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THE HIGHER COEFFICIENTS FOR BAZILEVIČ FUNCTIONS $\mathcal{B}_1(\alpha)$

MARJONO, SA'ADATUL FITRI, AND KRISNA ADILIA DANISWARA

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DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICS AND NATURAL SCIENCES, BRAWIJAYA UNIVERSITY, MALANG JAWA TIMUR 65145 INDONESIA
marjono@ub.ac.id
saadatulfitri@ub.ac.id
krisnaadiliadaniswara@gmail.com

ABSTRACT. Let f be analytic in $\mathbb{D} = \{z : |z| < 1\}$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and normalized by the conditions $f(0) = f'(0) - 1 = 0$. We give sharp estimates for the seventh and eighth coefficients for the class of Bazilevič functions with logarithmic growth, $\mathcal{B}_1(\alpha)$, defined by $Re\left\{\dfrac{z^{1-\alpha}f'(z)}{f(z)^{1-\alpha}}\right\} > 0$ for $\alpha \geq 0$.

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{S}$ be the class of analytic normalized univalent functions f defined in $z \in \mathbb{D} = \{z : |z| < 1\}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Then for $\alpha \geq 0$, $f \in \mathcal{B}_1(\alpha) \subset \mathcal{S}$, if and only if, for $z \in \mathbb{D}$

$$Re\,\frac{z^{1-\alpha} f'(z)}{f(z)^{1-\alpha}} > 0. \tag{1.2}$$

The Bazilevič functions with logarithmic growth, $\mathcal{B}_1(\alpha) \subset \mathcal{S}$ have been extensively studied (see e.g. [3], [7], [8]). Some results have been obtained for the class $\mathcal{B}_1(\alpha)$. Amongst other results, Singh [6], found sharp estimates for the modulus of the first four coefficients, $a_2,\ a_3$ and a_4. The sharp bounds for modulus of the inverse coefficients A_2, A_3 and A_4 also were obtained in [9]. The higher coefficients becomes interesting. In 2017, Marjono et. al.[4] given sharp estimate for $|a_5|, |a_6|$ and $|A_5|$.

It is the purpose of this paper to give some sharp bounds of the modulus of the coefficients $a_7,\ a_8$ and the correction of the proof of the coefficient A_5 in [4].

2. PRELIMINARIES

Let $\mathcal{P}$, the class of function h satisfying $Re\ h(z) > 0$ for $z \in \mathbb{D}$ with Taylor expansion

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{2.1}$$

We shall use the following results concerning the coefficients c_n in $\mathcal{P}$.

Lemma 2.1. [2] *If $h \in \mathcal{P}$, then*

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}.$$

Lemma 2.2. [1] *If $h \in \mathcal{P}$, then $|c_n| \leq 2$ for $n \geq 1$, and*

$$\left|c_2 - \frac{\mu}{2}c_1^2\right| \leq \max\{2,\ 2|\mu - 1|\} = \begin{cases} 2, & 0 \leq \mu \leq 2, \\ 2|\mu - 1|, & \textit{elsewhere.} \end{cases}$$

Lemma 2.3. *Let $h \in \mathcal{P}$, then* [1]

$$|c_3 - 2c_1c_2 + c_1^3| \leq 2.$$

and [5]

$$|c_1^4 - 3c_1^2c_2 + c_2^2 + 2c_1c_3 - c_4| \leq 2.$$

3. THE SEVENTH AND EIGHTH COEFFICIENTS

It follows from (1.2) that we can write

$$z^{1-\alpha}f'(z) = f(z)^{1-\alpha}h(z), \tag{3.1}$$

where $h \in \mathcal{P}$, the class of function satisfying $Re\, h(z) > 0$ for $z \in \mathbb{D}$.
Equating coefficients in (3.1) gives

$$\begin{aligned}
a_2 &= \frac{c_1}{(1+\alpha)},\\
a_3 &= \frac{c_2}{(2+\alpha)} + \frac{(1-\alpha)c_1^2}{2(1+\alpha)^2},\\
a_4 &= \frac{c_3}{(3+\alpha)} + (1-\alpha)\left(\frac{c_1c_2}{(1+\alpha)(2+\alpha)} + \frac{(1-2\alpha)c_1^3}{6(1+\alpha)^3}\right),\\
a_5 &= \frac{c_4}{(4+\alpha)} + (1-\alpha)\left(\frac{(1-2\alpha)(1-3\alpha)c_1^4}{24(1+\alpha)^2} + \frac{(1-2\alpha)c_1^2c_2}{2(1+\alpha)^2(2+\alpha)}\right.\\
&\qquad \left.\frac{c_2^2}{2(2+\alpha)^2} + \frac{c_1c_3}{(1+\alpha)(3+\alpha)}\right),\\
a_6 &= \frac{c_5}{5+\alpha} + (1-\alpha)(32X_1 + 16X_2 + 8X_3 + 8X_4 + 4X_5 + 4X_6),\\
a_7 &= \frac{c_6}{6+\alpha} + (1-\alpha)(Y_1c_1^6 + Y_2c_1^4c_2 + Y_3c_1^2c_2^2 + Y_4c_2^3 + Y_5c_1^3c_3\\
&\qquad + Y_6c_1c_2c_3 + Y_7c_3^2 + Y_8c_1^2c_4 + Y_9c_2c_4 + Y_{10}c_1c_5),\\
a_8 &= \frac{c_7}{7+\alpha} + (1-\alpha)\big(Z_1c_1^7 + Z_2c_1^5c_2 + Z_3c_1^3c_2^2 + Z_4c_1c_2^3 + Z_5c_1^4c_3\\
&\qquad + Z_6c_1^2c_2c_3 + Z_7c_2^2c_3 + Z_8c_1c_3^2 + Z_9c_1^3c_4 + Z_{10}c_1c_2c_4\\
&\qquad + Z_{11}c_3c_4 + Z_{12}c_1^2c_5 + Z_{13}c_2c_5 + Z_{14}c_1c_6\big)
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
X_1 &= \frac{(1-2\alpha)(1-3\alpha)(1-4\alpha)}{120(1+\alpha)^5}, & X_2 &= \frac{(1-2\alpha)(1-3\alpha)}{6(\alpha+1)^3(\alpha+2)},\\
X_3 &= \frac{1-2\alpha}{2(\alpha+1)(\alpha+2)^2}, & X_4 &= \frac{1-2\alpha}{2(\alpha+1)^2(\alpha+3)},\\
X_5 &= \frac{1}{(\alpha+2)(\alpha+3)}, & X_6 &= \frac{1}{(\alpha+1)(\alpha+4)},\\
Y_1 &= \frac{(1-2\alpha)(1-3\alpha)(1-4\alpha)(1-5\alpha)}{720(\alpha+1)^6}, & Y_2 &= \frac{(1-2\alpha)(1-3\alpha)(1-4\alpha)}{24(\alpha+1)^4(\alpha+2)}\\
Y_3 &= \frac{(1-2\alpha)(1-3\alpha)}{4(\alpha+1)^2(\alpha+2)^2}, & Y_4 &= \frac{1-2\alpha}{6(\alpha+2)^3}\\
Y_5 &= \frac{(1-2\alpha)(1-3\alpha)}{6(\alpha+1)^3(\alpha+3)}, & Y_6 &= \frac{1-2\alpha}{(\alpha+1)(\alpha+2)(\alpha+3)}\\
Y_7 &= \frac{1}{2(\alpha+3)^2}, & Y_8 &= \frac{1-2\alpha}{2(\alpha+1)^2(\alpha+4)}\\
Y_9 &= \frac{1}{(\alpha+2)(\alpha+4)}, & Y_{10} &= \frac{1}{(\alpha+1)(\alpha+5)}
\end{aligned}$$

and

$$Z_1 = \frac{(1-2\alpha)(1-3\alpha)(1-4\alpha)(1-5\alpha)(1-6\alpha)}{5040(1+\alpha)^7}, \quad Z_2 = \frac{(1-2\alpha)(1-3\alpha)(1-4\alpha)(1-5\alpha)}{120(1+\alpha)^5(2+\alpha)}$$

$$Z_3 = \frac{(1-2\alpha)(1-3\alpha)(1-4\alpha)}{12(1+\alpha)^3(2+\alpha)^2}, \quad Z_4 = \frac{(1-2\alpha)(1-3\alpha)}{6(1+\alpha)(2+\alpha)^3}$$

$$Z_5 = \frac{(1-2\alpha)(1-3\alpha)(1-4\alpha)}{24(1+\alpha)^4(3+\alpha)}, \quad Z_6 = \frac{(1-2\alpha)(1-3\alpha)}{2(1+\alpha)^2(2+\alpha)(3+\alpha)}$$

$$Z_7 = \frac{1-2\alpha}{2(2+\alpha)^2(3+\alpha)}, \quad Z_8 = \frac{1-2\alpha}{2(1+\alpha)(3+\alpha)^2}$$

$$Z_9 = \frac{(1-2\alpha)(1-3\alpha)}{6(1+\alpha)^3(4+\alpha)}, \quad Z_{10} = \frac{1-2\alpha}{(1+\alpha)(2+\alpha)(4+\alpha)}$$

$$Z_{11} = \frac{1}{(3+\alpha)(4+\alpha)}, \quad Z_{12} = \frac{1-2\alpha}{2(1+\alpha)^2(5+\alpha)}$$

$$Z_{13} = \frac{1}{(2+\alpha)(5+\alpha)}, \quad Z_{14} = \frac{1}{(1+\alpha)(6+\alpha)}$$

For completeness, we include the coefficients $a_2,\ a_3,\ a_4,\ a_5$ and a_6(without proof) proven in [4] and [6].

Theorem 3.1. *If $f \in \mathcal{B}_1(\alpha)$ and is given by (1.1), then*

$$\begin{aligned}
|a_2| &\le \frac{2}{1+\alpha} \text{ for } \alpha \ge 0,\\
|a_3| &\le \frac{2(3+\alpha)}{(1+\alpha)^2(2+\alpha)} \text{ for } 0 \le \alpha \le 1,\\
&\le \frac{2}{2+\alpha} \text{ for } \alpha \ge 1,\\
|a_4| &\le \frac{2}{3+\alpha} + \frac{4(1-\alpha)(2+5\alpha+\alpha^2)}{3(1+\alpha)^3(2+\alpha)} \text{ for } 0 \le \alpha \le 1\\
&\le \frac{2}{3+\alpha} \text{ for } \alpha \ge 1\\
|a_5| &\le \frac{2}{4+\alpha} + \frac{2(1-\alpha)(3\alpha^5+22\alpha^4+68\alpha^3+113\alpha^2+121\alpha+81)}{3(1+\alpha)^4(2+\alpha)^2(3+\alpha)} \text{ for } 0 \le \alpha \le \alpha_1\\
|a_6| &\le \frac{2}{5+\alpha} + (1-\alpha)(32X_1+16X_2+8X_3+8X_4+4X_5+4X_6) \text{ for } 0 \le \alpha \le 1/2,
\end{aligned}$$

where $\alpha_1 = 0.96942...$ is the smallest positive root of the equation

$$12(1+\alpha)^4 - 24(1+\alpha)^2(2\alpha-1)(2+\alpha) + 5(3\alpha-1)(2\alpha-1)(2+\alpha)^2 = 0$$

All the inequalities are sharp.

We next proof the seventh and eighth coefficients.

Theorem 3.2. *Let $f \in \mathcal{B}_1(\alpha)$ is given by (1.1). Then for $\alpha \in [0, \frac{1}{16}\left(\sqrt{393}-13\right)]$*

$$|a_7| \le \frac{2}{6+\alpha} + (1-\alpha)(64Y_1+32Y_2+16Y_3+8Y_4+16Y_5+8Y_6+4Y_7+8Y_8+4Y_9+4Y_{10})$$

Proof. From (3.2), we have

$$a_7 = \frac{c_6}{6+\alpha} + (1-\alpha)(Y_1c_1^6 + Y_2c_1^4c_2 + Y_3c_1^2c_2^2 + Y_4c_2^3 + Y_5c_1^3c_3 + Y_6c_1c_2c_3 + Y_7c_3^2 + Y_8c_1^2c_4 + Y_9c_2c_4 + Y_{10}c_1c_5). \tag{3.3}$$

We divide the interval into 4 cases: $\alpha \in [0, \frac{1}{5}]$, $\alpha \in [\frac{1}{5}, \frac{1}{4}]$, $\alpha \in [\frac{1}{4}.\frac{1}{3}]$ and $\alpha \in [\frac{1}{3}, \frac{1}{16}(-13 + \sqrt{393})]$.

Note first that for $\alpha \in [0, \frac{1}{5}]$, we have $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Y_{10} \geq 0$. Then applying $|c_n| \leq 2$ in (3.3) to obtain the result.

For case $\alpha \in [\frac{1}{5}, \frac{1}{4}]$, write (3.3) as

$$a_7 = \frac{c_6}{6+\alpha} + (1-\alpha)\left\{\left(Y_1 + \frac{Y_2}{2}\right)c_1^6 + Y_2c_1^4\left(c_2 - \frac{c_1^2}{2}\right) + Y_3c_1^2c_2^2 + Y_4c_2^3 + Y_5c_1^3c_3 + Y_6c_1c_2c_3 + Y_7c_3^2 + Y_8c_1^2c_4 + Y_9c_2c_4 + Y_{10}c_1c_5\right\}.$$

Since $\left(Y_1 + \frac{Y_2}{2}\right) \geq 0$ when $\alpha \in [\frac{1}{5}, \frac{1}{4}]$, then applying Lemma 2.1 and $|c_n| \leq 2$ for $n = 1, 2, 3, 4, 5, 6$ to obtain

$$\begin{aligned} |a_7| &\leq \frac{2}{6+\alpha} + (1-\alpha)\left\{\left(Y_1 + \frac{Y_2}{2}\right)|c_1|^6 + Y_2|c_1|^4\left(2 - \frac{|c_1|^2}{2}\right) + 4Y_3|c_1|^2 + 8Y_4 \right. \\ &\qquad \left. + 2Y_5|c_1|^3 + 4Y_6|c_1| + 4Y_7 + 2Y_8|c_1|^2 + 4Y_9 + 2Y_{10}|c_1|\right\} \\ &= \frac{2}{6+\alpha} + (1-\alpha)\left\{Y_1|c_1|^6 + 2Y_2|c_1|^4 + 4Y_3|c_1|^2 + 8Y_4 \right. \\ &\qquad \left. + 2Y_5|c_1|^3 + 4Y_6|c_1| + 4Y_7 + 2Y_8|c_1|^2 + 4Y_9 + 2Y_{10}|c_1|\right\} \\ &\leq \frac{2}{6+\alpha} + (1-\alpha)\left\{64Y_1 + 32Y_2 + 16Y_3 + 8Y_4 \right. \\ &\qquad \left. + 16Y_5 + 8Y_6 + 4Y_7 + 8Y_8 + 4Y_9 + 4Y_{10}\right\} \end{aligned}$$

For interval $\frac{1}{4} \leq \alpha \leq \frac{1}{3}$, we write (3.3) as

$$a_7 = \frac{c_6}{6+\alpha} + (1-\alpha)\left\{Y_1c_1^6 + \left(Y_2 + \frac{Y_3}{2}\right)c_1^4c_2 + Y_3c_1^2c_2\left(c_2 - \frac{c_1^2}{2}\right) + Y_4c_2^3 + Y_5c_1^3c_3 + Y_6c_1c_2c_3 + Y_7c_3^2 + Y_8c_1^2c_4 + Y_9c_2c_4 + Y_{10}c_1c_5\right\}.$$

We observe that $Y_2 \leq 0$ and $Y_1, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Y_{10} \geq 0$ for $\alpha \in [\frac{1}{4}.\frac{1}{3}]$.
Furthermore,

$$\left(Y_2 + \frac{Y_3}{2}\right) = \frac{(1-2\alpha)(1-3\alpha)(5-\alpha^2-\alpha)}{24(1+\alpha)^4(2+\alpha)^2} \geq 0, \quad \alpha \in [\frac{1}{4}.\frac{1}{3}].$$

Therefore, by Lemma 2.1 and $|c_n| \leq 2$ for $n = 1, 2, 3, 4, 5, 6$, we have

$$
\begin{aligned}
|a_7| &\leq \frac{2}{6+\alpha} + (1-\alpha)\left\{Y_1|c_1|^6 + 2\left(Y_2 + \frac{Y_3}{2}\right)|c_1|^4 + 2Y_3|c_1|^2\left(2 - \frac{|c_1|^2}{2}\right) + 8Y_4\right. \\
&\qquad \left. +2Y_5|c_1|^3 + 4Y_6|c_1| + 4Y_7 + 2Y_8|c_1|^2 + 4Y_9 + 2Y_{10}|c_1|\right\} \\
&= \frac{2}{6+\alpha} + (1-\alpha)\left\{Y_1|c_1|^6 + 2Y_2|c_1|^4 + 4Y_3|c_1|^2 + 8Y_4\right. \\
&\qquad \left. +2Y_5|c_1|^3 + 4Y_6|c_1| + 4Y_7 + 2Y_8|c_1|^2 + 4Y_9 + 2Y_{10}|c_1|\right\} \\
&\leq \frac{2}{6+\alpha} + (1-\alpha)\left\{64Y_1 + 32Y_2 + 16Y_3 + 8Y_4\right. \\
&\qquad \left. +16Y_5 + 8Y_6 + 4Y_7 + 8Y_8 + 4Y_9 + 4Y_{10}\right\}
\end{aligned}
$$

For the last interval, we observe that $Y_1, Y_3, Y_5 \leq 0$ and $Y_2, Y_4, Y_6, Y_7, Y_8, Y_9, Y_{10} \geq 0$. Then we write (3.3) as

$$
\begin{aligned}
a_7 = \frac{c_6}{6+\alpha} + (1-\alpha)&\left\{\left(Y_1 + \frac{Y_2}{2}\right)c_1^6 + Y_2c_1^4\left(c_2 - \frac{c_1^2}{2}\right) + \left(Y_3 + \frac{Y_4}{2}\right)c_1^2c_2^2\right. \\
&+ Y_4c_2^2\left(c_2 - \frac{c_1^2}{2}\right) + \left(Y_5 + \frac{Y_6}{2}\right)c_1^3c_3 + Y_6c_1c_3\left(c_2 - \frac{c_1^2}{2}\right) \\
&\left. +Y_7c_3^2 + Y_8c_1^2c_4 + Y_9c_2c_4 + Y_{10}c_1c_5\right\}.
\end{aligned}
$$

We also check that

$$
\begin{aligned}
\left(Y_1 + \frac{Y_2}{2}\right) &= \frac{(1-2\alpha)(1-3\alpha)(1-4\alpha)\left(10\alpha^2 + 21\alpha + 17\right)}{720(1+\alpha)^6(2+\alpha)} \geq 0, \\
\left(Y_3 + \frac{Y_4}{2}\right) &= \frac{(1-2\alpha)\left(-8\alpha^2 - 13\alpha + 7\right)}{12(1+\alpha)^2(2+\alpha)^3} \geq 0, \\
\left(Y_5 + \frac{Y_6}{2}\right) &= \frac{(5+\alpha)(1-2\alpha)}{6(1+\alpha)^3(2+\alpha)(3+\alpha)} \geq 0,
\end{aligned}
$$

for $\alpha \in [\frac{1}{3}, \frac{1}{16}(-13 + \sqrt{393})]$. Then we apply Lemma 2.1 and $|c_n| \leq 2$ for $n = 1, 2, 3, 4, 5, 6$ to get

$$
\begin{aligned}
|a_7| &\leq \frac{2}{6+\alpha} + (1-\alpha)\left\{\left(Y_1 + \frac{Y_2}{2}\right)|c_1|^6 + Y_2|c_1|^4\left(2 - \frac{|c_1|^2}{2}\right) + 4\left(Y_3 + \frac{Y_4}{2}\right)|c_1|^2\right. \\
&\qquad + 4Y_4\left(2 - \frac{|c_1|^2}{2}\right) + 2\left(Y_5 + \frac{Y_6}{2}\right)|c_1|^3 + 2Y_6|c_1|\left(2 - \frac{|c_1|^2}{2}\right) \\
&\qquad \left. +4Y_7 + 2Y_8|c_1|^2 + 4Y_9 + 2Y_{10}|c_1|\right\} \\
&= \frac{2}{6+\alpha} + (1-\alpha)\left\{Y_1|c_1|^6 + 2Y_2|c_1|^4 + 4Y_3|c_1|^2 + 8Y_4 + 2Y_5|c_1|^3\right. \\
&\qquad \left. +4Y_6|c_1| + 4Y_7 + 2Y_8|c_1|^2 + 4Y_9 + 2Y_{10}|c_1|\right\} \\
&\leq \frac{2}{6+\alpha} + (1-\alpha)\left\{64Y_1 + 32Y_2 + 16Y_3 + 8Y_4 + 16Y_5\right. \\
&\qquad \left. +8Y_6 + 4Y_7 + 8Y_8 + 4Y_9 + 4Y_{10}\right\}
\end{aligned}
$$

The sharpness of the bound for $|a_7|$ when $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 2$. ∎

Theorem 3.3. *Let* $f \in \mathcal{B}_1(\alpha)$ *is given by* (1.1)*. Then for* $\alpha \in [0, 1/3]$

$$|a_8| \leq \frac{2}{7+\alpha} + (1-\alpha)(128Z_1 + 64Z_2 + 32Z_3 + 16Z_4 + 32Z_5 + 16Z_6 + 8Z_7 + 8Z_8 + 16Z_9 + 8Z_{10} + 4Z_{11} + 8Z_{12} + 4Z_{13} + 4Z_{14})$$

Proof. From (3.2), we have

$$(3.4)\quad a_8 = \frac{c_7}{7+\alpha} + (1-\alpha)\big(Z_1c_1^7 + Z_2c_1^5c_2 + Z_3c_1^3c_2^2 + Z_4c_1c_2^3 + Z_5c_1^4c_3 + Z_6c_1^2c_2c_3 + Z_7c_2^2c_3 + Z_8c_1c_3^2 + Z_9c_1^3c_4 + Z_{10}c_1c_2c_4 + Z_{11}c_3c_4 + Z_{12}c_1^2c_5 + Z_{13}c_2c_5 + Z_{14}c_1c_6\big).$$

We consider each of the cases: $\alpha \in [0, \frac{1}{6}]$, $\alpha \in [\frac{1}{6}, \frac{1}{5}]$, $\alpha \in [\frac{1}{5}. \frac{1}{4}]$ and $\alpha \in [\frac{1}{4}, \frac{1}{3}]$.

For first case, we have

$$Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}, Z_{13}, Z_{14} \geq 0.$$

Then applying $|c_n| \leq 2$ in (3.4) to obtain the result.

We now consider on interval $[\frac{1}{6}, \frac{1}{5}]$. We write (3.4) as

$$a_8 = \frac{c_7}{7+\alpha} + (1-\alpha)\left(\left(Z_1 + \frac{Z_2}{2}\right)c_1^7 + Z_2c_1^5\left(c_2 - \frac{c_1^2}{2}\right) + Z_3c_1^3c_2^2 + Z_4c_1c_2^3 + Z_5c_1^4c_3 + Z_6c_1^2c_2c_3 + Z_7c_2^2c_3 + Z_8c_1c_3^2 + Z_9c_1^3c_4 + Z_{10}c_1c_2c_4 + Z_{11}c_3c_4 + Z_{12}c_1^2c_5 + Z_{13}c_2c_5 + Z_{14}c_1c_6\right).$$

Since

$$\left(Z_1 + \frac{Z_2}{2}\right) = \frac{(1-2\alpha)(1-3\alpha)(1-4\alpha)(1-5\alpha)(23+31\alpha+15\alpha^2)}{5040(1+\alpha)^7(2+\alpha)} \geq 0$$

when $\alpha \in [0, \frac{1}{5}]$, then applying Lemma 2.1 and $|c_n| \leq 2$ for $n = 1, 2, 3, 4, 5, 6, 7$ to obtain

$$\begin{aligned}
|a_8| &\leq \frac{2}{7+\alpha} + (1-\alpha)\left(\left(Z_1 + \frac{Z_2}{2}\right)|c_1|^7 + Z_2|c_1|^5\left(2 - \frac{|c_1|^2}{2}\right) + 4Z_3|c_1|^3 + 8Z_4|c_1|\right. \\
&\quad + 2Z_5|c_1|^4 + 4Z_6|c_1|^2 + 8Z_7 + 4Z_8|c_1| + 2Z_9|c_1|^3 \\
&\quad \left. +4Z_{10}c_1 + 4Z_{11} + 2Z_{12}|c_1|^2 + 4Z_{13} + 2Z_{14}|c_1|\right) \\
&= \frac{2}{7+\alpha} + (1-\alpha)\big(Z_1|c_1|^7 + 2Z_2|c_1|^5 + 4Z_3|c_1|^3 + 8Z_4|c_1| \\
&\quad + 2Z_5|c_1|^4 + 4Z_6|c_1|^2 + 8Z_7 + 4Z_8|c_1| + 2Z_9|c_1|^3 \\
&\quad +4Z_{10}c_1 + 4Z_{11} + 2Z_{12}|c_1|^2 + 4Z_{13} + 2Z_{14}|c_1|\big) \\
&\leq \frac{2}{7+\alpha} + (1-\alpha)(128Z_1 + 64Z_2 + 32Z_3 + 16Z_4 + 32Z_5 + 16Z_6 + 8Z_7 \\
&\quad + 8Z_8 + 16Z_9 + 8Z_{10} + 4Z_{11} + 8Z_{12} + 4Z_{13} + 4Z_{14}).
\end{aligned}$$

For interval $\dfrac{1}{5} \leq \alpha \leq \dfrac{1}{4}$, we write (3.4) as

$$a_8 = \frac{c_7}{7+\alpha} + (1-\alpha)\left(Z_1 c_1^7 + \left(Z_2 + \frac{Z_3}{2}\right) c_1^5 c_2 + Z_3 c_1^3 c_2 \left(c_2 - \frac{c_1^2}{2}\right) + Z_4 c_1 c_2^3 \right.$$
$$+ Z_5 c_1^4 c_3 + Z_6 c_1^2 c_2 c_3 + Z_7 c_2^2 c_3 + Z_8 c_1 c_3^2 + Z_9 c_1^3 c_4$$
$$\left. + Z_{10} c_1 c_2 c_4 + Z_{11} c_3 c_4 + Z_{12} c_1^2 c_5 + Z_{13} c_2 c_5 + Z_{14} c_1 c_6 \right).$$

We observe that $Z_2 \leq 0$ and

$$Z_1, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}, Z_{13}, Z_{14} \geq 0$$

when $\alpha \in [\frac{1}{5}.\frac{1}{4}]$. Furthermore,

$$\left(Z_2 + \frac{Z_3}{2}\right) = \frac{(7+\alpha)(1-2\alpha)(1-3\alpha)(1-4\alpha)}{120(1+\alpha)^5(2+\alpha)^2} \geq 0, \quad \alpha \in [\frac{1}{5}.\frac{1}{4}].$$

Therefore, by Lemma 2.1 and $|c_n| \leq 2$ for $n = 1, 2, 3, 4, 5, 6, 7$, we have

$$|a_8| \leq \frac{2}{7+\alpha} + (1-\alpha)\left(Z_1 |c_1|^7 + 2\left(Z_2 + \frac{Z_3}{2}\right)|c_1|^5 + 2Z_3 |c_1|^3 \left(2 - \frac{|c_1|^2}{2}\right)\right.$$
$$+ 8Z_4|c_1| + 2Z_5|c_1|^4 + 4Z_6|c_1|^2 + 8Z_7 + 4Z_8|c_1| + 2Z_9|c_1|^3$$
$$\left. + 4Z_{10}|c_1| + 4Z_{11} + 2Z_{12}|c_1|^2 + 4Z_{13} + 2Z_{14}|c_1|\right)$$
$$= \frac{2}{7+\alpha} + (1-\alpha)\left(Z_1|c_1|^7 + 2Z_2|c_1|^5 + 4Z_3|c_1|^3 + 8Z_4|c_1|\right.$$
$$+ 2Z_5|c_1|^4 + 4Z_6|c_1|^2 + 8Z_7 + 4Z_8|c_1| + 2Z_9|c_1|^3$$
$$\left. + 4Z_{10}|c_1| + 4Z_{11} + 2Z_{12}|c_1|^2 + 4Z_{13} + 2Z_{14}|c_1|\right)$$
$$\leq \frac{2}{7+\alpha} + (1-\alpha)(128Z_1 + 64Z_2 + 32Z_3 + 16Z_4 + 32Z_5 + 16Z_6 + 8Z_7$$
$$+ 8Z_8 + 16Z_9 + 8Z_{10} + 4Z_{11} + 8Z_{12} + 4Z_{13} + 4Z_{14}).$$

For the last interval, we observe that $Z_1, Z_3, Z_5 \leq 0$ and

$$Z_2, Z_4, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, Z_{12}, Z_{13}, Z_{14} \geq 0.$$

Then we write (3.4) as

$$a_8 = \frac{c_7}{7+\alpha} + (1-\alpha)\left(\left(Z_1 + \frac{Z_2}{2}\right)c_1^7 + Z_2 c_1^5\left(c_2 - \frac{c_1^2}{2}\right) + \left(Z_3 + \frac{Z_4}{2}\right)c_1^3 c_2^2\right.$$
$$+ Z_4 c_1 c_2^2 \left(c_2 - \frac{c_1^2}{2}\right) + \left(Z_5 + \frac{Z_6}{2}\right) c_1^4 c_3 + Z_6 c_1^2 c_3 \left(c_2 - \frac{c_1^2}{2}\right)$$
$$+ Z_7 c_2^2 c_3 + Z_8 c_1 c_3^2 + Z_9 c_1^3 c_4 + Z_{10} c_1 c_2 c_4$$
$$\left. + Z_{11} c_3 c_4 + Z_{12} c_1^2 c_5 + Z_{13} c_2 c_5 + Z_{14} c_1 c_6\right).$$

We also check that

$$\left(Z_1 + \frac{Z_2}{2}\right) = \frac{(1-2\alpha)(1-3\alpha)(1-4\alpha)(1-5\alpha)(23+31\alpha+15\alpha^2)}{5040(1+\alpha)^7(2+\alpha)} \geq 0,$$

$$\left(Z_3 + \frac{Z_4}{2}\right) = \frac{(1-2\alpha)(1-3\alpha)(3-5\alpha-3\alpha^2)}{12(1+\alpha)^2(2+\alpha)^3} \geq 0,$$

$$\left(Z_5 + \frac{Z_6}{2}\right) = \frac{(1-2\alpha)(1-3\alpha)(8+5\alpha+2\alpha^2)}{24(1+\alpha)^4(2+\alpha)(3+\alpha)} \geq 0,$$

for $\alpha \in [\frac{1}{4}, \frac{1}{3}]$. Then we applying Lemma 2.1 and $|c_n| \leq 2$ for $n = 1, 2, 3, 4, 5, 6, 7$ to get

$$\begin{aligned} |a_8| \leq \frac{2}{7+\alpha} &+ (1-\alpha)\left(\left(Z_1 + \frac{Z_2}{2}\right)|c_1|^7 + Z_2|c_1|^5\left(2 - \frac{|c_1|^2}{2}\right) + 4\left(Z_3 + \frac{Z_4}{2}\right)|c_1|^3\right. \\ &+ 4Z_4|c_1|\left(2 - \frac{|c_1|^2}{2}\right) + 2\left(Z_5 + \frac{Z_6}{2}\right)|c_1|^4 + 2Z_6|c_1|^2\left(2 - \frac{|c_1|^2}{2}\right) \\ &+ 8Z_7 + 4Z_8|c_1| + 2Z_9|c_1|^3 + 4Z_{10}|c_1| \\ &\left. + 4Z_{11} + 2Z_{12}|c_1|^2 + 4Z_{13} + 2Z_{14}|c_1|\right) \\ = \frac{2}{7+\alpha} &+ (1-\alpha)\left(Z_1|c_1|^7 + 2Z_2|c_1|^5 + 4Z_3|c_1|^3 + 8Z_4|c_1| + 2Z_5|c_1|^4\right. \\ &+ 4Z_6|c_1|^2 + 8Z_7 + 4Z_8|c_1| + 2Z_9|c_1|^3 + 4Z_{10}|c_1| \\ &\left. + 4Z_{11} + 2Z_{12}|c_1|^2 + 4Z_{13} + 2Z_{14}|c_1|\right) \\ \leq \frac{2}{7+\alpha} &+ (1-\alpha)(128Z_1 + 64Z_2 + 32Z_3 + 16Z_4 + 32Z_5 + 16Z_6 + 8Z_7 \\ &\qquad + 8Z_8 + 16Z_9 + 8Z_{10} + 4Z_{11} + 8Z_{12} + 4Z_{13} + 4Z_{14}). \end{aligned}$$

The sharpness of the bound for $|a_8|$ when $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = 2$. ∎

4. INVERSE COEFFICIENTS

Suppose that $\mathcal{B}_1(\alpha)^{-1}$ is the set of inverse functions f^{-1} of $\mathcal{B}_1(\alpha)$, then we can write

$$f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + ...$$

valid in some disc $|\omega| < r_0(f)$, where $r_0(f) \geq 1/4$.

Since $f(f^{-1}(\omega)) = \omega$, equating coefficients gives

$$\begin{aligned} A_2 &= -a_2, \\ A_3 &= 2a_2^2 - a_3, \\ A_4 &= -5a_3^2 + 5a_2a_3 - a_4, \\ A_5 &= 14a_2^4 - 21a_2^2a_3 + 3a_3^2 + 6a_2a_4 - a_5. \end{aligned} \tag{4.1}$$

Singh has been studied the inverse coefficients A_n for $n = 2, 3, 4$ for $f \in \mathcal{B}_1(\alpha)$ with $0 \leq \alpha \leq 1$[6]. But, Thomas has been succeeded prove it for all $\alpha \geq 0$[9]. The next observation is inverse coefficient A_5 by Marjono et.al. for $0 \leq \alpha \leq 1$[4]. We rewrite again the theorem of the inverse coefficient A_5 together with the proof and show the mistake of this proof. For completeness, we include the theorem of the inequalities of $|A_2|$, $|A_3|$ and $|A_4|$ for all $\alpha \geq 0$(without proof).

Theorem 4.1. *Let $f \in \mathcal{B}_1(\alpha)$ and $f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + ...$, then*

$$\begin{aligned}
|A_2| &\leq \frac{2}{1+\alpha}, \\
|A_3| &\leq \frac{2}{2+\alpha} \; for \; \alpha \geq 1, \\
&\leq \frac{2(5+3\alpha)}{(1+\alpha)^2(2+\alpha)} \; for \; 0 \leq \alpha \leq 1 \\
|A_4| &\leq \frac{2}{3+\alpha} \; for \; \alpha \geq 1, \\
&\leq \frac{2(\alpha^4+5\alpha^3+41\alpha^2+139\alpha+126)}{3(1+\alpha)^3(2+\alpha)(3+\alpha)} \; for \; 0 \leq \alpha \leq 1.
\end{aligned}$$

All the inequalities are sharp.

We now give a correction for Theorem 4.1 in [4].

Theorem 4.2. *Let $f \in \mathcal{B}_1(\alpha)$ and $f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \ldots$, then*

$$|A_5| \leq \frac{2\left(10\alpha^6+104\alpha^5+553\alpha^4+2075\alpha^3+4981\alpha^2+6245\alpha+3024\right)}{3(1+\alpha)^4(2+\alpha)^2(3+\alpha)}$$

for $0 \leq \alpha < \alpha_1$, with $\alpha_1 = 0.2687\ldots$ is the smallest positive root of the equation

$$18 - 49x - 66x^2 - 5x^3 + 6x^4 = 0.$$

The inequalities is sharp.

Proof. From (4.1) and (3.2), we have

$$\begin{aligned}
(4+\alpha)A_5 = &\left(\frac{(4+\alpha)(5+\alpha)(5+2\alpha)(5+3\alpha)}{24(1+\alpha)^4}\right)c_1^4 - \left(\frac{(4+\alpha)(5+\alpha)(5+2\alpha)}{2(1+\alpha)^2(2+\alpha)}\right)c_1^2c_2 \\
&+ \left(\frac{(4+\alpha)(5+\alpha)}{2(2+\alpha)^2}\right)c_2^2 + \left(\frac{(4+\alpha)(5+\alpha)}{(1+\alpha)(3+\alpha)}\right)c_1c_3 - c_4.
\end{aligned}$$

Write this as

$$\begin{aligned}
(4+\alpha)A_5 = &(c_1^4 - 3c_1^2c_2 + c_2^2 + 2c_1c_3 - c_4) \\
&+ \left(\frac{14+\alpha-\alpha^2}{(1+\alpha)(3+\alpha)}\right)c_1(c_1^3 - 2c_1c_2 + c_3) \\
&- \left(\frac{152+77\alpha-8\alpha^2-5\alpha^3}{2(1+\alpha)^2(2+\alpha)(3+\alpha)}\right)c_1^2\left(c_2 - \frac{\mu}{2}c_1^2\right) \\
&+ \left(\frac{(4-\alpha)(3+\alpha)}{2(2+\alpha)^2}\right)c_2^2,
\end{aligned}$$

with

$$\mu = \frac{(2+\alpha)(1092+1331\alpha+251\alpha^2-161\alpha^3-23\alpha^4+6\alpha^5)}{6(1+\alpha)^2(152+77\alpha-8\alpha^2-5\alpha^3)}.$$

Note that $\mu > 2$ when $0 \leq \alpha < \alpha_1$. Then we use Lemma 2.2 and Lemma 2.3 together with the inequality $|c_1| \leq 2$ and $|c_2| \leq 2$ to obtain

$$|A_5| \leq \frac{2\left(10\alpha^6 + 104\alpha^5 + 553\alpha^4 + 2075\alpha^3 + 4981\alpha^2 + 6245\alpha + 3024\right)}{3(1+\alpha)^4(2+\alpha)^2(3+\alpha)}$$

The inequality sharp when $c_1 = c_2 = c_3 = c_4 = 2$. So that for $\alpha_1 \leq \alpha \leq 1$ remains an open problem. ∎

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