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FRACTIONAL $\exp(-\phi(\xi))-$ EXPANSION METHOD AND ITS APPLICATION TO SPACE–TIME NONLINEAR FRACTIONAL EQUATIONS

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ABSTRACT. In this paper, we mainly suggest a new method that depends on the fractional derivative proposed by Katugampola for solving nonlinear fractional partial differential equations. Using this method, we obtained numerous useful and surprising solutions for the space–time fractional nonlinear Whitham–Broer–Kaup equations and space–time fractional generalized nonlinear Hirota–Satsuma coupled KdV equations. The solutions obtained varied between hyperbolic, trigonometric, and rational functions, and we hope those interested in the real-life applications of the previous two equations will find this approach useful.

Key words and phrases: Fractional $\exp(-\phi(\xi))-$ expansion method; Space–time fractional nonlinear Whitham–Broer–Kaup equations; Space–time fractional generalized nonlinear Hirota–Satsuma coupled KdV equations.

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1. INTRODUCTION

Mathematics has long been allied with physics, and while physicists have been studying and modelling various phenomena, mathematicians have been engaged in finding appropriate methods to obtain solutions to earlier mathematical models. Once we obtain such solutions, they are used to explain the original phenomenon from which they originated and to find possible realistic solutions. There are numerous physical phenomena, and we can model some of them in the form of nonlinear partial differential equations, and thus the methods for solving these equations have gained significant importance for those interested in this field. In addition, many effective and direct methods for generating analytical solutions to nonlinear partial differential equations been developed, such as the $\left(\frac{G'}{G}\right)$ expansion method [1, 2], double auxiliary equations method [3, 4], generalized $\exp\left(-\phi\left(\xi\right)\right)-$ expansion method [5, 6], $\coth_a\left(\xi\right)$ expansion method [7], $F-$ expansion method [8, 9], and others [10, 11, 12]. Several definitions of what is now known as the fractional derivative have emerged, such as the Riemann–Liouville fractional derivative, Caputo fractional derivative, and Atangana–Baleanu derivative. Using these derivatives, physicists have been able to study numerous important phenomena in the form of nonlinear fractional partial differential equations (NFPDEs). By contrast, numerous effective methods have been found to solve these equations, such as the modified extended tanh-function method and fractional sub-equation method proposed by Zhang and Zhang [13], which was developed later by Wangi and Xu [14]. The previous methods depend on the following definition of a modified Riemann–Liouville fractional derivative, which was proposed by Jumarie [15, 16, 17]:

$$f^{(\alpha)}\left(t\right)=\frac{1}{\Gamma\left(1-\alpha\right)}\frac{d}{dt}\int_0^t\left(t-x\right)^{-\alpha}\left(f\left(x\right)-f\left(0\right)\right)dx. \tag{1.1}$$

Numerous formulas have been provided by Jumarie, including the following rule (chain rule), which has mainly been used in two previous methods:

$$\left(f\left(u\left(t\right)\right)\right)^{(\alpha)}=f'_u u^{(\alpha)}\left(t\right). \tag{1.2}$$

However, Liu surprisingly published articles (counterexamples of Jumarie's basic fractional calculus formulae) in which the author outlined the invalidity of the previous chain rule, through which the corresponding results on differential equations were shown to be untrue [18, 19]. The findings by Liu led us to search for a new definition of a fractional derivative that satisfies the product, quotient, and chain rules for obtaining the exact traveling wave solutions for nonlinear fractional partial differential equations. Indeed, we found an extremely interesting definition of the fractional derivative proposed by Katugampola [20], which satisfies all of the previous rules, particularly the chain rule. Using this definition, we suggested a new method called the new fractional $\exp\left(-\phi\left(\xi\right)\right)-$ method for obtaining novel and more general exact traveling wave solutions for nonlinear fractional partial differential equations. In [21], the authors applied a modified extended tanh-function method for solving the space–time fractional nonlinear Whitham–Broer-Kaup equations, and space–time fractional generalized nonlinear Hirota–Satsuma coupled KdV equations, and because all of these solutions are incorrect, we applied our proposed method to solve the previous equations and thereby demonstrated the effectiveness and strength of the method, providing others with correct solutions, which were checked using Maple. The remainder of this paper is organized as follows: Section 2 provides some definitions and properties of the fractionals calculus proposed by Katugampola, and describes the fractional $\exp\left(-\phi\left(\xi\right)\right)-$ expansion method. Section 3 describes the application of this method to solving the space–time fractional nonlinear Whitham–Broer-Kaup equations, whereas Section 4 describes its application to solving the space–time fractional generalized nonlinear Hirota–Satsuma coupled KdV equations. Finally, Section 5 provides some concluding remarks regarding this research.

2. DESCRIPTION OF THE FRACTIONAL $\exp(-\phi(\xi))-$ EXPANSION METHOD FOR SOLVING NONLINEAR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

In this section, we present the basic definition of the fractional derivative presented by Katugampola [20], and the most important rules related to this definition, from which the proposed approach has benefited.

Definition 2.1. Let $f:[0,\infty)\to\mathbb{R}$ and $t>0$. The *fractional derivative* of f of order α is defined as follows:

$$D^{\alpha}(f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(te^{\varepsilon t^{-\alpha}}\right)-f(t)}{\varepsilon}, \tag{2.1}$$

for all $t>0, \alpha\in(0,1]$. If f is $\alpha-$differentiable in some $(0,a)\,, a>0$, and $\lim_{t\to 0^+}D^{\alpha}(f)(t)$ exist, then define the following:

$$D^{\alpha}(f)(0)=\lim_{t\to 0^+}D^{\alpha}(f)(t)\,. \tag{2.2}$$

The following theorem is the main result of the previous definition:

Theorem 2.1. Let $\alpha\in(0,1]$ and f,g be $\alpha-$ differentiable at a point $t>0$. Then,

i. $D_t^{\alpha}(af+bg)(t)=aD_t^{\alpha}f(t)+bD_t^{\alpha}g(t)$, for all $a,b\in\mathbb{R}$.
ii. $D_t^{\alpha}(t^n)=nt^{n-\alpha}$, for all $n\in\mathbb{R}$.
iii. $D_t^{\alpha}(c)=0$, for all constant functions $f(t)=c$.
iv. $D_t^{\alpha}(fg)(t)=D_t^{\alpha}f(t)\,g(t)+D_t^{\alpha}g(t)\,f(t)$.
v. $D_t^{\alpha}\left(\frac{f}{g}\right)(t)=\frac{D_t^{\alpha}f(t)g(t)-D_t^{\alpha}(g)f(t)}{g(t)^2}$.
vi. $D_t^{\alpha}(f\circ g)(t)=D_g^{\alpha}f(g(t))\,D_t^{\alpha}g(t)$.
vii. $D_t^{\alpha}\left(\frac{1}{\alpha}t^{\alpha}\right)=1$.

The main steps of the new fractional $\exp(-\phi(\xi))-$ expansion method are described as follows:

Step 1. It is assumed that a nonlinear fractional partial differential equation, i.e., in the independent variables x and t, is given as follows:

$$F(u,u_t,u_x,D_t^{\alpha}u,D_x^{\alpha}u,...)=0,\qquad 0<\alpha\leq 1, \tag{2.3}$$

where $u=u(x,t)$ is an unknown function, F is a polynomial in $u=u(x,t)$, and their various partial derivatives including fractional derivatives, $D_t^{\alpha}u$ and $D_x^{\alpha}u$ are Katugampola's fractional derivative.

Step 2. The traveling wave transformation is used:

$$u(x,t)=U(\xi)\ ,\ \ \xi=\left(\frac{k_1}{\alpha}\right)x^{\alpha}+\left(\frac{k_2}{\alpha}\right)t^{\alpha}, \tag{2.4}$$

where k_1,k_2 are arbitrary constants to be determined later, the nonlinear FDE (2.3) is reduced to an nonlinear fractional ordinary differential equation for $U=U(\xi)$ in the following form:

$$P\left(U,D_{\xi}^{\alpha}U,D_{\xi}^{2\alpha}U,D_{\xi}^{3\alpha}U,...\right)=0. \tag{2.5}$$

Step 3. By balancing the highest derivatives and nonlinear terms in Eq. (2.5), and using the following relationship, the value of the positive integer (m) is determined:

$$Degree\left[U^{p}\left(\frac{d^{q}U}{d\xi^{q}}\right)^{s}\right]=mp+s\left(m+q\right), \tag{2.6}$$

Step 4. It is assumed that the solution to Eq. (2.5) can be expressed as follows:

$$U\left(\xi\right)=\sum_{i=-m}^{m}l_{i}\left(\exp\left(-\phi\left(\xi\right)\right)\right)^{i}, \tag{2.7}$$

where $l_i\left(i=-m,...,m\right)$ are constants to be determined, such that $l_i\neq 0$ and $\phi\left(\xi\right)$ satisfy the following fractional differential equation:

$$D_{\xi}^{\alpha}\phi\left(\xi\right)=\exp\left(-\phi\left(\xi\right)\right)+\mu\exp\left(\phi\left(\xi\right)\right)\lambda. \tag{2.8}$$

Fortunately, Eq. (2.8) allows several of the following types of solutions:

Family 1. When $\mu\neq 0,(\lambda^2-4\mu)>0,$

$$\phi_1\left(\xi\right)=\ln\left(\frac{-\lambda-\sqrt{\lambda^2-4\mu}\tanh\left(\frac{(\xi^{\alpha}+\alpha c)\sqrt{\lambda^2-4\mu}}{2\alpha}\right)}{2\mu}\right),c\in\mathbb{R}. \tag{2.9}$$

Family 2. When $\mu\neq 0,(\lambda^2-4\mu)<0,$

$$\phi_2\left(\xi\right)=\ln\left(\frac{-\lambda+\sqrt{4\mu-\lambda^2}\tan\left(\frac{(\xi^{\alpha}+\alpha c)\sqrt{4\mu-\lambda^2}}{2\alpha}\right)}{2\mu}\right),c\in\mathbb{R}. \tag{2.10}$$

Family 3. When $\mu=0,\lambda\neq 0,$

$$\phi_3\left(\xi\right)=\frac{\lambda}{\alpha}\left(\xi^{\alpha}+\alpha c\right)+\ln\left(\frac{1-e^{-\frac{\lambda}{\alpha}(\xi^{\alpha}+\alpha c)}}{\lambda}\right),c\in\mathbb{R}. \tag{2.11}$$

Family 4. When $\mu\neq 0,\lambda\neq 0,(\lambda^2-4\mu)=0,$

$$\phi_4\left(\xi\right)=\ln\left(-\frac{2}{\lambda}-\frac{4\alpha}{\lambda^2\left(\xi^{\alpha}+\alpha c\right)}\right),c\in\mathbb{R}. \tag{2.12}$$

Family 5. When $\mu=0,\lambda=0,$

$$\phi_5\left(\xi\right)=\ln\left(\frac{\xi^{\alpha}+\alpha c}{\alpha}\right),c\in\mathbb{R}. \tag{2.13}$$

Step 5. Substituting Eqs. (2.7) and (2.5) into Eq. (2.8), respectively, and setting all coefficients of $\left(\exp\left(-\phi\left(\xi\right)\right)\right)^{i}$ of the resulting systems to zero, a system of algebraic equations for k_1,k_2 and $l_i\ \left(i=-m,...,m\right)$ is yielded.

Step 6. By solving the algebraic equations obtained in Step 4, substituting k_1,k_2,l_i and the solutions of Eq. (2.8) into Eq. (2.7), we immediately obtain the explicit solutions to Eq. (2.5).

3. EXACT SOLUTION TO SPACE–TIME FRACTIONAL NONLINEAR WHITHAM–BROER–KAUP EQUATIONS

In this section, we construct the exact solutions of the space–time fractional nonlinear Whitham–Broer–Kaup equations by using the fractional $\exp(-\phi(\xi))-$ expansion method. These equations are well-known [21] and have the following forms:

$$D_t^{\alpha}u+uD_x^{\alpha}u+D_x^{\alpha}v+\beta D_x^{2\alpha}u=0, \tag{3.1}$$

$$D_t^{\alpha}v+D_x^{\alpha}(uv)-\beta D_x^{2\alpha}v+\gamma D_x^{3\alpha}u=0, \tag{3.2}$$

where $0<\alpha\leq 1$, u and v are the functions of (x,t), and γ,β are constants. To solve Eqs. (3.1) and (3.2) using the proposed method, we utilize the following travelling wave transformations:

$$u(x,t)=U(\xi)\,,v(x,t)=V(\xi)\,,\ \ \xi=\left(\frac{k_1}{\alpha}\right)x^{\alpha}+\left(\frac{k_2}{\alpha}\right)t^{\alpha}, \tag{3.3}$$

where k_1,k_2 are constants. Eqs. (3.1) and (3.2) are carried into the following nonlinear fractional ordinary differential equations:

$$\beta k_1^2D_{\xi}^{2\alpha}U+k_1UD_{\xi}^{\alpha}U+k_1D_{\xi}^{\alpha}V+k_2D_{\xi}^{\alpha}U=0, \tag{3.4}$$

$$\gamma k_1^3D_{\xi}^{3\alpha}U-\beta k_1^2D_{\xi}^{2\alpha}V+k_1UD_{\xi}^{\alpha}V+k_1VD_{\xi}^{\alpha}U+k_2D_t^{\alpha}V=0. \tag{3.5}$$

Balancing the highest-order derivatives and highest nonlinear terms in Eqs. (3.4) and (3.5), we have the following formal solutions:

$$U(\xi)=\alpha_0+\alpha_1\exp(\phi(\xi))+\alpha_2\exp(-\phi(\xi))\,, \tag{3.6}$$

$$V(\xi)=\beta_0+\beta_1\exp(\phi(\xi))+\beta_2\exp(-\phi(\xi))+\beta_3\exp(2\phi(\xi))+\beta_4\exp(-2\phi(\xi))\,, \tag{3.7}$$

where $\alpha_i(i=0,1,2),\beta_j(j=0,1,2,3,4)$ are constants to be determined later. Substituting (3.6) and (3.7) along with Eq. (2.8) into Eqs. (3.4) and (3.5), the left-hand side is converted into polynomials in $(\exp(-\phi(\xi)))^i$, $(i=0,\pm1,\pm2,....)$. By collecting each coefficient of these resulted polynomials to zero, we obtain a set of simultaneous algebraic equations, which are not presented herein for the sake of clarity, for $\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2,\beta_3,\beta_4,k_1$ and k_2. Solving these algebraic equations with the help of the algebraic software Maple, we obtain the following three cases:

Case 1

$$\begin{cases}\alpha_0=\left(\frac{\lambda k_1^2\left(\beta^2+\gamma\right)-k_2\sqrt{\beta^2+\gamma}}{k_1\sqrt{\beta^2+\gamma}}\right),\alpha_1=\mu\alpha_2,\alpha_2=2k_1\sqrt{\beta^2+\gamma},\\ k_1=k_1,k_2=k_2,\beta_0=0,\beta_1=\lambda\beta_3,\beta_2=\lambda\beta_4,\\ \beta_3=-2\mu k_1^2\left(\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right),\beta_4=-2k_1^2\left(-\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right).\end{cases} \tag{3.8}$$

Substituting Eq. (3.8) into Eqs. (3.6) and (3.7), we have the following:

$$(3.9)\quad \begin{cases} U(\xi) = \left(\frac{\lambda k_1^2\left(\beta^2+\gamma\right)-k_2\sqrt{\beta^2+\gamma}}{k_1\sqrt{\beta^2+\gamma}}\right) + 2k_1\sqrt{\beta^2+\gamma}\left(\mu\exp\left(\phi\left(\xi\right)\right)+\exp\left(-\phi\left(\xi\right)\right)\right), \\ V(\xi) = \begin{pmatrix} -2\mu k_1^2\left(\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)\left(\lambda\exp\left(\phi\left(\xi\right)\right)+\exp\left(2\phi\left(\xi\right)\right)\right) \\ -2k_1^2\left(-\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)\left(\lambda\exp\left(-\phi\left(\xi\right)\right)+\exp\left(-2\phi\left(\xi\right)\right)\right) \end{pmatrix}, \\ \xi = \left(\frac{k_1}{\alpha}\right)x^\alpha + \left(\frac{k_2}{\alpha}\right)t^\alpha. \end{cases}$$

Case 2

$$(3.10)\quad \begin{cases} \alpha_0 = \left(\frac{\lambda k_1^2\left(\beta^2+\gamma\right)-k_2\sqrt{\beta^2+\gamma}}{k_1\sqrt{\beta^2+\gamma}}\right), \alpha_1 = 0, \alpha_2 = 2k_1\sqrt{\beta^2+\gamma}, k_1 = k_1, k_2 = k_2, \\ \beta_0 = \mu\beta_4, \beta_1 = 0, \beta_2 = \lambda\beta_4, \beta_3 = 0, \beta_4 = 2k_1^2\left(\beta\sqrt{\beta^2+\gamma}-\left(\beta^2+\gamma\right)\right). \end{cases}$$

Substituting (3.10) into (3.6) and (3.7), we have the following:

$$(3.11)\quad \begin{cases} U(\xi) = \left(\frac{\lambda k_1^2\left(\beta^2+\gamma\right)-k_2\sqrt{\beta^2+\gamma}}{k_1\sqrt{\beta^2+\gamma}}\right) + 2k_1\sqrt{\beta^2+\gamma}\exp\left(-\phi\left(\xi\right)\right), \\ V(\xi) = 2k_1^2\left(\beta\sqrt{\beta^2+\gamma}-\left(\beta^2+\gamma\right)\right)\left(\mu+\lambda\exp\left(-\phi\left(\xi\right)\right)+\exp\left(-2\phi\left(\xi\right)\right)\right), \\ \xi = \left(\frac{k_1}{\alpha}\right)x^\alpha + \left(\frac{k_2}{\alpha}\right)t^\alpha. \end{cases}$$

Case 3

$$(3.12)\quad \begin{cases} \alpha_0 = \left(\frac{\lambda k_1^2\left(\beta^2+\gamma\right)-k_2\sqrt{\beta^2+\gamma}}{k_1\sqrt{\beta^2+\gamma}}\right), \alpha_1 = 2\mu k_1\sqrt{\beta^2+\gamma}, \alpha_2 = 0, k_1 = k_1, k_2 = k_2, \\ \beta_0 = -2\mu k_1^2\left(\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right), \beta_1 = \lambda\beta_0, \beta_2 = 0, \beta_3 = \mu\beta_0, \beta_4 = 0. \end{cases}$$

Substituting Eq. (3.12) into Eqs. (3.6) and (3.7), we have the following:

$$(3.13)\quad \begin{cases} U(\xi) = \left(\frac{\lambda k_1^2\left(\beta^2+\gamma\right)-k_2\sqrt{\beta^2+\gamma}}{k_1\sqrt{\beta^2+\gamma}}\right) + 2\mu k_1\sqrt{\beta^2+\gamma}\exp\left(\phi\left(\xi\right)\right), \\ V(\xi) = -2\mu k_1^2\left(\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)\left(1+\lambda\exp\left(\phi\left(\xi\right)\right)+\mu\exp\left(2\phi\left(\xi\right)\right)\right), \\ \xi = \left(\frac{k_1}{\alpha}\right)x^\alpha + \left(\frac{k_2}{\alpha}\right)t^\alpha. \end{cases}$$

In particular, using (3.9) and the solution to Eq. (2.8), we can find the following exact solutions of the space–time fractional nonlinear Whitham–Broer–Kaup equations:

Family 1. When $\mu \neq 0, (\lambda^2 - 4\mu) > 0$,

(3.14)

$$\begin{cases} U_1(\xi) = \begin{pmatrix} \left(\frac{\lambda k_1^2\left(\beta^2+\gamma\right)-k_2\sqrt{\beta^2+\gamma}}{k_1\sqrt{\beta^2+\gamma}}\right) - \left(\frac{4\mu k_1\sqrt{\beta^2+\gamma}}{\lambda+\sqrt{\lambda^2-4\mu}\tanh\left(\frac{(\xi^\alpha+\alpha c)\sqrt{\lambda^2-4\mu}}{2\alpha}\right)}\right) \\ -k_1\sqrt{\beta^2+\gamma}\left(\lambda+\sqrt{\lambda^2-4\mu}\tanh\left(\frac{(\xi^\alpha+\alpha c)\sqrt{\lambda^2-4\mu}}{2\alpha}\right)\right) \end{pmatrix}, \\ V_1(\xi) = \begin{pmatrix} \lambda k_1^2\left(\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)\left(\lambda+\sqrt{\lambda^2-4\mu}\tanh\left(\frac{(\xi^\alpha+\alpha c)\sqrt{\lambda^2-4\mu}}{2\alpha}\right)\right) \\ -\frac{k_1^2\left(\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)}{2\mu}\left(\lambda+\sqrt{\lambda^2-4\mu}\tanh\left(\frac{(\xi^\alpha+\alpha c)\sqrt{\lambda^2-4\mu}}{2\alpha}\right)\right)^2 \\ +\frac{4\lambda\mu k_1^2\left(-\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)}{\lambda+\sqrt{\lambda^2-4\mu}\tanh\left(\frac{(\xi^\alpha+\alpha c)\sqrt{\lambda^2-4\mu}}{2\alpha}\right)} - \frac{8\mu^2 k_1^2\left(-\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)}{\left(\lambda+\sqrt{\lambda^2-4\mu}\tanh\left(\frac{(\xi^\alpha+\alpha c)\sqrt{\lambda^2-4\mu}}{2\alpha}\right)\right)^2} \end{pmatrix}, \\ \xi = \left(\frac{k_1}{\alpha}\right)x^\alpha + \left(\frac{k_2}{\alpha}\right)t^\alpha. \end{cases}$$

Family 2. When $\mu \neq 0, (\lambda^2 - 4\mu) < 0$,

(3.15)

$$\begin{cases} U_2(\xi) = \begin{pmatrix} \left(\frac{\lambda k_1^2\left(\beta^2+\gamma\right)-k_2\sqrt{\beta^2+\gamma}}{k_1\sqrt{\beta^2+\gamma}}\right) + \frac{4\mu k_1\sqrt{\beta^2+\gamma}}{-\lambda+\sqrt{4\mu-\lambda^2}\tan\left(\frac{(\xi^\alpha+\alpha c)\sqrt{4\mu-\lambda^2}}{2\alpha}\right)} \\ +k_1\sqrt{\beta^2+\gamma}\left(-\lambda+\sqrt{4\mu-\lambda^2}\tan\left(\frac{(\xi^\alpha+\alpha c)\sqrt{4\mu-\lambda^2}}{2\alpha}\right)\right) \end{pmatrix}, \\ V_2(\xi) = \begin{pmatrix} -\lambda k_1^2\left(\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)\left(-\lambda+\sqrt{4\mu-\lambda^2}\tan\left(\frac{(\xi^\alpha+\alpha c)\sqrt{4\mu-\lambda^2}}{2\alpha}\right)\right) \\ -\frac{k_1^2\left(\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)}{2\mu}\left(-\lambda+\sqrt{4\mu-\lambda^2}\tan\left(\frac{(\xi^\alpha+\alpha c)\sqrt{4\mu-\lambda^2}}{2\alpha}\right)\right)^2 \\ -\frac{4\lambda\mu k_1^2\left(-\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)}{-\lambda+\sqrt{4\mu-\lambda^2}\tan\left(\frac{(\xi^\alpha+\alpha c)\sqrt{4\mu-\lambda^2}}{2\alpha}\right)} - \frac{8\mu^2 k_1^2\left(-\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)}{\left(-\lambda+\sqrt{4\mu-\lambda^2}\tan\left(\frac{(\xi^\alpha+\alpha c)\sqrt{4\mu-\lambda^2}}{2\alpha}\right)\right)^2} \end{pmatrix}, \\ \xi = \left(\frac{k_1}{\alpha}\right)x^\alpha + \left(\frac{k_2}{\alpha}\right)t^\alpha. \end{cases}$$

Family 3. When $\mu = 0, \lambda \neq 0$,

(3.16)

$$\begin{cases} U_3(\xi) = \left(\frac{\lambda k_1^2(\beta^2+\gamma)-k_2\sqrt{\beta^2+\gamma}}{k_1\sqrt{\beta^2+\gamma}}\right) + 2k_1\sqrt{\beta^2+\gamma}\exp\left(\ln\left(\frac{\lambda}{1-e^{-\frac{\lambda}{\alpha}(\xi^\alpha+c\alpha)}}\right) - \frac{\lambda}{\alpha}(\xi^\alpha+c\alpha)\right), \\ V_3(\xi) = \begin{pmatrix} -2\lambda k_1^2\left(-\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)\exp\left(\ln\left(\frac{\lambda}{1-e^{-\frac{\lambda}{\alpha}(\xi^\alpha+c\alpha)}}\right) - \frac{\lambda}{\alpha}(\xi^\alpha+c\alpha)\right) \\ -2k_1^2\left(-\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)\exp\left(\ln\left(\frac{\lambda}{1-e^{-\frac{\lambda}{\alpha}(\xi^\alpha+c\alpha)}}\right)^2 - \frac{2\lambda}{\alpha}(\xi^\alpha+c\alpha)\right) \end{pmatrix}, \\ \xi = \left(\frac{k_1}{\alpha}\right)x^\alpha + \left(\frac{k_2}{\alpha}\right)t^\alpha. \end{cases}$$

Family 4. When $\mu \neq 0, \lambda \neq 0, (\lambda^2 - 4\mu) = 0$,

(3.17)

$$\begin{cases} U_4(\xi) = \left(\frac{\lambda k_1^2(\beta^2+\gamma)-k_2\sqrt{\beta^2+\gamma}}{k_1\sqrt{\beta^2+\gamma}}\right) - 2\mu k_1\sqrt{\beta^2+\gamma}\left(\frac{2}{\lambda}+\frac{4\alpha}{\lambda^2(\xi^\alpha+\alpha c)}\right) - \frac{2k_1\sqrt{\beta^2+\gamma}}{\left(\frac{2}{\lambda}+\frac{4\alpha}{\lambda^2(\xi^\alpha+\alpha c)}\right)}, \\ V_4(\xi) = \begin{pmatrix} 2\lambda\mu k_1^2\left(\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)\left(\frac{2}{\lambda}+\frac{4\alpha}{\lambda^2(\xi^\alpha+\alpha c)}\right) \\ -2\mu k_1^2\left(\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)\left(\frac{2}{\lambda}+\frac{4\alpha}{\lambda^2(\xi^\alpha+\alpha c)}\right)^2 \\ +\frac{2\lambda k_1^2\left(-\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)}{\left(\frac{2}{\lambda}+\frac{4\alpha}{\lambda^2(\xi^\alpha+\alpha c)}\right)} - \frac{2k_1^2\left(-\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)}{\left(\frac{2}{\lambda}+\frac{4\alpha}{\lambda^2(\xi^\alpha+\alpha c)}\right)^2} \end{pmatrix}, \\ \xi = \left(\frac{k_1}{\alpha}\right)x^\alpha + \left(\frac{k_2}{\alpha}\right)t^\alpha. \end{cases}$$

Family 5. When $\mu = 0, \lambda = 0$,

$$\begin{cases} U_5(\xi) = -\frac{k_2}{k_1} + 2k_1\sqrt{\beta^2+\gamma}\left(\frac{\alpha}{\xi^\alpha+\alpha c}\right), \\ V_5(\xi) = -2k_1^2\left(-\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma\right)\left(\frac{\alpha}{\xi^\alpha+\alpha c}\right)^2, \\ \xi = \left(\frac{k_1}{\alpha}\right)x^\alpha + \left(\frac{k_2}{\alpha}\right)t^\alpha. \end{cases} \tag{3.18}$$

Similarly, using cases 2 and 3 we can obtain more exact solutions of the space–time fractional nonlinear Whitham–Broer–Kaup equations, which we do not list in their entirety herein for simplicity.

4. EXACT SOLUTION FOR THE SPACE–TIME FRACTIONAL GENERALIZED NONLINEAR HIROTA–SATSUMA COUPLED KDV EQUATIONS

Space–time fractional generalized Hirota–Satsuma coupled KdV equations are well-known [21], and have the following forms:

$$D_t^{\alpha} g - \frac{1}{2} D_x^{3\alpha} g + 3g D_x^{\alpha} g - 3D_x^{\alpha}(hw) = 0, \tag{4.1}$$

$$D_t^{\alpha} h + D_x^{3\alpha} h - 3g D_x^{\alpha} h = 0, \tag{4.2}$$

$$D_t^{\alpha} w + D_x^{3\alpha} w - 3g D_x^{\alpha} w = 0, \tag{4.3}$$

where $0 < \alpha \leq 1, g, h$ and w are the functions of (x, t).
By considering the traveling wave transformation, we have the following:

$$g(x,t) = G(\xi),\ h(x,t) = H(\xi),\ w(x,t) = W(\xi),\ \xi = \left(\frac{k_1}{\alpha}\right) x^{\alpha} + \left(\frac{k_2}{\alpha}\right) t^{\alpha}, \tag{4.4}$$

Eqs. (4.1), (4.2), and (4.3) can be reduced to the following nonlinear fractional ordinary differential equations:

$$k_2 D_\xi^{\alpha} G - \frac{1}{2} k_1^3 D_\xi^{3\alpha} G + 3k_1 G D_\xi^{\alpha} G - 3k_1 W D_\xi^{\alpha} H - 3k_1 H D_\xi^{\alpha} W = 0, \tag{4.5}$$

$$k_2 D_\xi^{\alpha} H + k_1^3 D_\xi^{3\alpha} H - 3k_1 G D_\xi^{\alpha} H = 0, \tag{4.6}$$

$$k_2 D_\xi^{\alpha} W + k_1^3 D_\xi^{3\alpha} W - 3k_1 G D_\xi^{\alpha} W = 0, \tag{4.7}$$

By balancing the highest-order derivative and nonlinear terms in the previous equations, we have the following formal solutions:

$$G(\xi) = \alpha_0 + \alpha_1 \exp(\phi(\xi)) + \alpha_2 \exp(-\phi(\xi)) + \alpha_3 \exp(2\phi(\xi)) + \alpha_4 \exp(-2\phi(\xi)), \tag{4.8}$$

$$H(\xi) = \beta_0 + \beta_1 \exp(\phi(\xi)) + \beta_2 \exp(-\phi(\xi)) + \beta_3 \exp(2\phi(\xi)) + \beta_4 \exp(-2\phi(\xi)), \tag{4.9}$$

$$W(\xi) = \gamma_0 + \gamma_1 \exp(\phi(\xi)) + \gamma_2 \exp(-\phi(\xi)) + \gamma_3 \exp(2\phi(\xi)) + \gamma_4 \exp(-2\phi(\xi)), \tag{4.10}$$

where $\alpha_i, \beta_i, \gamma_i (i = 0, 1, 2, 3, 4)$ are constants to be determined later. Substituting (4.8), (4.9), and (4.10) along with Eq. (2.8) into Eqs. (4.5), (4.6), and (4.7), all of the terms with the same power of $(\exp(-\phi(\xi)))^i$, $(i = 0, \pm 1, \pm 2, ...)$ are collected. Setting all coefficients of $(\exp(-\phi(\xi)))^i$, $(i = 0, \pm 1, \pm 2, ...)$ to zero, we can obtain a set of algebraic equations for $\alpha_i, \beta_i, \gamma_i (i = 0, 1, 2, 3, 4), k_1$ and k_2. Solving these algebraic equations using Maple, we have the following cases:

Case 1

$$\begin{cases} \alpha_0 = \frac{k_2 + k_1^3\left(\lambda^2 + 8\mu\right)}{3k_1}, \alpha_1 = 2\lambda\mu k_1^2, \alpha_2 = 2\lambda k_1^2, \alpha_3 = 2\mu^2 k_1^2, \alpha_4 = 2k_1^2, \\ \beta_0 = \frac{k_1\mu\left(\gamma_1\lambda k_1^3\left(\lambda^2+20\mu\right) - \gamma_0\mu k_1^3\left(\lambda^2+8\mu\right) - 4k_2(\gamma_0\mu - \gamma_1\lambda)\right)}{3\gamma_1^2}, \beta_1 = \mu\beta_2, \beta_2 = \frac{k_1\mu\left(4k_2 + k_1^3\left(\lambda^2+8\mu\right)\right)}{3\gamma_1} \\ \beta_3 = 0, \beta_4 = 0, \gamma_0 = \gamma_0, \gamma_1 = \gamma_1, \gamma_2 = \frac{\gamma_1}{\mu}, \gamma_3 = 0, \gamma_4 = 0, k_1 = k_1, k_2 = k_2, \end{cases} \tag{4.11}$$

Substituting (4.11) into Eqs. (4.8), (4.9), and (4.10), we have the following:

(4.12)
$$\begin{cases} G(\xi) = \begin{pmatrix} \frac{k_2+k_1^3\left(\lambda^2+8\mu\right)}{3k_1} + \\ 2\lambda k_1^2\left(\mu\exp\left(\phi\left(\xi\right)\right)+\exp\left(-\phi\left(\xi\right)\right)\right)+2k_1^2\left(\mu^2\exp\left(2\phi\left(\xi\right)\right)+\exp\left(-2\phi\left(\xi\right)\right)\right) \end{pmatrix}, \\ H(\xi) = \begin{pmatrix} \left(\frac{k_1\mu\left(\gamma_1\lambda k_1^3\left(\lambda^2+20\mu\right)-\gamma_0\mu k_1^3\left(\lambda^2+8\mu\right)-4k_2\left(\gamma_0\mu-\gamma_1\lambda\right)\right)}{3\gamma_1^2}\right) \\ +\frac{k_1\mu\left(4k_2+k_1^3\left(\lambda^2+8\mu\right)\right)}{3\gamma_1}\left(\mu\exp\left(\phi\left(\xi\right)\right)+\exp\left(-\phi\left(\xi\right)\right)\right) \end{pmatrix}, \\ W(\xi) = \gamma_0+\gamma_1\exp\left(\phi\left(\xi\right)\right)+\frac{\gamma_1}{\mu}\exp\left(-\phi\left(\xi\right)\right), \\ \xi = \left(\frac{k_1}{\alpha}\right)x^\alpha+\left(\frac{k_2}{\alpha}\right)t^\alpha. \end{cases}$$

Case 2

$$\begin{cases} \alpha_0 = \frac{k_2+k_1^3\left(\lambda^2+8\mu\right)}{3k_1}, \alpha_1 = 4\lambda\mu k_1^2, \alpha_2 = 0, \alpha_3 = 4\mu^2k_1^2, \alpha_4 = 0, \\ \beta_0 = \beta_0, \beta_1 = \frac{\lambda\beta_3}{\mu}, \beta_2 = 0, \beta_3 = \beta_3, \beta_4 = 0, k_1 = k_1, k_2 = k_2, \\ \gamma_0 = \frac{2k_1\mu^2\left(\beta_3k_1^3\left(\lambda^2+8\mu\right)+4\beta_3k_2-6\beta_0k_1^3\mu^2\right)}{3\beta_3^3}, \gamma_1 = \frac{4\lambda\mu^3k_1^4}{\beta_3}, \gamma_2 = 0, \gamma_3 = \frac{4\mu^4k_1^4}{\beta_3}, \gamma_4 = 0. \end{cases} \tag{4.13}$$

Substituting (4.13) into Eqs. (4.8), (4.9), and (4.10), the following is derived:

(4.14)
$$\begin{cases} G(\xi) = \frac{k_2+k_1^3\left(\lambda^2+8\mu\right)}{3k_1}+4\lambda\mu k_1^2\exp\left(\phi\left(\xi\right)\right)+4\mu^2k_1^2\exp\left(2\phi\left(\xi\right)\right), \\ H(\xi) = \beta_0+\frac{\lambda\beta_3}{\mu}\exp\left(\phi\left(\xi\right)\right)+\beta_3\exp\left(2\phi\left(\xi\right)\right), \\ W(\xi) = \frac{2k_1\mu^2\left(\beta_3k_1^3\left(\lambda^2+8\mu\right)+4\beta_3k_2-6\beta_0k_1^3\mu^2\right)}{3\beta_3^3}+\frac{4\lambda\mu^3k_1^4}{\beta_3}\exp\left(\phi\left(\xi\right)\right)+\frac{4\mu^4k_1^4}{\beta_3}\exp\left(2\phi\left(\xi\right)\right), \\ \xi = \left(\frac{k_1}{\alpha}\right)x^\alpha+\left(\frac{k_2}{\alpha}\right)t^\alpha. \end{cases}$$

Case 3

(4.15)
$$\begin{cases} \alpha_0 = \frac{k_2+k_1^3\left(\lambda^2+2\mu\right)}{3k_1}, \alpha_1 = 2\lambda\mu k_1^2, \alpha_2 = 0, \alpha_3 = 2\mu^2k_1^2, \alpha_4 = 0, \\ \beta_0 = \frac{k_1\mu\left(\lambda\gamma_1-\mu\gamma_0\right)\left(4k_2+k_1^3\left(\lambda^2-4\mu\right)\right)}{3\gamma_1^2}, \beta_1 = \frac{k_1\mu^2\left(4k_2+k_1^3\left(\lambda^2-4\mu\right)\right)}{3\gamma_1}, \beta_2 = 0, \beta_3 = 0, \beta_4 = 0, \\ \gamma_0 = \gamma_0, \gamma_1 = \gamma_1, \gamma_2 = 0, \gamma_3 = 0, \gamma_4 = 0, k_1 = k_1, k_2 = k_2. \end{cases}$$

Substituting (4.15) into Eqs. (4.8), (4.9), and (4.10), results in the following:

$$
\left\{
\begin{array}{l}
G(\xi) = \frac{k_2+k_1^3\left(\lambda^2+2\mu\right)}{3k_1} + 2\lambda\mu k_1^2 \exp(\phi(\xi)) + 2\mu^2 k_1^2 \exp(2\phi(\xi)), \\
H(\xi) = \frac{k_1\mu(\lambda\gamma_1-\mu\gamma_0)\left(4k_2+k_1^3\left(\lambda^2-4\mu\right)\right)}{3\gamma_1^2} + \frac{k_1\mu^2\left(4k_2+k_1^3\left(\lambda^2-4\mu\right)\right)}{3\gamma_1} \exp(\phi(\xi)), \\
W(\xi) = \gamma_0 + \gamma_1 \exp(\phi(\xi)), \\
\xi = \left(\frac{k_1}{\alpha}\right) x^\alpha + \left(\frac{k_2}{\alpha}\right) t^\alpha.
\end{array}
\right. \tag{4.16}
$$

Case 4

$$
\left\{
\begin{array}{l}
\alpha_0 = \frac{k_2+k_1^3\left(\lambda^2+2\mu\right)}{3k_1}, \alpha_1 = 0, \alpha_2 = 2\lambda k_1^2, \alpha_3 = 0, \alpha_4 = 2k_1^2, \\
\beta_0 = \beta_0, \beta_1 = 0, \beta_2 = \beta_2, \beta_3 = 0, \beta_4 = 0, k_1 = k_1, k_2 = k_2, \\
\gamma_0 = \frac{k_1(\lambda\beta_2-\beta_0)\left(4k_2+k_1^3\left(\lambda^2-4\mu\right)\right)}{3\beta_2^2}, \gamma_1 = 0, \gamma_2 = \frac{4k_1k_2+k_1^4\left(\lambda^2-4\mu\right)}{3\beta_2}, \gamma_3 = 0, \gamma_4 = 0.
\end{array}
\right. \tag{4.17}
$$

Substituting (4.17) into Eqs. (4.8), (4.9), and (4.10), gives us the following:

$$
\left\{
\begin{array}{l}
G(\xi) = \frac{k_2+k_1^3\left(\lambda^2+2\mu\right)}{3k_1} + 2\lambda k_1^2 \exp(-\phi(\xi)) + 2k_1^2 \exp(-2\phi(\xi)), \\
H(\xi) = \beta_0 + \beta_2 \exp(-\phi(\xi)), \\
W(\xi) = \frac{k_1(\lambda\beta_2-\beta_0)\left(4k_2+k_1^3\left(\lambda^2-4\mu\right)\right)}{3\beta_2^2} + \frac{4k_1k_2+k_1^4\left(\lambda^2-4\mu\right)}{3\beta_2} \exp(-\phi(\xi)), \\
\xi = \left(\frac{k_1}{\alpha}\right) x^\alpha + \left(\frac{k_2}{\alpha}\right) t^\alpha.
\end{array}
\right. \tag{4.18}
$$

Case should 5

$$
\left\{
\begin{array}{l}
\alpha_0 = \frac{k_2+k_1^3\left(\lambda^2+8\mu\right)}{3k_1}, \alpha_1 = 0, \alpha_2 = 4\lambda k_1^2, \alpha_3 = 0, \alpha_4 = 4k_1^2, \\
\beta_0 = \beta_0, \beta_1 = 0, \beta_2 = \beta_2, \beta_3 = 0, \beta_4 = \frac{\beta_2}{\lambda}, k_1 = k_1, k_2 = k_2, \\
\gamma_0 = \frac{2\lambda k_1\left(\beta_2k_1^3\left(\lambda^2+8\mu\right)+4\beta_2k_2-6\beta_0\lambda k_1^3\right)}{3\beta_2^2}, \gamma_1 = 0, \gamma_2 = \frac{4\lambda^2k_1^4}{\beta_2}, \gamma_3 = 0, \gamma_4 = \frac{4\lambda k_1^4}{\beta_2}.
\end{array}
\right. \tag{4.19}
$$

Substituting (4.19) into Eqs. (4.8), (4.9), and (4.10), we have the following:

$$
\left\{
\begin{array}{l}
G(\xi) = \frac{k_2+k_1^3\left(\lambda^2+8\mu\right)}{3k_1} + 4\lambda k_1^2 \exp(-\phi(\xi)) + 4k_1^2 \exp(-2\phi(\xi)), \\
H(\xi) = \beta_0 + \beta_2 \exp(-\phi(\xi)) + \frac{\beta_2}{\lambda} \exp(-2\phi(\xi)), \\
W(\xi) = \frac{2\lambda k_1\left(\beta_2k_1^3\left(\lambda^2+8\mu\right)+4\beta_2k_2-6\beta_0\lambda k_1^3\right)}{3\beta_2^2} + \frac{4\lambda^2k_1^4}{\beta_2} \exp(-\phi(\xi)) + \frac{4\lambda k_1^4}{\beta_2} \exp(-2\phi(\xi)), \\
\xi = \left(\frac{k_1}{\alpha}\right) x^\alpha + \left(\frac{k_2}{\alpha}\right) t^\alpha.
\end{array}
\right. \tag{4.20}
$$

Using (4.12) and the solutions to Eq. (2.8), we can find the following exact solutions to the space–time fractional generalized nonlinear Hirota–Satsuma coupled KdV equations:

Family 1. When $\mu \neq 0, (\lambda^2 - 4\mu) > 0$,

$$
(4.21)\quad \begin{cases}
G_1(\xi) = \left(\begin{array}{l}
\left(\frac{k_2 + k_1^3\left(\lambda^2+8\mu\right)}{3k_1}\right) \\[2ex]
+2\lambda k_1^2 \left(\begin{array}{l}
\mu \left(\frac{-\lambda - \sqrt{\lambda^2-4\mu}\tanh\left(\frac{(\xi^\alpha+\alpha c)\sqrt{\lambda^2-4\mu}}{2\alpha}\right)}{2\mu} \right) \\[2ex]
+ \left(\frac{2\mu}{-\lambda - \sqrt{\lambda^2-4\mu}\tanh\left(\frac{(\xi^\alpha+\alpha c)\sqrt{\lambda^2-4\mu}}{2\alpha}\right)} \right)
\end{array} \right) \\[2ex]
+2k_1^2 \left(\begin{array}{l}
\mu^2 \left(\frac{-\lambda - \sqrt{\lambda^2-4\mu}\tanh\left(\frac{(\xi^\alpha+\alpha c)\sqrt{\lambda^2-4\mu}}{2\alpha}\right)}{2\mu} \right)^2 \\[2ex]
+ \left(\frac{2\mu}{-\lambda - \sqrt{\lambda^2-4\mu}\tanh\left(\frac{(\xi^\alpha+\alpha c)\sqrt{\lambda^2-4\mu}}{2\alpha}\right)} \right)^2
\end{array} \right)
\end{array} \right), \\[4ex]
H_1(\xi) = \left(\begin{array}{l}
\frac{k_1\mu\left(\gamma_1\lambda k_1^3\left(\lambda^2+20\mu\right) - \gamma_0\mu k_1^3\left(\lambda^2+8\mu\right) - 4k_2(\gamma_0\mu - \gamma_1\lambda)\right)}{3\gamma_1^2} + \\[2ex]
+\frac{k_1\mu^2\left(4k_2+k_1^3\left(\lambda^2+8\mu\right)\right)}{3\gamma_1} \left(\frac{-\lambda - \sqrt{\lambda^2-4\mu}\tanh\left(\frac{(\xi^\alpha+\alpha c)\sqrt{\lambda^2-4\mu}}{2\alpha}\right)}{2\mu} \right) \\[2ex]
+\frac{k_1\mu\left(4k_2+k_1^3\left(\lambda^2+8\mu\right)\right)}{3\gamma_1} \left(\frac{2\mu}{-\lambda - \sqrt{\lambda^2-4\mu}\tanh\left(\frac{(\xi^\alpha+\alpha c)\sqrt{\lambda^2-4\mu}}{2\alpha}\right)} \right)
\end{array} \right), \\[4ex]
W_1(\xi) = \left(\begin{array}{l}
\gamma_0 + \gamma_1 \left(\frac{-\lambda - \sqrt{\lambda^2-4\mu}\tanh\left(\frac{(\xi^\alpha+\alpha c)\sqrt{\lambda^2-4\mu}}{2\alpha}\right)}{2\mu} \right) \\[2ex]
+\frac{\gamma_1}{\mu} \left(\frac{2\mu}{-\lambda - \sqrt{\lambda^2-4\mu}\tanh\left(\frac{(\xi^\alpha+\alpha c)\sqrt{\lambda^2-4\mu}}{2\alpha}\right)} \right)
\end{array} \right), \\[4ex]
\xi = \left(\frac{k_1}{\alpha}\right) x^\alpha + \left(\frac{k_2}{\alpha}\right) t^\alpha.
\end{cases}
$$

Family 2. When $\mu \neq 0, (\lambda^2 - 4\mu) < 0$,

$$(4.22)\quad \begin{cases} G_2(\xi) = \left(\begin{array}{l} \left(\frac{k_2 + k_1^3(\lambda^2+8\mu)}{3k_1}\right) \\ +2\lambda k_1^2 \left(\begin{array}{l} \mu \left(\frac{-\lambda + \sqrt{4\mu-\lambda^2}\tan\left(\frac{(\xi^\alpha+\alpha c)\sqrt{4\mu-\lambda^2}}{2\alpha}\right)}{2\mu} \right) \\ + \left(\frac{2\mu}{-\lambda + \sqrt{4\mu-\lambda^2}\tan\left(\frac{(\xi^\alpha+\alpha c)\sqrt{4\mu-\lambda^2}}{2\alpha}\right)} \right) \end{array} \right) \\ +2k_1^2 \left(\begin{array}{l} \mu^2 \left(\frac{-\lambda + \sqrt{4\mu-\lambda^2}\tan\left(\frac{(\xi^\alpha+\alpha c)\sqrt{4\mu-\lambda^2}}{2\alpha}\right)}{2\mu} \right)^2 \\ + \left(\frac{2\mu}{-\lambda + \sqrt{4\mu-\lambda^2}\tan\left(\frac{(\xi^\alpha+\alpha c)\sqrt{4\mu-\lambda^2}}{2\alpha}\right)} \right)^2 \end{array} \right) \end{array} \right), \\ H_2(\xi) = \left(\begin{array}{l} \left(\frac{k_1\mu\left(\gamma_1\lambda k_1^3(\lambda^2+20\mu) - \gamma_0\mu k_1^3(\lambda^2+8\mu) - 4k_2(\gamma_0\mu - \gamma_1\lambda)\right)}{3\gamma_1^2} \right) \\ + \frac{k_1\mu^2\left(4k_2 + k_1^3(\lambda^2+8\mu)\right)}{3\gamma_1} \left(\frac{-\lambda + \sqrt{4\mu-\lambda^2}\tan\left(\frac{(\xi^\alpha+\alpha c)\sqrt{4\mu-\lambda^2}}{2\alpha}\right)}{2\mu} \right) \\ + \frac{k_1\mu\left(4k_2 + k_1^3(\lambda^2+8\mu)\right)}{3\gamma_1} \left(\frac{2\mu}{-\lambda + \sqrt{4\mu-\lambda^2}\tan\left(\frac{(\xi^\alpha+\alpha c)\sqrt{4\mu-\lambda^2}}{2\alpha}\right)} \right) \end{array} \right), \\ W_2(\xi) = \left(\begin{array}{l} \gamma_0 + \gamma_1 \left(\frac{-\lambda + \sqrt{4\mu-\lambda^2}\tan\left(\frac{(\xi^\alpha+\alpha c)\sqrt{4\mu-\lambda^2}}{2\alpha}\right)}{2\mu} \right) \\ + \frac{\gamma_1}{\mu} \left(\frac{2\mu}{-\lambda + \sqrt{4\mu-\lambda^2}\tan\left(\frac{(\xi^\alpha+\alpha c)\sqrt{4\mu-\lambda^2}}{2\alpha}\right)} \right) \end{array} \right), \\ \xi = \left(\frac{k_1}{\alpha}\right) x^\alpha + \left(\frac{k_2}{\alpha}\right) t^\alpha. \end{cases}$$

Family 3. When $\mu \neq 0, \lambda \neq 0, (\lambda^2 - 4\mu) = 0,$

(4.23)

$$\begin{cases} G_3(\xi) = \left(\begin{array}{l} \frac{k_2 + k_1^3(\lambda^2+8\mu)}{3k_1} + 2\lambda k_1^2 \left(\mu \left(-\frac{2}{\lambda} - \frac{4\alpha}{\lambda^2(\xi^\alpha + \alpha c)} \right) + \left(-\frac{2}{\lambda} - \frac{4\alpha}{\lambda^2(\xi^\alpha + \alpha c)} \right)^{-1} \right) \\ +2k_1^2 \left(\mu^2 \left(-\frac{2}{\lambda} - \frac{4\alpha}{\lambda^2(\xi^\alpha + \alpha c)} \right)^2 + \left(-\frac{2}{\lambda} - \frac{4\alpha}{\lambda^2(\xi^\alpha + \alpha c)} \right)^{-2} \right) \end{array} \right), \\ H_3(\xi) = \left(\begin{array}{l} \left(\frac{k_1\mu\left(\gamma_1\lambda k_1^3(\lambda^2+20\mu) - \gamma_0\mu k_1^3(\lambda^2+8\mu) - 4k_2(\gamma_0\mu - \gamma_1\lambda)\right)}{3\gamma_1^2} \right) \\ + \frac{k_1\mu^2\left(4k_2 + k_1^3(\lambda^2+8\mu)\right)}{3\gamma_1} \left(-\frac{2}{\lambda} - \frac{4\alpha}{\lambda^2(\xi^\alpha + \alpha c)} \right) \\ + \frac{k_1\mu\left(4k_2 + k_1^3(\lambda^2+8\mu)\right)}{3\gamma_1} \left(-\frac{2}{\lambda} - \frac{4\alpha}{\lambda^2(\xi^\alpha + \alpha c)} \right)^{-1} \end{array} \right), \\ W_3(\xi) = \gamma_0 + \gamma_1 \left(-\frac{2}{\lambda} - \frac{4\alpha}{\lambda^2(\xi^\alpha + \alpha c)} \right) + \frac{\gamma_1}{\mu} \left(-\frac{2}{\lambda} - \frac{4\alpha}{\lambda^2(\xi^\alpha + \alpha c)} \right)^{-1}, \\ \xi = \left(\frac{k_1}{\alpha}\right) x^\alpha + \left(\frac{k_2}{\alpha}\right) t^\alpha. \end{cases}$$

Similarly, using formulas (4.14), (4.16), (4.18), and (4.20), we can obtain a greater set of solutions to the space–time fractional generalized nonlinear Hirota–Satsuma coupled KdV equations.

5. CONCLUSION

In this study, the fractional $\exp(-\phi(\xi)) -$ expansion method was successfully applied to solve the space–time fractional nonlinear Whitham–Broer–Kaup equations and space–time fractional generalized nonlinear Hirota–Satsuma coupled KdV equations. We obtained several useful solutions for both equations, which can be used in practical applications. Previous applications have shown that the proposed method is effective and has the ability to demonstrate various forms of solutions. In addition, we are confident of its ability to find solutions to numerous nonlinear fractional partial differential equations in mathematical physics.

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