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ATTEMPTS TO DEFINE A BAUM–CONNES MAP VIA LOCALIZATION OF CATEGORIES FOR INVERSE SEMIGROUPS

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ABSTRACT. An induction functor in inverse semigroup equivariant KK-theory is considered, and together with restriction functors certain results similar to those known from the Mackey machinery are shown. It is also verified that for any so-called E-continuous inverse semigroup its equivariant KK-theory satisfies the universal property and is a triangulated category.

Key words and phrases: Baum–Connes map; inverse semigroup; induction functor; triangulated category; universal property.

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1. INTRODUCTION

In [23], Meyer and Nest found an alternative description of the Baum–Connes map

$$\lim_{Y \subseteq \underline{E}G} KK(C_0(Y), A) \to K(A \rtimes_r G)$$

with coefficients [1], where G denotes a locally compact, second countable group and A a G-algebra. (It was even achieved for groupoids of the form $G \ltimes X$.) Fundamental for this approach is a work by Chabert and Echterhoff [10], and the nontrivial "observation" that Kasparov's category KK^G may be viewed as a triangulated category. By using Brown's representability theorem for triangulated categories [25], a weakly isomorphic, so-called Dirac element $D \in KK^G(B, A)$ is constructed such that B is a G-algebra in the localizing subcategory of KK^G generated by G-algebras of the form $\mathrm{Ind}_H^G(F)$ (induction in the sense of Green [12]) for a compact subgroup $H \subseteq G$ and H-algebra F. If G is compact then one will choose $B = \mathrm{Ind}_G^G(A) = A$ and $D = id$, and for non-compact G one hopes that the compactly induced algebras approximate A sufficiently enough via D, like one approximates functions vanishing at infinity by compactly supported functions. The Baum–Connes map turns out to be equivalent to the map $K(B \rtimes_r G) \to K(A \rtimes_r G)$ induced by $j_r^G(D) \in KK(B \rtimes_r G, A \rtimes_r G)$ for the descent homomorphism j_r^G. Clearly, if for example the morphism D was an isomorphism then the functor image $j_r(D)$ would be an isomorphism as well and the Baum–Connes map bijective.

Let us observe the usefulness of this approach. Assume for the moment that B takes the particular simple form $B = \mathrm{Ind}_H^G(F)$. Then the left hand side of the new formulated Baum–Connes map is potentially computable via

$$K(B \rtimes_r G) = K\big(\mathrm{Ind}_H^G(F) \rtimes_r G\big) \cong K(F \rtimes_r H) \cong KK^H(\mathbb{C}, F) \tag{1.1}$$

by Green's imprimitivity theorem [12] and the Green–Julg isomorphism [15]. Arbitrary B might then be treated by homological means in triangulated categories.

In this paper we try to adapt the above method to unital, countable inverse semigroups G. The compact subgroups are then the finite subinverse semigroups $H \subseteq G$. In a former paper, [9], we proved a Green imprimitivity theorem $\mathrm{Ind}_H^G(F) \widehat{\rtimes} G \cong F \widehat{\rtimes} H$ (Sieben's crossed product [28]) for such Hs. Together with the Green–Julg isomorphism for inverse semigroups we get an analog identity to (1.1). The next fundamental step is to show that Kasparov's category KK^G is a triangulated category. Most of this goes literally through as in Meyer and Nest's paper [23], and we collect the definitions and facts in Section 6. However, there is one exception. To achieve that every morphism of KK^G fits into an exact triangle, one needs a Cuntz-picture of KK^G by representing morphisms as $*$-homomorphisms. This was done in group equivariant KK-theory by Meyer [22], and we adapt his proof in Section 5. One problem is that we need a model of a compatible $\ell^2(G)$-space, and to construct it we need to impose a transparent (see Lemma 5.1), but properly restricting condition on G which we call E-continuity.

The next step is to define an induction functor $\mathrm{Ind}_H^G : KK^H \to KK^G$ for finite subinverse semigroups $H \subseteq G$. We do this in Section 4. In Section 2 we recall the definitions of KK^G-theory and fix other notions we shall need. In Section 3 we discuss Bott periodicity for KK^G. In Section 9 we believed that we had defined a Dirac element $D \in KK^G(P, \mathbb{C})$ by an adaption of the corresponding proof in [23]. Unfortunately, however, when finishing this paper closely in this form, we have realized that we had a flaw in the proof of the fundamental identity

$$KK^G(\mathrm{Ind}_H^G A, B) \;\cong\; KK^H(A, \mathrm{Res}_G^H B), \tag{1.2}$$

which holds for discrete groups, see line (20) in [23]. It is even wrong, see Remark 4.1. On a sufficiently big subcategory there exists a right adjoint functor to Ind_H^G by theoretical results of Neeman, see Definition 8.5, but it is not the restriction functor. We have no concrete realization of it and consequently we cannot analyse it like the restriction functor.

Nevertheless, in the last Section 9 we shall work with the correct right adjoint functor instead of the restriction functor and prove the existence of a Dirac morphism under assumptions on the inverse semigroup which evidently hold for discrete groups at least, see Proposition 9.2. We remark that the existence of the Dirac morphism is the main obstacle. After having it, one could easily construct a Baum–Connes map as in [23], see Remark 9.1.

In the meanwhile, we have computed a right adjoint functor for the induction functor for a special subclass of G-algebras called fibered G-algebras in [2] and established a Baum–Connes map for them. This is however not the complete solution to the Baum–Connes map, as fibered G-algebras are not KK^G-equivalent to such important examples of G-algebras like $C_0(X)$.

Also, in the meanwhile we have verified that E-continuous inverse semigroups are exactly those whose associated groupoid is Hausdorff, see [4]. This strengthens that the technical assumptions of Proposition 9.2 might be fulfilled for E-continuous inverse semigroups as we would expect the existence of such a Baum–Connes map in that case.

On the way of our attempt of proving the existence of a Dirac morphism we also showed a number of lemmas in connection with restriction and induction functors which might be of independent interest and are collected in Sections 7 and 8.

2. G-EQUIVARIANT KK-THEORY

Let G denote a countable unital inverse semigroup. We write $E(G)$ (or simply E) for the set of projections of G. We shall denote the involution on G both by $g \mapsto g^*$ and $g \mapsto g^{-1}$ (determined by $gg^{-1}g = g$). A semigroup homomorphism is said to be *unital* if it preserves the identity $1 \in G$ and the zero element $0 \in G$ provided G has such elements, respectively. We consider G-equivariant KK-theory as defined in [6] (in its final form in Section 7 of [6]) but make a slight adaption by making this theory *compatible* in the following sense. We require that all G-Hilbert A, B-bimodules $\mathcal{E}$ of Kasparov cycles satisfy $e(a)\xi = ae(\xi)$ and $\xi e(b) = e(\xi)b$ for all $e \in E, a \in A, b \in B$ and $\xi \in \mathcal{E}$. Since the only constructions of Hilbert modules in [6] out of given ones are done by forming tensor products, direct sums, or taking the Hilbert module $\mathbb{C}$, and these constructions respect these modifications, we readily can accept this modified, compatible KK^G-theory to hold true with all its properties like the existence of the Kasparov product as in [6]. Since the additional properties of inverse semigroups as compared to semi-multiplicative sets in [6] slightly simplify the formal definitions of equivariant KK-theory (see for instance [7, Corollary 4.6]), we are going to recall the polished definitions for convenience of the reader.

Definition 2.1. A *G-algebra* (A, α) is a $\mathbb{Z}/2$-graded C^*-algebra A with a unital semigroup homomorphism $\alpha : G \to \text{End}(A)$ such that α_g respects the grading and $\alpha_{gg^{-1}}(x)y = x\alpha_{gg^{-1}}(y)$ for all $x, y \in A$ and $g \in G$.

Definition 2.2. A *G-Hilbert B-module* $\mathcal{E}$ is a $\mathbb{Z}/2$-graded Hilbert module over a G-algebra (B, β) endowed with a unital semigroup homomorphism $G \to \text{Lin}(\mathcal{E})$ (linear maps on $\mathcal{E}$) such that U_g respects the grading and $\langle U_g(\xi), U_g(\eta)\rangle = \beta_g(\langle \xi, \eta\rangle)$, $U_g(\xi b) = U_g(\xi)\beta_g(b)$, and $U_{gg^{-1}}(\xi)b = \xi\beta_{gg^{-1}}(b)$ for all $g \in G, \xi, \eta \in \mathcal{E}$ and $b \in B$.

In the last definition, $U_{gg^{-1}}$ is automatically a self-adjoint projection in the center of $\mathcal{L}(\mathcal{E})$, and the action $G \to \text{End}(\mathcal{L}(\mathcal{E}))$, $g(T) = U_g T U_{g^{-1}}$ turns $\mathcal{L}(\mathcal{E})$ to a G-algebra ($g \in G$ and $T \in \mathcal{L}(\mathcal{E})$). A G-algebra (A, α) is a G-Hilbert module over itself under the inner product $\langle a, b\rangle = a^*b$ and $U := \beta := \alpha$ in the last definition. A $*$-homomorphism between G-algebras is called *G-equivariant* if it intertwines the G-action. Usually the G-action on a G-algebra is denoted by $g(a) := \alpha_g(a)$. The complex numbers $\mathbb{C}$ are endowed with the trivial G-action

$g(1) = 1$ for all $g \in G$. A *G-Hilbert A, B-bimodule* over G-algebras A and B is a G-Hilbert B-module $\mathcal{E}$ equipped with a G-equivariant $*$-homomorphism $A \to \mathcal{L}(\mathcal{E})$.

Definition 2.3. Let A and B be G-algebras. We define a Kasparov cycle $(\mathcal{E}, T)$, where $\mathcal{E}$ is a G-Hilbert A, B-bimodule, to be an ordinary Kasparov cycle (without G-action) (see [16, 17]) satisfying $U_g T U_g^* - T U_{gg^{-1}} \in \{S \in \mathcal{L}(\mathcal{E}) | \, aS, Sa \in \mathcal{K}(\mathcal{E}) \text{ for all } a \in A\}$ for all $g \in G$. The Kasparov group $KK^G(A, B)$ is defined to be the collection $\mathbb{E}^G(A, B)$ of these cycles divided by homotopy induced by $\mathbb{E}^G(A, B[0, 1])$.

We write C^*_G for the category of ungraded, separable G-algebras as objects and G-equivariant $*$-homomorphisms as morphisms, and KK^G for the additive category consisting of ungraded, separable G-algebras as objects and $KK^G(A, B)$ as the morphism set from object A to object B, together with the Kasparov product $KK^G(A, B) \times KK^G(B, C) \to KK^G(A, C)$ as composition of morphisms. Define $C_G : C^*_G \to KK^G$ to be the well known functor which is identical on objects and satisfies $C_G(f) := f_*(1_A) \in KK^G(A, B)$ for morphisms $f : A \to B$, where $1_A := [(A, 0)] \in KK^G(A, A)$ denotes the unit.

Definition 2.4 (See Definition 25 of [6])**.** For a σ-unital G-algebra D we denote by τ_D : $KK^G(A, B) \to KK^G(A \otimes D, B \otimes D)$ the map induced by $(\mathcal{E}, T) \mapsto (\mathcal{E} \otimes D, T \otimes 1)$.

Occasionally we shall still refer to *incompatible* KK^G-theory as defined in [6] and denote it by IK^G. The class of underlying G-Hilbert modules is richer, but the G-algebras are the same. KK^G and their Hilbert modules are sometimes accompanied by the word *compatible*, to stress the difference to IK^G. It is often useful to compare IK^G and KK^G by the isomorphism $IK^G(A, B) \cong KK^G(A \rtimes E, B \rtimes E)$ from [8, Theorem 5.3] for *finite* G. Also remark that there exists a canonical functor $KK^G \to IK^G$ defined by the identity map on cycles.

Given a G-algebra A, we denote by $A \rtimes G$ the universal crossed product [19], and by $A \widehat{\rtimes} G$ Sieben's crossed product [28]. We identify G as a subset of $\mathbb{C} \rtimes G$, and denote by $\tilde{G} \subseteq \mathbb{C} \rtimes G$ the inverse semigroup generated by G and all projections $p \in \mathbb{C} \rtimes G$ of the form $p = e_0(1 - e_1) \ldots (1 - e_n)$ for $e_i \in E$ and $n \geq 0$. Note that every element of $\tilde{G}$ is of the form gp with $g \in G$ and p as before.

Every G-action α on a G-algebra (or G-Hilbert module) extends to a $\tilde{G}$-action by linearity, that is, $\alpha_{gp} = \alpha_g \alpha_{e_0}(\alpha_1 - \alpha_{e_1}) \ldots (\alpha_1 - \alpha_{e_n})$, where p is as before (see [9, Lemma 2.1]). We sometimes extend G-actions to $\tilde{G}$-actions in this way implicitly without saying. We shall also consider discrete groupoids $H \subseteq \tilde{G}$, and we may regard them as inverse semigroups $H \cup \{0\} \subseteq \tilde{G}$ with zero element in order to consistently redefine the known notion of H-equivariant KK-theory KK^H via the inverse semigroup $H \cup \{0\}$, where 0 is understood to act always as zero. Provided is here however that the H-algebras are defined in the groupoid sense, that is, that they are also $C_0(H^{(0)})$-algebras, see [17, Definition 1.5]. (Cf. also [3].)

Let $G \subseteq L \subseteq \tilde{G}$ be a subinverse semigroup. Then we have

$$KK^G(A, B) = KK^L(A, B) = KK^{\tilde{G}}(A, B) \tag{2.1}$$

via the identity map on cycles when using the above mentioned extension of G-actions for all G-algebras A and B. (A $\tilde{G}$-Hilbert B-module inherits the linearly extended $\tilde{G}$-action from B by compatibility.) Denote by X or X_G the totally disconnected, locally compact Hausdorff space such that $C_0(X)$ is the universal commutative C^*-algebra $C^*(E)$ generated by the commuting projections E. (Actually X is compact since E is unital.) $C_0(X)$ is endowed with the G-action $g(1_e) = 1_{geg^*}$ for $e \in E$ and $g \in G$. Every G-algebra A may be regarded as a $C_0(X)$-algebra (see Kasparov [17, Section 1.5]) by $\pi : C_0(X) \to Z(\mathcal{M}(A))$ with $\pi(1_e)(a) = e(a)$ since E has a unit. Write $A \otimes^X B$ for the balanced tensor product ($A \otimes B$ divided by all elements of the form $e(a) \otimes b - a \otimes e(b)$ where $e \in E$), see Le Gall [21] or [17, Section 1.6].

Definition 2.5. The *groupoid* $H \subseteq \tilde{G}$ *associated to a given finite subinverse semigroup* $H' \subseteq G$ is defined to be the finite groupoid $H = \{hp \in \tilde{G} \,|\, h \in H', p \in E(\widetilde{H'})$ is a minimal projection, $h^* h \geq p\}$.

Observe that $KK^{H'}(A, B) = KK^H(A, B)$ for all H'-algebras or H-algebras A and B by the equivalence of $C^*_{H'}$ and C^*_H, and $KK^{H'}$ and KK^H, respectively, see [3]. (Our notion $KK^{H'}$ coincides with $\widehat{KK^{H'}}$ of [3].) All subinverse semigroups of G are assumed to contain the *unit* of G! By regarding G as a discrete inverse semigroup, we often say compact instead of finite subinverse semigroup.

3. BOTT PERIODICITY

This section works both in IK^G and KK^G.

Definition 3.1. Define $KK_n^G(A, B) := KK^G(A \otimes C_{n,0}, B)$, where $C_{n,m}$ denotes the Clifford algebras of Kasparov [16, Sections 2.11 and 2.13] for $n, m \geq 0$. (The G-action on $C_{n,0}$ is trivial.)

Theorem 3.1 (Bott periodicity)**.** *Let the G-action on $C_0(\mathbb{R}^n)$ be trivial. Then*

$$KK_{i+n}^G(A \otimes C_0(\mathbb{R}^n), B) \cong KK_i^G(A, B) \cong KK_{i-n}^G(A, B \otimes C_0(\mathbb{R}^n))$$

Proof. The proof is a slight adaption of Kasparov's [16, §5, Theorem 7]. Note that Kasparov discusses in his proof the "real" case to be definite, and so our $\mathbb{R}^n$ appears as $\mathbb{R}^{p,q}$ in his proof; so we "identify" these two. In line (4) on page 547 of [16] he states that there exists elements $\beta_V \in KK^{Spin(V)}(\mathbb{C}, C_0(\mathbb{R}^n) \otimes C_V)$ and $\alpha_V \in KK^{Spin(V)}(C_0(\mathbb{R}^n) \otimes C_V, \mathbb{C})$ such that

$$a)\ \beta_V \otimes_{C_0(\mathbb{R}^n) \otimes C_V} \alpha_V = c_1; \qquad b)\ \beta_V \otimes_{\mathbb{C}} \alpha_V = \tau_{C_0(\mathbb{R}^n) \otimes C_V}(c_1), \tag{3.1}$$

where $c_1 := (id, \mathbb{C}, 0) \in KK^{Spin(V)}(\mathbb{C}, \mathbb{C})$ is the unit element, and the Kasparov products in (3.1) are the Kasparov's cup-cap product. As Kasparov remarks, a direct application of (3.1) to [16, §4, Theorem 6, 2)] yields the desired Bott periodicity [16, §5, Theorem 5].

We now regard β_V and α_V as elements in G-equivariant KK-theory KK^G by putting them into the canonical map $KK^{Spin(V)}(C, D) \to KK^G(C, D)$ ($\forall C, D$) by regarding $Spin(V)$-Kasparov cycles as G-Kasparov cycles via the trivial semigroup homomorphism $triv : G \to Spin(V) : g \mapsto 1$ ($\forall g \in G$). We can then also apply (3.1) to [16, §4, Theorem 6, 2)], but now in the G-equivariant setting. ∎

Corollary 3.2. *We have $KK^G(A \otimes C(\mathbb{R}^2), B) \cong KK^G(A, B) \cong KK^G(A, B \otimes C(\mathbb{R}^2))$ for all G-algebras A and B.*

Proof. The Clifford algebra $C_{0,0}$ is $\mathbb{C}$, so that KK_0^G is simply KK^G. The result follows then from Theorem 3.1 and the formal Bott periodicity [16, Theorem 5.5] (which works literally in our setting as the G-actions on the vector spaces V appearing there are trivial), which states that KK_n is periodic in n with period 2. ∎

4. INDUCTION AND RESTRICTION FUNCTORS

Given a compact subinverse semigroup $H' \subseteq G$, in [9] we defined an induced algebra and showed Green imprimitivity theorems. This was done by switching at first from H' to its associated finite subgroupoid $H \subseteq \tilde{G}$, proving everything for H, and at the end switching back to H' in notation. That H was induced by an inverse semigroup was extraneous. Hence we may, and shall, start here somewhat more generally with a finite groupoid like in Definition 4.1 below and still can use the results from [9].

Before we need however fix some notions. For an assertion $\mathcal{A}$ we let $[\mathcal{A}]$ be the real number 0 if $\mathcal{A}$ is false, and 1 if $\mathcal{A}$ is true. Let $H \subseteq \tilde{G}$ be a finite subgroupoid. Set

$$G_H := \{gp \in \tilde{G} \,|\, g \in G,\, p \in H^{(0)},\, g^*g \geq p\}.$$

We endow G_H with an equivalence relation: $g \equiv h$ if and only if there exists $t \in H$ such that $gt = h$ ($g, h \in G_H$). We denote by G_H/H the discrete, set-theoretical quotient of G_H by $\equiv$. The delta function δ_g in $C_0(G_H)$ and $C_0(G_H/H)$ is denoted by g ($g \in G_H$). The commutative C^*-algebras $C_0(G_H)$ and $C_0(G_H/H)$ are endowed with the G-action $g(h) := [gh \in G_H]\; gh$, where $g \in G$ and $h \in G_H$ (of course, $gh \in G_H$ is equivalent to $g^*g \geq hh^*$).

Definition 4.1. Let $H \subseteq \tilde{G}$ be a finite subgroupoid and D a H-algebra. Define, similar as in [18, §5 Def. 2],

$$\begin{aligned} \mathrm{Ind}_H^G(D) \;\; := \;\; & \{f : G_H \to D \,|\, \forall g \in G_H, t \in H \text{ with } gt \in G_H : f(gt) = t^{-1}(f(g)), \\ & \|f(g)\| \to 0 \text{ for } gH \to \infty \text{ in } G_H/H\,\}. \end{aligned}$$

It is a C^*-algebra under the pointwise operations and the supremum's norm and becomes a G-algebra under the G-action $(gf)(h) := [g^{-1}h \in G_H]\; f(g^{-1}h)$ for $g \in G$, $h \in G_H$ and $f \in \mathrm{Ind}_H^G(D)$.

Definition 4.2. Let $H \subseteq \tilde{G}$ be a finite subgroupoid. Define a functor $\mathcal{I}_H^G : C_H^* \to C_G^*$ by $\mathcal{I}_H^G(A) = \mathrm{Ind}_H^G(A)$ for objects A in C_H^* and $\mathcal{I}_H^G(f) : \mathrm{Ind}_H^G(A) \to \mathrm{Ind}_H^G(B)$ by $\mathcal{I}_H^G(f)(x) = f(x(g))$ for morphisms $f : A \to B$ in C_H^*, where $x \in \mathrm{Ind}_H^G(A)$ and $g \in G_H$.

Lemma 4.1. *The functor $\mathcal{I}_H^G$ is exact, and canonically intertwines direct sums (i.e. $\mathrm{Ind}_H^G(\bigoplus_i A_i) \cong \bigoplus_i \mathrm{Ind}_H^G(A_i)$), tensoring with a nuclear C^*-algebra B such that $e(A \otimes B) = e(A) \otimes B$ for all $e \in E$ (i.e. more precisely, $\mathcal{I}_H^G((A \otimes B, \tau)) \cong (\mathcal{I}_H^G(A) \otimes B, \theta)$, where τ is a given H-action and θ is a suitable chosen G-action), and the mapping cone (see (6.1)) (i.e. $\mathrm{Ind}_H^G(\mathrm{cone}(f)) \cong \mathrm{cone}(\mathrm{Ind}_H^G(f))$).*

Proof. The proof is straightforward, only the tensor product needs discussion. Ignoring any G-action on $\mathcal{I}_H^G(D)$, we have a $*$-isomorphism $\phi : \bigoplus_{g \in X} g^*g(D) \to \mathcal{I}_H^G(D)$ by $\phi(d)(gh) = h^{-1}(d(g))$ for all $g \in X \subseteq G_H, h \in H$ such that $gh \in G_H$, and where X is a fixed complete system of representatives of G_H/H. Hence, $\mathcal{I}_H^G(A \otimes B) \cong \mathcal{I}_H^G(A) \otimes B$ without G-action. We choose now θ such that this isomorphism becomes G-equivariant. ∎

Define $C_0(G_H/H, B)$ to be the G-invariant ideal of $C_0(G_H/H) \otimes B$ which is the closure of the linear span of all elements of the form $g \otimes gg^*(b)$ ($g \in G_H, b \in B$). Similarly, denote by $p \in Z(\mathcal{L}(\mathrm{Ind}_H^G(A) \otimes B))$ (center) the central projection $p(g \otimes a \otimes b) := g \otimes a \otimes gg^*(b)$ for $g \in G_H, a \in g^*g(A)$ and $b \in B$. We have a direct sum decomposition

$$\mathrm{Ind}_H^G(A) \otimes B \;\; \cong \;\; p\big(\mathrm{Ind}_H^G(A) \otimes B\big) \oplus (1-p)\big(\mathrm{Ind}_H^G(A) \otimes B\big), \tag{4.1}$$

and we denote the first summand (and ideal) by $\mathrm{Ind}_H^G(A) \overset{\rightarrow}{\otimes} B$.

Lemma 4.2 (Cf. line (17) in [23])**.** *Let B be a G-algebra and $H \subseteq \tilde{G}$ a finite subgroupoid. Then there is a G-equivariant $*$-isomorphism*

$$\Theta : \mathrm{Ind}_H^G \mathrm{Res}_G^H(B) \longrightarrow C_0(G_H/H, B), \quad \Theta(f) \;=\; \sum_{g \in G_H/H} g \otimes g(f(g))$$

for all $f \in \mathrm{Ind}_H^G \mathrm{Res}_G^H(B) \subseteq C_0(G_H) \otimes B$. (The sum is understood that we choose for every equivalence class in G_H/H exactly one arbitrary representative $g \in G_H$.)

Proof. The proof is straightforward. ∎

Lemma 4.3 (Cf. line (16) in [23]). *Let $H \subseteq \tilde{G}$ be a finite subgroupoid, A a H-algebra and B a G-algebra. Then there is a G-equivariant $*$-isomorphism*

$$\Theta : \mathit{Ind}_H^G\big(A \otimes^{X_H} \mathit{Res}_G^H(B)\big) \longrightarrow \mathit{Ind}_H^G(A) \overset{\rightarrow}{\otimes} B, \quad \Theta(g \otimes a \otimes b) = g \otimes a \otimes g(b)$$

for all $g \in G_H, a \in g^*g(A)$ and $b \in g^*g(B)$.

Proof. The tensor product $A \otimes^{X_H} \mathrm{Res}_G^H(B)$ denotes the balanced groupoid tensor product and is endowed with the diagonal H-action. In other words, we may regard A and $\mathrm{Res}_G^H(B)$ as $H \cup \{0\}$-inverse semigroup algebras and take the usual diagonal inverse semigroup action for the tensor product $A \otimes^{X_{H\cup\{0\}}} \mathrm{Res}_{\tilde{G}}^{H\cup\{0\}}(B)$.

Note that we have $gt \otimes t^*(a \otimes b) = gt \otimes t^*(a) \otimes t^*(b)$ in $\mathrm{Ind}_H^G\big(A \otimes^{X_H} \mathrm{Res}_G^H(B)\big) \subseteq C_0(G_H) \otimes A \otimes B$ for all $g \in G_H, t \in H, a \in A$ and $b \in B$ with $gt \in G_H$, so we can achieve the required format in the argument of Θ when setting $t := g^*g$. Surjectivity of Θ is obvious. That Θ is isometric is also clear as the transition $g^*gB \to gB$ by Θ is a $*$-isomorphism. ∎

From now on we restrict ourselves to trivially graded G-algebras.

Lemma 4.4. *The functor $F = C_G \circ \mathcal{I}_H^G$ from the category C_H^* to the additive category KK^G is a stable, split exact and homotopy invariant functor. (Stability means that $F(f : A \to A \otimes \mathcal{K})$ is an isomorphism for every corner embedding f, where $A \otimes \mathcal{K}$ is allowed to be equipped with any H-action.)*

Proof. By Higson [13, Section 4.4], we need to show that the functor $L : C_H^* \to Ab$ determined by $L(B) = KK^G(A, \mathcal{I}_H^G(B))$ for objects B and $L(f) = \mathcal{I}_H^G(f)_* : KK^H(A, \mathcal{I}_H^G(B_1)) \to KK^H(A, \mathcal{I}_H^G(B_2))$ for morphisms $f : B_1 \to B_2$ is a stable, split exact and homotopy invariant functor for all objects A in KK^G in the sense of [5]. This follows from Lemma 4.1 and [5, Proposition 1.1], which says that the functor $B \mapsto KK^G(A, B)$ is stable, split exact and homotopy invariant. With respect to stability, the condition $e(A \otimes \mathcal{K}) = e(A) \otimes \mathcal{K}$ of Lemma 4.1 is met by the fact that the image of e is an ideal in $A \otimes \mathcal{K}$ and e is in the center of the multiplier algebra of $A \otimes \mathcal{K}$. ∎

Because F is stable, split exact and homotopy invariant, it factors through KK^H by [5, Theorem 1.3] and this gives us a new functor defined next. We remark that [5, Theorem 1.3] works also for countable discrete groupoids H, as pointed out in [5], by regarding $H \cup \{0\}$ as an inverse semigroup with zero element.

Definition 4.3. Let $H \subseteq \tilde{G}$ be a finite subgroupoid. We define the *induction functor* $\mathrm{Ind}_H^G : KK^H \to KK^G$ as the unique functor satisfying $C_G \circ \mathcal{I}_H^G = \mathrm{Ind}_H^G \circ C_H$, see [5, Theorem 1.3] and Lemma 4.4.

If $H' \subseteq G$ is a finite subinverse semigroup then we consider its associated finite subgroupoid $H \subseteq \tilde{G}$ and define induction by $\mathrm{Ind}_{H'}^G := \mathrm{Ind}_H^G$; usually we regard it, however, as a functor $\mathrm{Ind}_{H'}^G : KK^{H'} \to KK^G$.

Definition 4.4. Let $H \subseteq G$ be a subinverse semigroup or $H \subseteq \tilde{G}$ a finite subgroupoid. The *restriction functor* $\mathrm{Res}_G^H : KK^G \to KK^H$ is defined by restricting G-actions (or $\tilde{G}$-action for the groupoid H) to H-actions in G-algebras and G-Hilbert modules of cycles. Additionally, every restricted H-algebra is cut-down to the form $\mathrm{Res}_G^H(A) = 1_H(A)$ in case that H is a groupoid ($1_H := \sum_{x \in H^{(0)}} x$) or H should not contain the identity of G.

Remark 4.1. Identity (1.2) is wrong in KK^G. Take for example a finite, unital inverse semigroup G where no other projection than 1 is connected with 1. Set $H = \{1\}$, and $A = B = \mathbb{C}$ endowed with the trivial G-action. Then $KK^G(\mathrm{Ind}_H^G \mathbb{C}, \mathbb{C}) = 0$, because a cycle $(\mathcal{E}, T)$ satisfies

$a\xi 1(b) = a\xi p(b) = p(a)\xi b = 0$ for all $a \in \mathrm{Ind}_H^G(\mathbb{C}), b \in \mathbb{C}, \xi \in \mathcal{E}$ and any projection $p < 1$ in E. But $KK^H(\mathbb{C}, \mathrm{Res}_G^H \mathbb{C}) = \mathbb{Z}$.

Identity (1.2) is also wrong in IK^G. Let $G = E$ be finite and consist only of projections. Set $H = \{e\}$, where e denotes the minimal projection of E. Then $\mathrm{Ind}_H^E \mathbb{C} \cong \mathbb{C}$ and thus $IK^E(\mathrm{Ind}_H^E \mathbb{C}, \mathbb{C}) \cong K(\mathbb{C} \rtimes E) \cong \mathbb{Z}^m$ by the Green–Julg isomorphism in [8]. But $IK^H(\mathbb{C}, \mathrm{Res}_G^H \mathbb{C}) \cong KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$.

5. REALIZING MORPHISMS IN KK^G BY $*$-HOMOMORPHISMS

Generalizing the Cuntz picture of KK-theory, [11], to equivariant KK-theory, Meyer showed in [22, Theorem 6.5] that for every locally compact second countable group G and for every morphism $x \in KK^G(A, B)$ there exist G-algebras A' and B', isomorphisms $y \in KK^G(A, A')$ and $z \in KK^G(B, B')$, and a $*$-homomorphism $f : A' \to B'$ (also interpreted as an morphism in KK^G) such that $x = z \circ f \circ y^{-1}$. That is, we may rewrite morphisms in KK^G as $*$-homomorphisms. We will adapt Meyer's proof to the case of an inverse semigroup G (see Theorem 5.6). To this end, we need a model for an $\ell^2(G)$-space, since it plays a central role in Meyer's work [22]. However, a direct translation from a group G to an inverse semigroup G does not work, even not if taking the $\ell^2(G)$ from Khoshkam and Skandalis [19], since it is a useful *incompatible* $\mathbb{C}$-module, however, we need a *compatible* model for $\ell^2(G)$, that is, a compatible G-Hilbert $C_0(X)$-modul. This is necessary as to achieve that the action gg^{-1} ($g \in G$) is in the center of $\mathcal{L}(\mathcal{E})$ in all derived spaces $\mathcal{E}$ from $\ell^2(G)$ and consequently the G-action on $\mathcal{L}(\mathcal{E})$ is multiplicative and so a G-action. Hence constructions like $q_s A := q(\mathbb{K}(G\mathbb{N})A)$ in [22] or Definitions 5.6 and 5.7 become indeed G-algebras as required.

In the next few paragraphs (until Definition 5.3) we shall identify elements $e \in E$ with its characteristic function 1_e in $C_0(X)$. Write $\mathrm{Alg}^*(E)$ for the dense $*$-subalgebra of $C_0(X)$ generated by the characteristic functions 1_e for all $e \in E$. Moreover, write $\bigvee_i f_i \in \mathbb{C}^X$ for the pointwise supremum of a family of functions $f_i : X \to \mathbb{C}$. We shall use the order relation on G defined by $g \le h$ iff $g = eh$ for some $e \in E$.

Definition 5.1. An inverse semigroup G is called *E-continuous* if the function $\bigvee\{e \in E | \, e \le g\} \in \mathbb{C}^X$ is a *continuous* function in $C_0(X)$ for all $g \in G$.

Lemma 5.1. *An inverse semigroup G is E-continuous if and only if for every $g \in G$ there exists a finite subset $F \subseteq E$ such that $\bigvee\{e \in E | \, e \le g\} = \bigvee\{e \in F | \, e \le g\}$.*

Proof. If $\bigvee\{e \in E | \, e \le g\} = 1_K \in C_0(X)$ for a clopen subset $K \subseteq X$ then K must be compact. Hence $K = \bigcup\{ \text{carrier}(1_e) \subseteq X \,|\, e \in E, \, e \le g\}$ allows a finite subcovering, where carrier denotes the usual carrier of a function on a locally compact space. ∎

Definition 5.2 (Compatible $L^2(G)$-space)**.** Let G be an E-continuous inverse semigroup. Write c for the linear span of all functions $\varphi_g : G \to \mathbb{C}$ (in the linear space $\mathbb{C}^G$) defined by

$$\varphi_g(t) \ := \ [t \le g]$$

for all $g, t \in G$. Endow c with the G-action $g(\varphi_h) := \varphi_{gh}$ for all $g, h \in G$. Turn c to an $\mathrm{Alg}^*(E)$-module by setting $\xi e := e(\xi)$ for all $\xi \in c$ and $e \in E$. Define an $\mathrm{Alg}^*(E)$-valued inner product on c by

$$\langle \varphi_g, \varphi_h \rangle \ := \ \bigvee\{e \in E \,|\, eg = eh, \, e \le gg^{-1}hh^{-1}\}. \tag{5.1}$$

The norm completion of c is a G-Hilbert $C_0(X)$-module denoted by $\widehat{\ell^2}(G)$.

We discuss the last definition. At first notice that $\langle \varphi_g, \varphi_h \rangle = gg^{-1} \bigvee\{e \in E | \, e = ehg^{-1}\}$ (observe that $e = ehg^{-1}$ implies $e \le hg^{-1}gh^{-1}$), so that by E-continuity $\langle \varphi_g, \varphi_h \rangle$ is in $C_0(X)$

and actually even in $\mathrm{Alg}^*(E)$ by Lemma 5.1, and $e \in E$ in (5.1) can be replaced by $e \in F$ for some finite subset $F \subseteq E$. The identities $\langle \varphi_g, \varphi_h \rangle = \langle \varphi_h, \varphi_g \rangle$, $\langle \varphi_g, \varphi_h f \rangle = \langle \varphi_g f, \varphi_h \rangle = \langle \varphi_g, \varphi_h \rangle f$, $j(\langle \varphi_g, \varphi_h \rangle) = \langle j(\varphi_g), j(\varphi_h) \rangle$ for all $g, h, j \in G$ and $f \in E$ are easy to check. We note that (5.1) is positive definite. Indeed, assume $\langle x, x \rangle = 0$ for $x = \sum_{i=1}^n \lambda_i \varphi_{g_i}$ with nonzero $\lambda_i \in \mathbb{C}$ and $g_i \in G$ mutually different. Choose g_j such that no other g_i satisfies $g_j g_j^{-1} < g_i g_i^{-1}$. Hence, $\langle \varphi_{g_j}, \varphi_{g_j} \rangle = g_j g_j^{-1}$ but $\langle \varphi_{g_i}, \varphi_{g_k} \rangle \neq g_j g_j^{-1}$ for all combinations where $i \neq k$. By linear independence of the projections E in $\mathrm{Alg}^*(E)$ λ_j must be zero; contradiction. The last proof also shows the following lemma.

Lemma 5.2. *The vectors $(\varphi_g)_{g \in G} \subseteq \widehat{\ell^2}(G)$ are linearly independent.*

Definition 5.3. Let $\mathcal{E}$ be a G-Hilbert B-module. Then $\widehat{\ell^2}(G, \mathcal{E}) := \widehat{\ell^2}(G) \otimes^X \mathcal{E}$ is a G-Hilbert B-module, where $\otimes^X$ denotes the $C_0(X)$-balanced exterior tensor product as defined by Le Gall [21, Definition 4.2] (or in this case equivalently, the internal tensor product $\otimes_{C_0(X)}$).

Everywhere in [22] we have to replace $L^2(G)$ (see [22, Section 2]) by $\widehat{\ell^2}(G)$ and $L^2(G, \mathcal{E})$ (see [22, Section 2.1.1]) by $\widehat{\ell^2}(G, \mathcal{E})$. These definitions have to go further.

Definition 5.4. Every separable G-Hilbert space $\mathcal{H}$ in Meyer [22] has to be replaced by a countably generated G-Hilbert $C_0(X)$-module $\mathcal{H}$. Every occurrence of the Hilbert space $\mathbb{C}$ in [22] has to be substituted by the G-Hilbert $C_0(X)$-module $C_0(X)$. For every G-Hilbert B-module or G-algebra $\mathcal{E}$, $\ell^2(\mathcal{H}) \otimes \mathcal{E}$ in [22] has to be replaced by the compatible tensor product $\ell^2(\mathcal{H}) \otimes^X \mathcal{E}$, and likewise $\mathbb{K}(\mathcal{H}) \otimes \mathcal{E}$ in [22] by $\mathbb{K}(\mathcal{H}) \otimes^X \mathcal{E}$.

In the beginning of Section 3 of [22] we have the following adaption.

Definition 5.5. Let A and B be σ-unital G_2-C^*-algebras and let $\mathcal{H}$ be a countably generated G_2-Hilbert $C_0(X)$-module. A Kasparov triple $(\mathcal{E}, \phi, F)$ is called *$\mathcal{H}$-special* iff

(i) F is a G-equivariant symmetry (*G-equivariance* means that the function $F : \mathcal{E} \to \mathcal{E}$ commutes with the G-action $U_g : \mathcal{E} \to \mathcal{E}$ for all $g \in G$), and

(ii) $\mathcal{H} \otimes^X \mathcal{E} \subseteq \hat{\mathcal{H}}_B$.

Lemma 5.3. *Lemma 3.1 of* [22] *holds true also for an inverse semigroup G.*

Proof. Let $(\mathcal{E}, \phi, F)$ be an essential Kasparov triple for A, B. Rather than the definition $F' : C_c(G, \mathcal{E}) \to C_c(G, \mathcal{E})$ ($(F'f)(g) = g(F)(f(g))$, $g \in G, f \in C_c(G, \mathcal{E})$) in Meyer [22] we have to use the following one. Define $F' : \widehat{\ell^2}(G) \otimes^X \mathcal{E} \to \widehat{\ell^2}(G) \otimes^X \mathcal{E}$ by

$$F'(\varphi_g \otimes \xi) \;:=\; \varphi_g \otimes g(F)(\xi)$$

for $g \in G, \xi \in \mathcal{E}$. We show that F' is G-equivariant (see Definition 5.5). For $h \in G$ we have

$$\begin{aligned} h\big(F'(\varphi_g \otimes \xi)\big) &= h\varphi_g \otimes hgFg^{-1}h^{-1}h(\xi) \\ &= \varphi_{hg} \otimes hg(F)(h(\xi)) \\ &= F'\big(h(\varphi_g \otimes \xi)\big), \end{aligned}$$

because $h^{-1}h \in \mathcal{L}(\mathcal{E})$ is in the center.

We have to check that F' is an F-connection (see [22, Section 2.5]) when writing $\widehat{\ell^2}(G, \mathcal{E}) \cong \widehat{\ell^2}(G, A) \otimes_A \mathcal{E}$ (because ϕ is essential). Write τ for the grading automorphisms on A and $\widehat{\ell^2}(G, A)$. Let $\xi := \varphi_g \otimes a \in \widehat{\ell^2}(G, A)$ for $g \in G$ and $a \in A$ with $gg^{-1}(a) = a$ without loss of generality. Set $K := T_\xi F - F' T_{\xi^\tau} : \mathcal{E} \to \widehat{\ell^2}(G, \mathcal{E})$ (see [22, Section 2.5]) for $T_\xi(\eta) = \xi \otimes \eta$ and $\eta \in \mathcal{E}$. Then we have

$$K\eta \;=\; \varphi_g \otimes \phi(a) F\eta - \varphi_g \otimes g(F)\phi\tau(a)\eta \;=\; \varphi_g \otimes K_g(\eta)$$

in the space $\widehat{\ell^2}(G) \otimes^X \mathcal{E}$ for all $\eta \in \mathcal{E}$, where

$$K_g \;:=\; \phi(a)gg^{-1}(F) - g(F)\phi\tau(a) \;=\; [\phi(a), F] + \big(gg^{-1}(F) - g(F)\big)\phi\tau(a),$$

because $a = gg^{-1}(a)$ and $gg^{-1} \in \mathcal{L}(\mathcal{E})$ is in the center and so $\phi(a)F = \phi(a)gg^{-1}(F)$. Since $(\mathcal{E}, \phi, F)$ is a Kasparov triple, $K_g \in \mathcal{K}(\mathcal{E})$. Assuming for the moment that K_g was an elementary compact operator $\theta_{\alpha,\beta}$ for $\alpha, \beta \in \mathcal{E}$, we would have $K = \varphi_g \otimes \theta_{\alpha,\beta} = \theta_{\varphi_g \otimes \alpha, \beta} \in \mathcal{K}(\mathcal{E}, \widehat{\ell^2}(G, \mathcal{E}))$ as required. This is also true for general K_g by approximation. ∎

Definition 5.6. Instead of $\mathbb{K}(G)A := \mathbb{K}(L^2(G)) \otimes A$ in Proposition 3.2 (and Section 2.1.1) of Meyer's paper [22] we have to use $\mathbb{K}(G)A := \mathbb{K}(\widehat{\ell^2}(G)) \otimes^X A$.

Note $\mathbb{K}(G)A$ is a G-algebra. We have also an isomorphism of G-algebras

$$\psi : \mathbb{K}(G)A \cong \mathbb{K}\big(\widehat{\ell^2}(G)\big) \otimes^X \mathbb{K}(A) \cong \mathbb{K}\big(\widehat{\ell^2}(G) \otimes^X A\big) = \mathbb{K}\big(\widehat{\ell^2}(G, A)\big) \tag{5.2}$$

as used in [22, Proposition 3.2]. This proposition goes essentially through unchanged but uses also this lemma by Mingo and Phillips [24].

Lemma 5.4 (Cf. Lemma 2.3 of [24]). *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are G-Hilbert A-modules which are isomorphic as Hilbert A-modules then $\widehat{\ell^2}(G, \mathcal{E}_1)$ and $\widehat{\ell^2}(G, \mathcal{E}_2)$ are isomorphic as G-Hilbert A-modules.*

Proof. Let $u \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ be a unitary operator. Then it can be checked that $V : \widehat{\ell^2}(G, \mathcal{E}_1) \to \widehat{\ell^2}(G, \mathcal{E}_2)$ given by $V(\varphi_g \otimes \xi) := \varphi_g \otimes gug^{-1}(\xi)$ defines an isomorphism of G-Hilbert A-modules. Note that V is defined like F' in Lemma 5.3, so we can take the equivariance proof from there. For the inner product note that $\langle \varphi_g, \varphi_h \rangle = \sum_{f \in F} f$ for a finite set $F \subseteq E$ with $fg = fh$ and $f \leq gg^* hh^*$ by Lemma 5.1, so that

$$\begin{aligned}\langle V(\varphi_g \otimes \xi), V(\varphi_h \otimes \eta) \rangle &= \sum_{f \in F} f \otimes \langle fgug^* f(\xi), fhuh^* f(\eta) \rangle \\ &= \langle \varphi_g \otimes \xi, \varphi_h \otimes \eta \rangle.\end{aligned}$$

∎

The last lemma implies also the validity of an literally identical version of [24, Theorem 2.4] ($L^2(G, \mathcal{E})^\infty \cong L^2(G, A)^\infty$ G-equivariantly) in our setting by the same proof.

In [22, Lemma 4.3] some homotopy results with $\mathcal{F}^\infty$ are recalled. The canonical proofs, using $L^2([0,1])$ (see [14, Lemma 1.3.7]) work also inverse semigroup equivariantly. In [22, Lemma 4.4] we note that we have to replace $\big(g(F) - F\big)\phi(a)$ by $\big(g(F) - gg^{-1}(F)\big)\phi(a)$. We recall that gg^{-1} is in the center of $\mathcal{L}(\mathcal{E})$ so that $\mathcal{E}' := J \cdot \mathcal{E}$ is G-invariant because $g(J \cdot \mathcal{E}) = g(J) \cdot g(\mathcal{E}) \subseteq \mathcal{E}'$. Everything goes through unchanged.

Section 5.1 in [22] can be ignored since we do not need it. In [22, Section 5.2] we have to replace $QA := A * A$ by the compatible free product $QA := A *^X A$ by identifying $e(a) * b$ and $a * e(b)$ in $A * A$ for all $a, b \in A$ and $e \in E$. Because of this identification, the diagonal action $g(a_1 * \cdots * a_n) := g(a_1) * \ldots * g(a_n)$ turns QA to a G-algebra. The kernel of the canonical G-equivariant $*$-homomorphism $A *^X A \to A$ is denoted by $q(A)$.

Definition 5.7. For a G-algebra A we define

$$\mathbb{K}(G\mathbb{N})A \;:=\; \mathbb{K}\big(\ell^2(\mathbb{N}) \otimes \big(\widehat{\ell^2}(G) \otimes^X A\big)\big) \;\cong\; \mathbb{K}\big((L^2(G, A))^\infty\big)$$

(by $\mathcal{E}^\infty := \ell^2(\mathbb{N}) \otimes \mathcal{E}$ in [22, Section 2.1.1]). (Confer also (5.2).)

In accordance to the rules of Definition 5.4 we may also write $\mathbb{K}(G\mathbb{N})A = \mathbb{K}\big(C_0(X)^\infty \otimes^X \big(\widehat{\ell^2}(G) \otimes^X A\big)\big)$.

In the last paragraph of the proof of [22, Proposition 5.4] one rewrites a special Kasparov triple $(\mathcal{E}, \phi, F)$ as the Kasparov triple $(\mathcal{E}^+ \oplus \mathcal{E}^+, \phi^+ \oplus \phi^-, P)$ by using the grading on $\mathcal{E}$ and identifying $\mathcal{E}^-$ with $\mathcal{E}^+$ via F; P is then the flip operator. Here we need Definition 5.5 that F commutes with the G-action such that F restricts to a G-equivariant Hilbert module isomorphism between $\mathcal{E}^-$ and $\mathcal{E}^+$, and thus $\phi^- : A \to \mathcal{L}(\mathcal{E}^+)$ is G-equivariant.

Definition 5.8. For G-algebras A and B set $[A, B]_s := [\mathbb{K}(G\mathbb{N})A, \mathbb{K}(G\mathbb{N})B]$, where $[A, B]$ denotes the homotopy group of $*$-homomorphisms from A to B. Denote by $[C^*_G]_s$ the category of separable G-algebras as objects and morphism sets $[A, B]_s$ between objects A and B.

Definition 5.9. A functor $F : C^*_G \to \mathcal{C}$ into a category $\mathcal{C}$ is called *stable* iff the map $F(\mathbb{K}(\mathcal{H})A) \to F(\mathbb{K}(\mathcal{H} \oplus \mathcal{H}')A)$ induced by the inclusion $\mathcal{H} \subseteq \mathcal{H} \oplus \mathcal{H}'$ is an isomorphism for all countably generated G-Hilbert $C_0(X)$-modules $\mathcal{H}, \mathcal{H}'$ and all separable G-algebras A.

Note that in [22, Proposition 6.1] $\mathbb{C} \oplus L^2(G\mathbb{N})$ has to be replaced by $C_0(X) \oplus L^2(G\mathbb{N})$.

Proposition 5.5 (Cf. Proposition 6.3 of [22])**.** *The canonical functor $C^*_G \to KK^G$ is a split exact stable homotopy functor.*

Proof. We only remark stability and may prove this like in [29, Lemma 3.1]. Consider $\mathcal{H}$ and $\mathcal{H}'$ as in Definition 5.9, and prove that the two cycles $(\iota, \mathbb{K}(\mathcal{H}\oplus\mathcal{H}'), 0) \in KK^G(\mathbb{K}(\mathcal{H}), \mathbb{K}(\mathcal{H}\oplus\mathcal{H}'))$ (ι induced by the inclusion $\mathcal{H} \subseteq \mathcal{H}\oplus\mathcal{H}'$) and $(id, \mathbb{K}(\mathcal{H}\oplus\mathcal{H}')p, 0) \in KK^G(\mathbb{K}(\mathcal{H}\oplus\mathcal{H}'), \mathbb{K}(\mathcal{H}))$ are inverses to each other, where $p \in \mathcal{L}(\mathcal{H}\oplus\mathcal{H}')$ is the canonical projection onto the first factor $\mathcal{H}$, because $\mathbb{K}(\mathcal{H} \oplus \mathcal{H}')p \otimes_{\mathbb{K}(\mathcal{H})} \mathbb{K}(\mathcal{H} \oplus \mathcal{H}') \cong \mathbb{K}(\mathcal{H} \oplus \mathcal{H}')$ via $a \otimes b \mapsto ab$. We apply then the compatible version $\tilde{\tau}_A$ of Definition 2.4 to these isomorphisms, where $\otimes$ is replaced by the compatible tensor product $\otimes^X$, to get isomorphisms with $\otimes^X A$. ∎

Theorem 5.6 (Adaption of Theorem 6.5 of [22])**.** *Assume that G is E-continuous. Let A and B separable (ungraded) G-algebras. Define $q_s A := q(\mathbb{K}(G\mathbb{N})A)$. The canonical functor $C^*_G \to KK^G$ factors through a functor $\sharp : [C^*_G]_s \to KK^G$. There is a morphism $\pi^s_A \in [q_s A, A]_s$ (see* [22]*), such that $\sharp(\pi^s_A) \in KK^G(q_s A, A)$ is invertible. Then the map*

$$\Delta : [q_s A, q_s B]_s \to KK^G(A, B), \qquad \Delta(f) = \sharp(\pi^s_B) \circ \sharp(f) \circ \sharp(\pi^s_A)^{-1}$$

is a natural isomorphism. Hence the Kasparov product on KK^G corresponds to the composition of homomorphisms.

By composing the functor Δ with the canonical functor $KK^G \to IK^G$ we see that we can rewrite morphisms in $IK^G(A, B)$ which are represented by compatible cycles also as $*$-homomorphisms in IK-theory.

6. $\widetilde{KK}^G$ IS A TRIANGULATED CATEGORY

In this Section we recall the facts which show that $\widetilde{KK}^G$ is a triangulated category. Everything from groups G to inverse semigroups G goes literally and canonically through and needs no adaption, the only exception from this being axiom (TR1) which is essentially Theorem 5.6. Actually we shall work with a slightly different category, the category $\widetilde{KK}^G$, rather than the category KK^G as we might expect. However, both categories are equivalent.

Definition 6.1. Define $\widetilde{KK}^G$ (see [23, Section 2.1]) to be the category where the objects are pairs (A, n) for all separable G-algebras A and $n \in \mathbb{Z}$, and the morphism set between two objects (A, n) and (B, m) is defined to be

$$\widetilde{KK}^G\big((A,n),(B,m)\big) \;:=\; \lim_{p\in\mathbb{N}} KK^G(\Sigma^{n+p}A, \Sigma^{m+p}B).$$

The maps in the direct limit are the maps $\tau_{C_0(\mathbb{R})}$ and of course we require $n+p, m+p \geq 0$. The composition of the morphisms is canonically via the Kasparov product.

By Bott periodicity $\tau_{C_0(\mathbb{R})}$ is an isomorphism, and so we may omit the direct limit. However, it is needed at least to make desuspension, defined next.

Definition 6.2. Define a suspension functor Σ from $\widetilde{KK}^G$ to $\widetilde{KK}^G$ by $\Sigma(A,n) := (A, n+1)$ and $\Sigma(x) := \tau_{C_0(\mathbb{R})}(x) \in KK^G(\Sigma^{n+p+1}A, \Sigma^{m+p+1}B) \subseteq \widetilde{KK}^G\big((A,n+1),(B,m+1)\big)$ for all $x \in KK^G(\Sigma^{n+p}A, \Sigma^{m+p}B) \subseteq \widetilde{KK}^G\big((A,n),(B,m)\big)$.

The desuspension functor Σ^{-1} on $\widetilde{KK}^G$ is defined to precisely reverse the functor Σ, and we have $\Sigma\circ\Sigma^{-1} = \Sigma^{-1}\circ\Sigma = id_{\widetilde{KK}^G}$, so Σ is an isomorphism of categories. The canonical map $KK^G \to \widetilde{KK}^G$ sending A to $(A,0)$ is an equivalence of categories. Indeed, by Bott periodicity, $KK^G(\Sigma^{2n}A, B) \cong KK^G(A,B)$, every element (A,n) is isomorphic to some $(B,0)$ in $\widetilde{KK}^G$. (We have $(A,2n) \cong (A,0)$ and $(A,2n+1) \cong (\Sigma A, 0)$.) Most of the time it is sufficient to think of $\widetilde{KK}^G$ just as KK^G.

Having now a suspension functor Σ, we further need distinguished triangles to turn $\widetilde{KK}^G$ into a triangulated category.

Definition 6.3. Let A and B G-algebras. Then to an equivariant $*$-homomorphism $f: A \to B$ we associate the *mapping cone* (cf. [23, Section 2.1]), which is the G-algebra

$$\text{cone}(f) \;:=\; \{(a,b)\in A\times C_0\big((0,1],B\big) \mid f(a) = b(1)\}, \tag{6.1}$$

and the *mapping cone triangle,* which is the sequence of equivariant $*$-homomorphisms

$$\Sigma B \xrightarrow{\iota} \text{cone}(f) \xrightarrow{\epsilon} A \xrightarrow{f} B\ , \tag{6.2}$$

where ι is the canonical inclusion (setting the coordinate a to zero) and ϵ is the canonical projection onto A.

Definition 6.4. A diagram $\Sigma B' \to C' \to A' \to B'$ in $\widetilde{KK}^G$ is called an *exact triangle* (see [23, Section 2.1]) if it is isomorphic to a mapping cone triangle (6.2) in $\widetilde{KK}^G$, that is, there exists an equivariant $*$-homomorphism $f: A \to B$ and a commutative diagram

$$\begin{tikzcd} \Sigma B \arrow[r, "\iota"] \arrow[d, "\Sigma\beta"] & \text{cone}(f) \arrow[r, "\epsilon"] \arrow[d, "\gamma"] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ \Sigma B' \arrow[r] & C' \arrow[r] & A' \arrow[r] & B' \end{tikzcd}$$

where α, β and γ are isomorphisms and the suspension $\Sigma\beta$ of β is of course also an isomorphism.

For convenience of the reader we recall the definition of extension triangles, which are exact triangles in the sense of Definition 6.4, and which are technically used in the proof that $\widetilde{KK}^G$ is a triangulated category.

Definition 6.5 (Definition 2.3 in [23]). Let $\mathcal{E} : 0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be an extension of G-algebras and associate to it the commuting diagram (without the indicated map μ)

$$\begin{tikzcd} \Sigma C \arrow[r, "\mu"] \arrow[d, "id"] & A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "p"] \arrow[d, "id"] & C \arrow[d, "id"] \\ \Sigma C \arrow[r, "\iota"] & \text{cone}(p) \arrow[r, "\epsilon"] & B \arrow[r, "p"] & C \end{tikzcd} \tag{6.3}$$

where $\text{cone}(p) \subseteq B \times C_0((0,1], C)$, $\iota(c) := (0, c)$, $\epsilon(b, c) := b$ and $\alpha(a) := (i(a), 0)$ for all $c \in C_0((0,1), C)$, $b \in B$ and $a \in A$. The extension $\mathcal{E}$ is called *admissible* if α is an isomorphism in $\widetilde{KK}^G$. In this case we have an obvious morphism $\mu := \alpha^{-1} \circ i$ which makes the diagram (6.3) to an isomorphism of exact triangles in $\widetilde{KK}^G$ in the sense of Definition 6.4 (since the second line is obviously a mapping cone triangle), and in this case we call the first line of (6.3), which is an exact triangle, also the *extension triangle* of $\mathcal{E}$.

We shall not need the following lemma but state it as an interesting observation in its own. It is proved like in the last paragraph of [23, Section 2.3].

Lemma 6.1 (Section 2.3 in [23]). *Every exact triangle is isomorphic to an extension triangle in $\widetilde{KK}^G$.*

Proposition 6.2 (Proposition 2.1 and Appendix A of [23]). *Suppose that G is E-continuous. The category $\widetilde{KK}^G$ endowed with the translation functor Σ^{-1} (the suspension functor in a triangulated category) and exact triangles from Definition 6.4 is a triangulated category.*

Proof. One of the axioms of an triangulated category, the axiom (TR1) of [26], requires that every morphism $f : A \to B$ in $\widetilde{KK}^G$ fits into an exact triangle $\Sigma B \to C \to A \xrightarrow{f} B$. If f is actually a $*$-homomorphism then we may take the mapping cone triangle as an exact triangle (see Definitions 6.3 and 6.4). Given a general morphism $f \in KK^G(A, B)$ we rewrite it as the image of the map Δ of Theorem 5.6, that is $f = x \circ g \circ y$, where $g : q_s A \to q_s B$ is an equivariant $*$-homomorphism, and $x \in KK^G(q_s A, A)$ and $y \in KK^G(B, q_s B)$ are isomorphisms in KK^G, and take the mapping cone triangle for g.

The rest of the axioms are proved in Appendix A of [23] directly by using canonical equivariant $*$-homomorphisms including homotopies, and extension triangles as in Definition 6.5. This canonical proof goes literally through also in our setting. ∎

Like in [23], in the remainder of this paper we sloppily do not distinguish between the equivalent categories KK^G and $\widetilde{KK}^G$ and shall work practically exclusively with KK^G.

7. SOME LEMMAS WITH RESTRICTION AND INDUCTION

In this section we present a mix of lemmas which deal with restriction and induction functors and which might be of independent interest and are reminiscent to some similar computations in the group equivariant Mackey machinery. They may be particularly interesting as they handle equivalence relations on inverse semigroups, which are less often considered, and projections which do not appear in groups at all. We shall often leave out notating the restriction functor Res_H^G where it is obviously there for better readability.

The following Lemma 7.1 prepares Lemma 7.2. They deal with expressions where induction and restriction functors come together.

Lemma 7.1. *Let $U' \subseteq G$ a finite subinverse semigroup of G and $U \subseteq \tilde{G}$ its associated finite groupoid. Let $L \subseteq G$ be a subinverse semigroup of G. Let D be G-algebra. Let $g \in G_U$ (that is, $g = g_0 u_0$ for some $g_0 \in G$ and $u_0 \in U^{(0)}$).*

Define L' as the subinverse semigroup of G generated by $L \cup g_0 \cdot E(U') \cdot g_0^* \subseteq G$ and set $M := (gg^*Lgg^* \cap gUg^*)\backslash\{0\} \subseteq \tilde{G}$. Then we have an isomorphism of L-algebras

$$\theta : \mathit{Ind}_M^{L'}\mathit{Res}_G^M(D) \longrightarrow \{f \in \mathit{Ind}_U^G\mathit{Res}_G^U(D) \,|\, f \text{ has carrier in } LgU \cap G_U\}$$

via $\theta(f)(lgu) = u^*g^*(f(lgg^*))$ for all $f \in \mathit{Ind}_M^{L'}(D)$, $l \in L$ and $u \in U$.

Proof. We may write $g = g_0u_0$ for some $g_0 \in G$ and $u_0 \in U^{(0)}$, and note that $g_0^*g_0 \geq u_0$ and $g^*g = u_0$. Note that $M \subseteq \widetilde{L'}$ since $gg^* = g_0u_0g_0^*$ can be expressed in $\widetilde{L'}$. Of course, every element of M has source and range projection $gug^*gu^*g^* = gg^* \in \widetilde{G}$, so M is a subgroupoid (or even subgroup) of $\widetilde{L'}$. If there is $l \in L$ such that $l^*l \geq gg^*$ then the indicated image of θ is nonempty, if and only if $gl^*lg^* = gg^* \in M$, if and only M is nonempty, the case we are considering now, because otherwise θ is, correctly, the empty function. Every element $l' \in L'$ may be written in the form

$$l' = (g_0u_1g_0^*)l_1(g_0u_2g_0^*)l_2(g_0u_3g_0^*)\ldots l_n(g_0u_ng_0^*) = lp \tag{7.1}$$

for some $u_i \in E(U')$, $l_i, l \in L$ and $p \in E(L')$. Then an element is in $(L')_M \subseteq \tilde{G}$ if and only if it is of the form $l'gg^*$ with $l' \in L'$ and $l'^*l' \geq gg^*$. We may write $l'gg^* = lp(gg^*) = lgg^*$ by (7.1), and because the source projection of $l'gg^*$ is gg^*, we also have $l^*l \geq gg^*$. Hence we have obtained

$$(L')_M = \{lgg^* \in \tilde{G} |\, l \in L,\, l^*l \geq gg^*\}. \tag{7.2}$$

To show that θ is well defined, consider an ambiguously represented element $lgu = l'gu' \in LgU \cap G_U$ for $l, l' \in L$ and $u, u' \in U$. Notice that $l^*l, l'^*l' \geq gg^*$ (because of G_U), and that source and range projections of u and u' are the same. Thus $guu'^*g^* = l^*l'gg^*$ is in M. Hence

$$\begin{aligned} \theta(f)(l'gu') &= u'^*g^*\big(f(l'gg^*)\big) = u'^*g^*\big(f(lguu'^*g^*)\big) \\ &= u'^*g^*(guu'^*g^*)^*\big(f(lg)\big) = u^*g^*\big(f(lg)\big) = \theta(f)(lgu). \end{aligned}$$

Injectivity of θ follows from $gu(\theta(f)(lgu)) = gg^*(f(lgg^*)) = f(lgg^*gg^*)$ (because $gg^* \in M$) and identity (7.2). To check surjectivity of θ, write a given $j \in \mathrm{Ind}_U^G(D)$ with carrier in $LgU \cap G_U$ as $j = \theta(f)$ for the $f \in \mathrm{Ind}_M^{L'}(D)$ determined by $f(lgg^*) := g(j(lg))$ for all $l \in L$ (confer also (7.2)). In verifying L-invariance of θ, we compute

$$\begin{aligned} \theta\big(h(f)\big)(lgu_0) &= g^*\big(h(f)(lgg^*)\big) = g^*\big(f(h^*lgg^*)\big)\,[hh^* \geq lgg^*l^*] \\ &= \theta(f)(h^*lg)\,[hh^* \geq lgg^*l^*] = h\big(\theta(f)\big)(lgu_0) \end{aligned}$$

for all $h, l \in L$. ∎

Lemma 7.2. *Let H' a finite subinverse semigroup of G and H its associated finite subgroupoid of $\tilde{G}$. Let L be a subinverse semigroup of G. Let D be a G-algebra. Then there is an L-equivariant $*$-isomorphism*

$$\mathit{Res}_G^L\mathit{Ind}_H^G\mathit{Res}_G^H(D) \;\cong\; \bigoplus_{g \in J}\mathit{Res}_{L'_g}^L\mathit{Ind}_{M_g}^{L'_g}\mathit{Res}_G^{M_g}(D),$$

where $J \subseteq G$ is a subset and M_g is the set M of Lemma 7.1 for $U' := H'$.

Proof. Say that two elements $g, g' \in G_H$ are L-equivalent if $lg = g'$ for some $l \in L$ with $l^*l \geq gg^*$. This relation is reflexive as $1 \in L$, symmetric because $l^*lg = g = l^*g'$ and $ll^* \geq lgg^*l^* = g'g'^*$, and transitive because $lg = g' = l''g''$ implies $g = l^*l''g''$ and $l''^*ll^*l'' \geq$

$l''^* lgg^* l^* l'' = l''^* l'' gg^* l''^* l'' = gg^*$. Similarly, two elements in $g, g' \in G_H$ are said to be L, H-equivalent if $lgh = g'$ for some $l \in L$ with $l^* l \geq gg^*$ and some $h \in H$, and this is also an equivalence relation. Its equivalence classes are exactly of the form $LgH \cap G_H \subseteq G_H$ (the intersection taken in $\widetilde{G}$) for all $g \in G$

For every $g \in G$ apply Lemma 7.1 for $U' := H'$, and denote θ of Lemma 7.1 more precisely by θ_g, the image of θ_g by F_g, M by M_g and L' by L'_g. Note that F_g is a L-invariant C^*-subalgebra of $\mathrm{Ind}_H^G(D)$. Choose from every L, H-equivalence class exactly one representative $g \in G$ and denote their collection by $J \subseteq G$. (We remove those g for which F_g is empty.) Of course, we have a canonical $*$-isomorphism of L-algebras

$$\mathrm{Res}_G^L \mathrm{Ind}_H^G \mathrm{Res}_G^H(D) \quad \cong \quad \bigoplus_{g \in J} F_g \quad \cong \quad \bigoplus_{g \in J} \mathrm{Res}_{L'_g}^L \mathrm{Ind}_{M_g}^{L'_g} \mathrm{Res}_G^{M_g}(D),$$

the last isomorphism being the one induced by the θ_gs. ∎

The idea of the next lemma is to get rid off the $\mathrm{Res}_{L'_g}^L$-term appearing in the last lemma, where L and L'_g distinguish only by projections which could not appear in a group.

Lemma 7.3. *Let $L \subseteq G$ be a finite subinverse semigroup and $P \subseteq G$ a subset of projections. Let $L' \subseteq G$ denote the subinverse semigroup generated by $L \cup P$. Assume that L' is E-unitary. Let A be a finite dimensional, commutative L'-algebra. Let B be a L-algebra. Then there exists an $n \geq 1$ and a L'-action on a (quite canonical) subalgebra $B' \subseteq B^n$ such that*

$$KK^L(Res_{L'}^L A, B) \cong KK^{L'}(A, B').$$

The assignment $B \mapsto B'$ commutes canonically with all (infinite) direct sums.

Proof. Note that $L' = \{lp \in L' | \, l \in L, \, p \in E(L')\}$. Similarly, writing $W := \widetilde{L'}$, $W = \{lp \in W | \, l \in L, \, p \in E(W)\}$. Let α denote the L'-action on A and β the L-action on B. Note that A is of the form $\mathbb{C}^n = C_0(\{1, \ldots, n\})$ and so the L'-action can only cancel or permute the factors $\mathbb{C}$. Consider the finite set $\alpha(E(W)) \subseteq \mathcal{L}(A)$ of projections, which is already a refined set of projections, and enumerate by $(p_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n_i}$ all their minimal projections, where $p_i := \sum_{j=1}^{n_i} p_{i,j}$ denotes the minimal projections of the smaller projection set $\alpha(E(\tilde{L}))$. Choose a selection (lift) $\sigma : \{p_{i,j}\} \to W$ such that $\alpha \circ \sigma = id$, and write $q_{i,j} := \sigma(p_{i,j})$ for simplicity. Also, denote by $q_1, \ldots, q_m \in W$ the minimal projections of $E(\tilde{L})$.

Let us be given a cycle $(\pi, \mathcal{E}, T)$ in $KK^L(A, B)$. We want to mirror the W-structure of the A-side to the B-side. By a well known cut-down of a cycle, we may assume without loss of generality that $\pi(1) = 1_{\mathcal{L}(\mathcal{E})}$. Denote the L-action on $\mathcal{E}$ by γ. Set $B_i := \beta(q_i)B \subseteq B$ for $1 \leq i \leq m$. Note that $B \cong B_1 \oplus \ldots \oplus B_m$. (Also observe that $\mathcal{E}$ has an analog, associated decomposition $\mathcal{E} = \pi(p_1(1))\mathcal{E} \oplus \ldots \oplus \pi(p_m(1))\mathcal{E}$ by L-equivariance of π.) Define $B' := \bigoplus_{i=1}^m B_i^{n_i} = \bigoplus_{i=1}^m \bigoplus_{j=1}^{n_i} B_i$. Denote these summands by $B_{i,j}$. We want to define a cycle $(\pi', \mathcal{E}', T')$ in $KK^{L'}(A, B')$. Let $\mathcal{E}'$ denote an identical copy of $\mathcal{E}$ as a graded vector space. We define a B'-valued inner product on $\mathcal{E}'$ by

$$\langle \xi, \eta \rangle_{\mathcal{E}'} := \oplus_{i,j} \big\langle \pi\big(p_{i,j}(1)\big)\xi, \pi\big(p_{i,j}(1)\big)\eta \big\rangle_{\mathcal{E}} \quad \in \; B' = \oplus_{i,j} B_{i,j}$$

for all $\xi, \eta \in \mathcal{E}'$, and the B'-module multiplication on $\mathcal{E}'$ by $\xi(\oplus_{i,j} b_{i,j}) := \sum_{i,j} (\pi((p_{i,j}(1)))\xi) b_{i,j}$ (the last $b_{i,j}$ regarded in B_i). Define a L'-action γ' on $\mathcal{E}'$ by $\gamma'(lp) := \gamma(l)\pi\big(\alpha(p)(1)\big)$ for all $l \in L$ and $p \in E(L')$. Because L' is E-unitary, the presentation lp with $p \leq l^* l$ is unique and thus γ' well-defined. Define a W-action β' on B' by $\beta'(lp)(\oplus_{i,j} b_{i,j}) = \oplus_{i,j} 1_{\{i = i_1\}} 1_{\{j = j_1\}} \beta(l)(b_{i_0, j_0})$ if $\alpha(p) = p_{i_0, j_0}$ and $\alpha(lp)$ has source projection p_{i_0, j_0} and range projection p_{i_1, j_1}. We extend this definition to a W-action by additivity, that is, $\beta'(\sum_{i,j} \lambda_{i,j} l q_{i,j}) := \sum_{i,j} \lambda_{i,j} \beta'(l q_{i,j})$ for $l \in L$ and $\lambda_{i,j} \in \{0, 1\}$.

Noting that $\sum_{i,j}\pi(p_{i,j}(1)) = 1_{\mathcal{L}(\mathcal{E})}$, we may write T in matrix form $(T_{(i,j),(i',j')})_{(i,j),(i',j')}$. Since $[T,\pi(p_{i,j}(1))] \in \mathcal{K}(\mathcal{E})$, all off-diagonal elements of T are compact operators and so by a compact perturbation we may replace T by its diagonal matrix T' (canceling the off-diagonal terms of T) without changing the cycle, that is, $[(\pi,\mathcal{E},T)] = [(\pi,\mathcal{E},T')]$. Note that the identical map $\mathcal{L}(\mathcal{E}) \cap$ diagonal matrices $\to \mathcal{L}(\mathcal{E}')$ is an isomorphism, which restricts to a bijection $\mathcal{K}(\mathcal{E}) \cap$diagonal matrices $\to \mathcal{K}(\mathcal{E}')$ because $\pi(p_{i,j}(1))\mathcal{E} \cong \pi(p_{i,j}(1))\mathcal{E}'$ for all i,j. We set $\pi' := \pi$. The desired cycle in $KK^{L'}(A,B')$ is $(\pi',\mathcal{E}',T')$.

Let us reversely be given a cycle $(\pi',\mathcal{E}',T')$ in $KK^{L'}(A,B')$. Define $\pi := \pi'$, $T := T'$ and $\mathcal{E}$ an identical copy of $\mathcal{E}'$ as a graded vector space. (Note that $\mathcal{E} \cong \oplus_{i,j}\pi(p_{i,j}(1))\mathcal{E}$ corresponding to B' by L'-equivariance of π.) Set

$$\langle \xi,\eta\rangle_{\mathcal{E}} := \bigoplus_{i=1}^{m}\sum_{j=1}^{n_i}\big\langle \pi\big(p_{i,j}(1)\big)\xi,\pi\big(p_{i,j}(1)\big)\eta\big\rangle_{\mathcal{E}'} \quad \in\ B = B_1\oplus\ldots\oplus B_m$$

for all $\xi,\eta\in\mathcal{E}$, the B-module product on $\mathcal{E}$ by $\xi(\oplus_i b_i) := \sum_i\sum_j \pi(p_{i,j}(1))\xi b_i$ (the last b_i regarded in $B_{i,j}$), and the L-action on $\mathcal{E}$ to be the restriction of the L'-action on $\mathcal{E}'$. It is easy to see that both constructed assignments $(\pi,\mathcal{E},T') \leftrightarrow (\pi',\mathcal{E}',T')$ are reverses to each others. The detailed, tedious verifications we left out in this proof are left to the reader. ∎

The next lemma deals with the question how to remove Res_G^{pG}.

Lemma 7.4. *Let $p\in G$ be a projection in the center. Then $KK^{pG}(\mathrm{Res}_G^{pG}A,\mathrm{Res}_G^{pG}B) \cong KK^G(pA,pB)\cong KK^G(pA,B)\cong KK^G(A,pB)$.*

Proof. The first isomorphism is just the identity on cycles; a cycle $(\mathcal{E},T)$ in $KK^G(pA,pB)$ degenerates to $(p\mathcal{E},pT)$; a pG-action extends to a G action by $g\mapsto pg$. Also recall that $\mathrm{Res}_G^{pG}(A) = pA$. For the second isomorphism we decompose $B\cong pB\oplus(1-p)B$ and note that $KK^G(pA,(1-p)B) = 0$ since $p(a)\xi(1-p)(b) = 0$ for $a\in A,\xi\in\mathcal{E}$ and $b\in B$, where $(\mathcal{E},T)$ is a cycle. ∎

The next lemma is similar to the fact that the K-theory group $KK(\mathbb{C},B) = K(B)$ is countable. It is immediately evidently true in IK-theory by the Green–Julg isomorphism $IK^H(\mathbb{C},A)\cong K(A\rtimes H)$ in [8].

Lemma 7.5. *For all compact subinverse semigroups $H\subseteq G$ $KK^H(\mathrm{Res}_G^H\mathbb{C},B)$ is countable for all $B\in KK^G$ and commutes with countable direct sums in the variable B.*

Proof. Let $f:\mathbb{C}\to C_0(X_H)$ be the map $f(1) = 1_e$, where e denotes the minimal projection in $E(H)$, so is also in X_H. Reversely, let $p: C_0(X_H)\to\mathbb{C}$ be the projection $p(1_e) = 1$. Both f and p are G-equivariant $*$-homomorphisms, because $g(1_e) = 1_{geg^*} = 1_e$ since geg^* is both in X_H and in $E(H)$, so must be e again. The map $f^*: KK^H(C_0(X_H),B)\to KK^H(\mathbb{C},B)$ is surjective and p^* is injective because $f^*p^* = (pf)^* = id$. Hence, noting that the K-theory of a separable C^*-algebra is countable, $KK^H(\mathrm{Res}_G^H\mathbb{C},B)$ is countable since it is the image of f^* of the countable abelian group

$$KK^H(C_0(X_H),\mathrm{Res}_G^H B)\ \cong\ K\big(\mathrm{Res}_G^H(B)\widehat{\rtimes}H\big), \tag{7.3}$$

where this is essentially the Green–Julg isomorphism for groupoids, see Tu [30, Proposition 6.25], or directly apply [3, Corollary 5.4]. Both diagrams

$$\begin{array}{ccc} \bigoplus_i KK^H(C_0(X_H),B_i) & \longrightarrow & KK^H\big(C_0(X_H),\bigoplus_i B_i\big) \\ {\scriptstyle \oplus_i p^*}\big\uparrow\big\downarrow{\scriptstyle \bigoplus_i f^*} & & {\scriptstyle p^*}\big\uparrow\big\downarrow{\scriptstyle f^*} \\ \bigoplus_i KK^H(\mathbb{C},B_i) & \longrightarrow & KK^H\big(\mathbb{C},\bigoplus_i B_i\big) \end{array} \tag{7.4}$$

commute (one with f^* and another with p^*) and because the first line is an isomorphism because of (7.3) (K-theory respects direct sums), the second line is also one. ∎

8. Some specialized results

In this section we shall prove some specialized results with induction and restriction.

Definition 8.1. Set

$$\mathcal{CI}_1 := \{\mathrm{Ind}_{H_n}^G \mathrm{Res}_G^{H_n} \ldots \mathrm{Ind}_{H_1}^G \mathrm{Res}_G^{H_1}(\mathbb{C}) \,|\, H_i \subseteq G \,\text{compact subinverse s.}, \, n \ge 1\}.$$

Considering for example an object in $\mathcal{CI}_1$ for $n = 3$, we may write it as

$$\mathrm{Ind}_{H_3}^G \mathrm{Res}_G^{H_3} \mathrm{Ind}_{H_2}^G \mathrm{Ind}_{H_1}^G \mathbb{C} \;=\; \mathrm{Ind}_{H_3}^G \bigoplus_{g \in J} \mathrm{Res}_{L_g'}^{H_3} \mathrm{Ind}_{M_g}^{L_g'} \mathrm{Res}_G^{M_g} \mathrm{Ind}_{H_1}^G \mathbb{C} \tag{8.1}$$

by an application of Lemma 7.2. Go back to Lemma 7.1 and define $V \subseteq G$ to be the finite subinverse semigroup $g_0 U'' g_0^*$, where $U'' \subseteq U'$ denotes the finite subinverse semigroup consisting of those elements $u \in U'$ such that u commutes with u_0 and $gug^* \in M \cup \{0\}$. Note that $M \subseteq \widetilde{V}$ since $E(U') \subseteq U''$ and so $g_0 u_0 g_0^* \in \widetilde{V}$. Observe that gg^* is in the center of $\tilde{V}$ and $\tilde{V} gg^* = V gg^* = M \cup \{0\}$. Write V_g for the V of M_g. Continue (8.1) with

$$= \bigoplus_{g \in J} \mathrm{Ind}_{H_3}^G \mathrm{Res}_{L_g'}^{H_3} \mathrm{Ind}_{M_g}^{L_g'} \mathrm{Res}_{V_g}^{M_g} \mathrm{Res}_G^{V_g} \mathrm{Ind}_{H_1}^G \mathrm{Res}_{H_1}^G \mathbb{C} \tag{8.2}$$

$$= \bigoplus_{g \in J} \bigoplus_{h \in J_g} \mathrm{Ind}_{H_3}^G \mathrm{Res}_{L_g'}^{H_3} \mathrm{Ind}_{M_g}^{L_g'} \mathrm{Res}_{V_g}^{M_g} \mathrm{Res}_{L_{g,h}'}^{V_g} \mathrm{Ind}_{M_{g,h}}^{L_{g,h}'} \mathrm{Res}_G^{M_{g,h}} \mathbb{C} \tag{8.3}$$

by another application of Lemma 7.2.

Note that every summand in (8.3) is of the form $\mathrm{Ind}_{H_3}^G A$ for some finite dimensional, commutative H_3-algebra A. (Because $(L')_M$ is finite by (7.2).) Similarly, by a successive n-fold application of Lemma 7.2 write $\mathrm{Ind}_{H_n}^G \ldots \mathrm{Ind}_{H_1}^G \mathbb{C}$ as a countable direct sum of G-algebras of the form $\mathrm{Ind}_{H_n}^G A$ for some finite dimensional, commutative H_n-algebras A.

Definition 8.2. Varying over all $n \ge 1$ and $H_1, \ldots, H_n \subseteq G$, denote by $\mathcal{CI}_0$ the countable collection of all G-algebras of the form $\mathrm{Ind}_{H_n}^G A$ as just described (where, recall, A is some finite dimensional, commutative H_n-algebras).

Corollary 8.1. *Every G-algebra of $\mathcal{CI}_1$ is a direct sum of G-algebras of $\mathcal{CI}_0$.*

From here we shall assume that G is E-continuous, for KK^G to be a triangulated category in the sense of Proposition 6.2.

Definition 8.3. A subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is called a *triangulated subcategory* (see [20, Section 4.5]) if it is nonempty, full, closed under suspension and desuspension, and, whenever for a given exact sequence $A \to B \to C \to SC$ two objects of $\{A, B, C\}$ are in $\mathcal{S}$ then also the third one. $\mathcal{S}$ is also called *thick* (see [20, Section 4.5]) if every retract (summand) of an object in $\mathcal{S}$ is also in $\mathcal{S}$, and *localizing* (see [20, Section 6.2]) if it is thick and closed under coproducts in $\mathcal{T}$.

Definition 8.4. For a class $\mathcal{G}$ of objects in $\mathcal{T}$ we write $\langle \mathcal{G} \rangle$ for the smallest localizing subcategory of $\mathcal{T}$ containing $\mathcal{G}$, cf. [23, Section 2.5].

Note that in KK^G coproducts are direct sums, and we only allow *countable* direct sums. In the next definition we *assume* that the used results by A. Neeman hold true under this countability restriction for coproducts.

Definition 8.5. Suppose that G is E-continuous. Fix a compact subinverse semigroup $H \subseteq G$. Let $\mathcal{F}_H$ denote the set of all finite dimensional, commutative H-algebras which are compact objects of the category KK^H in the sense of [25, Definition 1.6]. (For instance, $\mathbb{C} \in \mathcal{F}_H$ by Lemma 7.5.) The set $\Sigma\mathcal{F}_H \cup \mathcal{F}_H$ is closed under suspension by Bott periodicity and consists of compact objects. By [26, Proposition 8.4.1] it is a generating set for $\langle \mathcal{F}_H \rangle$. Hence $\langle \mathcal{F}_H \rangle$ is a compactly generated triangulated category in the sense of [25, Definition 1.7]. By Lemma 4.1 and [25, Theorem 4.1], the restricted induction functor $\mathrm{Ind}_H^L : \langle \mathcal{F}_H \rangle \to KK^L$ has a right adjoint functor $\mathrm{Right}_L^H : KK^L \to \langle \mathcal{F}_H \rangle$ for every subinverse semigroup $L \subseteq G$.

Note that if G is a discrete group then Right_L^H is just the ordinary restriction functor Res_L^H.

Corollary 8.2. *Assume that an inverse semigroup G is such that G is E-unitary, E-continuous and the functors Right$_L^H$ respect countable direct sums. (In the worst case scenario, if G is a group, see* [23].*) Then for all $A \in \mathcal{CI}_0$, $KK^G(A,B)$ is countable for all $B \in KK^G$ and commutes with countable direct sums in the variable B.*

Moreover, every algebra in $\mathcal{CI}_0$ is of the form Ind$_H^G(A)$ for some $A \in \mathcal{F}_H$.

Remark 8.1. It appears natural that $KK^H(A,B)$ is countable and commutes with countable direct sums in B for all finite subinverse semigroups $H \subseteq G$ and finite-dimensional, commutative H-algebras A. (The Künneth theorem comes into mind, but is difficult even for $G = \mathbb{Z}/2$, see Rosenberg [27].) But then the claim of Corollary 8.2 would follow alone from Definition 8.2 and the assumption that Right respects countable direct sums.

Proof of Corollary 8.2. To demonstrate the proof of Corollary 8.2, assume A is one of the summands of (8.3). We go inductively from right to left in (8.3). The first algebra $A_1 := \mathrm{Res}_G^{M_{g,h}} \mathbb{C}$ of (8.3) satisfies the claim of Corollary 8.2 when replacing A by A_1 by Lemma 7.5. The next algebra $A_2 := \mathrm{Ind}_{M_{g,h}}^{L'_{g,h}} A_1$ satisfies the claim of Corollary 8.2 because now evidently $A_1 \in \mathcal{F}_{M_{g,h}}$ and we assume that $\mathrm{Right}_{L'_{g,h}}^{M_{g,h}}$ respects countable direct sums, whence A_2 satisfies the claim by putting Ind to the other side as Right, cf. [25, Theorem 5.1]. Going back how we deduced identity (8.3) from Lemma 7.2, a check shows that both expressions $\mathrm{Res}_{L'_g}^{H_3}$ and $\mathrm{Res}_{L'_{g,h}}^{V_g}$ of (8.3) are of the form $\mathrm{Res}_{L'}^{L}$, where L' and L are the notions from Lemma 7.1 and additionally L is finite. But from Lemma 7.3 we know that

$$KK^L(\mathrm{Res}_{L'}^L A_2, B) \cong KK^{L'}(A_2, B'). \tag{8.4}$$

Since A_2 satisfies the assumption, $\mathrm{Res}_{L'}^L A_2 = \mathrm{Res}_{L'_{g,h}}^{V_g} A_2 =: A_3$ does it also because of (8.4). Recall that gg^* is in the center of $\widetilde{V}_g$ and $\widetilde{V}_g gg^* = M_g \cup \{0\}$. (See before (8.2).) Consequently we have

$$KK^{M_g}(\mathrm{Res}_{V_g}^{M_g} A_3, \mathrm{Res}_{V_g}^{M_g} B) \cong KK^{V_g}(A_3, gg^* B) \tag{8.5}$$

for every V_g-algebra B by Lemma 7.4 and (2.1). Hence, since A_3 satisfies the assumption, the algebra $A_4 := \mathrm{Res}_{V_g}^{M_g} A_3$ appearing in (8.3) does it also by (8.5). Successively we proceed in the same vein for the final three expressions $\mathrm{Ind}_{H_3}^G$, $\mathrm{Res}_{L'_g}^{H_3}$ and $\mathrm{Ind}_{M_g}^{L'_g}$ in (8.3) until the assumption is verified for A. The proof for arbitrary $A \in \mathcal{CI}_0$ is analog. The last claim follows evidently from this proof. ∎

9. OUTLOOK TOWARDS A POTENTIAL BAUM–CONNES MAP

In this section we shall switch from the restriction functors to the Right-functors of Definition 8.5. We shall prove the existence of simplicial approximations and even a Dirac morphism and a Baum–Connes map for all coefficient algebras under some theoretical technical assumptions which are motivated by the last section. Because of Proposition 6.2 it is assumed that G is E-continuous.

Let $\mathcal{F}_H$ be the set of Definition 8.5 or any other countable set of compact objects of KK^H; by Definition 8.5 there exists a right adjoint functor Right_G^H for Ind_H^G.

We are going to introduce analogous sets to $\mathcal{CI}_1$ and $\mathcal{CI}_0$ of the last section by replacing the Res-functors by the Right-functors. As a motivation for the next definition also recall Corollary 8.2.

Definition 9.1. Let us be given a G-algebra Z. Set

$$\mathcal{CJ}_1 := \{\mathrm{Ind}_{H_n}^G \mathrm{Right}_G^{H_n} \dots \mathrm{Ind}_{H_1}^G \mathrm{Right}_G^{H_1}(Z) \,|\, H_i \subseteq G \,\text{comp. sub. s.}, \, n \geq 1\}$$

and $\mathcal{CJ}_0$ the countable set of objects of the form $\mathrm{Ind}_H^G A$, where H is a finite subinverse semigroup of G and $A \in \mathcal{F}_H$.

The following corollary is a slight modification of Brown's representability theorem.

Corollary 9.1 (Cf. Lemma 6.3 of [23])**.** *Assume that for all $A \in \mathcal{CJ}_0$ $KK^G(A,B)$ is countable for all $B \in KK^G$ and commutes with countable direct sums in the variable B. Then for any object B in KK^G there exist an object $\tilde{B}$ in $\langle \mathcal{CJ}_0 \rangle$ and a morphism $f \in KK^G(\tilde{B}, B)$ such that $f_* : KK^G(A, \tilde{B}) \to KK^G(A,B)$ ($f_*(x) := f \circ x$ for $x \in KK^G(A,\tilde{B})$) is an isomorphism for all objects A in $\langle \mathcal{CJ}_0 \rangle$.*

Definition 9.2 (Cf. Definition 4.1 of [23])**.** An object A in KK^G is called *compactly induced* if there exists an object B in KK^G and a compact subinverse semigroup $H \subseteq G$ such that A is isomorphic to $\mathrm{Ind}_H^G(B)$ in KK^G. The full subcategory of KK^G of compactly induced objects is denoted by $\mathcal{CJ}$.

Definition 9.3 (Cf. Definition 4.5 of [23])**.** Let Z be a G-algebra. A *$\mathcal{CJ}$-simplicial approximation* for Z is an element $f \in KK^G(B,Z)$ for some object B in $\langle \mathcal{CJ} \rangle$ such that $\mathrm{Right}_G^H(f)$ is invertible in KK^H for all compact subinverse semigroups H of G. If $Z = C_0(X)$ we particularly call f a *Dirac morphism*.

As a motivation for the assumptions of the next proposition recall the analogous results Corollaries 8.1 and 8.2 and Remark 8.1. Also note that the right adjoint functor R_G^H of [2] commutes with direct sums and satisfies $R_G^H(\varepsilon(E)) \in \mathcal{F}_H$ for H finite.

Proposition 9.2 (Cf. Proposition 4.6 of [23])**.** *Assume that G is an E-continuous inverse semigroup and Z a G-algebra such that the following assumptions hold true (for example, they hold true if G is a discrete group (where Right = Res), $Z = \mathbb{C}$ and $\mathcal{F}_H = \{\mathbb{C}\}$):*

(a) Assume that $KK^H(A,B)$ is countable and commutes with countable direct sums in B for all finite subinverse semigroups H of G and $A \in \mathcal{F}_H$ (see Def. 8.5).

(b) Assume that the Right-functors commute with countable direct sums.

(c) Assume that $\mathrm{Right}_G^H(Z) \in \mathcal{F}_H$ for all finite subsemigroups H of G.

(d) Suppose that every object of $\mathcal{CJ}_1$ can be expressed as a countable direct sum of objects of $\mathcal{CJ}_0$ up to KK^G-equivalence.

Then Z has a $\mathcal{CJ}$-simplicial approximation.

Proof. Assume without loss of generality that for every $A \in \mathcal{F}_H$, $\mathrm{Ind}_H^G A \in \mathcal{CJ}_0$ appears as a summand of some $B \in \mathcal{CJ}_1$; if not so, simply restrict $\mathcal{F}_H$ to a smaller set. Notice that our assumptions imply the validity of the assumption of Corollary 9.1, see Remark 8.1.

Apply Corollary 9.1 to $B := Z$ and obtain an object $P \in \langle \mathcal{CJ}_0 \rangle \subseteq KK^G$ and a morphism $D \in KK^G(P,Z)$ (where $P := \tilde{B}$ and $D := f$ from Corollary 9.1) such that

$$D_* : KK^G(A,P) \to KK^G(A,Z) \tag{9.1}$$

is a group isomorphism for all $A \in \langle \mathcal{CJ}_0 \rangle$. We want to show that $\mathrm{Right}_G^H(D)$ is an isomorphism for every compact subinverse semigroup H of G (see Definition 9.3); so fix any such H. To this end it is sufficient to show that both induced group homomorphisms

$$\mathrm{Right}_G^H(D)_* : KK^H(\mathrm{Right}_G^H P, \mathrm{Right}_G^H P) \to KK^H(\mathrm{Right}_G^H P, \mathrm{Right}_G^H Z)$$

and

$$\mathrm{Right}_G^H(D)_* : KK^H(\mathrm{Right}_G^H Z, \mathrm{Right}_G^H P) \to KK^H(\mathrm{Right}_G^H Z, \mathrm{Right}_G^H Z)$$

are isomorphisms. For verifying that the first stated $\mathrm{Right}_G^H(D)_*$ is an isomorphism it is sufficient to show that

$$\mathrm{Right}_G^H(D)_* : KK^H(\mathrm{Right}_G^H A, \mathrm{Right}_G^H P) \to KK^H(\mathrm{Right}_G^H A, \mathrm{Right}_G^H Z) \tag{9.2}$$

is an isomorphism for all $A \in \mathcal{CJ}_0$ because $P \in \langle \mathcal{CJ}_0 \rangle$.

We consider first the case that $A \in \mathcal{CJ}_1$. Applying on both ends of (9.2) the adjointness relation between Ind and Right, (9.2) turns to

$$D_* : KK^G(\mathrm{Ind}_H^G \mathrm{Right}_G^H A, P) \to KK^G(\mathrm{Ind}_H^G \mathrm{Right}_G^H A, Z). \tag{9.3}$$

But since $\mathrm{Ind}_H^G \mathrm{Right}_G^H A$ is in $\mathcal{CJ}_1$, and hence a countable direct sum of objects in $\mathcal{CJ}_0$ by assumption, $\mathrm{Ind}_H^G \mathrm{Right}_G^H A$ is also in $\langle \mathcal{CJ}_0 \rangle$ by Definitions 8.3 and 8.4, and hence (9.3) and so (9.2) are isomorphisms by (9.1).

We may write $A \cong \bigoplus_j B_j$ KK^G-equivalently by assumption, where $B_j \in \mathcal{CJ}_0$. The canonical injection and projection $\mathrm{Right}_G^H B_j \xrightarrow{p} \mathrm{Right}_G^H A \xrightarrow{f} \mathrm{Right}_G^H B_j$ to the jth coordinate satisfy $id = (fp)^* = p^* f^*$, and an analog diagram as in (7.4) shows that the isomorphism (9.2) is also an isomorphism for $A := B_j$. By varying over all $A \in \mathcal{CJ}_1$ and all coordinate projections j, we see that (9.2) is an isomorphism for all $A \in \mathcal{CJ}_0$.

That the second homomorphism $\mathrm{Right}_G^H(D)_*$ is an isomorphism follows from (9.1) applied to $A := \mathrm{Ind}_H^G \mathrm{Right}_G^H Z \in \mathcal{CJ}_0$. ∎

The last proposition might offer a chance for defining a Baum–Connes map:

Remark 9.1. If the assumptions of Proposition 9.2 hold true for an inverse semigroup and $Z = C_0(X)$ then its application yields a $\mathcal{CJ}$-simplicial approximation and thus a Baum–Connes map for all coefficient algebras A (by tensoring a $\mathcal{CJ}$-simplicial approximation D for $C_0(X)$ with A, that is, forming $D \otimes^{C_0(X)} A$); see [23] or [2, Section 10] for the concept. The *Baum–Connes map* with coefficient algebra A is then defined to be the homomorphism $K(B \widehat{\rtimes} G) \to K(A \widehat{\rtimes} G)$ (Sieben's crossed product) induced by taking the Kasparov product with the element $\widehat{j^G}(D) \in KK(B \widehat{\rtimes} G, A \widehat{\rtimes} G)$ (descent homomorphism) for any $\mathcal{CJ}$-simplicial approximation $D \in KK(B, A)$ of A.

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