Aust. J. Math. Anal. Appl.
Vol. **17** (2020), No. 2, Art. 6, 18 pp.
AJMAA

NUMERICAL APPROXIMATION BY THE METHOD OF LINES WITH FINITE-VOLUME APPROACH OF A SOLUTE TRANSPORT EQUATION IN PERIODIC HETEROGENEOUS POROUS MEDIUM

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Received 29 March, 2020; accepted 2 August, 2020; published 19 August, 2020.

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ABSTRACT. In this paper we are interested in the numerical approximation of a two-dimensional solute transport equation in heterogeneous porous media having periodic structures. It is a class of problems which has been the subject of various works in the literature, where different methods are proposed for the determination of the so-called homogenized problem. We are interested in this paper, in the direct resolution of the problem, and we use the method of lines with a finite volume approach to discretize this equation. This discretization leads to an ordinary differential equation (ODE) that we discretize by the Euler implicit scheme. Numerical experiments comparing the obtained solution and the homogenized problem solution are presented. They show that the precision and robustness of this method depend on the ratio between, the mesh size and the parameter involved in the periodic homogenization.

Key words and phrases: Porous media, Homogenization, Parabolic Equation, Finite Volume Method, Method of Lines, Ordinary Differential Equation (ODE), Implicit Numerical Scheme, Solute Transport.

2000 *Mathematics Subject Classification.* Primary 46C05, 46C99. Secondary 26D15, 26D10.

ISSN (electronic): 1449-5910

1. INTRODUCTION

There are many practical computational problems with highly oscillatory solutions e.g. computation of flow in heterogeneous porous media for petroleum and groundwater reservoir simulation (see, e.g., [9] and the bibliographies therein). If a porous medium with a periodic structure is considered, with the size of the period is small enough compared to the size of the reservoir, and denoting their ratio by ε an asymptotic analysis, as $\varepsilon \to 0$, can be used.
In this paper we consider a two-dimensional parabolic equation modeling a diffusion process (for example, a solute diffusion process) in a periodic medium (for example, an heterogeneous domain obtained by mixing periodically two different phases, one being the matrix and the other the inclusions).
To fix ideas, the periodic domain is called Ω, a bounded polygonal convex domain in $\mathbb{R}^2$, with a smooth boundary Γ. Its period ε (a positive number wich is assumed to be very small in comparison with size of the domain), and the rescaled unit periodic cell $Y = (0,1)^2$.
For a final time $T > 0$, a source term s, and an initial condition u^0, we consider the Cauchy problem:

$$(P_\varepsilon) \quad \begin{cases} \Phi^\varepsilon \dfrac{\partial u^\varepsilon}{\partial t} - \operatorname{div}\left(K^\varepsilon(x)\nabla u^\varepsilon\right) = s(x,t) & \text{in } \Omega \times (0,T), \\ u^\varepsilon(x,t) = 0 & \text{on } \Gamma \times (0,T), \\ u^\varepsilon(x,0) = u^0(x) & \text{in } \Omega. \end{cases}$$

In flow in porous medium, Φ^ε denotes, the medium porosity, K^ε its absolute permeabilities tensor, and u^ε denotes the solute concentration. We will make the following assumptions:

$$(A) \quad \begin{cases} K^\varepsilon(x) := K\left(\dfrac{x}{\varepsilon}\right),\ K \text{ is a second order and symetric matrix with} \\ K_{ij} \in L^\infty(Y)\ \ 1 \leq i,j \leq 2, \\ \forall \xi \in \mathbb{R}^2\ \lambda \mid \xi \mid^2 \leq < K(x)\xi, \xi > \leq\ \mu \mid \xi \mid^2 \ \text{ a.e. } x\ \in \Omega,\ 0 < \lambda \leqslant \mu < +\infty, \\ \Phi^\varepsilon(x) := \Phi\left(\dfrac{x}{\varepsilon}\right),\ \Phi \text{ is a bounded positive, } Y\text{-periodic function with} \\ 0 < \Phi^- \leq \Phi(y) \leq \Phi^+ < +\infty, \text{and } s \in L^2(\Omega \times (0,T)),\ \ u^0 \in L^2(\Omega). \end{cases}$$

$< .,. >$ denotes the standard inner product on $\mathbb{R}^2$. Under (A) assumptions, it is a well-known result that there exists a unique solution u^ε of P_ε.
Using the homogenization tools (see, e.g., [3], [6]-[7], [9] [10], [11], [12], [18]) the original equation describing this problem can be replaced by an effective or homogenized following problem modeling some average quantity without the oscillations.

$$\begin{cases} \Phi^* \dfrac{\partial u}{\partial t} - \operatorname{div}\left(K^*(x)\nabla u\right) = s(x,t) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma \times (0,T), \\ u(x,0) = u^0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where the entries of the matrix K^* are given by resolution of the so-called or cell problems and

$$\Phi^* = \int_Y \Phi(y)dy.$$

We should notice that K^* is still symmetric and positive definite but in general cases, even with K^ε isotropic, we may have K^* anisotropic. One can refer e.g., to [1] and [10] for numerical computation of K^*.

Whenever homogenized equations are applicable they are very useful for computational purposes. However there are many situations for which ε is not sufficiently small, similarly it is difficult to know a priori, for any matrix K^ε and any function Φ^ε, from which value, ε can be considered small enough, so that the homogenized solution correctly reproduces the behavior of the solution of problem (P_ε). In this cases we would like to approximate the original equation directly.

Numerical approximation of partial differential equations with highly oscillating coefficients has been a problem of interest for many years. Elliptic equations case is the most studied and many methods have been developed (see, e.g., [2], [5], [14]-[17], and the bibliographies therein).

Parabolic equations case (in particulary Cauchy problem) have however been fairly widely approached for the determination of the homogenized equations (see, e.g. [3], [12], [18], and the bibliographies therein) but works addressing the numerical resolution, of the original problem are difficult to find out in the literature, and in [14] the problem was approached only in the monodimensional case with discontinuous coefficients.

In this paper we study the two-dimensional problem with continuous coefficients by using the method of lines with finite-volume approach to discretize the problem. This discretization leads to an ordinary differential equation that we discretize by a Euler implicit numerical scheme.

The outlines of the remainder of this paper is as follows. In Section 2, a description of used methods is presented. Section 3 is devoted to numerical simulations. Lastly, some concluding remarks are presented in Section 4.

2. METHODS

2.1. **Variational problem.**

We define space

$$\mathbb{W} = \left\{ u : u \in L^2\left(0,T; H_0^1(\Omega)\right), u \in H^1\left(0,T; H^{-1}(\Omega)\right) \right\}.$$

Variational problem of (P_ε) is to seek $u^\varepsilon(x,t) \in \mathbb{W}$, for almost every $t \in (0,T)$, $u^\varepsilon(.,t) \in H_0^1(\Omega)$ such that:

(2.1)

$$< \Phi^\varepsilon \frac{\partial u^\varepsilon(t)}{\partial t}, v > + < K^\varepsilon \nabla u^\varepsilon, \nabla v > = < s, v >, \; \forall v \in H_0^1(\Omega), \; t \in (0,T) \text{ and } u^\varepsilon(0) = u^0.$$

Proposition 2.1. *Let $u^0 \in L^2(\Omega)$ and $s \in C\left([0,T], L^2(\Omega)\right)$. Then, for the solution $u^\varepsilon(t)$ of 2.1 the following estimate holds:*

$$\|u^\varepsilon(t)\|_{L^2(\Omega)} \;\le\; \left\|u^0\right\|_{L^2(\Omega)} e^{-\frac{\lambda t}{\Phi^- C_\Omega}} + \int_0^t \left\| \frac{s(\gamma)}{\Phi^-} \right\|_{L^2(\Omega)} e^{-\frac{\lambda(t-\gamma)}{\Phi^- C_\Omega}} d\gamma, \tag{2.2}$$

where λ and Φ^- are the constants given in the assumptions (A) and C_Ω is a constant depending only to Ω.

Proof. The proof of this proposition is inspired by ([13], page 307). Thus, the following equations are valid almost everywhere in $(0,T)$. Setting $v = u^\varepsilon(t)$, 2.1 reads as

$$< \Phi^\varepsilon \frac{\partial u^\varepsilon(t)}{\partial t}, u^\varepsilon(t) > + < K^\varepsilon \nabla u^\varepsilon, \nabla u^\varepsilon(t) > = < f, u^\varepsilon(t) > .$$

Using the relations

$$< \Phi^\varepsilon \frac{\partial u^\varepsilon(t)}{\partial t}, u^\varepsilon(t) > = \frac{1}{2}\frac{\partial}{\partial t} < \Phi^\varepsilon\, u^\varepsilon(t), u^\varepsilon(t) > + < K^\varepsilon \nabla u^\varepsilon, \nabla u^\varepsilon(t) > \ge \lambda \left\|u^\varepsilon(t)\right\|^2_{H_0^1(\Omega)}$$

and using the assumptions (A) on the function Φ^ε, it follows that

$$\Phi^- \left\| u^\varepsilon(t) \right\|_{L^2(\Omega)} \frac{\partial}{\partial t} \left\| u^\varepsilon(t) \right\|_{L^2(\Omega)} + \lambda \left\| u^\varepsilon(t) \right\|^2_{H_0^1(\Omega)} \le < f, u^\varepsilon(t) > .$$

Now, from Poincaré inequality

$$\left\| u^\varepsilon(t) \right\|_{L^2(\Omega)} \le C_\Omega \left\| u^\varepsilon(t) \right\|_{H_0^1(\Omega)} ,$$

and Cauchy-Schwarz inequality

$$< s, u^\varepsilon(t) > \le \left\| s(t) \right\|_{L^2(\Omega)} \left\| u^\varepsilon(t) \right\|_{L^2(\Omega)} ,$$

yield after division by $\left\| u^\varepsilon(t) \right\|_{L^2(\Omega)}$ the estimate

$$\Phi^- \frac{\partial}{\partial t} \left\| u^\varepsilon(t) \right\|_{L^2(\Omega)} + \frac{\lambda}{C_\Omega} \left\| u^\varepsilon(t) \right\|_{L^2(\Omega)} \le \left\| s(t) \right\|_{L^2(\Omega)} .$$

Multipling this relation by $e^{\frac{\lambda t}{\Phi^- C_\Omega}}$, the relation

$$\frac{\partial}{\partial t} \left(\Phi^- e^{\frac{\lambda t}{\Phi^- C_\Omega}} \left\| u^\varepsilon(t) \right\|_{L^2(\Omega)} \right) = \Phi^- e^{\frac{\lambda t}{\Phi^- C_\Omega}} \frac{\partial}{\partial t} \left\| u^\varepsilon(t) \right\|_{L^2(\Omega)} + e^{\frac{\lambda t}{\Phi^- C_\Omega}} \frac{\lambda}{C_\Omega} \left\| u^\varepsilon(t) \right\|_{L^2(\Omega)}$$

leads to

$$\frac{\partial}{\partial t} \left(c^- e^{\frac{\lambda t}{c^- C_\Omega}} \left\| u^\varepsilon(t) \right\|_{L^2(\Omega)} \right) \le e^{\frac{\lambda t}{c^- C_\Omega}} \left\| s(t) \right\|_{L^2(\Omega)} .$$

Integrating over $(0, t)$, and taking into consideration the initial condition, we get:

$$c^- e^{\frac{\lambda t}{c^- C_\Omega}} \left\| u^\varepsilon(t) \right\|_{L^2(\Omega)} - \Phi^- \left\| u^0 \right\|_{L^2(\Omega)} \le \int_0^t \left\| s(\gamma) \right\|_{L^2(\Omega)} e^{\frac{\lambda \gamma}{c^- C_\Omega}} d\gamma, \ \ t \in (0, t).$$

Multipling this by $\dfrac{e^{-\frac{\lambda t}{\Phi^- C_\Omega}}}{\Phi^-}$

$$\left\| u^\varepsilon(t) \right\|_{L^2(\Omega)} \le \left\| u^0 \right\|_{L^2(\Omega)} e^{-\frac{\lambda t}{c^- C_\Omega}} + \int_0^t \left\| \frac{s(\gamma)}{\Phi^-} \right\|_{L^2(\Omega)} e^{-\frac{\lambda (t-\gamma)}{\Phi^- C_\Omega}} d\gamma. \ \square$$

2.2. **Semi-discretization.** For simplification reasons, we limit ourselves in this paper to the case of a diagonal K matrix, and we consider $\Omega = (0, 1) \times (0, 1)$. We are interested in the case where K has continuous coefficients (the non-diagonal, and discontinuous coefficients cases are being processed in [4]). (P_ε) can, so be simply written as follows:

$$\text{(2.3)} \qquad \begin{cases} Q_T := \{0 < x < 1, 0 < y < 1, 0 < t \le T\}, \\ \Phi^\varepsilon(x, y) u_t^\varepsilon - \left(K_{11}^\varepsilon u_x^\varepsilon \right)_x - \left(K_{22}^\varepsilon u_y^\varepsilon \right)_y = s(x, y, t) \ \text{ in } \ Q_T, \\ u^\varepsilon(0, y, t) = u^\varepsilon(x, 0, t) = u^\varepsilon(1, y, t) = u^\varepsilon(x, 1, t) = 0, \ t \in]0, T], \\ u^\varepsilon(x, y, 0) = u^0(x, y), \ (x, y) \in]0, 1[\times]0, 1[. \end{cases}$$

By semi-discretization we mean discretization only in space, not in time. This approach is also called method of lines.

We now introduce the grid for the cell-centered finite volume scheme for which we partition $\Omega = (0, 1) \times (0, 1)$ in the x and y directions as

$$0 = x_{\frac{1}{2}} < x_1 < x_{\frac{3}{2}} < x_2 < ... < x_{i-\frac{1}{2}} < x_i < x_{N+\frac{1}{2}} = 1,$$
$$0 = y_{\frac{1}{2}} < y_1 < y_{\frac{3}{2}} < y_2 < ... < y_{i-\frac{1}{2}} < y_i < y_{N+\frac{1}{2}} = 1.$$

We then define the cells (finite volumes) to be the square:

$$V_{i,j} = \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}} \right) \times \left(y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}} \right), i = 1, ..., N, j = 1, ..., N,$$

with the center (x_i, y_j) and nodes of half indices. Let

(2.4)
$$h = x_{i+1} - x_i = y_{j+1} - y_j,\ 1 \leq i,j \leq N-1,\ h = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}},\ 1 \leq i,j \leq N.$$

The discrete cell-centered finite volume scheme uniform mesh Ω_h is then defined as follows:

$$\Omega_h = \{\ V_{i,j},\ \ 1 \leq i,j \leq N\}\,. \tag{2.5}$$

In this paper, we construct the scheme and derive the analysis on a square uniform mesh for simplicity. Some modifications are required to extend the approach to rectangular or non-rectangular meshes.

We next discretize the problem (2.3) by the finite volume scheme, only in space, not in time. In cell-centered finite volume method the unknowns approximate the average of the solution over a grid cell. More precisely, we let be $u^{\varepsilon}_{i,j}(t)$ the approximation

$$u^{\varepsilon}_{i,j}(t) := \frac{1}{h^2}\int_{V_{i,j}} u^{\varepsilon}(x,y,t)dxdy. \tag{2.6}$$

To enforce the boundary conditions across the outer boundaries, we define the fictitious boundary cells where the ghost values:

$$\left\{u^{\varepsilon}_{0,j}(t),\ u^{\varepsilon}_{N+1,j}(t),\ j=1,...,N\right\},\ \text{and}\ \left\{u^{\varepsilon}_{i,0}(t),\ u^{\varepsilon}_{i,N+1}(t),\ i=1,...,N\right\}\ \text{are defined.}$$

We set:

$$V_{0,j} = \left(x_{-\frac{1}{2}}, x_{\frac{1}{2}}\right) \times \left(y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}\right),\ V_{N+1,j} = \left(x_{N+\frac{1}{2}}, x_{N+\frac{3}{2}}\right) \times \left(y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}\right),$$

$$V_{i,0} = \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right) \times \left(y_{-\frac{1}{2}}, y_{\frac{1}{2}}\right),\ V_{i,N+1} = \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right) \times \left(y_{N+\frac{1}{2}}, y_{N+\frac{3}{2}}\right),$$

where $1 \leq i,j \leq N$ and

$$x_{-\frac{1}{2}} = -x_{\frac{3}{2}},\ \ x_{N+\frac{3}{2}} = 2 - x_{N-\frac{1}{2}},\ y_{-\frac{1}{2}} = -y_{\frac{3}{2}},\ \ y_{N+\frac{3}{2}} = 2 - y_{N-\frac{1}{2}}.$$

Let

$$\Phi^{\varepsilon}_{ij}u^{\varepsilon}_{ij}(t) := \frac{1}{h^2}\int_{V_{i,j}} \Phi^{\varepsilon}u^{\varepsilon}(x,y,t)dxdy, \tag{2.7}$$

where

$$\Phi^{\varepsilon}_{ij} := \Phi^{\varepsilon}\left(x_i, y_j\right),\ \text{and}\ u^{\varepsilon}_{ij}\ \text{ is given by}\ (2.6). \tag{2.8}$$

Using the divergence theorem to integrate the first equation of (2.3) over "volume" $V_{i,j}$, dividing by h^2 and using (2.7) we get:

(2.9)
$$\begin{aligned}\Phi^{\varepsilon}_{ij}\frac{du^{\varepsilon}_{ij}(t)}{dt} &= \frac{1}{h^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} K^{\varepsilon}_{11}(x_{i+\frac{1}{2}},y)u^{\varepsilon}_x(x_{i+\frac{1}{2}},y,t)dy - \frac{1}{h^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} K^{\varepsilon}_{11}(x_{i-\frac{1}{2}},y)u^{\varepsilon}_x(x_{i-\frac{1}{2}},y,t)dy\\ &+\frac{1}{h^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} K^{\varepsilon}_{22}(x,y_{j+\frac{1}{2}})u^{\varepsilon}_y(x,y_{j+\frac{1}{2}},t)dx - \frac{1}{h^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} K^{\varepsilon}_{22}(x,y_{j-\frac{1}{2}})u^{\varepsilon}_y(x,y_{j-\frac{1}{2}},t)dx\\ &+\frac{1}{h^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} s(x,y,t)dxdy\end{aligned}$$

The last term of the right hand side can be approximated by $s_{i,j}(t) = s\left(x_i, y_j, t\right)$.
We're only going to approximate the first integral on the right hand side in (2.9) , the others terms can be treat by the same way.

(2.10)
$$\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} K_{11}^{\varepsilon}(x_{i+\frac{1}{2}}, y) u_x^{\varepsilon}(x_{i+\frac{1}{2}}, y, t) dy \approx \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} K_{11}^{\varepsilon}(x_{i+\frac{1}{2}}, y) \left(\frac{u^{\varepsilon}\left(x_{i+1}, y, t\right) - u^{\varepsilon}\left(x_i, y, t\right)}{h}\right) dy$$
$$\approx K_{11}^{\varepsilon}(x_{i+\frac{1}{2}}, y_j) \left(u_{i+1,j}^{\varepsilon} - u_{i,j}^{\varepsilon}\right), \text{ on interior cells } (2 \leq i, j \leq N-1).$$

Note that we have made two approximations at this step. Firstly, a piecewise constant approximation to the $y-$dependence of $K_{11}^{\varepsilon}(x, y)$. Along each volume boundary we have replaced $K_{11}^{\varepsilon}(x_{i+\frac{1}{2}}, y)$ by the midpoint (in y) value $K_{11}^{\varepsilon}(x_{i+\frac{1}{2}}, y_j)$. Secondly, we have made a linear approximation to the $x-$dependence of u^{ε}. By similar reasoning, we get, on interior cells:

$$\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} K_{11}^{\varepsilon}(x_{i-\frac{1}{2}}, y) u_x^{\varepsilon}(x_{i-\frac{1}{2}}, y, t) dy \approx K_{11}^{\varepsilon}(x_{i-\frac{1}{2}}, y_j) \left(u_{i,j}^{\varepsilon} - u_{i-1,j}^{\varepsilon}\right), \tag{2.11}$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} K_{22}^{\varepsilon}(x, y_{j+\frac{1}{2}}) u_y^{\varepsilon}(x, y_{j+\frac{1}{2}}, t) dx \approx K_{22}^{\varepsilon}(x_i, y_{j+\frac{1}{2}}) \left(u_{i,j+1}^{\varepsilon} - u_{i,j}^{\varepsilon}\right) \tag{2.12}$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} K_{22}^{\varepsilon}(x, y_{j-\frac{1}{2}}) u_x^{\varepsilon}(x_{i-\frac{1}{2}}, y, t) dx \approx K_{22}^{\varepsilon}(x_i, y_{j-\frac{1}{2}}) \left(u_{i,j}^{\varepsilon} - u_{i,j-1}^{\varepsilon}\right). \tag{2.13}$$

Using (2.9)-(2.13), we get :

$$\frac{du_{ij}^{\varepsilon}(t)}{dt} = \frac{K_{11}^{\varepsilon}(x_{i+\frac{1}{2}}, y_j)}{\Phi_{ij}^{\varepsilon} h^2} \left(u_{i+1,j}^{\varepsilon} - u_{i,j}^{\varepsilon}\right) - \frac{K_{11}^{\varepsilon}(x_{i-\frac{1}{2}}, y_j)}{\Phi_{ij}^{\varepsilon} h^2} \left(u_{i,j}^{\varepsilon} - u_{i-1,j}^{\varepsilon}\right) \tag{2.14}$$
$$+ \frac{K_{22}^{\varepsilon}(x_i, y_{j+\frac{1}{2}})}{\Phi_{ij}^{\varepsilon} h^2} \left(u_{i,j+1}^{\varepsilon} - u_{i,j}^{\varepsilon}\right) - \frac{K_{22}^{\varepsilon}(x_i, y_{j-\frac{1}{2}})}{\Phi_{ij}^{\varepsilon} h^2} \left(u_{i,j}^{\varepsilon} - u_{i,j-1}^{\varepsilon}\right) + \frac{s_{ij}(t)}{\Phi_{ij}^{\varepsilon}}$$

After grouping simular terms in (2.14) we get, on interior cells ($2 \leq i, j \leq N-1$) :

(2.15)
$$\frac{du_{i,j}^{\varepsilon}(t)}{dt} = \frac{K_{11}^{\varepsilon}(x_{i-\frac{1}{2}}, y_j)}{\Phi_{ij}^{\varepsilon} h^2} u_{i-1,j}^{\varepsilon} + \frac{K_{22}^{\varepsilon}(x_i, y_{j-\frac{1}{2}})}{\Phi_{ij}^{\varepsilon} h^2} u_{i,j-1}^{\varepsilon} + \frac{K_{22}^{\varepsilon}(x_i, y_{j+\frac{1}{2}})}{\Phi_{ij}^{\varepsilon} h^2} u_{i,j+1}^{\varepsilon} + \frac{K_{11}^{\varepsilon}(x_{i+\frac{1}{2}}, y_j)}{\Phi_{ij}^{\varepsilon} h^2} u_{i+1,j}^{\varepsilon}$$
$$- \frac{K_{11}^{\varepsilon}(x_{i+\frac{1}{2}}, y_j) + K_{11}^{\varepsilon}(x_{i-\frac{1}{2}}, y_j) + K_{22}^{\varepsilon}(x_i, y_{j+\frac{1}{2}}) + K_{22}^{\varepsilon}(x_i, y_{j-\frac{1}{2}})}{\Phi_{ij}^{\varepsilon} h^2} u_{i,j}^{\varepsilon} + \frac{s_{i,j}(t)}{\Phi_{ij}^{\varepsilon}}.$$

To enforce the boundary conditions, a standard approach is to make use of ghost cells with ghost values defined as (like in ([8])):

$$u_{0,j}^{\varepsilon} = -u_{1,j}^{\varepsilon},\ u_{i,0}^{\varepsilon} = -u_{i,1}^{\varepsilon},\ u_{N+1,j}^{\varepsilon} = -u_{N,j}^{\varepsilon},\ u_{i,N+1}^{\varepsilon} = -u_{i,N}^{\varepsilon}\ 1 \leq i, j \leq N. \tag{2.16}$$

To complete the scheme (2.15), we need update formula also for the boundary cells by using (2.16). For $i = 1$, the formula (2.15) becomes:

(2.17)
$$\frac{du_{1,j}^{\varepsilon}(t)}{dt} = \frac{K_{11}^{\varepsilon}(x_{1-\frac{1}{2}}, y_j)}{\Phi_{1j}^{\varepsilon} h^2} u_{0,j}^{\varepsilon} + \frac{K_{22}^{\varepsilon}(x_1, y_{j-\frac{1}{2}})}{\Phi_{1j}^{\varepsilon} h^2} u_{1,j-1}^{\varepsilon} + \frac{K_{22}^{\varepsilon}(x_1, y_{j+\frac{1}{2}})}{\Phi_{1j}^{\varepsilon} h^2} u_{1j+1}^{\varepsilon} + \frac{K_{11}^{\varepsilon}(x_{1+\frac{1}{2}}, y_j)}{\Phi_{1j}^{\varepsilon} h^2} u_{2,j}^{\varepsilon}$$
$$- \frac{K_{11}^{\varepsilon}(x_{1+\frac{1}{2}}, y_j) + K_{11}^{\varepsilon}(x_{1-\frac{1}{2}}, y_j) + K_{22}^{\varepsilon}(x_1, y_{j+\frac{1}{2}}) + K_{22}^{\varepsilon}(x_1, y_{j-\frac{1}{2}})}{\Phi_{1j}^{\varepsilon} h^2} u_{1,j}^{\varepsilon} + \frac{s_{1,j}(t)}{\Phi_{1j}^{\varepsilon}}.$$

So, using (2.16)-(2.17), we get :

$$(2.18)\quad \frac{du^\varepsilon_{1,1}(t)}{dt} = -\frac{2K^\varepsilon_{11}(x_{1-\frac{1}{2}},y_1) + 2K^\varepsilon_{22}(x_1,y_{1-\frac{1}{2}}) + K^\varepsilon_{11}(x_{1+\frac{1}{2}},y_1) + K^\varepsilon_{22}(x_1,y_{1+\frac{1}{2}})}{\Phi^\varepsilon_{11}h^2}u^\varepsilon_{1,1}$$
$$+\frac{K^\varepsilon_{11}(x_{1+\frac{1}{2}},y_1)}{\Phi^\varepsilon_{11}h^2}u^\varepsilon_{2,1} + \frac{K^\varepsilon_{22}(x_1,y_{1+\frac{1}{2}})}{\Phi^\varepsilon_{11}h^2}u^\varepsilon_{1,2} + \frac{s_{1,1}(t)}{\Phi^\varepsilon_{11}},$$

$$(2.19)\quad \frac{du^\varepsilon_{1,j}(t)}{dt} = -\frac{K^\varepsilon_{11}(x_{1+\frac{1}{2}},y_j) + 2K^\varepsilon_{11}(x_{1-\frac{1}{2}},y_j) + K^\varepsilon_{22}(x_1,y_{j+\frac{1}{2}}) + K^\varepsilon_{22}(x_1,y_{j-\frac{1}{2}})}{\Phi^\varepsilon_{1j}h^2}u^\varepsilon_{1,j}$$
$$+\frac{K^\varepsilon_{11}(x_{1+\frac{1}{2}},y_j)}{\Phi^\varepsilon_{1j}h^2}u^\varepsilon_{2,j} + \frac{K^\varepsilon_{22}(x_1,y_{j-\frac{1}{2}})}{\Phi^\varepsilon_{1j}h^2}u^\varepsilon_{1,j-1} + \frac{K^\varepsilon_{22}(x_1,y_{j+\frac{1}{2}})}{\Phi^\varepsilon_{1j}h^2}u^\varepsilon_{1,j+1} + + \frac{s_{1,j}(t)}{\Phi^\varepsilon_{1j}},\ 2 \le j \le N-1.$$

(2.20)
$$\frac{du^\varepsilon_{1,N}(t)}{dt} = -\frac{2K^\varepsilon_{11}(x_{1-\frac{1}{2}},y_N) + 2K^\varepsilon_{22}(x_1,y_{N+\frac{1}{2}}) + K^\varepsilon_{11}(x_{1+\frac{1}{2}},y_N) + K^\varepsilon_{22}(x_1,y_{N-\frac{1}{2}})}{c^\varepsilon_{1,N}h^2}u^\varepsilon_{1,N}$$
$$+\frac{K^\varepsilon_{22}(x_1,y_{N-\frac{1}{2}})}{\Phi^\varepsilon_{1N}h^2}u^\varepsilon_{1,N-1} + \frac{K^\varepsilon_{11}(x_{1+\frac{1}{2}},y_N)}{\Phi^\varepsilon_{1N}h^2}u^\varepsilon_{2,N} + \frac{s_{1,N}(t)}{\Phi^\varepsilon_{1N}}.$$

For $i = N$, the formula (2.15) becomes:

$$(2.21)\quad \frac{du^\varepsilon_{N,j}(t)}{dt} = \frac{K^\varepsilon_{11}(x_{N-\frac{1}{2}},y_j)}{\Phi^\varepsilon_{Nj}h^2}u^\varepsilon_{N-1,j} + \frac{K^\varepsilon_{22}(x_N,y_{j-\frac{1}{2}})}{\Phi^\varepsilon_{Nj}h^2}u^\varepsilon_{N,j-1}$$
$$-\frac{K^\varepsilon_{11}(x_{N+\frac{1}{2}},y_j) + K^\varepsilon_{11}(x_{N-\frac{1}{2}},y_j) + K^\varepsilon_{22}(x_N,y_{j+\frac{1}{2}}) + K^\varepsilon_{22}(x_N,y_{j-\frac{1}{2}})}{\Phi^\varepsilon_{Nj}h^2}u^\varepsilon_{N,j} + \frac{s_{N,j}(t)}{\Phi^\varepsilon_{Nj}}$$
$$+\frac{K^\varepsilon_{22}(x_N,y_{j+\frac{1}{2}})}{\Phi^\varepsilon_{Nj}h^2}u^\varepsilon_{N,j+1} + \frac{K^\varepsilon_{11}(x_{N+\frac{1}{2}},y_j)}{\Phi^\varepsilon_{Nj}h^2}u^\varepsilon_{N+1,j}.$$

So, using (2.16), we get :

(2.22)
$$\frac{du^\varepsilon_{N,1}(t)}{dt} = -\frac{2K^\varepsilon_{11}(x_{N+\frac{1}{2}},y_1) + 2K^\varepsilon_{22}(x_N,y_{1-\frac{1}{2}}) + K^\varepsilon_{11}(x_{N-\frac{1}{2}},y_1) + K^\varepsilon_{22}(x_N,y_{1+\frac{1}{2}})}{\Phi^\varepsilon_{N1}h^2}u^\varepsilon_{N,1}$$
$$+\frac{K^\varepsilon_{11}(x_{N-\frac{1}{2}},y_1)}{\Phi^\varepsilon_{N,1}h^2}u^\varepsilon_{N-1,1} + \frac{K^\varepsilon_{22}(x_N,y_{1+\frac{1}{2}})}{\Phi^\varepsilon_{N1}h^2}u^\varepsilon_{N,2} + \frac{s_{N,1}(t)}{\Phi^\varepsilon_{N1}}.$$

(2.23)
$$\frac{du^\varepsilon_{N,N}(t)}{dt} = -\frac{2K^\varepsilon_{22}(x_N,y_{N+\frac{1}{2}}) + 2K^\varepsilon_{11}(x_{N+\frac{1}{2}},y_N) + K^\varepsilon_{11}(x_{N-\frac{1}{2}},y_N) + k^\varepsilon_{22}(x_N,y_{N-\frac{1}{2}})}{\Phi^\varepsilon_{NN}h^2}u^\varepsilon_{N,N}$$
$$+\frac{K^\varepsilon_{11}(x_{N-\frac{1}{2}},y_N)}{\Phi^\varepsilon_{N,N}h^2}u^\varepsilon_{N-1,N} + \frac{K^\varepsilon_{22}(x_N,y_{N-\frac{1}{2}})}{\Phi^\varepsilon_{NN}h^2}u^\varepsilon_{N,N-1} + \frac{s_{N,N}(t)}{\Phi^\varepsilon_{NN}}.$$

(2.24)
$$\frac{du^\varepsilon_{N,j}(t)}{dt} = -\frac{2K^\varepsilon_{11}(x_{N+\frac{1}{2}},y_j) + K^\varepsilon_{11}(x_{i-\frac{1}{2}},y_j) + K^\varepsilon_{22}(x_N,y_{j+\frac{1}{2}}) + K^\varepsilon_{22}(x_N,y_{j-\frac{1}{2}})}{\Phi^\varepsilon_{Nj}h^2}u^\varepsilon_{N,j}$$
$$+\frac{k^\varepsilon_{11}(x_{N-\frac{1}{2}},y_j)}{\Phi^\varepsilon_{Nj}h^2}u^\varepsilon_{N-1,j} + \frac{K^\varepsilon_{22}(x_N,y_{j-\frac{1}{2}})}{\Phi^\varepsilon_{Nj}h^2}u^\varepsilon_{N,j-1} + \frac{k^\varepsilon_{22}(x_N,y_{j+\frac{1}{2}})}{\Phi^\varepsilon_{Nj}h^2}u^\varepsilon_{N,j+1} + \frac{s_{N,j}(t)}{\Phi^\varepsilon_{Nj}},\ 2 \le j \le N-1.$$

for $j = 1$ and $2 \leq i \leq N-1$

$$\frac{du^{\varepsilon}_{i,1}(t)}{dt} = -\frac{K^{\varepsilon}_{11}(x_{i+\frac{1}{2}}, y_1) + K^{\varepsilon}_{11}(x_{i-\frac{1}{2}}, y_1) + K^{\varepsilon}_{22}(x_i, y_{1+\frac{1}{2}}) + 2K^{\varepsilon}_{22}(x_i, y_{1-\frac{1}{2}})}{\Phi^{\varepsilon}_{i1} h^2} u^{\varepsilon}_{i,1} \tag{2.25}$$

$$+\frac{K^{\varepsilon}_{11}(x_{i-\frac{1}{2}}, y_1)}{\Phi^{\varepsilon}_{i1} h^2} u^{\varepsilon}_{i-1,1} + \frac{K^{\varepsilon}_{22}(x_i, y_{1+\frac{1}{2}})}{\Phi^{\varepsilon}_{i1} h^2} u^{\varepsilon}_{i,2} + \frac{K^{\varepsilon}_{11}(x_{i+\frac{1}{2}}, y_1)}{\Phi^{\varepsilon}_{i1} h^2} u^{\varepsilon}_{i+1,1} + \frac{s_{i,1}(t)}{\Phi^{\varepsilon}_{i1}}.$$

for $j = N$ and $2 \leq i \leq N-1$

$$\frac{du^{\varepsilon}_{i,N}(t)}{dt} = -\frac{K^{\varepsilon}_{11}(x_{i+\frac{1}{2}}, y_N) + K^{\varepsilon}_{11}(x_{i-\frac{1}{2}}, y_N) + 2k^{\varepsilon}_{22}(x_i, y_{N+\frac{1}{2}}) + K^{\varepsilon}_{22}(x_i, y_{N-\frac{1}{2}})}{\Phi^{\varepsilon}_{iN} h^2} u^{\varepsilon}_{i,N} \tag{2.26}$$

$$+\frac{k^{\varepsilon}_{11}(x_{i-\frac{1}{2}}, y_N)}{\Phi^{\varepsilon}_{iN} h^2} u^{\varepsilon}_{i-1,N} + \frac{K^{\varepsilon}_{22}(x_i, y_{N-\frac{1}{2}})}{\Phi^{\varepsilon}_{iN} h^2} u^{\varepsilon}_{i,N-1} + \frac{k^{\varepsilon}_{11}(x_{i+\frac{1}{2}}, y_N)}{\Phi^{\varepsilon}_{iN} h^2} u^{\varepsilon}_{i+1,N} + \frac{s_{i,N}(t)}{\Phi^{\varepsilon}_{iN}}.$$

from (2.15), (2.18), (2.19), (2.20),(2.22), (2.23), (2.25) and (2.26), we get the following ODE

$$\begin{cases} \frac{du^{\varepsilon h}(t)}{dt} = K^{\varepsilon h} u^{\varepsilon h}(t) + s^{\varepsilon h}(t) \text{ in } \Omega_h \\ u^{\varepsilon h}(0) = u^0 \end{cases} \tag{2.27}$$

where

$$u^{\varepsilon h}(t) = \begin{bmatrix} u^{\varepsilon}_{1,1}(t) \\ u^{\varepsilon}_{2,1}(t) \\ . \\ . \\ . \\ u^{\varepsilon}_{N-1,N}(t) \\ u^{\varepsilon}_{N,N}(t) \end{bmatrix}, \quad s^{\varepsilon h}(t) = \begin{bmatrix} \frac{s_{1,1}(t)}{\Phi^{\varepsilon}_{11}} \\ \frac{s_{2,1}(t)}{\Phi^{\varepsilon}_{21}} \\ . \\ . \\ . \\ \frac{s_{N-1,N}(t)}{\Phi^{\varepsilon}_{N-1N}} \\ \frac{s_{N,N}(t)}{\Phi^{\varepsilon}_{NN}} \end{bmatrix}, \tag{2.28}$$

$K^{\varepsilon h}$ is a square matrix of order N^2, given as follows:
We count all the cells $V_{i,j} (1 \leq i, j \leq N)$ in a counterclockwise direction. If l denotes the one or the other cells number, then we have $1 \leq l \leq N^2$, and the coefficients $K^{\varepsilon,h}_{ll}$ of the matrix

$K^{\varepsilon,h}$ are given as follows:

(2.29)

$$\begin{cases} -\dfrac{2K_{11}^{\varepsilon}(x_{l-\frac{1}{2}},y_l)+2K_{22}^{\varepsilon}(x_l,y_{l-\frac{1}{2}})+K_{11}^{\varepsilon}(x_{l+\frac{1}{2}},y_l)+K_{22}^{\varepsilon}(x_l,y_{l+\frac{1}{2}})}{\Phi_{ll}^{\varepsilon}h^2}, l=1,\\ -\dfrac{K_{11}^{\varepsilon}(x_{i-\frac{1}{2}},y_j)+K_{11}^{\varepsilon}(x_{i+\frac{1}{2}},y_j)+K_{22}^{\varepsilon}(x_i,y_{j+\frac{1}{2}})+K_{22}^{\varepsilon}(x_i,y_{j-\frac{1}{2}})}{\Phi_{ij}^{\varepsilon}h^2}, 2\le i,j\le N-1,\\ -\dfrac{K_{11}^{\varepsilon}(x_{i-\frac{1}{2}},y_1)+K_{11}^{\varepsilon}(x_{i+\frac{1}{2}},y_1)+K_{22}^{\varepsilon}(x_i,y_{1+\frac{1}{2}})+2K_{22}^{\varepsilon}(x_i,y_{1-\frac{1}{2}})}{\Phi_{i1}^{\varepsilon}h^2}, 2\le i\le N-1,\\ -\dfrac{K_{11}^{\varepsilon}(x_{i-\frac{1}{2}},y_N)+K_{11}^{\varepsilon}(x_{i+\frac{1}{2}},y_N)+K_{22}^{\varepsilon}(x_i,y_{N-\frac{1}{2}})+2K_{22}^{\varepsilon}(x_i,y_{N+\frac{1}{2}})}{\Phi_{iN}^{\varepsilon}h^2}, 2\le i\le N-1,\\ -\dfrac{K_{11}^{\varepsilon}(x_{1+\frac{1}{2}},y_j)+2K_{11}^{\varepsilon}(x_{1-\frac{1}{2}},y_j)+K_{22}^{\varepsilon}(x_1,y_{j+\frac{1}{2}})+K_{22}^{\varepsilon}(x_1,y_{j-\frac{1}{2}})}{\Phi_{1j}^{\varepsilon}h^2}, 2\le j\le N-1,\\ -\dfrac{2K_{11}^{\varepsilon}(x_{N+\frac{1}{2}},y_j)+K_{11}^{\varepsilon}(x_{N-\frac{1}{2}},y_j)+K_{22}^{\varepsilon}(x_N,y_{j+\frac{1}{2}})+K_{22}^{\varepsilon}(x_N,y_{j-\frac{1}{2}})}{\Phi_{Nj}^{\varepsilon}h^2}, 2\le j\le N-1,\\ -\dfrac{2K_{11}^{\varepsilon}(x_{i-\frac{1}{2}},y_j)+2K_{22}^{\varepsilon}(x_i,y_{j+\frac{1}{2}})+K_{11}^{\varepsilon}(x_{i+\frac{1}{2}},y_j)+K_{22}^{\varepsilon}(x_i,y_{j-\frac{1}{2}})}{\Phi_{ij}^{\varepsilon}h^2}, i=1,j=N,\\ -\dfrac{2K_{11}^{\varepsilon}(x_{i+\frac{1}{2}},y_j)+2K_{22}^{\varepsilon}(x_i,y_{j-\frac{1}{2}})+K_{11}^{\varepsilon}(x_{i-\frac{1}{2}},y_1)+K_{22}^{\varepsilon}(x_i,y_{j+\frac{1}{2}})}{\Phi_{ij}^{\varepsilon}h^2}, i=N,j=1,\\ -\dfrac{2K_{22}^{\varepsilon}(x_i,y_{j+\frac{1}{2}})+2K_{11}^{\varepsilon}(x_{i+\frac{1}{2}},y_j)+K_{11}^{\varepsilon}(x_{i-\frac{1}{2}},y_j)+K_{22}^{\varepsilon}(x_i,y_{j-\frac{1}{2}})}{\Phi_{ij}^{\varepsilon}h^2}, i=N,j=N, \end{cases}$$

$$\begin{cases} K_{1\,2}^{\varepsilon,h}=\dfrac{K_{11}^{\varepsilon}(x_{1+\frac{1}{2}},y_1)}{\Phi_{11}^{\varepsilon}h^2},\ K_{1\,1+N}^{\varepsilon,h}=\dfrac{K_{22}^{\varepsilon}(x_1,y_{1+\frac{1}{2}})}{\Phi_{11}^{\varepsilon}h^2},\\ K_{l\,l-1}^{\varepsilon,h}=\dfrac{K_{11}^{\varepsilon}(x_{l-\frac{1}{2}},y_1)}{\Phi_{l1}^{\varepsilon}h^2},\ K_{l\,l+N}^{\varepsilon,h}=\dfrac{K_{22}^{\varepsilon}(x_l,y_{1+\frac{1}{2}})}{\Phi_{l1}^{\varepsilon}h^2},\ l=N,\\ K_{l\,l-1}^{\varepsilon,h}=\dfrac{K_{11}^{\varepsilon}(x_{l-\frac{1}{2}},y_1)}{\Phi_{l1}^{\varepsilon}h^2},\ K_{l\,l+1}^{\varepsilon,h}=\dfrac{K_{11}^{\varepsilon}(x_{l+\frac{1}{2}},y_1)}{\Phi_{l1}^{\varepsilon}h^2}, 2\le l\le N-1,\\ K_{l\,l+N}^{\varepsilon,h}=\dfrac{K_{22}^{\varepsilon}(x_l,y_{1+\frac{1}{2}})}{\Phi_{l1}^{\varepsilon}h^2},\ 2\le l\le N-1,\\ K_{l\,l-1}^{\varepsilon,h}=\dfrac{K_{11}^{\varepsilon}(x_{i-\frac{1}{2}},y_N)}{\Phi_{iN}^{\varepsilon}h^2},\ K_{l\,l+1}^{\varepsilon,h}=\dfrac{K_{11}^{\varepsilon}(x_{i+\frac{1}{2}},y_N)}{\Phi_{iN}^{\varepsilon}h^2}, 2\le i\le N-1,\\ K_{l\,l-N}^{\varepsilon,h}=\dfrac{K_{22}^{\varepsilon}(x_i,y_{N-\frac{1}{2}})}{\Phi_{iN}^{\varepsilon}h^2},\ 2\le i\le N-1,\\ K_{l\,l-N}^{\varepsilon,h}=\dfrac{K_{22}^{\varepsilon}(x_1,y_{j-\frac{1}{2}})}{\Phi_{1j}^{\varepsilon}h^2},\ K_{l\,l+N}^{\varepsilon,h}=\dfrac{K_{22}^{\varepsilon}(x_1,y_{j+\frac{1}{2}})}{\Phi_{1j}^{\varepsilon}h^2}, 2\le j\le N-1,\\ K_{l\,l+1}^{\varepsilon,h}=\dfrac{K_{11}^{\varepsilon}(x_{1+\frac{1}{2}},y_j)}{\Phi_{1j}^{\varepsilon}h^2},\ 2\le j\le N-1, \end{cases} \tag{2.30}$$

$$(2.31)\quad \begin{cases} K^{\varepsilon,h}_{l\,l-N} = \dfrac{K^{\varepsilon}_{22}(x_N, y_{j-\frac{1}{2}})}{\Phi^{\varepsilon}_{Nj}h^2},\ K^{\varepsilon,h}_{l\,l+N} = \dfrac{K^{\varepsilon}_{22}(x_N, y_{j+\frac{1}{2}})}{\Phi^{\varepsilon}_{Nj}h^2}, 2 \le j \le N-1, \\ K^{\varepsilon,h}_{l\,l-1} = \dfrac{K^{\varepsilon}_{11}(x_{N-\frac{1}{2}}, y_j)}{\Phi^{\varepsilon}_{Nj}h^2},\ 2 \le j \le N-1, \\ K^{\varepsilon,h}_{l\,l-1} = \dfrac{K^{\varepsilon}_{11}(x_{i-\frac{1}{2}}, y_j)}{\Phi^{\varepsilon}_{ij}h^2}, 2 \le i,j \le N-1,\ N+2 \le l \le N^2-1, \\ K^{\varepsilon,h}_{l\,l+1} = \dfrac{K^{\varepsilon}_{11}(x_{i+\frac{1}{2}}, y_j)}{\Phi^{\varepsilon}_{ij}h^2},\ 2 \le i,j \le N-1,\ N+2 \le l \le N^2-1, \\ K^{\varepsilon,h}_{l\,l-N} = \dfrac{K^{\varepsilon}_{22}(x_i, y_{j-\frac{1}{2}})}{\Phi^{\varepsilon}_{ij}h^2}, 2 \le i,j \le N-1,\ N+2 \le l \le N^2-1, \\ K^{\varepsilon,h}_{l\,l+N} = \dfrac{K^{\varepsilon}_{22}(x_i, y_{j+\frac{1}{2}})}{\Phi^{\varepsilon}_{ij}h^2},\ 2 \le i,j \le N-1,\ N+2 \le l \le N^2-1, \\ K^{\varepsilon,h}_{l\,l-N} = \dfrac{K^{\varepsilon}_{22}(x_i, y_{j-\frac{1}{2}})}{\Phi^{\varepsilon}_{ij}h^2},\ K^{\varepsilon,h}_{l\,l+1} = \dfrac{K^{\varepsilon}_{11}(x_{i+\frac{1}{2}}, y_j)}{\Phi^{\varepsilon}_{ij}h^2},\ i=1, j=N, \\ K^{\varepsilon,h}_{N^2\,N^2-N} = \dfrac{K^{\varepsilon}_{22}(x_N, y_{N-\frac{1}{2}})}{\Phi^{\varepsilon}_{N,N}h^2},\ K^{\varepsilon,h}_{N^2\,N^2-1} = \dfrac{K^{\varepsilon}_{11}(x_{N-\frac{1}{2}}, y_N)}{\Phi^{\varepsilon}_{NN}h^2}. \end{cases}$$

2.3. **Full discretization.** The system (2.27) can be solved by different methods. In this paper we examine the numerical scheme corresponding to implicit Euler scheme. We note $\Delta t = \dfrac{T}{M}$ $(M > 0)$, the time step. The discretization of (2.27) by the implicit Euler scheme is given by the relation:

(2.32)
$$u^{\varepsilon h}_{n+1} = u^{\varepsilon h}_{n} + \Delta t\left[K^{\varepsilon h}u^{\varepsilon h}_{n+1} + s^{\varepsilon h}_{n+1}\right],\ u^{\varepsilon h}_{n} \approx u^{\varepsilon h}(t_n),\ \ s^{\varepsilon h}_{n} \approx s^{\varepsilon h}(t_n),\ t_n = n\Delta t,\ n = 0, ..., M.$$

Which equals:

$$(2.33)\qquad \left(I - \Delta t\, K^{\varepsilon h}\right) u^{\varepsilon h}_{n+1} = u^{\varepsilon h}_{n} + \Delta t\, s^{\varepsilon h}_{n+1}.$$

Proposition 2.2. *The implicit Euler scheme (2.33) is L^{∞}(endowed with the discrete norm)$stable$.*

Proof. We write the scheme (2.32) in the following form for the inner points $(2 \le i, j \le N-1)$ by using (2.15):

$$(2.34)\quad \frac{u^{\varepsilon}_{i,j}(n+1) - u^{\varepsilon}_{i,j}(n)}{\Delta t} = \frac{K^{\varepsilon}_{11}(x_{i-\frac{1}{2}}, y_j)}{\Phi^{\varepsilon}_{ij}h^2}u^{\varepsilon}_{i-1,j}(n+1) + \frac{K^{\varepsilon}_{22}(x_i, y_{j-\frac{1}{2}})}{\Phi^{\varepsilon}_{ij}h^2}u^{\varepsilon}_{i,j-1}(n+1)$$
$$+\frac{K^{\varepsilon}_{22}(x_i, y_{j+\frac{1}{2}})}{\Phi^{\varepsilon}_{ij}h^2}u^{\varepsilon}_{i,j+1}(n+1) + \frac{K^{\varepsilon}_{11}(x_{i+\frac{1}{2}}, y_j)}{\Phi^{\varepsilon}_{ij}h^2}u^{\varepsilon}_{i+1,j}(n+1)$$
$$-\frac{K^{\varepsilon}_{11}(x_{i+\frac{1}{2}}, y_j) + K^{\varepsilon}_{11}(x_{i-\frac{1}{2}}, y_j) + K^{\varepsilon}_{22}(x_i, y_{j+\frac{1}{2}}) + K^{\varepsilon}_{22}(x_i, y_{j-\frac{1}{2}})}{\Phi^{\varepsilon}_{ij}h^2}u^{\varepsilon}_{i,j}(n+1) + \frac{s_{i,j}(n+1)}{\Phi^{\varepsilon}_{ij}}.$$

(2.35)
$$\left[1 + \frac{\Delta t}{\Phi^{\varepsilon}_{ij}h^2}\left(K^{\varepsilon}_{11}(x_{i+\frac{1}{2}}, y_j) + K^{\varepsilon}_{11}(x_{i-\frac{1}{2}}, y_j) + K^{\varepsilon}_{22}(x_i, y_{j+\frac{1}{2}}) + K^{\varepsilon}_{22}(x_i, y_{j-\frac{1}{2}})\right)\right] u^{\varepsilon}_{i,j}(n+1)$$

$$-\frac{K^{\varepsilon}_{11}(x_{i-\frac{1}{2}},y_j)\Delta t}{\Phi^{\varepsilon}_{ij}h^2}u^{\varepsilon}_{i-1,j}(n+1)-\frac{K^{\varepsilon}_{22}(x_i,y_{j-\frac{1}{2}})\Delta t}{\Phi^{\varepsilon}_{ij}h^2}u^{\varepsilon}_{i,j-1}(n+1)$$

$$-\frac{K^{\varepsilon}_{22}(x_i,y_{j+\frac{1}{2}})\Delta t}{\Phi^{\varepsilon}_{ij}h^2}u^{\varepsilon}_{i,j+1}(n+1)-\frac{K^{\varepsilon}_{11}(x_{i+\frac{1}{2}},y_j)\Delta t}{\Phi^{\varepsilon}_{ij}h^2}u^{\varepsilon}_{i+1,j}(n+1)=u^{\varepsilon}_{i,j}(n)+\frac{\Delta t}{\Phi^{\varepsilon}_{ij}}s^{n+1}_{i,j}$$

$$2\leq i,j\leq N-1.$$

We are going to prove that:

$$\max_{1\leq i,j\leq N}u^{\varepsilon}_{i,\,j}(n)\leq\max_{1\leq i,j\leq N}u^{\varepsilon}_{i,\,j}(0)+\Delta t\sum_{p=0}^{n-1}\max_{1\leq i,j\leq N}\left(\frac{s^{p}_{i,j}}{\Phi^{\varepsilon}_{ij}}\right),\tag{2.36}$$

and

$$\min_{1\leq i,j\leq N}u^{\varepsilon}_{i,\,j}(n)\geq\min_{1\leq i,j\leq N}u^{\varepsilon}_{i,\,j}(0)+\Delta t\sum_{p=0}^{n-1}\min_{1\leq i,j\leq N}\left(\frac{s^{p}_{i,j}}{\Phi^{\varepsilon}_{ij}}\right).\tag{2.37}$$

Let i_0 and j_0 such as $u^{\varepsilon}_{i_0,\,j_0}(n+1)=\max\limits_{2\leq i,j\leq N-1}u^{\varepsilon}_{i,\,j}(n+1)$.

Then $u^{\varepsilon}_{i_0,\,j_0}(n+1)-u^{\varepsilon}_{i,\,j}\geq 0$, for $i=i_0\pm 1$ and $j=j_0\pm 1$. So, from (2.35), we have

$$u^{\varepsilon}_{i_0,\,j_0}(n+1)\leq u^{\varepsilon}_{i_0,\,j_0}(n)+\frac{\Delta t}{\Phi^{\varepsilon}_{i_0j_0}}s^{n+1}_{i_0,j_0}.$$

In particular

$$\max_{2\leq i,j\leq N-1}u^{\varepsilon}_{i,\,j}(n+1)\leq\max_{2\leq i,j\leq N-1}u^{\varepsilon}_{i,\,j}(n)+\Delta t\max_{2\leq i\leq N-1}\left(\frac{s^{n+1}_{i,j}}{\Phi^{\varepsilon}_{ij}}\right).\tag{2.38}$$

By recurrence and from (2.38) we get

$$\max_{2\leq i,j\leq N-1}u^{\varepsilon}_{i,\,j}(n)\leq\max_{2\leq i,j\leq N-1}u^{\varepsilon}_{i,\,j}(0)+\Delta t\sum_{p=1}^{n}\max_{2\leq i,j\leq N-1}\left(\frac{s^{p}_{i,j}}{\Phi^{\varepsilon}_{ij}}\right).\tag{2.39}$$

By adopting the same approach with border cells (by using relationships like (2.17) and (2.21) and using (2.39), we get (2.36). The proof of (2.37) can be done by simular way. Hence the L^{∞} stability of the scheme (2.33) is proven.

Remark 2.1. One can easily show that the matrix $\left(I-\Delta t\,K^{\varepsilon h}\right)$ of the linear system (2.33) is invertible. It suffices in fact to show that the kernel of the associated linear application is zero.

3. Numerical simumlations

In this section, we are going to present numerical experiments obtained by using the implicit scheme (2.33), and compare them with the homogenized problem solution (1.1). Two test examples are considered and the source functions $s(x,t)$ are chosen so as to have an analytical solution of the homogenized problem.

3.1. **Test Problem 1.**

The first test problem involves simulations with the coefficients:

$$K_{11}(x,y) = K_{22}(x,y) = (2+\sin(2\pi x))\,(4+\sin(2\pi y))\,, \quad (x,y) \in\,]0,1[\,\times\,]0,1[\,,$$

$$\Phi(x,y) = \frac{1}{(1+0.5\sin(2\pi x))\,(1+0.5\sin(2\pi y))}. \text{ So}$$

$$K^*_{11} = 4\sqrt{3}, \quad K^*_{22} = 2\sqrt{15}, \quad K^*_{12} = 0, \quad \text{and} \quad \Phi^* \approx 1.3333.$$

consider $T = 1$ and the following homogenized solution: $u(x,y,t) = \sin(\pi t/2)\sin(\pi x)\sin(\pi y)$.

The source function and initial are respectively: $s(x,y,t) = \Phi^*(0.5)\cos(0.5\pi t)(\pi)\sin(\pi x)\sin(\pi y)+$ $K^*_{11}\sin(0.5\pi t)\sin(\pi x)(\pi)^2\sin(\pi y)$ $+K^*_{22}\sin(0.5\pi t)\sin(\pi x)(\pi)^2\sin(\pi y)$ and $u^0(x,y) = 0$.

Below, in Figure 1 we represent the coefficients of the matrix K^ε and the porosity Φ^ε, for $\varepsilon = 9810^{-4}$.

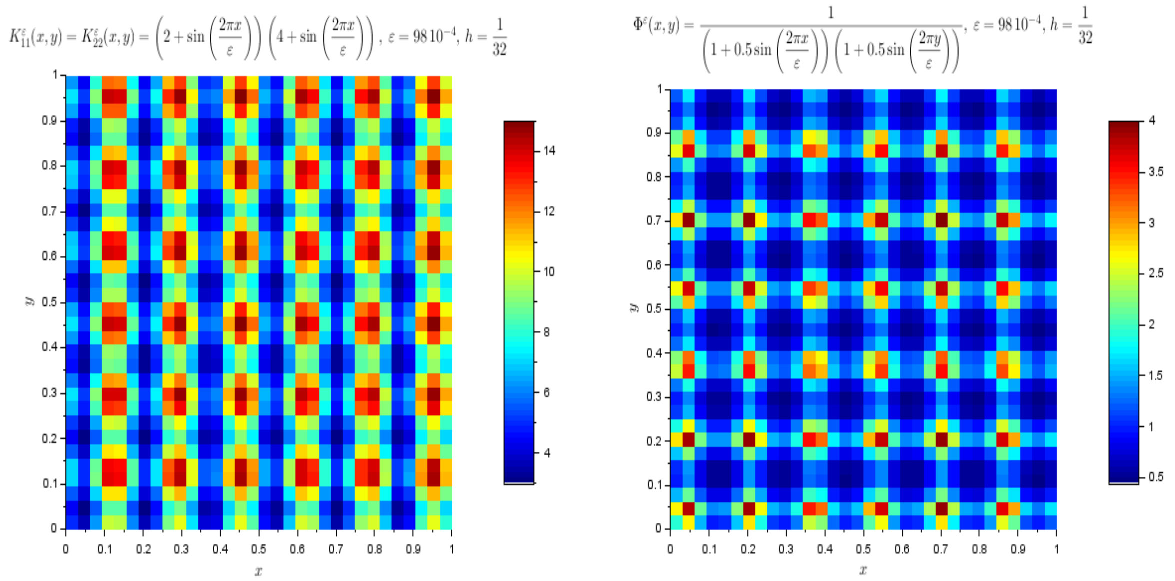

Figure 1: Test Problem 1: Coefficients K^ε and Φ^ε.

In the two error tables below (Table 3.1 and 3.2), and also the four graphs in Figures 2 and 3 we show a convergence of the numerical solution towards the homogenized solution, when ε is fixed and that h tends towards zero, with however a better convergence when $\varepsilon = 9810^{-4}$. Figure 4 is an example which confirms the stability of Euler's implicit scheme (2.33) for a fixed ε.

$\varepsilon = 98\,10^{-4}$, $\Delta t = 0.1$	$h = 0.125$	$h = 0.0625$	$h = 0.03125$	$h = 0.015625$
$\left\|u^{\varepsilon h} - u\right\|_{\infty,h}$	1.594223e-01	7.615822e-02	7.607865e-02	2.032251e-02
$\left\|u^{\varepsilon h} - u\right\|_{0,h}$	7.232259e-02	2.365037e-02	2.227364e-02	5.864612e-03
$\left\|u^{h} - u\right\|_{\infty,h}$	1.133885e-02	2.059973e-03	3.286629e-04	9.301633e-04
$\left\|u^{h} - u\right\|_{0,h}$	5.893743e-03	1.039978e-03	1.647280e-04	4.653619e-04

Table 3.1: Test Problem 1: Error table at the final time $(T = 1)$ for a "small" value of ε.

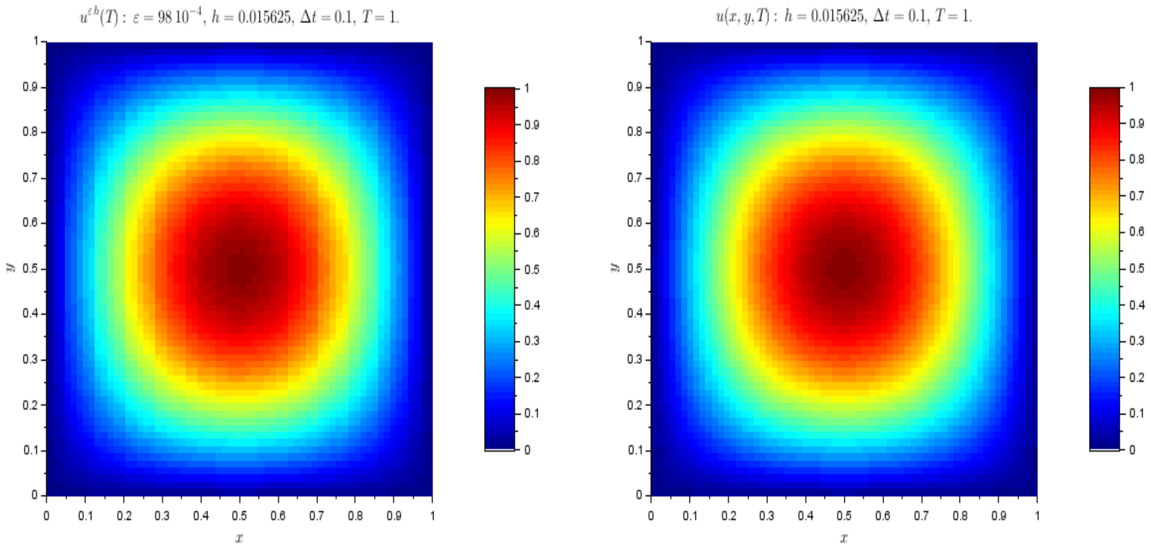

Figure 2: Test Problem 1: Numerical solution $u^{\varepsilon h}(x, y, T)$ and exact homogenized solution $u(x, y, T)$.

$\varepsilon = 98\,10^{-2}$, $\Delta t = 0.1$	$h = 0.125$	$h = 0.0625$	$h = 0.03125$	$h = 0.015625$
$\left\|u^{\varepsilon h} - u\right\|_{\infty,h}$	2.046393e-01	1.926291e-01	1.988044e-01	2.007435e-01
$\left\|u^{\varepsilon h} - u\right\|_{0,h}$	6.957719e-02	6.814815e-02	6.782822e-02	6.774657e-02

Table 3.2: Test Problem 1: Error table at the final time $(T = 1)$ for a "large" value of ε.

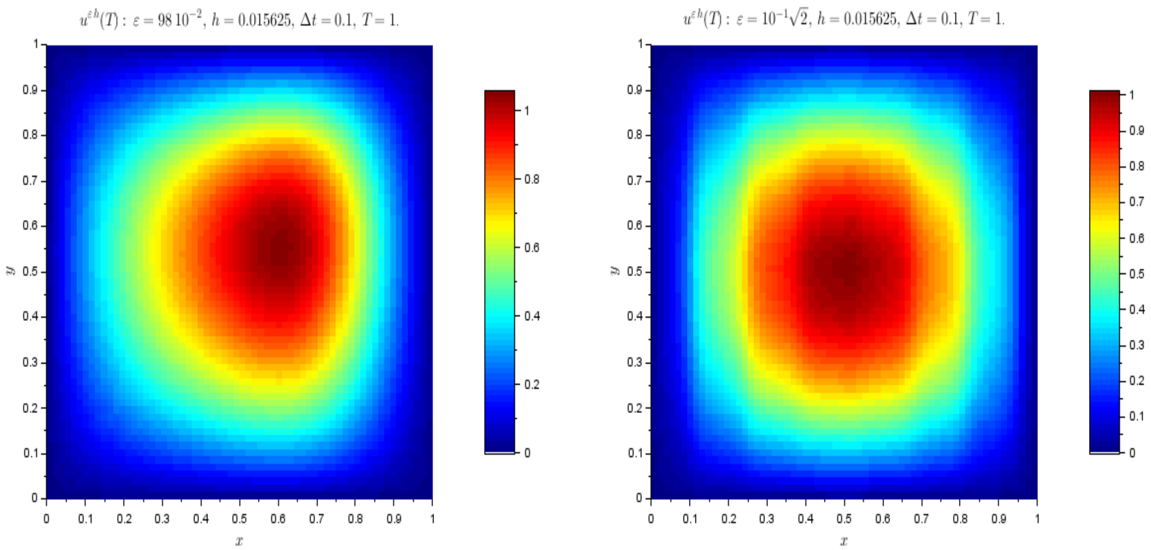

Figure 3: Test Problem 1: Numerical solution $u^{\varepsilon h}(x, y, T)$ for two different ε's values.

When h is fixed, the numerical solution tends towards the homogenized solution, when ε tends towards 0, as shown in Table 3.3.

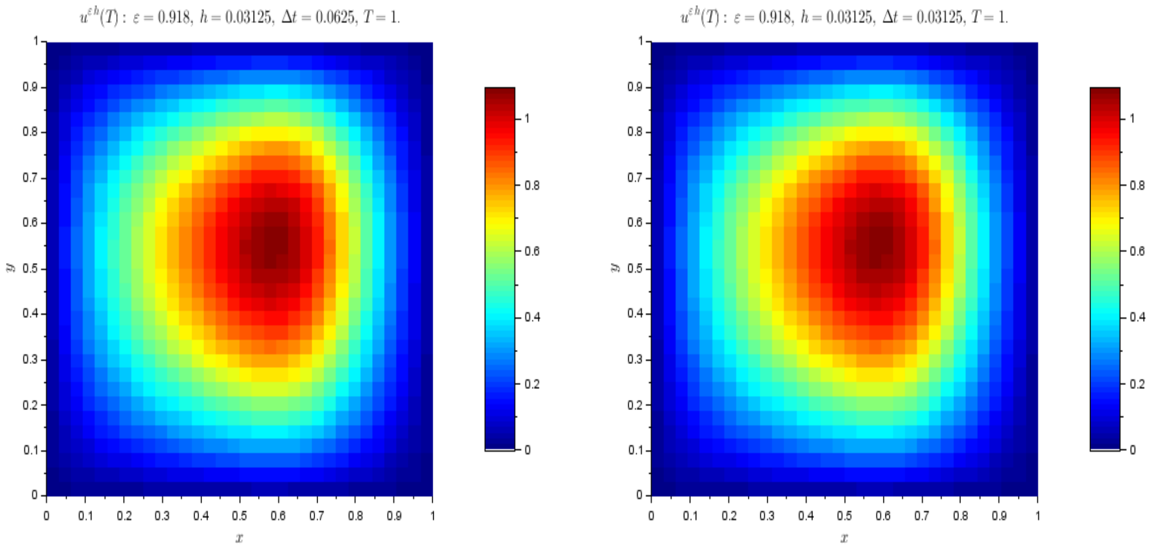

Figure 4: Test Problem 1: Numerical solution $u^{\varepsilon h}(x, y, T)$ for two different Δt's values.

$h = 0.015625,\ \Delta t = 0.1$	$\varepsilon = 10^{-1}\sqrt{2}$	$\varepsilon = 10^{-5}\sqrt{2}$	$\varepsilon = 10^{-10}\sqrt{2}$
$\left\|u^{\varepsilon h} - u\right\|_{\infty,h}$	6.437710e-02	4.638636e-02	2.333415e-02
$\left\|u^{\varepsilon h} - u\right\|_{0,h}$	1.845973e-02	1.227500e-02	6.023618e-03

Table 3.3: Test Problem 1: Error table at the final time $(T = 1)$ when ε decreases.

3.2. **Test Problem 2.** The second test problem involves simulations with the coefficients:

$$K_{11}(x, y) = K_{22}(x, y) = 2.1 + 2\sin(2\pi(x - y)), \quad (x, y) \in]0, 1[\times]0, 1[\,,$$

$$\Phi(x, y) = (1 + 0.5\sin(2\pi x))\,(1 + 0.5\sin(2\pi y))\,. \text{ So}$$

$$K^*_{11} = K^*_{22} = (0.64 + 2.1)/2., \quad K^*_{12} = -(0.64 - 2.1)/2., \text{ and } \Phi^* = 1.$$

We consider $T = 1$ and the following homogenized solution: $u(x, y, t) = (t + 1)(x - 1)(y - 1)xy$. The source function and initial are respectively:

$$s(x, y, t) = u(x, y, 1) - 2yK^*_{11}(t+1)(y-1) - 2xK^*_{22}(t+1)(x-1) - 2K^*_{12}(t+1)(4xy - 2x - 2y + 1),$$

and $u^0(x, y) = (x - 1)(y - 1)xy$.

Below, in Figure 5 we represent the coefficients of the matrix K^ε and the porosity Φ^ε, for $\varepsilon = 10^{-1}\sqrt{2}$.

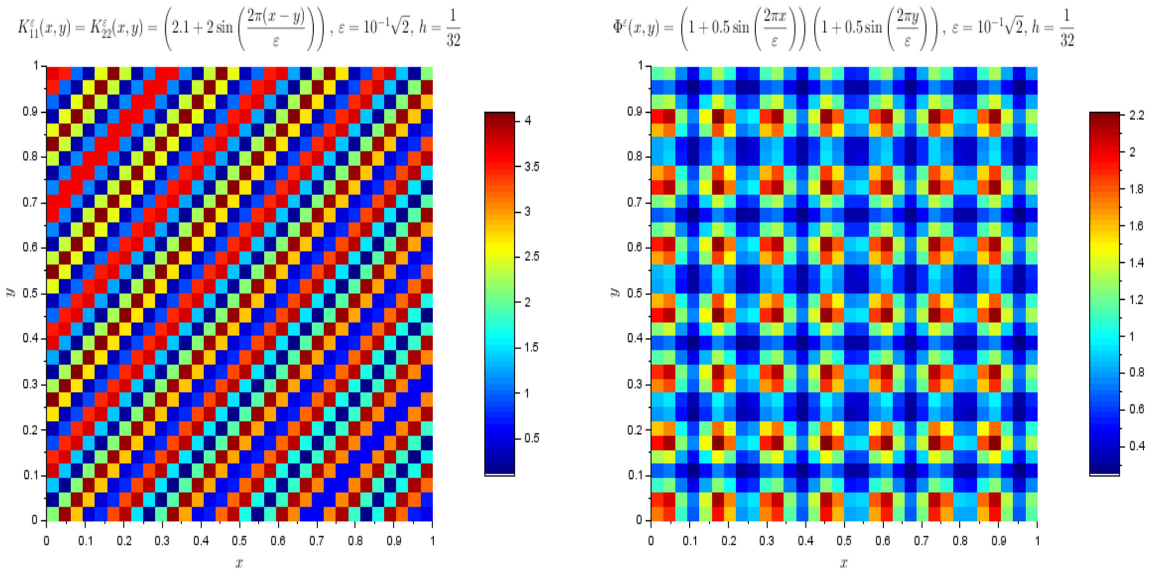

Figure 5: Test Problem 2: Coefficients K^{ε} and Φ^{ε}.

The four below graphs (see Figures 6 and 7) and also the table 3.4 confirm the fact that when ε is not "small enough", the original problem solution can be strongly oscillating and therefore completely different from the homogenized solution.

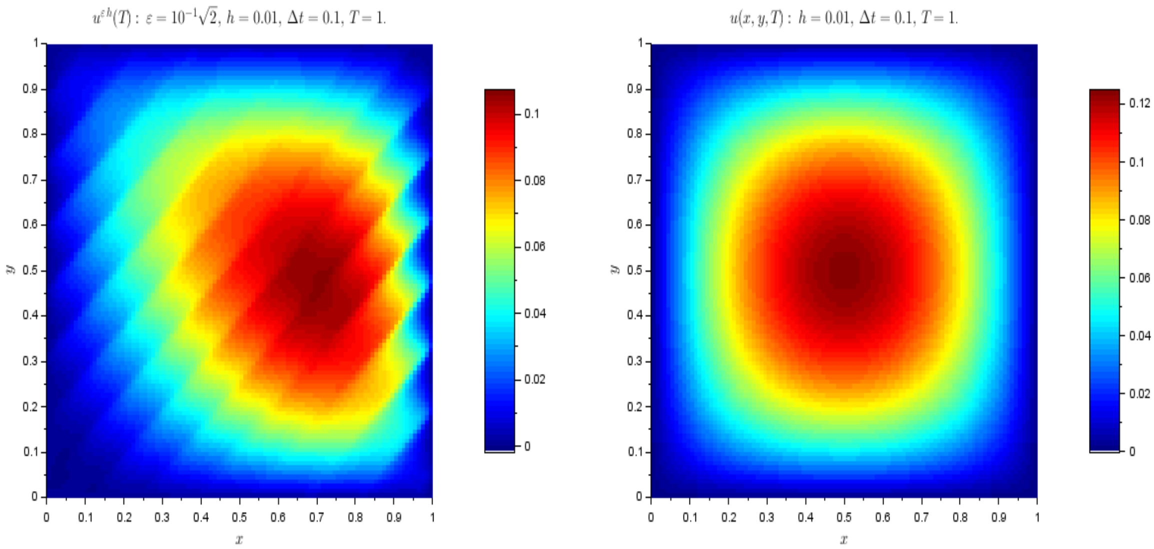

Figure 6: Test Problem 2: Numerical solution $u^{\varepsilon h}(x, y, T)$ and exact homogenized solution $u(x, y, T)$.

$\varepsilon = 10^{-1}\sqrt{2}$	$h = \frac{1}{32}$	$h = \frac{1}{64}$	$h = \frac{1}{80}$	$h = \frac{1}{96}$	$h = \frac{1}{100}$
$\left\|u^{\varepsilon h} - u\right\|_{\infty,h}$	7.575510e-02	6.547129e-02	5.679881e-02	5.181871e-02	5.095292e-02
$\left\|u^{\varepsilon h} - u\right\|_{0,h}$	3.457312e-02	2.885146e-02	2.503560e-02	2.242630e-02	2.179972e-02

Table 3.4: Test Problem 2: Error table at the final time $(T = 1)$ with $\varepsilon = 10^{-1}\sqrt{2}, \Delta t = 0.1$.

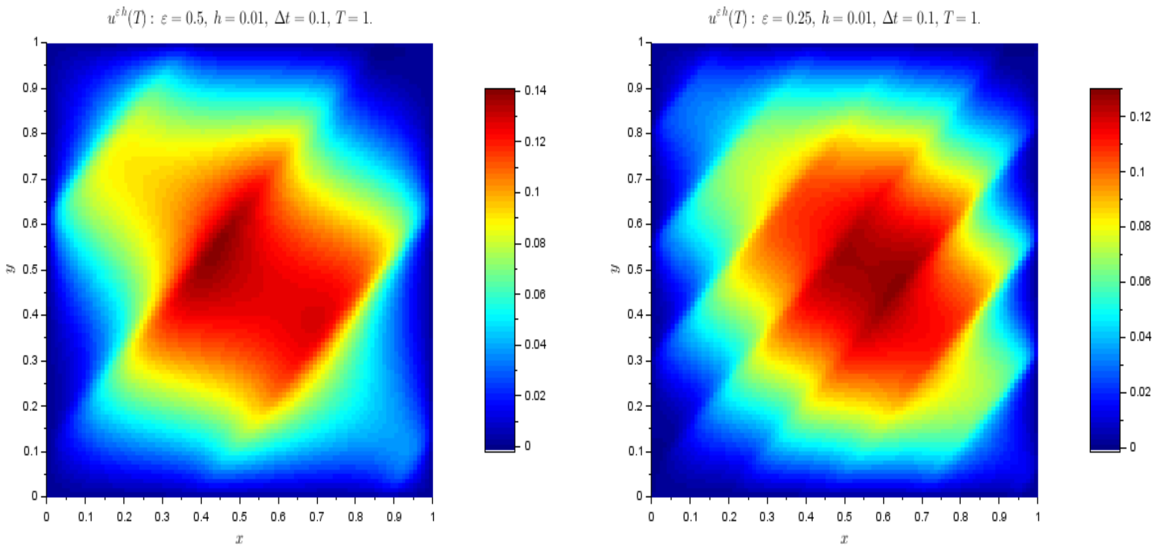

Figure 7: Test Problem 2: Numerical solution $u^{\varepsilon h}(x, y, T)$ for two different ε's values.

The following last graphic (Figure 8) shows that when $\frac{h}{\varepsilon}$ is a positive integer the numerical solution coincides with the false solution (solution obtained par remplacing K_{11} and K_{22} by 2.1). This resultat was alredy proved in the monodimensional cas on elliptic problems (see, [16]-[17]) for finite element methods.

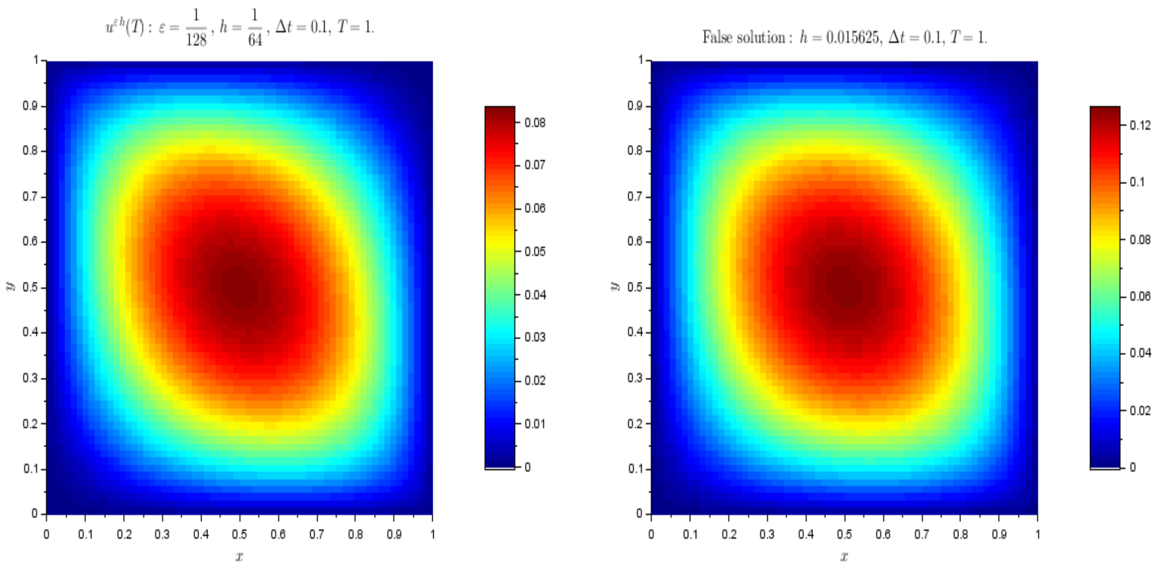

Figure 8: Test Problem 2: Numerical solution $u^{\varepsilon h}(x, y, T)$ when $\frac{h}{\varepsilon} = m \in \mathbb{N}^{*}$.

4. CONCLUDING REMARKS

The purpose of this paper was to apply the method of lines with a finite volumes approach for numerical approximation of a class of parabolic equations with continuous and oscillating coefficients. We used a cell-centered finite volumes scheme with an uniform grid and an implicit Euler scheme, for solving the obtained ODE. After proving that the implicit scheme is L^∞ stable, we presented numerical results, showing that this approach is well adapted for this class of problems under the condition that $\frac{h}{\varepsilon} \neq m \in \mathbb{N}^*$. This condition is well known for elliptic problems, with continuous and oscillating coefficients on others methods, like finite element methods (see e.g [2], [16]-[17]). The obtained results remain true if one use a rectangular grid or/and an another numerical scheme to solve this ODE. The numerical approximation of the problem, in discontinuous coefficients case is underway (see [4]).

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