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ANTIDERIVATIVES AND INTEGRALS INVOLVING INCOMPLETE BETA FUNCTIONS WITH APPLICATIONS

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ABSTRACT. In this paper, we prove that incomplete beta functions are antiderivatives of several products and powers of trigonometric functions, we give formulas for antiderivatives for products and powers of trigonometric functions in term of incomplete beta functions, and we evaluate integrals involving trigonometric functions using incomplete beta functions. Also, we extend some properties of the beta functions to the incomplete beta functions. As an application for the above results, we find the moments for certain probability distributions.

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1. INTRODUCTION

The theory of special functions have many applications, see [6]- [11]. In particular, it is needed to construct and to prove many properties of probability distributions (e.g. the moments for certain probability distributions), see [1]- [4]. As an application for special functions is to find the antiderivatives of products and powers of trigonometric functions, which are not always easy to find. In this paper, we show that the antiderivatives for products and powers of trigonometric functions can be found using incomplete beta functions. Moreover, we apply these findings to certain probability distributions.

Definition 1.1. *The incomplete beta function, which is a generalization of the beta function, is defined as follows: for* $0 \leq x \leq 1$, $\Re(s) > 0$ *and* $\Re(t) > 0$

$$\beta(s,t;x) = \int_0^x u^{s-1}(1-u)^{t-1}du.$$

and let

$$I(s,t;x) = \frac{\beta(s,t;x)}{\beta(s,t)}.$$

Clearly, $\beta(s,t,1) = \beta(s,t)$ and $I(s,t;1) = 1$.

Proposition 1.1. *For* $0 \leq x \leq 1$, $\Re(s) > 0$ *and* $\Re(t) > 0$, *we have*

$$\beta(s,t;x) = 2\int_0^{\arcsin(\sqrt{x})} \sin^{2s-1}(\theta)\cos^{2t-1}(\theta)d\theta.$$

Proof. The substitution $u = \sin^2(\theta)$ gives

$$\beta(s,t;x) = \int_0^x u^{s-1}(1-u)^{t-1}du = 2\int_0^{\arcsin(\sqrt{x})} \sin^{2s-1}(\theta)\cos^{2t-1}(\theta)d\theta.$$

∎

2. INCOMPLETE BETA FUNCTIONS AS ANTIDERIVATIVES FOR PRODUCTS AND POWERS OF TRIGONOMETRIC FUNCTIONS

Theorem 2.1. *For* $0 \leq x \leq 1, \Re(s) > 0$ *and* $\Re(t) > 0$. *The antiderivative of* $\sin^{2s-1}(x)\cos^{2t-1}(x)$ *is* $\frac{1}{2}\beta(s,t,\sin^2(x))$.

Proof.

$$\frac{d}{dx}\left(\frac{1}{2}\beta(s,t,\sin^2(x))\right) = \frac{1}{2}\frac{d}{dx}\left(\int_0^{\sin^2(x)} u^{s-1}(1-u)^{t-1}du\right) = \sin^{2s-1}(x)\cos^{2t-1}(x).$$

∎

Corollary 2.2. $\int \sin^{2s-1}(x)\cos^{2t-1}(x)dx = \frac{1}{2}\beta(s,t;\sin^2(x)) + C$, *where* C *is arbitrary constant.*

Corollary 2.3.

$$\int_a^b \sin^{2s-1}(\theta)\cos^{2t-1}(\theta)d\theta = \frac{1}{2}\left(\beta(s,t,\sin^2(b)) - \beta(s,t,\sin^2(a))\right).$$

The above result gives the well-known result

$$\int_0^{\pi/2} \sin^{2s-1}(\theta)\cos^{2t-1}(\theta)d\theta = \frac{1}{2}\left(\beta(s,t;\sin^2(\pi/2)) - \beta(s,t;\sin^2(0))\right) = \frac{1}{2}\beta(s,t).$$

Theorem 2.4. *For $\Re(s) > -1$. The antiderivative of $(a\sin(x) + b\cos(x))^s$ is $\frac{(a^2+b^2)^{s/2}}{2}\beta(\frac{s+1}{2}, \frac{1}{2}; \sin^2(x+\phi))$, where ϕ is the angle for which $\cos(\phi) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\phi) = \frac{b}{\sqrt{a^2+b^2}}$.*

Proof.

$$\begin{aligned}\frac{d}{dx}\frac{(a^2+b^2)^{s/2}}{2}\beta(\frac{s+1}{2}, \frac{1}{2}; \sin^2(x+\phi)) &= \frac{(a^2+b^2)^{s/2}}{2}\frac{d}{dx}\left(\int_0^{\sin^2(x+\phi)} u^{\frac{s-1}{2}}(1-u)^{\frac{-1}{2}}du\right)\\ &= (a^2+b^2)^{s/2}\sin^s(x+\phi)\\ &= (a^2+b^2)^{s/2}(\cos(\phi)\sin(x) + \sin(\phi)\cos(x))^s\\ &= (a^2+b^2)^{s/2}(\frac{a}{\sqrt{a^2+b^2}}\sin(x) + \frac{b}{\sqrt{a^2+b^2}}\cos(x))^s\\ &= (a\sin(x) + b\cos(x))^s\end{aligned}$$

∎

Corollary 2.5. *$\int(a\sin(x) + b\cos(x))^s dx = \frac{(a^2+b^2)^{s/2}}{2}\beta(\frac{s+1}{2}, \frac{1}{2}, \sin^2(x+\phi)) + C$, where C is arbitrary constant and ϕ is the angle for which $\cos(\phi) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\phi) = \frac{b}{\sqrt{a^2+b^2}}$.*

Corollary 2.6.

$$\begin{aligned}&\int_\gamma^\delta (a\sin(\theta) + b\cos(\theta))^s d\theta\\ = \quad &\frac{(a^2+b^2)^{s/2}}{2}\left(\beta(\frac{s+1}{2}, \frac{1}{2}, \sin^2(\delta+\phi)) - \beta(\frac{s+1}{2}, \frac{1}{2}, \sin^2(\gamma+\phi))\right).\end{aligned}$$

where ϕ is the angle for which $\cos(\phi) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\phi) = \frac{b}{\sqrt{a^2+b^2}}$.

This corollary proves the following Proposition

Proposition 2.7.

$$\int_{-\phi}^{\frac{\pi}{2}-\phi}(a\sin(\theta) + b\cos(\theta))^s d\theta = \frac{(a^2+b^2)^{s/2}}{2}\beta(\frac{s+1}{2}, \frac{1}{2}),$$

where ϕ is the angle for which $\cos(\phi) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\phi) = \frac{b}{\sqrt{a^2+b^2}}$.

Proof.

$$\begin{aligned}\int_{-\phi}^{\frac{\pi}{2}-\phi}(a\sin(\theta) + b\cos(\theta))^s d\theta &= \frac{(a^2+b^2)^{s/2}}{2}\left(\beta(\frac{s+1}{2}, \frac{1}{2}, \sin^2(\frac{\pi}{2})) - \beta(\frac{s+1}{2}, \frac{1}{2}, \sin^2(0))\right)\\ &= \frac{(a^2+b^2)^{s/2}}{2}\beta(\frac{s+1}{2}, \frac{1}{2}).\end{aligned}$$

∎

Example 2.1.

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}}\sqrt{\sin(\theta) + \sqrt{3}\cos(\theta)}d\theta = \frac{(4)^{1/4}}{2}\beta(\frac{3}{4}, \frac{1}{2}) \approx 1.69443.$$

3. ADDITIONAL PROPERTIES OF INCOMPLETE BETA FUNCTIONS

Theorem 3.1. *For* $\Re(s) > 0$, $\Re(t) > 0$ *and* $r_1 \leq a < b \leq r_2$

$$\int_a^b (x - r_1)^{s-1}(r_2 - x)^{t-1} dx = (r_2 - r_1)^{s+t-1} \left[\beta(s, t; \frac{b - r_1}{r_2 - r_1}) - \beta(s, t; \frac{a - r_1}{r_2 - r_1}) \right].$$

In particular,

$$\int_{r_1}^{r_2} (x - r_1)^{s-1}(r_2 - x)^{t-1} dx = (r_2 - r_1)^{s+t-1}\beta(s, t). \tag{3.1}$$

Proof. Using the substitution $w = \frac{x - r_1}{r_2 - r_1}$, we get

$$\int_a^b (x - r_1)^{s-1}(r_2 - x)^{t-1} dx = \int_{\frac{a-r_1}{r_2-r_1}}^{\frac{b-r_1}{r_2-r_1}} ((r_2 - r_1)w)^{s-1}(r_2 - r_1 - (r_2 - r_1)w)^{t-1}(r_2 - r_1)dw$$

$$= (r_2 - r_1)^{s+t-1} \int_{\frac{a-r_1}{r_2-r_1}}^{\frac{b-r_1}{r_2-r_1}} w^{s-1}(1 - w)^{t-1} dw$$
$$= (r_2 - r_1)^{s+t-1} \left(\beta(s, t; \frac{b - r_1}{r_2 - r_1}) - \beta(s, t; \frac{a - r_1}{r_2 - r_1}) \right).$$

Now, choose $b = r_2$ and $a = r_1$ to get the particular case. ∎

This theorem gives the following representation of the incomplete beta functions

Corollary 3.2.

$$\beta(s, t; \xi) = \frac{1}{(r_2 - r_1)^{s+t-1}} \int_{r_1}^{r_1+(r_2-r_1)\xi} (x - r_1)^{s-1}(r_2 - x)^{t-1} dx.$$

Theorem 3.3. *For* $\frac{r_1+r_2}{2} = \overline{r} \leq a < b \leq r_2$, $\Re(s) > 0$, *and* $\Re(t) > 0$

$$\int_a^b (x - r_1)^{s-1}(x - \overline{r})^{t-1}(r_2 - x)^{s-1} dx$$
$$= \frac{1}{2}(\frac{r_2 - r_1}{2})^{2s+t-2} \left[\beta(s, \frac{t}{2}; \left(\frac{2(b - \overline{r})}{r_2 - r_1}\right)^2) - \beta(s, \frac{t}{2}; \left(\frac{2(a - \overline{r})}{r_2 - r_1}\right)^2) \right].$$

In particular,

$$\int_{r_1}^{\overline{r}} (x - r_1)^{s-1}(\overline{r} - x)^{t-1}(r_2 - x)^{s-1} dx = \int_{\overline{r}}^{r_2} (x - r_1)^{s-1}(x - \overline{r})^{t-1}(r_2 - x)^{s-1} dx$$
$$= \frac{1}{2}(\frac{r_2 - r_1}{2})^{2s+t-2}\beta(s, \frac{t}{2}).$$

Proof. Using the substitution $w = \left(\frac{2(x-\overline{r})}{r_2-r_1}\right)^2$ and the fact that

$$\overline{r} - r_1 = r_2 - \overline{r} = \frac{r_2 - r_1}{2},$$

we get that

$$\begin{aligned}
&\int_a^b (x-r_1)^{s-1}(x-\overline{r})^{t-1}(r_2-x)^{s-1}dx \\
&= \int_{(\frac{2(a-\overline{r})}{r_2-r_1})^2}^{(\frac{2(b-\overline{r})}{r_2-r_1})^2} (\overline{r}-r_1+\frac{r_2-r_1}{2}\sqrt{w})^{s-1}(\frac{r_2-r_1}{2}\sqrt{w})^{t-1}(r_2-\overline{r}-\frac{r_2-r_1}{2}\sqrt{w})^{s-1}dw \\
&= \frac{1}{2}(\frac{r_2-r_1}{2})^{2s+t-2}\int_{(\frac{2(a-\overline{r})}{r_2-r_1})^2}^{(\frac{2(b-\overline{r})}{r_2-r_1})^2} (1-w)^{s-1}w^{\frac{t-1}{2}}dw.
\end{aligned}$$

Now, choose $b=r_2$ and $a=\overline{r}$ to get the particular case

$$\int_{\overline{r}}^{r_2} (x-r_1)^{s-1}(x-\overline{r})^{t-1}(r_2-x)^{s-1}dx = \frac{1}{2}(\frac{r_2-r_1}{2})^{2s+t-2}\beta(s,\frac{t}{2}).$$

∎

Now, following the same proof with $\sqrt{w}=\frac{2(\overline{r}-x)}{r_2-r_1}$ to prove the assertion

$$\int_{r_1}^{\overline{r}} (x-r_1)^{s-1}(\overline{r}-x)^{t-1}(r_2-x)^{s-1}dx = \frac{1}{2}(\frac{r_2-r_1}{2})^{2s+t-2}\beta(s,\frac{t}{2}).$$

Example 3.1. *using $r_1=1$, $r_2=3$, $s=t=3/2$ and $\overline{r}=2$ to get that*

$$\int_2^3 \sqrt{6+6x^2-11x-x^3}dx = \int_2^3 \sqrt{(3-x)(x-2)(x-1)}dx = \frac{1}{2}\beta(3/2,3/4).$$

Also, using $r_1=-1$, $r_2=3$, $t=7$, $s=1/2$ and $\overline{r}=1$ to get that

$$\int_{-1}^1 \frac{(1-x)^6}{\sqrt{3+2x-x^2}}dx = \int_{-1}^1 (1-x)^5(3-x)^{-1/2}(1+x)^{-1/2}dx = \frac{1}{2}(2)^6\beta(1/2,7/2) = 10\pi$$

4. THE INCOMPLETE BETA FUNCTIONS AND THE GENERALIZED ARCSINE RANDOM VARIABLES

As applications for the results in the section, we prove many properties of the generalized arcsine random variable with bounded support

Definition 4.1. *The random variable X has a **generalized arcsine random variable with bounded support** if the probability density function is*

$$f_X(x) = \frac{((x-r_1)(r_2-x))^{s-1}}{(r_2-r_1)^{2s-1}\beta(s,s)}, r_1<x<r_2. \tag{4.1}$$

The above argument shows that this function is a valid density function. As a special case, when $r_1=0, r_2=1$, and $s=1/2$, the generalized arcsine random variable X is said to have the standard arcsine distribution, i.e., X has the probability density function f given by

$$f_X(x) = \frac{1}{\pi\sqrt{x(1-x)}}, x\in(0,1).$$

In addition, in Equation 4.1, if $s=1/2$ then this distribution is called **the arcsine probability distribution with bounded support** (see [5]). For similar probability distributions related to gamma functions, see [1], [2], and [3].

Using Theorem 3.1 we have the result

Lemma 4.1. *If X has generalized arcsine random variable with bounded support, then the distribution function $F_X(x)$ of X is given by*

$$\begin{aligned}
F_X(x) &= I(s,s;\frac{x-r_1}{r_2-r_1}) \\
&= \frac{1}{2^{2s-1}\beta(s,s)}\int_0^{2\arcsin(\sqrt{\frac{x-r_1}{r_2-r_1}})} \sin^{2s-1}(\theta)d\theta,\ r_1\le x\le r_2 \\
&= \frac{1}{4^{s-1}\beta(s,s)}\int_0^{\arcsin(\sqrt{\frac{x-r_1}{r_2-r_1}})} \sin^{2s-1}(2\theta)d\theta,\ r_1\le x\le r_2.
\end{aligned}$$

Theorem 4.2. *The central moments $\mu_n = E(X-\overline{X})^n$ of a random variable with the generalized arcsine distribution with bounded support are*

$$\mu_n = \frac{((-1)^n+1)(r_2-r_1)^n\beta(s,\frac{n+1}{2})}{2^{2s+n}\beta(s,s)}.$$

Proof. Using (3.1)

$$\begin{aligned}
\overline{X} = E(X) &= \int_{r_1}^{r_2} x\frac{((x-r_1)(r_2-x))^{s-1}}{(r_2-r_1)^{2s-1}\beta(s,s)}dx \\
&= \int_{r_1}^{r_2}(x-r_1+r_1)\frac{((x-r_1)(r_2-x))^{s-1}}{(r_2-r_1)^{2s-1}\beta(s,s)}dx \\
&= \frac{1}{(r_2-r_1)^{2s-1}\beta(s,s)}\left(\int_{r_1}^{r_2}(x-r_1)^s(r_2-x)^{s-1}dx + r_1\int_{r_1}^{r_2}((x-r_1)(r_2-x))^{s-1}dx\right) \\
&= \frac{1}{(r_2-r_1)^{2s-1}\beta(s,s)}((r_2-r_1)^{2s}\beta(s+1,s)+r_1(r_2-r_1)^{2s-1}\beta(s,s)) = \frac{r_2+r_1}{2} = \overline{r}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mu_n = E(X-\overline{X})^n &= \int_{r_1}^{r_2}(x-\overline{X})^n\frac{((x-r_1)(r_2-x))^{s-1}}{(r_2-r_1)^{2s-1}\beta(s,s)}dx \\
&= \int_{r_1}^{r_2}(x-\overline{r})^n\frac{((x-r_1)(r_2-x))^{s-1}}{(r_2-r_1)^{2s-1}\beta(s,s)}dx
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mu_n = \frac{1}{(r_2-r_1)^{2s-1}\beta(s,s)}(&\int_{r_1}^{\overline{r}}(x-\overline{r})^n((x-r_1)(r_2-x))^{s-1}dx \\
&+\int_{\overline{r}}^{r_2}(x-\overline{r})^n((x-r_1)(r_2-x))^{s-1}dx).
\end{aligned}$$

Using Theorem 3.3,

$$\begin{aligned}
\mu_n = \frac{1}{(r_2-r_1)^{2s-1}\beta(s,s)}((-1)^n&\int_{r_1}^{\overline{r}}(\overline{r}-x)^n((x-r_1)(r_2-x))^{s-1}dx \\
&+\int_{\overline{r}}^{r_2}(x-\overline{r})^n((x-r_1)(r_2-x))^{s-1}dx).
\end{aligned}$$

Therefore, $\mu_n = \frac{((-1)^n+1)(r_2-r_1)^n\beta(s,\frac{n+1}{2})}{2^{2s+n}\beta(s,s)}$. In other words, for $n\in\mathbf{N}$, we have $\mu_{2n-1}=0$ and $\mu_{2n} = \frac{(r_2-r_1)^{2n}\beta(s,n+\frac{1}{2})}{2^{2s+2n-1}\beta(s,s)}$. ∎

5. Conclusion

In this study, we have proved that incomplete beta functions are antiderivatives for products and powers of trigonometric functions. Also, additional properties of incomplete beta functions are proved. Using these additional properties of incomplete beta functions, we find the central moments of a random variable with the generalized arcsine distribution.

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