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SOME PROPERTIES OF (CO)HOMOLOGY OF LIE ALGEBRA

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ABSTRACT. Lie algebra is one of the important types of algebras. Here, we studied lie algebra and discussed its properties. Moreover, we define the Dihedral homology of lie algebra and prove some relations among Hochschild, Cyclic and Dihedral homology of lie algebra. Finally, we prove the Mayer-vetories sequence on Dihedral homology of lie algebra and proved that $HC^n(g,M) \cong^- HD^n(g,M) \bigoplus^+ HD^n(g,M)$.

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1. INTRODUCTION

The dihedral group is the symmetry group with reflection and rotation. Dihedral groups are one from the simplest examples of limited group, and they assume an imperative role in geometry, group theory and science. The documentation for the dihedral group contrasts in abstract algebra and geometry. In algebra, the notation of dihedral group is D_{2n}.The dihedral homology and cohomology was studied in [8]. In [1], [2] and [3], Alaa H.N. discussed the dihedral (co)homology for schemes, dihedral homology of Banach algebras and dihedral homology of algebra, respectively. And the reflexive and dihedral (co)homology of $\mathbb{Z}/2$ Graded algebras was studied in [4].
Now, we illustrate the notation used here. Let $\mathfrak{g}$ be the lie algebra and $[-,-]$ denotes lie bracket. Tensor product of lie algebra denoted by $\mathfrak{g}^{\otimes n}$, the complex $C_*(\mathfrak{g},V)$ is the chevally-Eilenberg complex and $H^n(\mathfrak{g},M)$ is the n^{th}-cohomology of $\mathfrak{g}$. $HH_n(\mathfrak{g},M), HC_n(\mathfrak{g},M),\ {}^{\alpha}HD_n(\mathfrak{g},M)$ are Hochschild, Cyclic and dihedral homology of $\mathfrak{g}$, respectively.
In our work we study the dihedral cohomology of lie algebra. The present section, we discussed the basic definitions of lie algebra. And we used the references([6], [7] and [9]).

Definition 1.1. Let $\mathfrak{g}$ be a vector space over k with a binary operation $[-,-] : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$. Then $\mathfrak{g}$ is called lie algebra and denoted by $\mathfrak{g}L(k)$ and $[-,-]$ is called lie bracket since,$[x,y] = xy - yx$ and satisfy that:
(1) $[x,x] = 0$
(2) $[x,y] = -[y,x]$
(3) $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$
(4) $[ax + by, z] = a[x,z] + b[y,z], [z, ax + by] = a[z,x] + b[z,y]$.

Remark 1.1. Consider $\mathfrak{g}$ be an associative lie algebra, then
(1) $[a,b] + [b,a] = 0$
(2) $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$.

Proposition 1.1. *Let $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a morphism of lie algebra which is a linear map such that $f[a,b] = [f(a), f(b)]\ \forall a,b \in \mathfrak{g}_1$. Then:*
(1) $ker(f)$ is an ideal of $\mathfrak{g}_1$
(2) $im(f)$ is a subalgebra of $\mathfrak{g}_2$
(3) $im(f) \cong \mathfrak{g}_1/ker(f)$

Proof. (1) $ker(f) = \{a \in \mathfrak{g}_1\ s.h.\ f(a) = 0\}$, then we get that $ker(f)$ is an ideal of $\mathfrak{g}$ as:
let $a \in ker(f), b \in \mathfrak{g}_1$; $f[a,b] = [f(a), f(b)] = [0, f(b)] = 0$, then $[a,b] \in ker(f)$.
(2) $\forall a,b \in \mathfrak{g}_1 \exists\ c,d \in \mathfrak{g}_2\ s.h.\ f(a) = c, f(b) = d$. Then
$f[a,b] = [f(a), f(b)] = [c,d] \in \mathfrak{g}_2 \Rightarrow im(f) \subset \mathfrak{g}_2$
(3) $\forall a,b \in im(f) \subset \mathfrak{g}_2 \exists\ \acute{a}, \acute{b} \in \mathfrak{g}_1\ s.h.\ f(\acute{a}) = a, f(\acute{b}) = b$, then $[a,b] = [f(\acute{a}), f(\acute{b})] = [\acute{a}, \acute{b}]$, since $[\acute{a}, \acute{b}]$ is the pre-image of $[a,b]$. Then we easily find that $im(f) \cong \mathfrak{g}_1/ker(f)$ ∎

Definition 1.2. Let $\mathfrak{g}$ be a lie algebras and if a and b are two elements in $\mathfrak{g}$. Then a and b are said to be commute if $[a,b] = 0$. And if any two elements of $\mathfrak{g}$ are commute, then $\mathfrak{g}$ is said to be commutative.

Definition 1.3. The tensor product of lie algebra is denoted by $\mathfrak{g}^{\bigotimes n}$ and given by

$$[(g_1, , g_n), g] = \sum_{i=1} (g_1, , [g_i, g], , g_n) \tag{1.1}$$

And the exterior product of the lie algebra which denoted by $\Lambda^n\mathfrak{g}$ is given by

$$[(g_1 \wedge \ldots \wedge g_n), g] = \sum_{i=1}^{n}(g_1 \wedge \ldots \wedge [g_i, g] \wedge \ldots \wedge g_n) \tag{1.2}$$

Definition 1.4. Consider the lie algebra g over k and V be a $\mathfrak{g}$-module. Then we can define the chevally-Eilenbrg complex $C_*(\mathfrak{g}, V)$ as the form

$$\cdots \longrightarrow V \otimes \Lambda^n\mathfrak{g} \xrightarrow{d} V \otimes \Lambda^{n-1}\mathfrak{g} \xrightarrow{d} \cdots \xrightarrow{d} V \otimes \Lambda^1\mathfrak{g} \xrightarrow{d} V \tag{1.3}$$

Since $\Lambda^n\mathfrak{g}$ is the n^{th}-exterior product of $\mathfrak{g}$ and the map $d : V \otimes \Lambda^n\mathfrak{g} \longrightarrow V \otimes \Lambda^{n-1}\mathfrak{g}$ is defined as

$$\begin{aligned} d(v \otimes g_1 \wedge \ldots \wedge g_n) &= \sum_{1\leq j\leq n} (-1)^j [v, g_j] \otimes g_1 \wedge \ldots \wedge \widehat{g}_j \wedge \ldots \wedge g_n \\ &+ \sum_{1\leq i\leq j\leq n} (-1)^{i-j-1} v \otimes [g_i, g_j] \wedge g_1 \wedge \ldots \wedge \widehat{g}_i \wedge \ldots \wedge \widehat{g}_j \wedge \ldots \wedge g_n \end{aligned} \tag{1.4}$$

When $\widehat{g}_i$ means that g_i is deleted. And $d^2 = 0$,(to illustrate see [3] and [5]).

Definition 1.5. Consider $\mathfrak{g}$ is the lie algebra and M is $\mathfrak{g}$-module, then we can define the P-cochains on $\mathfrak{g}$ with coefficients in M as

$$C^p(\mathfrak{g}, M) = Hom_k(\Lambda^p\mathfrak{g}, M) \tag{1.5}$$

Since if $p = 0 \Rightarrow C^0(\mathfrak{g}, M) = M$. Then we get the cochain complex $(C^*(\mathfrak{g}, M), d)$ which is called the chevally-Eilenberg complex, where $dc \in C^{p+1}(\mathfrak{g}, M)$ and defined as

$$dc(x_1, \ldots, x_{p+1}) = \sum_{1\leq i\leq j\leq p+1} (-1)^{i+j} c([x_i, x_j], x_1, \ldots, \widehat{x}_i, \ldots, x_{p+1}) + \sum_{i=1}^{p+1} x_i \cdot c(x_1, \ldots, \widehat{x}_i, \ldots, x_{p+1}) \tag{1.6}$$

And we define the P-cocycles $Z^p(\mathfrak{g}, M)$ and P-coboundary $B^p(\mathfrak{g}, M)$ as

$$Z^p(\mathfrak{g}, M) = \{c \in C^p(\mathfrak{g}, M) : dc = 0\} = ker\ d : C^p(\mathfrak{g}, M) \to C^{p+1}(\mathfrak{g}, M) \tag{1.7}$$

and

$$B^p(\mathfrak{g}, M) = \{c \in C^p(\mathfrak{g}, M) : \exists\ \acute{c} \in C^{p-1}(\mathfrak{g}, M) : c = d\acute{c}\} = imd : C^{p-1}(\mathfrak{g}, M) \to C^p(\mathfrak{g}, M) \tag{1.8}$$

then the cohomology of $\mathfrak{g}$ with coeffients in M as

$$H^p(\mathfrak{g}, M) = Z^p(\mathfrak{g}, M)/B^p(\mathfrak{g}, M) \tag{1.9}$$

Definition 1.6. Let the exact sequence $0 \longrightarrow A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \longrightarrow 0$ with the connected homomorphisms, $\partial_n : H_n(C_\bullet) \longrightarrow H_{n-1}(A_\bullet)$. Then we get the long exact sequence as

$$\cdots \xrightarrow{g} H_{n+1}(C_\bullet) \xrightarrow{\partial} H_n(A_\bullet) \xrightarrow{f} H_n(B_\bullet) \xrightarrow{g} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \xrightarrow{f} \cdots \tag{1.10}$$

Similarly, let the exact sequence of cochain complexes is the form

$$0 \longrightarrow A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \longrightarrow 0 \tag{1.11}$$

With the connected homomorphisms $\partial^n : H^n(C^\bullet) \longrightarrow H^{n+1}(C^\bullet)$, then we get the long exact sequence

$$\cdots \xrightarrow{g} H^{n-1}(C^\bullet) \xrightarrow{\partial} H^n(A^\bullet) \xrightarrow{f} H^n(B^\bullet) \xrightarrow{g} H^n(C^\bullet) \xrightarrow{\partial} H^{n+1}(A^\bullet) \xrightarrow{f} \cdots \tag{1.12}$$

Theorem 1.2. *Let $\mathfrak{g}$ be a lie algebra and M is a bimodule over $\mathfrak{g}$, and $M_r(M)$ be a module of matrices of degree $r \times r$. Then the inclusion is defined as*

$$inc : M_r(M) \longrightarrow M_{r+1}(M) \tag{1.13}$$

as the form

$$\begin{bmatrix} & & & 0 \\ & \alpha & & \bullet \\ 0 & \bullet & 0 & 0 \end{bmatrix} \tag{1.14}$$

Since the trace map $tr : M_r(M) \longrightarrow M$ is given by $tr(\alpha) = \sum_{i=1}^{r} \alpha_{ii}$.
And we define the generalized trace map

$$tr : M_r(M) \otimes M_r(R)^{\otimes n} \longrightarrow M \otimes R^{\otimes n} \tag{1.15}$$

As

$$tr(\alpha \otimes \beta \otimes ... \otimes \eta) = \sum (\alpha_{i_0 i_1} \otimes \beta_{i_1 i_2} \otimes ... \otimes \eta_{i_n i_0}) \tag{1.16}$$

Theorem 1.3. *Let $\mathfrak{g}$ be a lie algebra for $r \geq 1$, the maps*

$$tr_* : H_*(M_r(\mathfrak{g}), M_r(M)) \longrightarrow H_*(\mathfrak{g}, M) \tag{1.17}$$

And

$$inc_* : H_*(\mathfrak{g}, M) \longrightarrow H_*(M_R(\mathfrak{g}), M_r(M)) \tag{1.18}$$

are isomorphisms and both of them is inverse to each other.

Theorem 1.4. *Suppose that $\mathfrak{g}$ be a lie algebra, A and B are subspaces of $\mathfrak{g}$ since whose interiors is cover of $\mathfrak{g}$. Then we can define the Mayer-vietoris sequence of $(\mathfrak{g}, A, B)$ as the long exact sequence which relates the Hocschild homology groups of $\mathfrak{g}, A, B$ with the intersection $A \cap B$*

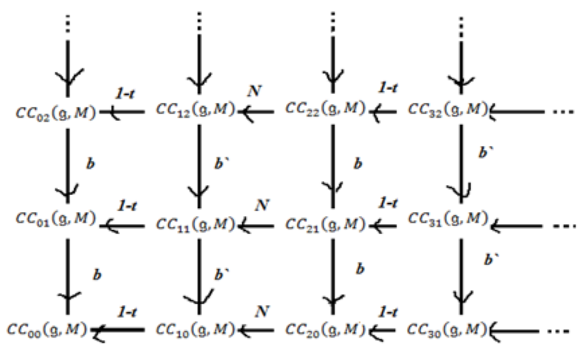
$$\begin{aligned} \cdots \longrightarrow H_{n+1}(\mathfrak{g}, M) \xrightarrow{\partial_*} H_n(A \cap B) \xrightarrow{(i_*, j_*)} H_n(A) \oplus H_n(B) \xrightarrow{k_* - l_*} H_n(\mathfrak{g}, M) \xrightarrow{\partial_*} \\ H_{n-1}(A \cap B) \longrightarrow \cdots \longrightarrow H_0(A) \oplus H_0(B) \xrightarrow{k_* - l_*} H_0(\mathfrak{g}, M) \longrightarrow 0 \end{aligned} \tag{1.19}$$

Where $i : A \cap B \hookrightarrow A, j : A \cap B \rightharpoonup B, k : A \hookrightarrow g$ and $l : B \hookrightarrow g$ and $\oplus$ means the direct sum of abelian group (to illustrate see [11]*).*

2. CYCLIC AND DIHEDRAL (CO)HOMOLOGY OF LIE ALGEBRA

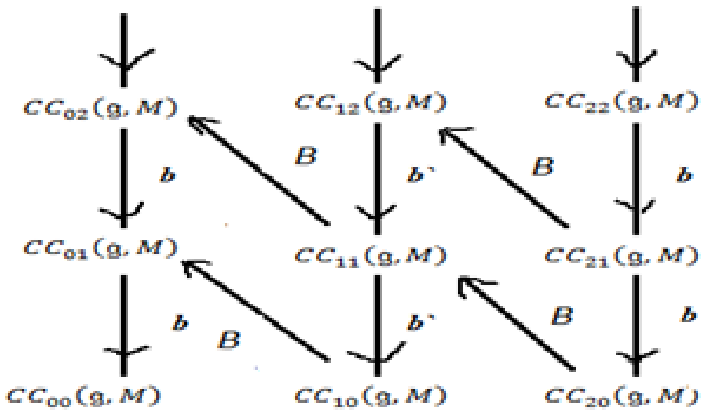
In the following part, we will give the Cyclic and Dihedral (co)homology of the Lie algebra and it‘s properties. Here, we use ([8],[9],[10]).

Definition 2.1. If we consider the lie algebra $\mathfrak{g}$ with unity and it's coefficients are in module M, then the homology group of lie algebra can be defined as the homology of the complex $TotCC_{**}(\mathfrak{g}, M)$

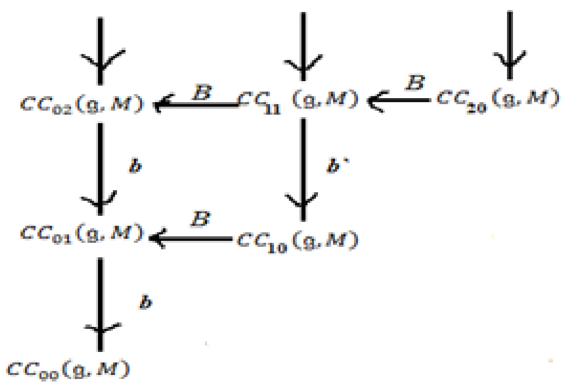


Since $HC_n(\mathfrak{g}, M) = H_n(TotCC_{**}(\mathfrak{g}, M))$ with the bi-complex $CC_{**}(\mathfrak{g}, M)$ and the vertical maps $b_*, \grave{b}_* : CC_{\alpha\ \beta} \longrightarrow CC_{\alpha\ \beta-1} s.h.b = \sum_{i=0}^{n}(-1)^i d_i$ and $\grave{b} = \sum_{i=0}^{n-1}(-1)^i d_i$ and horizontal maps $(1-t)_*, N_* : CC_{\alpha\ \beta} \longrightarrow CC_{\alpha-1\ \beta}\ s.h.\ N = \sum_{i=0}^{n} t_n^i$, Where $b(1-t) = (1-t)\grave{b}$ and $\grave{b}N = Nb$.

Proposition 2.1. *contentIn the bi-complex in definition (2.1), if we define the extra degeneracy* $S = S_{n+1} : C_n \longrightarrow C_{n+1}; S = (-1)^n t_n S_n$ *and* $S\grave{b} + \grave{b}S = id$. *By applying the composition between the maps* $B = (1-t)SN$*, we get the bi complex*



If we rearrange, we get the bi-complex



Definition 2.2. Since $\mathfrak{g}$ is the lie algebras, and M be a module, then the (co)homology of the complex $C(\mathfrak{g}, M)$ with the space $\mathfrak{g}(C, \delta, \sigma)$ is called the Hochschild (co)homology and denoted by $H_n(\mathfrak{g}, M)(H^n(\mathfrak{g}, M))$.

Similarly, the Cyclic (co)homology $HC_n(\mathfrak{g}, M)(HC^n(\mathfrak{g}, M))$ is the (co)homology of the complex $CC(\mathfrak{g}, M)$ with the space $\mathfrak{g}(C, \delta, \varrho, \tau)$.

Finally, the Dihedral (co)homology ${}^{\alpha}HD_n(\mathfrak{g}, M)({}^{\alpha}HD^n(\mathfrak{g}, M))$ and the Reflexive (co)homology ${}^{\alpha}HR_n(\mathfrak{g}, M)({}^{\alpha}HR^n(\mathfrak{g}, M))$ is the (co)homology of the complex $\mathfrak{g}(C, \delta, \sigma, \tau, \rho)(\mathfrak{g}(C, \delta, \sigma, \rho))$

where $\alpha = \pm 1$. Such that the following are satisfied:

$$
\begin{array}{cc}
\delta^j_{n+1}\delta^i_n = \delta^i_{n+1}\delta^{j-1}_n & if\ i < j \\
\sigma^j_n\sigma^i_{n+1} = \sigma^i_n\sigma^{j-1}_{n+1} & if\ i \leq j \\
\sigma^j_n\delta^i_{n+1} = \delta^i_{n-2}\sigma^{j-1}_{n-2} & if\ i < j \\
\sigma^j_n\delta^i_{n+1} = Id_{[n]} & if\ i = j\ or\ j+1 \\
\sigma^j_n\delta^i_{n+1} = \delta^{i-1}_{n+1}\sigma^j_n & if\ i > j+1 \\
\tau_n\delta^i_n = \delta^{i-1}_n\tau_{n-1} & 1 \leq i \leq n \\
\tau_n\sigma^i_n = \sigma^{j-1}_n\tau_{n+1} & 1 \leq j \leq n \\
\tau^{n+1}_n = Id_{[n]} & \\
\rho_n\delta^i_n = \delta^{i-1}_n\rho_{n-1} & 0 \leq i \leq n \\
\rho_n\sigma^j_n = \sigma^{j-1}_n\rho_{n+1} & 0 \leq j \leq n \\
\rho^2_n = Id_{[n]}, & \tau_n\rho_n = \rho_n\tau^{-1}_n
\end{array}
\tag{2.1}
$$

3. MAIN RESULTS

Lie algebra is one of important types of algebras. After we study the definitions of Hochschild, Cyclic and Dihedral (co)homology of Lie algebra, we study the relations among them.

Theorem 3.1. *Consider $\mathfrak{g}$ is lie algebra with unity and module M, and the complex $C(\mathfrak{g}, M)$, then we can relate between the Hochschild homology $HH(\mathfrak{g}, M)$ and the Cyclic homology $HC(\mathfrak{g}, M)$ in the form;*

(3.1)
$$\cdots \xrightarrow{B} HH_{n+1}(\mathfrak{g}, M) \xrightarrow{I} HC_{n+1}(\mathfrak{g}, M) \xrightarrow{s} HC_{n-1}(\mathfrak{g}, M) \xrightarrow{B} HH_n(\mathfrak{g}, M) \longrightarrow \cdots$$

Proof. If $CC(\mathfrak{g}, M)^{\{2\}}$ be the bi-complex which contains the first two columns of $CC(\mathfrak{g}, M)$, since $C[2, 0]_{pq} = C_{p-2,q}$.
Then the long exact sequence which we get from the exact sequence

$$0 \longrightarrow CC(\mathfrak{g}, M)[2, 0] \longrightarrow CC(\mathfrak{g}, M) \longrightarrow CC(\mathfrak{g}, M)^{\{2\}} \longrightarrow 0 \tag{3.2}$$

is the sequence which related between the Hochschild and Cyclic homology of lie group. And then we get the required. ∎

Theorem 3.2. *Let $\mathfrak{g}$ be the lie algebra with unity and involution, and module M, then the relation between the Cyclic homology $HC(\mathfrak{g}, M)$ and the Dihedral homology $HD(\mathfrak{g}, M)$ is defined as the sequence*

(3.3)
$$\cdots \longrightarrow {}^-HD_{n+1}(\mathfrak{g}, M) \longrightarrow {}^+HD_n(\mathfrak{g}, M) \xrightarrow{j^*} HC_n(\mathfrak{g}, M) \xrightarrow{i^*} {}^-HD_n\mathfrak{g}, M) \longrightarrow \cdots$$

Proof. We can get the required by getting the long exact sequence

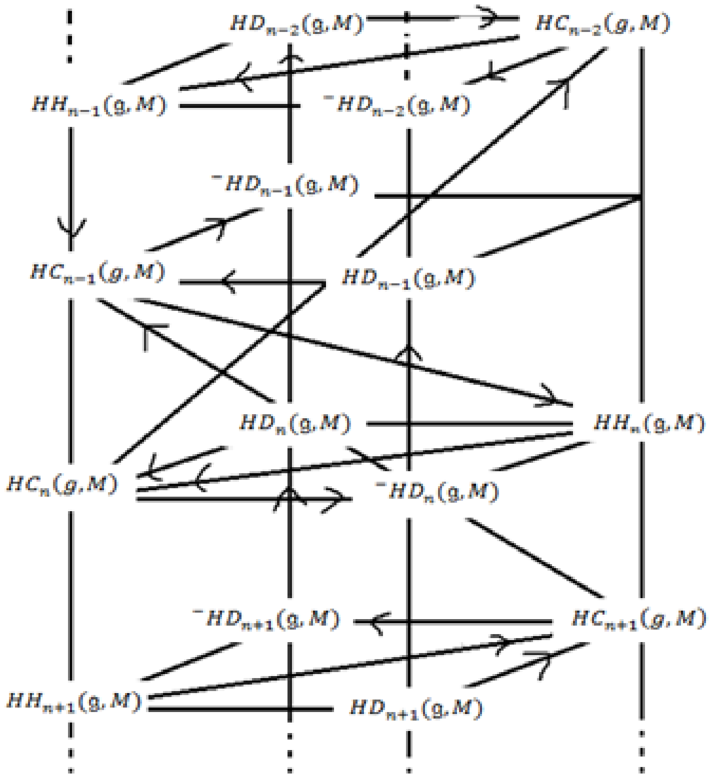
(3.4)
$$\cdots \longrightarrow {}^-HD_{n+1}(\mathfrak{g}, M) \longrightarrow {}^+HD_n(\mathfrak{g}, M) \xrightarrow{j^*} HC_n(\mathfrak{g}, M) \xrightarrow{i^*} {}^-HD_n\mathfrak{g}, M) \longrightarrow \cdots$$

From the short exact sequence

$$0 \longrightarrow Tot^-D(\mathfrak{g}, M) \longrightarrow Tot^+D(\mathfrak{g}, M) \longrightarrow Tot\ C(\mathfrak{g}, M) \longrightarrow 0 \tag{3.5}$$

∎

Theorem 3.3. *Since $\mathfrak{g}$ is the lie algebra, with unity and involution, and module M. Then the sequence which related among the Hochschild homology $HH(\mathfrak{g}, M)$, Cyclic homology $HC(\mathfrak{g}, M)$ and Dihedral homology $HD(\mathfrak{g}, M)$ is:*



Proof. By using
(3.6)

$$\cdots \longrightarrow {}^{-}HD_{n+1}(\mathfrak{g}, M) \longrightarrow {}^{+}HD_n(\mathfrak{g}, M) \xrightarrow{j^*} HC_n(\mathfrak{g}, M) \xrightarrow{i^*} {}^{-}HD_n\mathfrak{g}, M) \longrightarrow \cdots$$

and
(3.7)

$$\cdots \xrightarrow{B} HH_{n+1}(\mathfrak{g}, M) \xrightarrow{I} HC_{n+1}(\mathfrak{g}, M) \xrightarrow{s} HC_{n-1}(\mathfrak{g}, M) \xrightarrow{B} HH_n(\mathfrak{g}, M) \longrightarrow \cdots$$

we get the relation among $HD(\mathfrak{g}, M),\ HC(\mathfrak{g}, M)$ and $HH(\mathfrak{g}, M)$. ∎

Theorem 3.4. *for the dihedral cohomology $HD(\mathfrak{g}, M)$ and the cyclic cohomology $HC(\mathfrak{g}, M)$, we get*

$$HC^n(\mathfrak{g}, M) \cong {}^{-}HD^n(\mathfrak{g}, M) \oplus {}^{+}HD^n(\mathfrak{g}, M) \tag{3.8}$$

Proof. Consider the cyclic space $\mathfrak{g}(C, \delta, \sigma, \tau)$ and dihedral lie space $\mathfrak{g}(C, \delta, \sigma, \tau, \rho)$ with their associated lie $\mathfrak{g}$-module $C(\mathfrak{g}, M)$ as the diagram

$$\begin{array}{cccc}
0 \to {}^{+}HD^0(\mathfrak{g}, M) \xrightarrow{\delta^0} & {}^{+}HD^1(\mathfrak{g}, M) \to \cdots \to & {}^{+}HD^{n-1}(\mathfrak{g}, M) \xrightarrow{\delta^{n-1}} & {}^{+}HD^n(\mathfrak{g}, M) \to \cdots \\
i \uparrow \uparrow \mu & i \uparrow \uparrow \mu & i \uparrow \uparrow \mu & i \uparrow \uparrow \mu \\
0 \to HC^0(\mathfrak{g}, M) \xrightarrow{\delta^0} & HC^1(\mathfrak{g}, M) \to \cdots & \to HC^{n-1}(\mathfrak{g}, M) \xrightarrow{\delta^{n-1}} & HC^n(\mathfrak{g}, M) \to \cdots \\
i \downarrow \downarrow \gamma & i \downarrow \downarrow \gamma & i \downarrow \downarrow \gamma & i \downarrow \downarrow \gamma \\
0 \to {}^{-}HD^0(\mathfrak{g}, M) \xrightarrow{\delta^0} & {}^{-}HD^1(\mathfrak{g}, M) \to \cdots & \to HC^{n-1}(\mathfrak{g}, M) \xrightarrow{\delta^{n-1}} & HC^n(\mathfrak{g}, M) \to \cdots
\end{array}$$

Since $\mu : x \longrightarrow x + R_n \cdot x,\ \gamma : x \longrightarrow x - R_n \cdot x,\ R_n = (-1)^{\frac{n(n+1)}{2}} \rho_n$ and l is the natural imbedding. ∎

Theorem 3.5. ***(Mayer-vietories sequence)***
Consider the exact sequence

$$\cdots \longrightarrow C_{n+1}(\mathfrak{g}, M) \xrightarrow{d_n} C_n(\mathfrak{g}, M) \xrightarrow{d_{n-1}} C_{n-1}(\mathfrak{g}, M) \longrightarrow \cdots \tag{3.9}$$

and we have two subsequences A and B of C as

$$A : \cdots \xrightarrow{d_{n+1}} A_{n+1}(\mathfrak{g}, M) \xrightarrow{d_n} A_n(\mathfrak{g}, M) \xrightarrow{d_{n-1}} A_{n-1}(\mathfrak{g}, M) \longrightarrow \cdots \tag{3.10}$$

$$B : \cdots \xrightarrow{d_{n+1}} B_{n+1}(\mathfrak{g}, M) \xrightarrow{d_n} B_n(\mathfrak{g}, M) \xrightarrow{d_{n-1}} B_{n-1}(\mathfrak{g}, M) \longrightarrow \cdots \tag{3.11}$$

Then we have the Mayer-vetories sequence as a long exact sequence
(3.12)

$$\begin{aligned} \cdots \xrightarrow{f^*_{n+1}} HD_{n+1}(A \oplus B) \xrightarrow{g^*_{n+1}} HD_{n+1}(A + B) \xrightarrow{h^*_{n+1}} HD_n(A \cap B) \xrightarrow{f^*_n} HD_n(A \oplus B) \\ \xrightarrow{g^*_n} HD_n(A + B) \xrightarrow{h^*_n} HD_{n-1}(A \cap B) \xrightarrow{f^*_{n-1}} HD_{n-1}(A \oplus B) \xrightarrow{g^*_{n-1}} \cdots \end{aligned}$$

Proof. By studying the long exact sequence which relate the homology of two subcomplexes

$$A : \cdots \xrightarrow{d_{n+1}} A_{n+1}(\mathfrak{g}, M) \xrightarrow{d_n} A_n(\mathfrak{g}, M) \xrightarrow{d_{n-1}} A_{n-1}(\mathfrak{g}, M) \longrightarrow \cdots \tag{3.13}$$

and

$$B : \cdots \xrightarrow{d_{n+1}} B_{n+1}(\mathfrak{g}, M) \xrightarrow{d_n} B_n(\mathfrak{g}, M) \xrightarrow{d_{n-1}} B_{n-1}(\mathfrak{g}, M) \longrightarrow \cdots \tag{3.14}$$

Of the same complex

$$\cdots \longrightarrow C_{n+1}(\mathfrak{g}, M) \xrightarrow{d_n} C_n(\mathfrak{g}, M) \xrightarrow{d_{n-1}} C_{n-1}(\mathfrak{g}, M) \longrightarrow \cdots$$

to the homology of their sum and intersection, where the sequence $(A+B)$ is the subsequence of C with chain group (A_n+B_n), also $A \cap B$ is a subsequence of C with a chain group $A_n \cap B_n$. Then we get the Mayer-vietories sequence from the long exact sequence

$$\begin{aligned} \cdots \xrightarrow{f^*_{n+1}} HD_{n+1}(A \oplus B) \xrightarrow{g^*_{n+1}} HD_{n+1}(A + B) \xrightarrow{h^*_{n+1}} HD_n(A \cap B) \xrightarrow{f^*_n} HD_n(A \oplus B) \\ \xrightarrow{g^*_n} HD_n(A + B) \xrightarrow{h^*_n} HD_{n-1}(A \cap B) \xrightarrow{f^*_{n-1}} HD_{n-1}(A \oplus B) \xrightarrow{g^*_{n-1}} \cdots \end{aligned}$$

with the maps f^*_n such that $f_n : C_n(A \cap B) \longrightarrow C_n(A \oplus B)$ and defined as $f_n(a) = (a, -a), g^*_n \; s.h. \; g_n : C_n(A \oplus B) \longrightarrow C_n(A + B)$ since $g_n(x, y) = x + y$ and the map h^*_n from the homology class of z to $d_n(x)$, since $x \in X_n \; s.h. \; z - x \in Y_n$. ∎

4. CONCLUSION

We study and prove Some Properties of the (co)homology theory of lie algebra and we get:
1- Then relation between the Hochschild homology $HH(\mathfrak{g}, M)$ and the Cyclic homology $HC(\mathfrak{g}, M)$ of lie algebra exists and prove it.
2- The relation between the cylic and dihedral cohomology oh lie algebra exists and proving.
3- We get the Mayer-vietories sequence of the dihedral homology of lie algebra exists and proved.

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