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INEQUALITIES OF GAMMA FUNCTION APPEARING IN GENERALIZING PROBABILITY SAMPLING DESIGNS

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ABSTRACT. In this paper, we investigate the complete monotonicity of some functions involving gamma function. Using the monotonic properties of these functions, we derived some inequalities involving gamma and beta functions. Such inequalities can be used to generalize different probability distribution functions. Also, they can be used to generalize some statistical designs, e.g., the probability proportional to the size without replacement design.

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1. Introduction

Completely monotonic functions play a major role in Probability and Mathematical Analysis due to their monotonic properties [9, 10, 24]. A function $f(z)$ is called completely monotonic on an interval $I \subset \mathbb{R}$ if it has derivatives of any order $f^{(n)}(z),\ n = 0, 1, 2, 3, \cdots$, and if

$$(-1)^n f^{(n)}(z) \geq 0$$

for all $z \in I$ and all $n \geq 0$. Recently, complete monotonicity of functions involving gamma and beta functions have been considered in many articles, see, e.g., [8, 18, 19, 20, 26, 31]. Also, many articles have appeared to provide various inequalities of functions involving gamma and beta functions, see, e.g., [2, 3, 4, 5, 6, 7, 11, 12, 13, 14, 15, 17, 21, 22, 23, 25, 27, 28, 32]. In this paper, we investigate the complete monotonicity of some functions involving gamma function. Using the monotonic properties of these functions, we derive some inequalities involving gamma function, where gamma function is defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt,\ x > 0.$$

Due to the relation between gamma and beta function, we derived some inequalities that are involving beta function. Commonly, beta function is defined by

$$\mathcal{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt,\ x, y > 0,$$

and its relation with the gamma function is given by

$$\mathcal{B}(x, y) = \frac{\Gamma(x+y)}{\Gamma(x)\Gamma(y)},\ x, y > 0.$$

The layout of the paper: In Section 2, we prove the main results. Section 3 is devoted for discussions and , and Section 4 is devoted for concluding remarks.

2. The Main Results

In this section, we present some preliminarily results and we prove the main result in the paper. We start by defining the complete monotonic functions.

Definition 2.1. A function $f(z)$ is called completely monotonic on an interval I if it has derivatives of any order $f^{(n)}(z),\ n = 0, 1, 2, 3, \cdots$, and if

$$(-1)^n f^{(n)}(z) \geq 0$$

for all $z \in I$ and all $n \geq 0$. If the above inequality is strict for all $z \in I$ and all $n \geq 0$, then $f(z)$ is called strictly completely monotonic.

Definition 2.2. A function $f(z)$ is called logarithmically completely monotonic on an interval I if its logarithm has derivatives $[\ln f(z)]^{(n)}$ of orders $n \geq 1$, and if

$$(-1)^n [\ln f(z)]^{(n)} \geq 0$$

for all $z \in I$ and all $n \geq 1$. If the above inequality is strict for all $z \in I$ and all $n \geq 1$, then $f(z)$ is called strictly logarithmically completely monotonic.

Theorem 2.1. [26]. *Every (strict) logarithmically completely monotonic function is (strict) completely monotonic.*

Now, we turn to prove our main result. For this purpose, we prove a sequence of lemmas. First, we prove the following lemma. In its proof, we employ the approach improved in [20].

Lemma 2.2. *Let* $a, b \geq 0$. *Define* $f(z) = \dfrac{\Gamma(z+1)\Gamma(z-a-b+1)}{\Gamma(z-a+1)\Gamma(z-b+1)}$, $z > a+b-1$. *Then* $f(z)$ *is a completely monotonic function. Moreover, the the following inequality holds:*

$$\frac{\Gamma(z+1)\Gamma(z-a-b+1)}{\Gamma(z-a+1)\Gamma(z-b+1)} \geq 1, \quad z > a+b-1. \tag{2.1}$$

Proof. Let $f(z) = \dfrac{\Gamma(z+1)\Gamma(z-a-b+1)}{\Gamma(z-a+1)\Gamma(z-b+1)}$. Then $f(z) > 0$, and

$$\ln f(z) = \ln\Gamma(z+1) + \ln\Gamma(z-a-b+1) + \ln\Gamma(z-a+1) + \ln\Gamma(z-b+1). \tag{2.2}$$

By differentiating Eq. (2.2), we get

$$\begin{aligned}\frac{d}{dz}\ln f(z) &= \frac{\Gamma'(z+1)}{\Gamma(z+1)} + \frac{\Gamma'(z-a-b+1)}{\Gamma(z-a-b+1)} - \frac{\Gamma'(z-a+1)}{\Gamma(z-a+1)} - \frac{\Gamma'(z-b+1)}{\Gamma(z-b+1)} \\ &= \Psi(z+1) + \Psi(z-a-b+1) - \Psi(z-a+1) - \Psi(z-b+1),\end{aligned} \tag{2.3}$$

where $\Psi(z)$ is the Digamma function (the logarithmic differentiation of the $\Gamma-$function). By using the integral representation of Digamma function

$$\Psi(z) = -\gamma + \int_0^\infty \frac{(e^{-t} - e^{-zt})}{1 - e^{-t}} dt, \quad Re(z) > 0,$$

where $\gamma = 0.577218...$ is Euler's constant [1], we get

$$\begin{aligned}\frac{d}{dz}\ln f(z) &= -\int_0^\infty \frac{e^{-t(z-a-b+1)}}{1-e^{-t}} \left(e^{-(a+b)t} - e^{-at} - e^{-bt} + 1\right) dt \\ &= -\int_0^\infty \frac{e^{-t(z-a-b+1)}}{1-e^{-t}} \left(1 - e^{-at}\right)\left(1 - e^{-bt}\right) dt \\ &\leq 0, \quad \forall z > a+b-1.\end{aligned} \tag{2.4}$$

Therefore, $-\frac{d}{dz}\ln f(z) \geq 0$, $\forall z > a+b-1$. Inductively, we have

$$(-1)^n \frac{d^n}{dz^n}\ln f(z) = \int_0^\infty \frac{t^{n-1}e^{-t(z-a-b+1)}}{1-e^{-t}} \left(1 - e^{-at}\right)\left(1 - e^{-bt}\right) dt \geq 0, \quad \forall z > a+b-1. \tag{2.5}$$

Hence, by Theorem 2.1, $f(z)$ is a completely monotonic function. Since $f(z)$ is a completely monotonic function. Then it is a decreasing function. By applying the asymptotic relation, see, e.g., [16, 30],

$$\lim_{z\to\infty} z^{(b-a)} \,\frac{\Gamma(z+a)}{\Gamma(z+b)} = 1. \tag{2.6}$$

Then, the inequality holds.□

By using the same argument above, we can show that $f(z) = \dfrac{\Gamma(z+a)\Gamma(z-a)}{\Gamma(z)^2}$, for all $z > a \geq 0$, is a completely monotonic function. Thus, the inequality $\dfrac{\Gamma(z+a)\Gamma(z-a)}{\Gamma(z)^2} \geq 1$, for all $z > a \geq 0$, holds. In fact, this inequality has been proved in [15] by using some classical integral inequalities. Now, if we replace z by $z+1$ in the above inequality, then we get

$$\frac{\Gamma(z+a+1)\Gamma(z-a+1)}{\Gamma(z+1)^2} \geq 1, \tag{2.7}$$

for all $z > a-1$, and $a \geq 0$. Moreover, if we assume that $a \in [0, 1)$, then the following two inequalities hold:

I)

$$1 \leq \frac{\Gamma(z+a+1)\Gamma(z-a+1)}{\Gamma(z+1)^2} \leq \Gamma(1-a)\Gamma(1+a),\ z \geq 0. \tag{2.8}$$

II)

$$\frac{z^2}{z^2-a^2} \leq \frac{\Gamma(z+a)\Gamma(z-a)}{\Gamma(z)^2} \leq \frac{z^2}{z^2-a^2}(\Gamma(1-a)\Gamma(1+a)),\ z > a. \tag{2.9}$$

To get the upper bounds in the above two inequalities, we used the fact that $f(z) = \dfrac{\Gamma(z+a+1)\Gamma(z-a+1)}{\Gamma(z+1)^2}$, is a completely monotonic function for all $z > a-1$. Thus, it is a decreasing function. So, $f(z) \leq f(0) = \Gamma(1-a)\Gamma(1+a)$. Also, we get use from the well known identity $\Gamma(z+1) = z\Gamma(z)$.

Lemma 2.3. *Let $a, b > 0$, and let M be a positive number such that $a + b \leq M < \infty$. Moreover, let $f(z) = \dfrac{\Gamma(z+1)\Gamma(z-a-b+1)}{\Gamma(z-a+1)\Gamma(z-b+1)}, z \in [a+b, M]$. Then $f(z)$ is a strictly decreasing function on $[a+b, M]$.*

Proof. Define the function $\delta(z) = \dfrac{e^{-t(z-a-b+1)}}{1-e^{-t}}(1-e^{-at})(1-e^{-bt}),\ z \in [a+b-1, M],\ t \in [0, \infty)$. Then $\delta(z)$ is nonnegative and is not identically zero function on $z \in [a+b-1, M]$. Hence, as a consequence of the proof of Theorem 2.2, $f'(z) < 0, \forall z \in (a+b-1, M)$. Using this fact with the continuity of $f(z)$, we get that $f(z)$ is a strictly decreasing function on $[a+b, M]$. □

Remark 2.1. Similarly, we can prove that $g(z) = \dfrac{\Gamma(z-a+1)\Gamma(z-b+1)}{\Gamma(z+1)\Gamma(z-a-b+1)},\ a, b > 0$, is strictly increasing function on $[a+b, M]$, and $g'(z) > 0$ on $(a+b-1, M)$. Moreover, $g(z) \leq 1, \forall z \in (a+b-1, \infty)$.

Lemma 2.4. *Let $a, b > 0$, and define*

$$h(x,y) := f(x)g(y) = \frac{\Gamma(x+1)\Gamma(x-a-b+1)}{\Gamma(x-a+1)\Gamma(x-b+1)} \frac{\Gamma(y-a+1)\Gamma(y-b+1)}{\Gamma(y+1)\Gamma(y-a-b+1)},\ x, y \geq a+b.$$

Then $h(x,y)$ is positive and continuous function on the rectangular domain $\mathcal{D} = [a+b, \infty) \times [a+b, \infty)$. Moreover, $\lim_{\|(x,y)\| \to \infty} h(x,y) \to 1,\ \forall (x,y) \in \mathcal{D}$.

Proof. Clearly, $h(x,y)$ is continuous and positive from its definition. Using the asymptotic relation (2.6), we conclude that $\lim_{\|(x,y)\| \to \infty} h(x,y) \to 1,\ \forall (x,y) \in \mathcal{D}$. □

Now, we turn to prove the main result in the paper. Particularly, we prove the multi-variable inequality

$$\frac{\Gamma(x+1)\Gamma(x-a-b+1)}{\Gamma(x-a+1)\Gamma(x-b+1)} \frac{\Gamma(y-a+1)\Gamma(y-b+1)}{\Gamma(y+1)\Gamma(y-a-b+1)} \geq 1,\ 0 < a \leq b < a+b \leq x \leq y. \tag{2.10}$$

This inequality is useful to generalize some probability distributions such as the hypergeometric distribution, and it is also useful to generalize some statistical designs such as the probability proportional to size without replacement design.

Theorem 2.5. *Let $a, b > 0$, and let $\Omega = \{(x,y) \mid a+b \leq x \leq y\} \subset \mathcal{D}$. Then*

$$h(x,y) := \frac{\Gamma(x+1)\Gamma(x-a-b+1)}{\Gamma(x-a+1)\Gamma(x-b+1)} \frac{\Gamma(y-a+1)\Gamma(y-b+1)}{\Gamma(y+1)\Gamma(y-a-b+1)} \geq 1,\ \forall (x,y) \in \Omega.$$

Proof. Since $h(x,y)$ is positive, continuous, and $\lim_{\|(x,y)\|\to\infty} h(x,y) \to 1$. Then there exists a sufficiently large constant $M > 0$, such that

$$h(a+b, M) = \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)} \frac{\Gamma(M-a+1)\Gamma(M-b+1)}{\Gamma(M+1)\Gamma(M-a-b+1)} \geq 1.$$

Define the closed and bounded triangular region $\mathcal{R} = \{(x,y) |\ a+b \leq x \leq y \leq M\}$. Since $h(x,y)$ is continuous function on $\mathcal{R}$. Then it takes its absolute values on the boundaries of $\mathcal{R}$, or at the points in $\mathcal{R}$ where $\boldsymbol{\nabla} h(x,y) = \mathbf{0}$, here $\nabla h(x,y)$ is the gradient vector of $h(x,y)$. Clearly, $\boldsymbol{\nabla} h(x,y) = f'(x)g(y)\mathbf{i} + f(x)g'(y)\mathbf{j}$. Moreover, we have $f(x) > 0,\ g(y) > 0,\ g'(y) > 0$, and $f'(x) < 0$ in $\mathcal{R}$. Hence, $\boldsymbol{\nabla} h(x,y) \neq \mathbf{0}$ in $\mathcal{R}$. Hence, the absolute values of $h(x,y)$ must occur at the boundaries of $\mathcal{R}$. In fact, the boundaries of $\mathcal{R}$ are the line segments

(1) $l_1 = \{(x, M) \mid a+b \leq x \leq M\}$,
(2) $l_2 = \{(a+b, y) \mid a+b \leq y \leq M\}$, and
(3) $l_3 = \{(x, x) \mid a+b \leq x \leq M\}$.

Obviously, $h(x,y) = h(x,x) = 1$ on the line segment l_3. Moreover, on the line segment l_1, we have

$$h(x, M) = \frac{\Gamma(x+1)\Gamma(x-a-b+1)}{\Gamma(x-a+1)\Gamma(x-b+1)} \frac{\Gamma(M-a+1)\Gamma(M-b+1)}{\Gamma(M+1)\Gamma(M-a-b+1)},\ a+b \leq x \leq M,$$

which is a decreasing function in x with a negative derivative. Similarly, on the line segment l_1, we have

$$h(a+b, y) = \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)} \frac{\Gamma(y-a+1)\Gamma(y-b+1)}{\Gamma(y+1)\Gamma(y-a-b+1)},\ a+b \leq y \leq M,$$

which is an increasing function in y with a positive derivative. Hence, $h(x,y)$ takes its absolute values at the points $(a+b, a+b)$, (M, M), and $(a+b, M)$. Clearly, $h(a+b, a+b) = h(M, M) = 1$. At the point $(a+b, M)$, we have

$$h(a+b, M) = \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)} \frac{\Gamma(M-a+1)\Gamma(M-b+1)}{\Gamma(M+1)\Gamma(M-a-b+1)} \geq 1.$$

This implies that $h(x,y) \geq 1,\ \forall (x,y) \in \mathcal{R}$. By letting $M \to \infty$ and by using the limit $\lim_{\|(x,y)\|\to\infty} h(x,y) \to 1$, we get $h(x,y) \geq 1,\ \forall (x,y) \in \Omega$. This completes the proof. $\square$

Remark 2.2. Let $\psi(x,y) = \dfrac{\Gamma(x+1)\Gamma(y-a+1)}{\Gamma(y+1)\Gamma(x-a+1)},\ y > x > a > 0$. Let $x \in (a, y)$. Then there exists $\alpha_x > 0$, such that $x = y - \alpha_x$. Define

$$\xi(y) = \psi(y - \alpha_x, y) = \frac{\Gamma(y-\alpha_x+1)\Gamma(y-a+1)}{\Gamma(y+1)\Gamma(y-\alpha_x-a+1)},\ y > \alpha_x + a > a > 0.$$

Then, by (2.1), we have

$$\psi(y-\alpha_x, y) = \frac{\Gamma(y-\alpha_x+1)\Gamma(y-a+1)}{\Gamma(y+1)\Gamma(y-\alpha_x-a+1)} \leq 1, \ \ \forall y > \alpha_x + a > a > 0.$$

Therefore,

$$\psi(x,y) = \frac{\Gamma(x+1)\Gamma(y-a+1)}{\Gamma(y+1)\Gamma(x-a+1)} \leq 1, \ \ y > x > a > 0.$$

Thus,

$$\frac{\Gamma(x+1)\Gamma(y-a-b+1)}{\Gamma(y+1)\Gamma(x-a-b+1)} \leq 1,\ y > x > a+b > b \geq a > 0.$$

3. DISCUSSION AND CONCLUDING REMARKS

In this section, we consider some special cases of our results. Thus, we have the following sequence of concluding remarks:

Remark 3.1. Let $z = a + b,\ a, b \geq 0$ in (2.1). Then, we get

$$\frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)} \geq 1,\ \forall a, b \geq 0. \tag{3.1}$$

By using the fact that $\Gamma(z+1) = z\Gamma(z)$, we have

$$\frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)} = \frac{(a+b)}{ab}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \geq 1,\ \forall a, b > 0. \tag{3.2}$$

Hence,

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \geq \frac{ab}{(a+b)},\ \forall a, b > 0. \tag{3.3}$$

Remark 3.2. Since $\mathcal{B}(a, b) = \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$. Then, by Inequality (3.3), we have

$$\mathcal{B}(a, b) \leq \frac{(a+b)}{ab},\ \forall a, b > 0. \tag{3.4}$$

Moreover, by using the facts $\mathcal{B}(x, y+1) = \dfrac{y}{x}\mathcal{B}(x+1, y) = \dfrac{y}{x+y}\mathcal{B}(x, y)$, we get

$$\mathcal{B}(a, b+1) = \frac{b}{a}\mathcal{B}(a+1, b) = \frac{b}{a+b}\mathcal{B}(a, b) \leq \frac{1}{a},\ a, b > 0.$$

Hence, $\mathcal{B}(a, b+1) \leq \dfrac{1}{a}$ and $\mathcal{B}(a+1, b) \leq \dfrac{1}{b}$ for $a, b > 0$. In [15], the authors proved the following inequality:

$$\mathcal{B}(a, b) \leq \frac{1}{ab},\ 0 < a, b \leq 1. \tag{3.5}$$

Clearly, if $0 < a, b \leq 1$ and $a + b \leq 1$, the upper bound of $\mathcal{B}(a, b)$ given in (3.4) is better than the lower bound given in (3.5). Moreover, the inequality (3.4) is valid for any $a, b > 0$ while the inequality (3.5) is restricted on $a, b \in (0, 1]$.

Remark 3.3. Assume that $a, b > 0$ and $0 < z \leq 1$. Then inequality (2.1) implies that

$$\frac{\Gamma(z-a-b+1)}{\Gamma(z-a+1)\Gamma(z-b+1)} \geq \frac{1}{\Gamma(z+1)} \geq 1,\ a, b > 0,\ 0 < z \leq 1. \tag{3.6}$$

This is correct since $\Gamma(x+1) \leq 1, \forall z \in [0, 1]$. Moreover, we have

$$1 \leq \frac{\Gamma(z+1)\Gamma(z-a-b+1)}{\Gamma(z-a+1)\Gamma(z-b+1)} \leq \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)},\ z \geq a+b > b \geq a > 0. \tag{3.7}$$

Hence, for $z \geq a + b > 0$, we have

$$\frac{1}{\Gamma(z+1)} \leq \frac{\Gamma(z-a-b+1)}{\Gamma(z-a+1)\Gamma(z-b+1)} \leq \frac{1}{\Gamma(z+1)}\frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)}. \tag{3.8}$$

Therefore,

$$\lim_{z\to\infty} \frac{\Gamma(z-a-b+1)}{\Gamma(z-a+1)\Gamma(z-b+1)} = 0.$$

i.e., for large values of z, we have

$$\frac{\Gamma(z-a-b+1)}{\Gamma(z-a+1)\Gamma(z-b+1)} \leq 1.$$

Remark 3.4. Let $a=b$ in (2.1). Then, we get

$$\frac{\Gamma(z+1)\Gamma(z-2a+1)}{\Gamma(z-a+1)^2} \geq 1,\ z > 2a-1.$$

Equivalently,

$$\frac{\Gamma(z)\Gamma(z-2a)}{\Gamma(z-a)^2} \geq \frac{(z-a)^2}{z(z-2a)} = 1 + \frac{a^2}{z(z-2a)},\ z > 2a. \tag{3.9}$$

Remark 3.5. Let $z=a$ in (2.7). Then, we get

$$\frac{\Gamma(2a+1)}{\Gamma(a+1)^2} \geq 1,\ a \geq 0, \tag{3.10}$$

Particularly,

$$\frac{\Gamma(2a)}{\Gamma(a)^2} \geq \frac{a}{2},\ a > 0. \tag{3.11}$$

Assume that $0 < a \leq 1$. Then the following inequality

$$\frac{\Gamma(a)^2}{\Gamma(2a)} \geq \frac{2a-a^2}{a^2}, \tag{3.12}$$

holds [21]. Combining (3.11) and (3.12), we get

$$\frac{2a-a^2}{a^2} \leq \frac{\Gamma(a)^2}{\Gamma(2a)} \leq \frac{2}{a},\ 0 < a \leq 1.$$

Remark 3.6. Let $a \in (0,1)$ in (2.9). Since $\Gamma(1-a)\Gamma(1+a) = \frac{\pi a}{\sin \pi a}$, see, e.g., [29]. Then, we have

$$\frac{z^2}{z^2-a^2} \leq \frac{\Gamma(z+a)\Gamma(z-a)}{\Gamma(z)^2} \leq \frac{\pi a z^2}{\sin \pi a (z^2-a^2)},\ z > a.$$

For a particular case, consider $a = \frac{1}{2}$ in above inequality. Then, we have

$$\frac{4z^2}{4z^2-1} \leq \frac{\Gamma(z+\frac{1}{2})\Gamma(z-\frac{1}{2})}{\Gamma(z)^2} \leq \frac{2\pi z^2}{4z^2-1},\ z > \frac{1}{2}.$$

Similarly, if we let $a = \frac{1}{2}$ in (2.8), then we get

$$1 \leq \frac{\Gamma(z+\frac{3}{2})\Gamma(z+\frac{1}{2})}{\Gamma(z+1)^2} \leq \frac{\pi}{2},\quad z \geq 0.$$

Equivalently,

$$\left(\frac{2}{2z+1}\right)^{\frac{1}{2}} \leq \frac{\Gamma(z+\frac{1}{2})}{\Gamma(z+1)} \leq \left(\frac{\pi}{2z+1}\right)^{\frac{1}{2}},\quad z \geq 0.$$

More precisely, we have

$$\left(\frac{2z^2}{2z+1}\right)^{\frac{1}{2}} \leq \frac{\Gamma(z+\frac{1}{2})}{\Gamma(z)} \leq \left(\frac{\pi z^2}{2z+1}\right)^{\frac{1}{2}},\quad z > 0.$$

4. Conclusions

In this paper, we investigated the complete monotonicity of some ratio functions which are involving the gamma function. Using the monotonic properties of the completely monotonic functions and the properties of gamma function, we derived many inequalities involving gamma function. Such inequalities are necessary to generalize some probability distributions and statistical designs. For example, the hypergeometric distribution and the probability proportional to size without replacement design. The generalization of these statistical concepts is out of this paper scope, and we leave it for a future work.

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