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OPERATORS ON FRAMES

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ABSTRACT. In this paper, we first show the conditions under which an operator on a Hilbert space H can be represented as sum of two unitary operators. Then, it is concluded that a Riesz basis for a Hilbert space H can be written as a sum of two orthonormal bases. Finally, the study proves that if A is a normal maximal partial isometry on a Hilbert space H and if $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H, then $\{Ae_k\}_{k=1}^{\infty}$ is a 1-tight frame for H.

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1. INTRODUCTION

This section provides preliminaries from operators theory which will be needed them. Normally, $B(H,K)$ consists of all bounded operators from a Hilbert space H to a Hilbert space K, $B(H)$ denotes for which $H = K$, and $I \subseteq \mathbb{N}$. Throughout the paper, H denotes a separable Hilbert space.
Recall that an operator $T \in B(H)$ is an isometry if for all $x \in H$, $||Tx|| = ||x||$, and is a partial isometry if it is an isometry on the orthogonal complement of its kernel. Also, we define a unitary operator as a linear transformation which is a surjective isometry.

Definition 1.1. A maximal partial isometry, either itself or its adjoint is isometry.

The followings facts can be found in any standard text of operators theory (for example, see [5]).

Lemma 1.1. *$U \in B(H)$ is surjective if and only if U^* is bounded below.*

Theorem 1.2. *(**Polar Decomposition**) If $T \in B(H,K)$, then*
(i) it has a decomposition as $T = VP$ such that
1- $V \in B(H,K)$ is a partial isometry.
2- $P \in B(H)$ is a positive operator.
3- $kerV = kerP$.
(ii) Let $T = UA$ be an another decomposition as product of partial isometry U and positive operator A such that $kerU = kerA$. Then $U = V$ and $P = A = |T|$.
(iii) If $T = V|T|$, then $|T| = V^*T$.

Corollary 1.3. *If $T = VP$ is the polar decomposition of T, then*
(i) V is isometry if and only if T is injective.
(ii) V^* is isometry if and only if ImT is dense.

Proof. The proofs are based on the facts that:

$$kerP = kerT^*T = kerT$$

and also

$$kerV^* = (ranV)^\perp = (kerT)^\perp.$$

∎

It is known from operators theory that every separable Hilbert space H has an orthonormal basis, and if $U \in B(H)$ is a unitary operator and $\{e_k\}_{k=1}^\infty$ is an orthonormal basis for H, then $\{Ue_k\}_{k=1}^\infty$ is an orthonormal basis for H. The next theorem which can be found in any text of operators theory characterizes all orthonormal bases of a Hilbert space H with one basis.

Theorem 1.4. *Let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis for a Hilbert space H. Then orthonormal bases for H are precisely the sets $\{Ue_k\}_{k=1}^\infty$, where U is a unitary operator on H.*

2. FRAMES AND PRELIMINARIES

Frames were first utilized in 1952 by Duffin and Schaeffer [7]. The theory of frames plays significant roles in applied mathematics, science, and engineering today. The feature of a basis $\{f_k\}_{k=1}^\infty$ in a Hilbert space H is that every element $f \in H$ can be represented as an (infinite)linear combination of the elements f_k as follows:

$$f = \sum_{k=1}^{\infty} c_k(f) f_k, \tag{2.1}$$

where the coefficients $c_k(f)$ are unique.
The frames are an extension of bases in Hilbert spaces. In fact, a frame is a sequence $\{f_k\}_{k=1}^{\infty}$ in H which it allows every element $f \in H$ can be written as in the relation (2.1), whereas the coefficients are not unique. So, a frame need not a basis.

Definition 2.1. A frame for a Hilbert space H is a family of vectors $F = \{f_k\}_{k\in I}$ in H such that there are constants A and $B > 0$ satisfying:

$$A||f||^2 \leq \sum_{k\in I} | < f, f_k > |^2 \leq B||f||^2, \quad \forall\, f \in H.$$

The constants A and B are called lower and upper frame bounds, respectively, and they are not unique. If only the right-hand side inequality is assumed, it is called a B-Bessel sequence. If $A = B$, it is said to be an A-tight frame.
For any Bessel sequence $F = \{f_k\}_{k\in I}$ the pre-frame (synthesis) operator is defined by

$$T : l^2(I) \longrightarrow H, \qquad T(\{c_k\}) = \sum_{k\in I} c_k f_k.$$

The analysis operator for F is T^* and is given by $T^* f = \{< f, f_k >\}_{k\in I}$. The frame operator is $S = TT^*$ and it satisfies: $S_F f = \sum_{k\in I} < f, f_k > f_k, \;\; \forall f \in H.$

It is a fact that if $F = \{f_k\}_{k\in I}$ is an A-tight frame with the frame operator S, then $S = AI$, so for each f, we have $f = \frac{1}{A}\sum_{k\in I} < f, f_k > f_k$.
The next lemma can be seen in [4] gives some important properties of the frame operators S and S^{-1}:

Lemma 2.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame with the frame operator S and frame bounds A, B. Then the following holds:*
(i) S is bounded, invertible, self-adjoint, and positive.
(ii) $\{S^{-1}f_k\}_{k=1}^{\infty}$ is a frame with the frame operator S^{-1} and frame bounds B^{-1}, A^{-1}.
(iii) If A, B are the optimal frame bounds for $\{f_k\}_{k=1}^{\infty}$, then the bounds B^{-1}, A^{-1} are optimal for $\{S^{-1}f_k\}_{k=1}^{\infty}$.

The frame $\{S^{-1}f_k\}_{k=1}^{\infty}$ is called the canonical dual frame of $\{f_k\}_{k=1}^{\infty}$. It is well-known that the definition of a frame has several equivalents. It can be considered an equivalence relation between the frames and surjective operators; that is, if we have a theorem about frames, then we have a theorem about surjective operators and vice versa. The first theorem states an equivalent on frames. The second theorem characterizes the frames for a Hilbert space H and it is similar to the definition of a Riesz basis. All the following theorems can be found in [4].

Theorem 2.2. *A sequence $\{f_k\}_{k=1}^{\infty}$ in H is a frame for H if and only if there is a bounded surjective operator $U : l^2(N) \to H$ such that for all k, $Ue_k = f_k$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H.*

Theorem 2.3. *Let $\{e_k\}_{k=1}^{\infty}$ be an arbitrary orthonormal basis for H. The frames for H are precisely the family $\{Ue_k\}_{k=1}^{\infty}$, where $U : H \to H$ is a bounded surjective operator.*

Proof. Suppose that $\{\delta_k\}_{k=1}^{\infty}$ is the canonical basis for $l^2(N)$, $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H, and $\phi : H \to l^2(N)$ is the isometric isomorphism of the form $\phi e_k := \delta_k$.
If $\{f_k\}_{k=1}^{\infty}$ is a frame, then the pre-frame operator T is a bounded surjective operator, thus by Theorem 2.2 the family $\{Ue_k\}_{k=1}^{\infty}$ is a frame.
In other words, if $Ue_k = f_k$ and U is a bounded surjective operator, then we have

$$\sum_{k=1}^{\infty} | < f, f_k > |^2 = \sum_{k=1}^{\infty} | < f, Ue_k > |^2 = ||U^* f||^2, \quad \forall\, f \in H.$$

Since U is bounded and surjective, again by Theorem 2.2 the sequence $\{f_k\}_{k=1}^{\infty}$ is a frame. ∎

A special example of a frame (in fact, the motivation behind the definition) is an orthonormal basis for a Hilbert space H or isomorphism images of orthonormal bases which are Riesz bases. Theorem 1.4 characterized all orthonormal bases in terms of unitary operators acting on a single orthonormal basis. The definition of a Riesz basis appears by weakening the condition on the operator of it theorem:

Definition 2.2. A Riesz basis for a Hilbert space H is a family of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H and $U : H \longrightarrow H$ is a bounded bijective operator.

The next theorem shows that a Riesz basis is a frame, in fact, a Riesz basis is a basis.

Theorem 2.4. *If $\{f_k\}_{k=1}^{\infty} = \{Ue_k\}_{k=1}^{\infty}$ is a Riesz basis for H, then there exist constants $A, B > 0$ such that*

$$A||f||^2 \leq \sum_{k=1}^{\infty} | < f, f_k > |^2 \leq B||f||^2, \quad \forall\, f \in H.$$

The largest possible value for the constant A is $\frac{1}{||U^{-1}||^2}$, and the smallest possible value for B is $||U||^2$.

3. MAIN RESULTS

In this section, we first show the conditions under which an operator on a Hilbert space H can be represented as sum of two unitary operators. Then, it is concluded that a Riesz basis can be shown as sum of two orthonormal bases, whereas a frame cannot be shown as sum of two orthonormal bases. First, we prove a fact on operators.

Proposition 3.1. *Let $T \in B(H)$ be a self-adjoint positive operator. Then $I + T$ is a bounded invertible operator on H.*

Proof. We know that for any $h \in H$,

$$\begin{aligned} ||(I+T)h||^2 &= < (I+T)h, (I+T)h > \\ &= ||h||^2 + < h, Th > + < Th, h > + ||Th||^2, \end{aligned}$$

since two the middle terms of the last relation are nonnegative, hence for all $h \in H$, we get $||(I + T)h|| \geq ||h||$; that is, $I + T$ is bounded below, so by Lemma 1.1 it is injective and $(I + T)^* = I + T$ is surjective.
On the other hand, the inequality $||(I + T)h|| \geq ||h||, \forall h \in H$ implies that

$$||(I+T)^{-1}h|| \leq ||(I+T)(I+T)^{-1}h|| = ||h||.$$

Therefore, $I + T$ is invertible in $B(H)$. ∎

Example 3.1. *If $\phi = \{\varphi_i\}_{i\in I}$ is a frame for H, then $\phi + (-\phi)$ is not a frame.*

The next corollary shows that the summation of a frame and its canonical dual is a frame.

Corollary 3.2. *If $\phi = \{\varphi_i\}_{i\in I}$ is a frame (Riesz basis) for H with the frame operator S, then $\{(I + S)\varphi_i\}_{i\in I}$ is a frame (Riesz basis) as well.*

Similarly, the sequence $\{\varphi_i + S^{-1}\varphi_i\}_{i\in I}$ is a frame (Riesz basis) for H.

If $\{\varphi_k\}_{k\in I}$ is a frame for H and $T \in B(H)$, then $\{T\varphi_k\}_{k\in I}$ need not a frame. For example, if $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H and $T = 0$.

If $\{\varphi_k\}_{k\in I}$ is a frame for H with upper and lower bounds A and B, respectively, and $T \in B(H)$ is surjective, then for any $h \in H$, we get

$$\begin{aligned}\sum_{k\in I} |< h, T\varphi_k >|^2 &= \sum_{k\in I} |< T^*h, \varphi_k >|^2 \\ &\geq A||T^*h||^2 \geq AC||h||^2,\end{aligned}$$

where the last inequality holds by Lemma 1.1.
On the other hand, it is clear that

$$\begin{aligned}\sum_{k\in I} |< h, T\varphi_k >|^2 &= \sum_{k\in I} |< T^*h, \varphi_k >|^2 \\ &\leq B||T^*h||^2 \leq B||T||^2||h||^2.\end{aligned}$$

Therefore, $\{T\varphi_k\}_{i\in I}$ is a frame.
We now assume that $\{T\varphi_k\}_{k\in I}$ is a frame for H with the frame operator U, then by definition for all $f \in H$, we obtain

$$\begin{aligned}Uf &= \sum_{k\in I} < f, T\varphi_k > T\varphi_k \\ &= T(\sum_{k\in I} < T^*f, \varphi_k > \varphi_k) = TU(T^*f).\end{aligned}$$

That is, $U = TUT^*$. Since U is invertible, so it is concluded that T is surjective. Now we can summarise the above discussion as follows:

Proposition 3.3. *Let $\{\varphi_k\}_{k\in I}$ be a frame for a Hilbert space H with lower and upper frame bounds A and B, respectively, and $T \in B(H)$. Then the family $\{T\varphi_k\}_{k\in I}$ is a frame for H if and only if T is surjective.*

Corollary 3.4. *Let $\{\varphi_k\}_{k\in I}$ be a frame for H and $T \in B(H)$. Then the family $\{\varphi_k + T\varphi_k\}_{k\in I}$ is a frame if and only if $I + T$ is surjective.*

Lemma 3.5. *Every positive operator $P \in B(H)$ with $||P|| \leq 1$ can be represented as:*

$$P = \frac{1}{2}(U + U^*),$$

where $U = P + i\sqrt{1 - P^2}$ is a unitary operator.

Proof. The proof on based of the definition U is clear. ∎

Proposition 3.6. *If $A \in B(H)$ is invertible, then it can be written as a linear combination of two unitary operators.*

Proof. Suppose that $A = VP$ is the polar decomposition of A. Since A is injective, so by Corollary 1.3 the operator V is an isometry, in fact, V is a unitary. We now take

$$\acute{P} = \frac{2P}{3||P||}.$$

Because of $\acute{P}$ is a positive operator and $||\acute{P}|| \leq 1$, hence by the previous lemma we can write $\acute{P} = \frac{1}{2}(U + U^*)$, where U is a unitary operator. Therefore,

$$A = \frac{3||P||}{4}(VU + VU^*),$$

and the operators VU and VU^* are unitary. ∎

Proposition 3.7. *There exists a frame (not a Riesz basis) for a Hilbert space H so that it cannot be shown as a sum of two orthonormal bases.*

Proof. We consider the orthonormal basis $\{e_k\}_{k=1}^{\infty}$ for H and for fixed $m \in \mathbb{N}$, we define the sequence $\{f_k\}_{k=1}^{\infty}$ by

$$f_1 = f_2 = ... = f_m = 0 \; and \;\; f_{m+k} = me_k, \; k = 1, 2, ...$$

Hence, for any $h \in H$, we get

$$\sum_{k=1}^{\infty} | < h, f_k > |^2 = \sum_{k=1}^{\infty} | < h, me_k > |^2$$
$$= m^2 \sum_{k=1}^{\infty} | < h, e_k > |^2 = m^2 ||h||^2.$$

Thus, the sequence $\{f_k\}_{k=1}^{\infty}$ is a m-tight frame for H.
We now assume that there are two orthonormal bases $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ and also nonzero scalers α and β such that for each k, we have $f_k = \alpha g_k + \beta h_k$. Then the relation $\alpha g_k + \beta h_k = f_k = 0$, for $k = 1, 2, ...m$ results that

$$span\{g_k\}_{k=1}^{m} = span\{h_k\}_{k=1}^{m}.$$

This relation alone with

$$span\{g_k\}_{k=1}^{\infty} = span\{h_k\}_{k=1}^{\infty} = H$$

yields that

$$span\{g_k\}_{k=m+1}^{\infty} = span\{h_k\}_{k=m+1}^{\infty} \neq H.$$

On the other hand, since the sequences $\{g_k\}_{k=1}^{\infty}$, $\{h_k\}_{k=1}^{\infty}$, and $\{e_k\}_{k=1}^{\infty}$ are orthonormal bases, so we have

$$span\{g_k\}_{k=m+1}^{\infty} = span\{e_k\}_{k=1}^{\infty} = H.$$

But, two these the last relations contradict each other, so the proof completes. ∎

Proposition 3.8. *The frame $\Phi = \{\varphi_k\}_{k\in I}$ is a Riesz basis for a Hilbert space H if and only if it can be represented as a sum of two orthonormal bases.*

Proof. Let $\Phi = \{\varphi_k\}_{k\in I}$ be a Riesz basis for H, hence $Ue_k = \varphi_k$, where $U \in B(H)$ is a bijective operator. By Proposition 3.6 we can write $U = c(U_1 + U_2)$ and each U_i is unitary. So, $\varphi_k = c(U_1e_k + U_2e_k)$ and by Theorem 1.4, $\{U_ie_k\}_{k\in I}$ is an orthonormal basis for H.
Conversely, if $\varphi_k = c(f_k + g_k)$ is a frame and $\{f_k\}_{k\in I}$, $\{g_k\}_{k\in I}$ are orthonormal bases for H. Hence, by Theorem 1.4 we have $f_k = U_1e_k$ and $g_k = U_2e_k$, where $\{e_k\}_{k\in I}$ is an orthonormal basis for H and U_i is a unitary operator on H. Thus, $\varphi_k = c(U_1 + U_2)e_k$ and $c(U_1 + U_2)$ is a bounded bijective operator, therefore $\{\varphi_k\}_{k\in I}$ is a Riesz basis. ∎

Proposition 3.9. *If $\{f_k\}_{k=1}^{\infty} = \{Ue_k\}_{k=1}^{\infty}$ is a Riesz basis for a Hilbert space H with the frame operator S, then we have $S = UU^*$.*

Proof. We know that $Sf = \sum_{k\in I} < f, f_k > f_k, \;\; \forall \;\; f \in H$. On the other hand, since $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H, so for every f in H, we can write $f = \sum_{k\in I} < f, e_k > e_k$, hence

$$Uf = \sum_{k\in I} < f, e_k > Ue_k = \sum_{k\in I} < f, e_k > f_k.$$

Thus, we obtain

$$UU^*f = \sum_{k\in I} < U^*f, e_k > f_k = \sum_{k\in I} < f, Ue_k > f_k = \sum_{k\in I} < f, f_k > f_k.$$

Therefore, we conclude that for all $f \in H$, $Sf = UU^*f$ and the proof is complete. ∎

Proposition 3.10. *If $A \in B(H)$ is a normal maximal partial isometry and $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H, then $\{Ae_k\}_{k=1}^{\infty}$ is a 1-tight frame for H.*

Proof. Because of A is normal, so for all $h \in H$, we have $||Ah|| = ||A^*h||$.
If A^* is isometry, then we get

$$||h||^2 = ||A^*h||^2 = \sum_{k=1}^{\infty} |< A^*h, e_k >|^2$$
$$= \sum_{k=1}^{\infty} |< h, Ae_k >|^2, \;\; \forall\, h \in H.$$

If A is isometry, then for all $h \in H$, we obtain

$$||h||^2 = ||Ah||^2 = ||A^*h||^2 = \sum_{k=1}^{\infty} |< h, Ae_k >|^2.$$

Therefore, in each case it concludes that

$$\sum_{k=1}^{\infty} |< h, Ae_k >|^2 = ||h||^2, \quad \forall\, h \in H,$$

that is, $\{Ae_k\}_{k=1}^{\infty}$ is a 1-tight frame. ∎

Corollary 3.11. *If $A \in B(H)$ is a unitary and $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H, then $\{Ae_k\}_{k=1}^{\infty}$ is a 1-tight frame.*

Proposition 3.12. *Let $T \in B(H)$ so that T^* be an isometry. Let $\{\varphi_k\}_{k\in I}$ be a frame for a Hilbert space H with lower and upper bounds A and B, respectively. Then $\{T\varphi_k\}_{k\in I}$ is a frame with lower and upper bounds A and $B||T||^2$, respectively.*

Proof. The proof is based on which for all $h \in H$,we have

$$A||h||^2 = A||T^*h||^2 \leq \sum_{k\in I} |< T^*h, \varphi_k >|^2 = \sum_{k\in I} |< h, T\varphi_k >|^2$$

and also

$$\sum_{k\in I} |< h, T\varphi_k >|^2 = \sum_{k\in I} |< T^*h, \varphi_k >|^2$$
$$\leq B||T^*h||^2 \leq B||T||^2||h||^2.$$

∎

Corollary 3.13. *Let $T \in B(H)$ such that T^* be an isometry. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H. Then $\{Te_k\}_{k=1}^{\infty}$ is a 1-tight frame.*

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