1}{y^2}g(y) \geq \frac{g'_-(y)}{y}\left(1 - \frac{y}{x}\right)$$

for any $x, y \in (a, b)$.

Taking the integral mean over y we have

$$g(x)\frac{1}{b-a}\int_a^b \frac{1}{y^2}dy - \frac{1}{b-a}\int_a^b \frac{g(y)}{y^2}dy \geq \frac{1}{b-a}\int_a^b \frac{g'_-(y)}{y}dy - \frac{1}{x}\frac{1}{b-a}\int_a^b g'_-(y)\,dy$$

that is equivalent to

$$\frac{g(x)}{ab} - \frac{1}{b-a}\int_a^b \frac{g(y)}{y^2}dy \geq \frac{1}{b-a}\left(\frac{g(b)}{b} - \frac{g(a)}{a}\right) + \frac{1}{b-a}\int_a^b \frac{g(y)}{y^2}dy - \frac{1}{x}\frac{g(b)-g(a)}{b-a},$$

for any $x \in (a, b)$. This can be written as

$$\frac{1}{2}\left(\frac{1}{b-a}\left[g(b)\frac{b-x}{xb} + g(a)\frac{x-a}{ax}\right] + \frac{g(x)}{ab}\right) \geq \frac{1}{b-a}\int_a^b \frac{g(y)}{y^2}dy.$$

This is equivalent to the desired result (2.23). ∎

If $f : [a, b] \subset (0, \infty) \to (0, \infty)$ is an *HG*-convex function on the interval $[a, b]$, then $g := \ln f$ is *HA*-convex on $[a, b]$, and by (2.23) we have

$$\frac{1}{2x}\left(\frac{a(b-x)\ln f(b) + b(x-a)\ln f(a)}{b-a} + x\ln f(x)\right) \geq \frac{ab}{b-a}\int_a^b \frac{\ln f(y)}{y^2}dy$$

for any $x \in [a, b]$.

This is clearly equivalent to

$$\ln\left(\sqrt{[f(b)]^{\frac{a(b-x)}{x(b-a)}}[f(a)]^{\frac{b(x-a)}{x(b-a)}}}\sqrt{f(x)}\right) \geq \frac{ab}{b-a}\int_a^b \frac{\ln f(y)}{y^2}dy \tag{2.25}$$

for any $x \in [a, b]$.

If we take the exponential in (2.25), then we get the desired result (2.8).

CHAPTER 9

Inequalities for HH-Convex Functions

1. Some Preliminary Facts

Following [**1**] (see also [**27**]) we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is *HA-convex* if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t) f(x) + t f(y) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.1) is reversed, then f is said to be *HA-concave*.

If $I \subset (0,\infty)$ and f is convex and nondecreasing function then f is *HA*-convex and if f is *HA*-convex and nonincreasing function then f is convex.

If $[a,b] \subset I \subset (0,\infty)$ and if we consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$, defined by $g(t) = f\left(\frac{1}{t}\right)$, then we can state the following fact [**1**]:

Lemma 1.1. *The function f is HA-convex (concave) on $[a,b]$ if and only if g is convex (concave) in the usual sense on $\left[\frac{1}{b}, \frac{1}{a}\right]$.*

Therefore, as examples of *HA*-convex functions we can take $f(t) = g\left(\frac{1}{t}\right)$, where g is any convex function on $\left[\frac{1}{b}, \frac{1}{a}\right]$.

In the recent paper [**19**] we obtained the following characterization result as well:

Lemma 1.2. *Let $f, h : [a,b] \subset (0,\infty) \to \mathbb{R}$ be so that $h(t) = tf(t)$ for $t \in [a,b]$. Then f is HA-convex (concave) on the interval $[a,b]$ if and only if h is convex (concave) on $[a,b]$.*

Following [**1**], we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \to (0,\infty)$ is *HH-convex* if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \frac{f(x) f(y)}{(1-t) f(y) + t f(x)} \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.2) is reversed, then f is said to be *HH-concave*.

We observe that the inequality (1.2) is equivalent to

$$(1-t)\frac{1}{f(x)} + t\frac{1}{f(y)} \leq \frac{1}{f\left(\frac{xy}{tx+(1-t)y}\right)} \tag{1.3}$$

for all $x, y \in I$ and $t \in [0,1]$.

Therefore we have the following fact:

Lemma 1.3 (Dragomir, 2015 [**21**]). *The function $f : I \subset \mathbb{R} \setminus \{0\} \to (0,\infty)$ is HH-convex (concave) on I if and only if $g : I \subset \mathbb{R} \setminus \{0\} \to (0,\infty)$, $g(x) = \frac{1}{f(x)}$ is HA-concave (convex) on I.*

Taking into account the above lemmas, we can state the following result:

Proposition 1.4 (Dragomir, 2015 [**21**]). *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ and define the related functions $P_f : \left[\frac{1}{b}, \frac{1}{a}\right] \subset (0,\infty) \to (0,\infty)$, $P_f(x) = \frac{1}{f\left(\frac{1}{x}\right)}$ and $Q_f : [a,b] \subset (0,\infty) \to (0,\infty)$, $Q_f(x) = \frac{x}{f(x)}$. The following statements are equivalent:*

(i) The function f is HH-convex (concave) on $[a,b]$;
(ii) The function P_f is concave (convex) on $\left[\frac{1}{b},\frac{1}{a}\right]$;
(iii) The function Q_f is concave (convex) on $[a,b]$.

In this chapter we present some inequalities of Hermite-Hadamard type for *HH*-convex functions defined on positive intervals. Applications for special means are also provided.

2. INEQUALITIES FOR HH-CONVEX FUNCTIONS

We have the following result that can be obtained by the use of the regular Hermite-Hadamard inequality:

THEOREM 2.1 (Dragomir, 2015 [**21**]). *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a HH-convex (concave) function on* $[a,b]$ *. Then we have*

$$f\left(\frac{2ab}{a+b}\right)\geq(\leq)\frac{ab}{b-a}\int_a^b\frac{1}{t^2f(t)}dt\geq(\leq)\frac{f(b)+f(a)}{2} \tag{2.1}$$

and

$$\frac{\frac{a+b}{2}}{f\left(\frac{a+b}{2}\right)}\geq(\leq)\frac{1}{b-a}\int_a^b\frac{t}{f(t)}dt\geq(\leq)\frac{af(b)+bf(a)}{2f(a)f(b)}. \tag{2.2}$$

PROOF. Since f is *HH*-convex (concave) on $[a,b]$, then by the Proposition 1.4 we have that P_f is concave (convex) on $\left[\frac{1}{b},\frac{1}{a}\right]$. By Hermite-Hadamard inequality for P_f we have

$$f\left(\frac{1}{\frac{\frac{1}{a}+\frac{1}{b}}{2}}\right)\geq(\leq)\frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}}\frac{1}{f\left(\frac{1}{s}\right)}ds\geq(\leq)\frac{f\left(\frac{1}{\frac{1}{b}}\right)+f\left(\frac{1}{\frac{1}{a}}\right)}{2},$$

which is equivalent to

$$f\left(\frac{2ab}{a+b}\right)\geq(\leq)\frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}}\frac{1}{f\left(\frac{1}{s}\right)}ds\geq(\leq)\frac{f(b)+f(a)}{2}. \tag{2.3}$$

If we make the change of variable $\frac{1}{s}=t,$ then $s=\frac{1}{t}$ and $ds=-\frac{dt}{t^2}$ and from (2.3) we get (2.1).

Since f is *HH*-convex (concave) on $[a,b]$, then by the Proposition 1.4 we also have that Q_f is concave (convex) on $[a,b]$. By Hermite-Hadamard inequality for Q_f we have

$$\frac{\frac{a+b}{2}}{f\left(\frac{a+b}{2}\right)}\geq(\leq)\frac{1}{b-a}\int_a^b\frac{t}{f(t)}dt\geq(\leq)\frac{\frac{a}{f(a)}+\frac{b}{f(b)}}{2},$$

which is equivalent to (2.2). ∎

We have the following reverse inequalities:

THEOREM 2.2 (Dragomir, 2015 [**21**]). *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a HH-convex (concave) function on* $[a,b]$ *. Then we have*

$$\begin{aligned}0\geq(\leq)\frac{f(b)+f(a)}{2}&-\frac{ab}{b-a}\int_a^b\frac{1}{t^2f(t)}dt\\&\geq(\leq)\frac{1}{8ab}\left(\frac{a^2}{f^2(a)}f'_+(a)-\frac{b^2}{f^2(b)}f'_-(b)\right)(b-a),\end{aligned} \tag{2.4}$$

$$(2.5)\quad 0\geq(\leq)\frac{ab}{b-a}\int_a^b\frac{1}{t^2f(t)}dt-f\left(\frac{2ab}{a+b}\right)\geq(\leq)$$
$$\geq(\leq)\frac{1}{8ab}\left(\frac{a^2}{f^2(a)}f'_+(a)-\frac{b^2}{f^2(b)}f'_-(b)\right)(b-a),$$

$$(2.6)\quad 0\geq(\leq)\frac{af(b)+bf(a)}{2f(a)f(b)}-\frac{1}{b-a}\int_a^b\frac{t}{f(t)}dt$$
$$\geq(\leq)\frac{1}{8}\left(\frac{f(b)-bf'_-(b)}{f^2(b)}-\frac{f(a)-af'_+(a)}{f^2(a)}\right)(b-a)$$

and

$$(2.7)\quad 0\geq(\leq)\frac{1}{b-a}\int_a^b\frac{t}{f(t)}dt-\frac{\frac{a+b}{2}}{f\left(\frac{a+b}{2}\right)}$$
$$\geq(\leq)\frac{1}{8}\left(\frac{f(b)-bf'_-(b)}{f^2(b)}-\frac{f(a)-af'_+(a)}{f^2(a)}\right)(b-a).$$

PROOF. The first part in all inequalities (2.4)-(2.7) follow from Theorem 2.1.

Now, if we take the derivative of $P_f(x)$, then we have

$$P'_f(x)=\left(\frac{1}{f\left(\frac{1}{x}\right)}\right)'=\left(f^{-1}\left(\frac{1}{x}\right)\right)'=f^{-2}\left(\frac{1}{x}\right)f'\left(\frac{1}{x}\right)\left(\frac{1}{x^2}\right).$$

Therefore we have

$$P'_{+f}\left(\frac{1}{b}\right)=b^2f^{-2}(b)f'_-(b)=\frac{b^2}{f^2(b)}f'_-(b)$$

and

$$P'_{-f}\left(\frac{1}{a}\right)=a^2f^{-2}(a)f'_+(a)=\frac{a^2}{f^2(a)}f'_+(a)$$

and by the right hand side inequalities in Lemma 0.1 we get the corresponding inequalities in (2.4) and (2.5).

If we take the derivative of Q_f, we have

$$Q'_f(x)=\left(\frac{x}{f(x)}\right)'=\frac{f(x)-xf'(x)}{f^2(x)}.$$

Therefore

$$Q'_{+f}(a)=\frac{f(a)-af'_+(a)}{f^2(a)}\text{ and }Q'_{-f}(b)=\frac{f(b)-bf'_-(b)}{f^2(b)}$$

and by the right hand side inequalities in Lemma 0.1 we get the corresponding inequalities in (2.6) and (2.7). ∎

We have the following result:

THEOREM 2.3 (Dragomir, 2015 [**21**]). *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a HH-convex (concave) function on $[a,b]$. Then we have*

$$(2.8)\quad \frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt\leq(\geq)\frac{G^2(f(a),f(b))}{L(f(a),f(b))}.$$

PROOF. By the definition of *HH*-convex (concave) function, we have by integrating on $[0,1]$ over λ, that

$$\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)d\lambda \le (\ge) \int_0^1 \frac{f(a)f(b)}{(1-\lambda)f(b)+\lambda f(a)}d\lambda. \tag{2.9}$$

Consider the change of variable $\frac{ab}{(1-\lambda)b+\lambda a} = t$. Then $(1-\lambda)b+\lambda a = \frac{ab}{t}$ and $(b-a)\,d\lambda = \frac{ab}{t^2}dt$. Using this change of variable, we have

$$\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)d\lambda = \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt.$$

If $f(b) = f(a)$, then

$$\int_0^1 \frac{f(a)f(b)}{(1-\lambda)f(b)+\lambda f(a)}d\lambda = f(a).$$

If $f(b) \ne f(a)$, then by the change of variable $(1-\lambda)f(b)+\lambda f(a) = s$, then we have

$$\begin{aligned}\int_0^1 \frac{f(a)f(b)}{(1-\lambda)f(b)+\lambda f(a)}d\lambda &= \frac{f(a)f(b)}{f(a)-f(b)}\int_{f(b)}^{f(a)}\frac{ds}{s}\\ &= \frac{f(a)f(b)}{L(f(a),f(b))} = \frac{G^2(f(a),f(b))}{L(f(a),f(b))}.\end{aligned}$$

By making use of (2.9) we deduce the desired result (2.8). ∎

We also have:

THEOREM 2.4 (Dragomir, 2015 [**21**]). *Let* $f:[a,b]\subset(0,\infty)\to(0,\infty)$ *be a HH-convex (concave) function on* $[a,b]$. *Then we have*

$$f\left(\frac{2ab}{a+b}\right) \le (\ge) \frac{\int_a^b \frac{1}{t^2} f(t)\, f\left(\frac{abt}{(a+b)t-ab}\right)dt}{\int_a^b \frac{f(t)}{t^2}dt}. \tag{2.10}$$

PROOF. From the definition of *HH*-convex (concave) function we have

$$f\left(\frac{2xy}{x+y}\right) \le (\ge) \frac{2f(x)f(y)}{f(x)+f(y)} \tag{2.11}$$

for any $x,y\in[a,b]$.

If we take

$$x = \frac{ab}{(1-\lambda)b+\lambda a},\ y = \frac{ab}{(1-\lambda)a+\lambda b} \in [a,b],$$

then

$$\frac{2xy}{x+y} = \frac{2ab}{a+b}$$

and by (2.11) we get

$$f\left(\frac{2ab}{a+b}\right) \le (\ge) \frac{2f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)}{f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)+f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)},$$

which is equivalent to

$$f\left(\frac{2ab}{a+b}\right)\left[f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)+f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)\right] \leq (\geq)\, 2f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right), \tag{2.12}$$

for any $\lambda \in [0,1]$.

If we integrate the inequality over λ on $[0,1]$ we get

$$f\left(\frac{2ab}{a+b}\right)\left[\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)d\lambda+\int_0^1 f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda\right] \leq (\geq)\, 2\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda. \tag{2.13}$$

Now, we observe that

$$\int_0^1 f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda=\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)d\lambda = \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt \tag{2.14}$$

and

$$\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda = \int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{1}{\frac{1}{b}+\frac{1}{a}-\frac{((1-\lambda)b+\lambda a)}{ab}}\right)d\lambda. \tag{2.15}$$

If we change the variable $t=\frac{ab}{(1-\lambda)b+\lambda a}$, then we have

$$\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{1}{\frac{1}{b}+\frac{1}{a}-\frac{((1-\lambda)b+\lambda a)}{ab}}\right)d\lambda = \frac{ab}{b-a}\int_a^b \frac{1}{t^2}f(t)\,f\left(\frac{abt}{(a+b)t-ab}\right)dt. \tag{2.16}$$

On making use of (2.13) - (2.16) we deduce the desired result (2.10). ∎

REMARK 2.1. By Cauchy-Bunyakowsky-Schwarz integral inequality we have

$$\int_0^1 f\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda \leq \left(\int_0^1 f^2\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)d\lambda\right)^{1/2}\left(\int_0^1 f^2\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)d\lambda\right)^{1/2} = \int_0^1 f^2\left(\frac{ab}{(1-\lambda)b+\lambda a}\right)d\lambda = \int_a^b \frac{f^2(t)}{t^2}dt. \tag{2.17}$$

Now, if $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is a HH-convex function on $[a,b]$, then by (2.10) and (2.17) we get

$$f\left(\frac{2ab}{a+b}\right) \le \frac{\int_a^b \frac{1}{t^2} f(t) f\left(\frac{abt}{(a+b)t-ab}\right) dt}{\int_a^b \frac{f(t)}{t^2} dt} \le \frac{\int_a^b \frac{f^2(t)}{t^2} dt}{\int_a^b \frac{f(t)}{t^2} dt}. \tag{2.18}$$

The following lemma is of interest as well.

LEMMA 2.5 (Dragomir, 2015 [**21**]). *If* $f : [a,b] \subset (0,\infty) \to (0,\infty)$ *is HH-convex on* $[a,b]$, *then the associated function* $R_f : [a,b] \subset (0,\infty) \to (0,\infty)$, $R_f(x) = \frac{f(x)}{x}$ *is convex on* $[a,b]$. *The reverse is not true.*

PROOF. Let $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and $x, y \in [a,b]$. By the *HH*-convexity of f we have

$$\begin{aligned} R_f(\alpha x + \beta y) &= \frac{f(\alpha x + \beta y)}{\alpha x + \beta y} = \frac{f\left(\frac{1}{\frac{1}{\alpha x+\beta y}}\right)}{\alpha x + \beta y} \\ &= \frac{f\left(\frac{1}{\frac{\alpha x \frac{1}{x} + \beta y \frac{1}{y}}{\alpha x+\beta y}}\right)}{\alpha x + \beta y} \le \frac{\frac{1}{\frac{\alpha x \frac{1}{f(x)} + \beta y \frac{1}{f(y)}}{\alpha x+\beta y}}}{\alpha x + \beta y} = \frac{1}{\alpha \frac{x}{f(x)} + \beta \frac{y}{f(y)}}. \end{aligned} \tag{2.19}$$

By the weighted Cauchy-Bunyakowsky-Schwarz inequality we have

$$\begin{aligned} &\left(\alpha \frac{x}{f(x)} + \beta \frac{y}{f(y)}\right)\left(\alpha \frac{f(x)}{x} + \beta \frac{f(y)}{y}\right) \\ &\qquad = \left(\alpha \left(\sqrt{\frac{x}{f(x)}}\right)^2 + \beta \left(\sqrt{\frac{y}{f(y)}}\right)^2\right) \\ &\qquad\qquad \times \left(\alpha \left(\sqrt{\frac{f(x)}{x}}\right)^2 + \beta \left(\sqrt{\frac{f(y)}{y}}\right)^2\right) \ge (\alpha+\beta)^2 = 1, \end{aligned}$$

which implies that

$$\frac{1}{\alpha \frac{x}{f(x)} + \beta \frac{y}{f(y)}} \le \alpha \frac{f(x)}{x} + \beta \frac{f(y)}{y}$$

for any $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and $x, y \in [a,b]$.

By (2.19) we have

$$R_f(\alpha x + \beta y) \le \alpha \frac{f(x)}{x} + \beta \frac{f(y)}{y} = \alpha R_f(x) + \beta R_f(y)$$

for any $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and $x, y \in [a,b]$, which shows that R_f is convex on $[a,b]$.

Consider the function $f : [a,b] \subset (0,\infty) \to (0,\infty)$, $f(x) = x^p$, $p \neq 0$. The function $R_f(x) = x^{p-1}$ is convex iff $p \in (-\infty, 1) \cup [2,\infty)$. Since $Q_f(x) = x^{1-p}$ which is concave iff $p \in (0,1)$. By Proposition 1.4 we have that the function $f : [a,b] \subset (0,\infty) \to (0,\infty)$, $f(x) = x^p$ is *HH*-convex iff Q_f is concave, namely $p \in (0,1)$. Therefore R_f is convex and not *HH*-convex if $p \in (-\infty, 0) \cup [2,\infty)$. ∎

If we denote by $\mathcal{C}_I[a,b]$ the class of all positive functions f for which R_f is convex, then the class of *HH*-convex functions $f : [a,b] \subset (0,\infty) \to (0,\infty)$ on $[a,b]$ is strictly enclosed in $\mathcal{C}_I[a,b]$.

We have the following inequalities of Hermite-Hadamard type.

THEOREM 2.6 (Dragomir, 2015 [**21**]). *If $f \in \mathcal{C}_I[a,b]$, then we have*

$$\frac{f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}} \le \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt \le \frac{f(a)b+f(b)a}{2ab}, \tag{2.20}$$

$$0 \le \frac{f(a)b+f(b)a}{2ab} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt \le \frac{1}{8}\left[\frac{f'_-(b)b-f(b)}{b^2} - \frac{f'_+(a)a-f(a)}{a^2}\right](b-a) \tag{2.21}$$

and

$$0 \le \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt - \frac{f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}} \le \frac{1}{8}\left[\frac{f'_-(b)b-f(b)}{b^2} - \frac{f'_+(a)a-f(a)}{a^2}\right](b-a). \tag{2.22}$$

PROOF. By the Hermite-Hadamard inequalities for R_f we have

$$\frac{f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}} \le \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt \le \frac{\frac{f(a)}{a}+\frac{f(b)}{b}}{2}$$

and the inequality (2.20) is proved.

We have

$$R'_f(t) = \left(\frac{f(t)}{t}\right)' = \frac{f'(t)t-f(t)}{t^2}$$

and then

$$R'_{-f}(b) = \frac{f'_-(b)b-f(b)}{b^2} \text{ and } R'_{+f}(a) = \frac{f'_+(a)a-f(a)}{a^2}.$$

By Lemma 0.1 we have

$$0 \le \frac{\frac{f(a)}{a}+\frac{f(b)}{b}}{2} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt \le \frac{1}{8}\left[\frac{f'_-(b)b-f(b)}{b^2} - \frac{f'_+(a)a-f(a)}{a^2}\right](b-a)$$

and

$$0 \le \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt - \frac{f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}} \le \frac{1}{8}\left[\frac{f'_-(b)b-f(b)}{b^2} - \frac{f'_+(a)a-f(a)}{a^2}\right](b-a),$$

which are equivalent to the desired inequalities (2.21) and (2.22). ∎

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