Aust. J. Math. Anal. Appl.
Vol. **17** (2020), No. 1, Art. 8, 14 pp.
AJMAA

SIMPLICIAL (CO)-HOMOLOGY OF BAND SEMIGROUP

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Received 28 October, 2019; accepted 11 April, 2020; published 24 April, 2020.

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ABSTRACT. We consider the Banach algebra $\ell^1(S)$, with convolution, where S is a band semigroup. We prove directly, without using the cyclic cohomology, that the simplicial cohomology groups $\mathcal{H}^n(\ell^1(S), \ell^1(S)^*)$ vanish for all $n \geq 1$. This proceeds in three steps. In each step, we introduce a bounded linear map. By iteration in each step, we achieve our goal.

Key words and phrases: Simplicial homology; Banach algebra; Simplicial cohomology.

2000 *Mathematics Subject Classification.* Primary 46H20. Secondary 43A20, 16E40.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Computing the simplicial cohomology groups of $\ell^1(S)$, where S is a *normal band* semigroup (particular case of *band* semigroup), was done in [2, 3]. The general case, of band semigroup, was done in [1]. In this last paper, Choi et all have identified the cyclic cohomology groups, in three steps. Using the Connes-Tzygan exact sequence [8], they determined that the simplicial cohomology groups vanishes. This same strategy was used in [5, 6, 7].
Determining the simplicial cohomology directly will prove on effective method for its comprehensive. In this way, we provided in [4] an explicit contracting homotopy, which allows us to get an explicit result for the simplicial cohomology of the Banach algebra $\ell^1(\mathbb{Z}_+)$. Also, a direct method can aid in generalizing for other Banach algebras (especially those which do not respect Connes-Tzygan lemma). For this goal, we follow three steps. In the first step, the important step, we give two methods, Sections 3.2 and 3.2. The second and third steps (subsections 4.1 and 4.2 respectively) are compatible with the methods of the first step.

2. BACKGROUND AND DEFINITIONS

2.1. **Simplicial cohomology.** We now briefly establish our notation and recall some definitions. For a Banach algebra $\mathcal{A}$, we regard $\mathcal{A}^*$, the topological dual space of $\mathcal{A}$, as a Banach $\mathcal{A}$-bimodule in the usual way.

For $n \geq 1$, we denote the Banach space of bounded n-linear operator from $\mathcal{A}^n := \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{n}$ to $\mathcal{A}^*$ by $\mathcal{C}^n(\mathcal{A}, \mathcal{A}^*)$. We define the boundary operator $\delta^n : \mathcal{C}^n(\mathcal{A}, \mathcal{A}^*) \longrightarrow \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{A}^*)$ as the bounded linear operator given by

$$\begin{aligned}(\delta^n T)(a_1, \ldots, a_{n+1})(a_{n+2}) &= T(a_2, \ldots, a_{n+1})(a_{n+2}a_1) \\ &+ \sum_{j=1}^{n} (-1)^j T(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1})(a_{n+2}) \\ &+ (-1)^{n+1} T(a_1, \ldots, a_n)(a_{n+1}a_{n+2})\end{aligned}$$

where $T \in \mathcal{C}^n(\mathcal{A}, \mathcal{A}^*)$.

By convention, for $n = 0$, we have $\mathcal{C}^n(\mathcal{A}, \mathcal{A}^*) := \mathcal{A}^*$, and $\delta^0 : \mathcal{C}^0(\mathcal{A}, \mathcal{A}^*) \longrightarrow \mathcal{C}^1(\mathcal{A}, \mathcal{A}^*)$ is given by $\delta^0(T)(a_1)(a_2) = T(a_2 a_1) - T(a_1 a_2)$.

For $T \in \mathcal{C}^n(\mathcal{A}, \mathcal{A}^*)$ we say that T is an n-*cocycle* if $\delta^n T = 0$, and we say that T is n-*coboundary* if $T = \delta^{n-1} S$, for some $S \in \mathcal{C}^{n-1}(\mathcal{A}, \mathcal{A}^*)$. Let $\mathcal{Z}^n(\mathcal{A}, \mathcal{A}^*)$ the subspace of $n-$cocycles, and $\mathcal{B}^n(\mathcal{A}, \mathcal{A}^*)$ the subspace of $n-$coboundaries. Knowing that $\delta^n \circ \delta^{n-1} = 0$, the n^{th} *simplicial cohomology group of* $\mathcal{A}$ is the space $\mathcal{H}^n(\mathcal{A}, \mathcal{A}^*) := \frac{\mathcal{Z}^n(\mathcal{A}, \mathcal{A}^*)}{\mathcal{B}^n(\mathcal{A}, \mathcal{A}^*)}$.

Elements of $\mathcal{C}^n(\mathcal{A}, \mathcal{A}^*)$ may be regarded as bounded linear functional on the space $\mathcal{C}_n(\mathcal{A}, \mathcal{A}) := \mathcal{A}^{\hat{\otimes} n+1}$, the $(n+1)$-fold completed projective tensor product of $\mathcal{A}$. The bounded linear operator $\delta^n : \mathcal{C}^n(\mathcal{A}, \mathcal{A}^*) \longrightarrow \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{A}^*)$ is then the adjoint of the bounded linear operator $d^n : \mathcal{C}_{n+1}(\mathcal{A}, \mathcal{A}) \longrightarrow \mathcal{C}_n(\mathcal{A}, \mathcal{A})$ defined on elementary tensors $X = a_1 \otimes a_2 \otimes \cdots \otimes a_{n+2} \in \mathcal{C}_{n+1}(\mathcal{A}, \mathcal{A})$ by $\mathbf{d}^n(X) = \sum\limits_{i=0}^{n+1} d_i^n(X)$ where

$$\begin{aligned}d_0^n(X) &:= a_2 \otimes \cdots \otimes a_{n+1} \otimes a_{n+2}a_1, \\ d_i^n(X) &:= (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+2}, \quad i = 1, \ldots, n+1.\end{aligned}$$

2.2. Band semigroup.

Definition 2.1.

(1) *A band is a semigroup in which every element is idempotent.*
(2) *A rectangular band is a semigroup in which the identity $a = aba$ always holds.*
(3) *A semilattice is a commutative band semigroup.*

Note that, for $a, b, c \in S$, rectangular band semigroup, we have $abc = ac$. In particular, a rectangular band semigroup is a band semigroup. A characterization of band semigroup is given in [9, Theorem 4.4.5] by:

Theorem 2.2. *Any band semigroup S can be represented as a disjoint union $\coprod_{\alpha\in L} R_\alpha$, where L is a semilattice, each R_α is a rectangular band given by $A_\alpha \times B_\alpha$, the left and right index sets, and the following properties are satisfied:*

i $R_\alpha R_\beta \subseteq R_{\alpha\beta}$ *for all* $\alpha, \beta \in L$;
ii *for* $x = (a_1, b_1) \in R_\alpha$ *and* $y = (a_2, b_2) \in R_\beta$ *with* $\alpha \preceq \beta$, xy *and* y *have the same right index (i. e.* $xy = (., b_2)$*), while* yx *and* y *have the same left index* *(i. e.* $yx = (a_2, .)$*);*
iii *the product is associative.*

So for x in a band semigroup there exist a unique α, in a semilattice, such that $x \in R_\alpha$, rectangular band. And we define $[x] = \alpha$. Note that for α, β belong to semilattice, we write $\alpha \prec \beta$ if $\alpha\beta = \alpha$ and $\alpha \neq \beta$. Let S be a band semigroup. The *left coherent units* is a function denoted by $\langle .]$, from S to S and defined as follows: for each rectangular band R_α (α belongs to the semilattice L), fix an element $y_\alpha \in R_\alpha$ and define $\langle x] = xy_\alpha$, for each $x \in R_\alpha$. Then, the function $\langle .] : S \longrightarrow S$ has the following properties:

(1) for each $\alpha \in L$, $\langle R_\alpha] \subseteq R_\alpha$;
(2) for each $\alpha \in L$ and each $x \in R_\alpha$, $\langle x]x = x$;
(3) for each $\alpha, \beta \in L$ such that $\alpha \preceq \beta$, and each $x \in R_\alpha$ and $y \in R_\beta$, $\langle xy] = \langle x]$.

Using the same method $[x\rangle = y_\alpha x$. We use this notation in the subsection 4.2.

Notations. For the rest of the paper, S is a band semigroup. We denote simply by x the point mass at x, $\delta_x \in \ell^1(S)$. Then, $X = x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1}$ will denote a typical *elementary tensor* in $\ell^1(S)\hat{\otimes}\ell^1(S)\hat{\otimes}\ldots\hat{\otimes}\ell^1(S)$. If $i = (n+1)$, then $(i+1) := 1$ so $x_{i+1} := x_1$. Moreover, if $1 \leq i \leq (n+1)$ then $(n+1+i) := i$, and $x_{n+1+i} := x_i$. Also if $i = 1$ then $i - 1 := n+1$ and $x_{i-1} := x_{n+1}$.

From [1, Definition 5.5], we have:

Definition 2.3. *Let* $X = x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1} \in \ell^1(S)\hat{\otimes}\ell^1(S)\hat{\otimes}\ldots\hat{\otimes}\ell^1(S)$*:*

(1) *We say that X has a* minimal element *if there exist* $1 \leq i \leq n+1$ *such that* $[x_i] \preceq [x_k]$ *for all* $1 \leq k \leq n+1$.
(2) *If X is without minimal element. We say that a cyclic subtensor $x_k \otimes \cdots \otimes x_l$ of X has a* minimal left element *if $[x_k] \preceq [x_i]$ for all i in the cyclic interval $[k, l]$. A cyclic subtensor is a* left-block *if it has a minimal left element and is not strictly included in another cyclic subtensor which has a minimal left element.*

Notation For an elementary tensor without minimal element, it is easy to see that X has a unique decomposition into left-blocks, and, therefore, we can define

$$I_X = \{i \in \{1, 2, 3, \ldots, n+1\} : x_i \text{ is the first component of a left} - \text{block}\}.$$

It is easy to see that for an elementary tensor X without minimal element we have $2 \leq |I_X| \leq n+1$.

To prove that for each $T \in Ker\delta^n$ there exist $R \in \mathcal{C}^{n-1}(\ell^1(S), \ell^1(S)^*)$ such that $\delta^{n-1}(R) = T$, it is sufficient to prove that there exist bounded linear maps $r^k : \mathcal{C}_k(\ell^1(S), \ell^1(S)) \longrightarrow \mathcal{C}_{k+1}(\ell^1(S), \ell^1(S))$, $k \in \{n-1, n\}$, such that $\big(I - (r^{n-1}d^{n-1} + d^n r^n)\big)(X) = 0$, for all $X \in C_n(\ell^1(S), \ell^1(S))$ ******. Note that $\big(I - (r^{n-1}d^{n-1} + d^n r^n)\big)\big(I - (s^{n-1}d^{n-1} + d^n s^n)\big) = \big(I - (q^{n-1}d^{n-1} + d^n q^n)\big)$. So, we can prove the condition ****** with two steps or more. To achieve our goal, we can prove that there exist $r^k, s^k, q^k : \mathcal{C}_k(\ell^1(S), \ell^1(S)) \longrightarrow \mathcal{C}_{k+1}(\ell^1(S), \ell^1(S))$, $k \in \{n-1, n\}$, bounded linear maps such that:

(1) For all $X \in \mathcal{C}_k(\ell^1(S), \ell^1(S))$, we have $\big(I - (r^{n-1}d^{n-1} + d^n r^n)\big)(X) \in F_{n,me}$.
(2) For all $X \in F_{n,me}$, we have $\big(I - (s^{n-1}d^{n-1} + d^n s^n)\big)(X) \in F_{n,cons}$.
(3) For all $X \in F_{n,cons}$, we have $\big(I - (q^{n-1}d^{n-1} + d^n q^n)\big)(X) = 0$.

Where $F_{n,me} = lin\{X \in \mathcal{C}_n\big(\ell^1(S), \ell^1(S)\big) : X$ has minimal element$\}$, $F_{n,cons} = lin\{X \in \mathcal{C}_n\big(\ell^1(S), \ell^1(S)\big) : X = x_1 \otimes x_2 \cdots \otimes x_{n+1}$ and $[x_1] = [x_2] = \cdots = [x_{n+1}]\}$. In the rest of our work we need also the following subspace and definition: $F_n^j = lin\{X \in \mathcal{C}_n\big(\ell^1(S), \ell^1(S)\big) : X$ has at most j left-blocks$\}$.

Definition 2.4. *If T is a finite semilattice, and $\alpha \in T$, the height of α in T is the length of the longest descending chain in T which starts at α. That is,*

$$h_T(\alpha) := sup\{m : \text{there exist } t_0, t_1, \ldots, t_m \in T \text{ with } \alpha = t_m \succ t_{m-1} \succ \cdots \succ t_0\}.$$

If $X = x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1}$ is an elementary tensor in $\mathcal{C}_n\big(\ell^1(S), \ell^1(S)\big)$, let $T(X)$ be the finite semilattice that is generated by the set $\big\{[x_1], [x_2], \ldots, [x_{n+1}]\big\}$, and define the height of X to be

$$h(X) := \sum_{k=1}^{n+1} h_{T(X)}([x_k]).$$

It is clear that for X without minimal element we have $(n+1) \leq h(X) \leq n(n+1)$. We denote by $F_n^{j,h} = lin\Big\{X \in \mathcal{C}_n\big(\ell^1(S), \ell^1(S)\big) : X$ has at most j left-blocks with height at most h$\Big\}$. And we denote by $G_{n,j,h} = lin\big\{F_n^{j,h} + F_n^{j-1}\big\}$.

3. WITHOUT MINIMAL ELEMENT

In this section we will show that there exist $s^k : \mathcal{C}_k(\ell^1(S), \ell^1(S)) \longrightarrow \mathcal{C}_{k+1}(\ell^1(S), \ell^1(S))$, $k \in \{n-1, n\}$, bounded linear map such that for each elementary tensor $X \in \mathcal{C}_k(\ell^1(S), \ell^1(S))$, we have $\big(I - (s^{n-1}d^{n-1} + d^n s^n)\big)(X) \in F_{me}$. From [1, equation (5.3)] we have the following definition:

Definition 3.1. *For $k \in \{n-1, n\}$ and $1 \leq i \leq k+1$ let the linear bounded map $s_i^k : \mathcal{C}_k(\ell^1(S), \ell^1(S)) \longrightarrow \mathcal{C}_{k+1}(\ell^1(S), \ell^1(S))$ defined by*

$$s_i^k(x_1 \otimes x_2 \otimes \cdots \otimes x_{k+1}) = (-1)^i \, x_1 \otimes \cdots \otimes \langle x_i] \otimes x_i \otimes \cdots \otimes x_{k+1},$$

and

$$s^k(X) = \sum_{i \in I_X} s_i^k(X).$$

If X has a minimal element $(I_X = \emptyset)$, $s^k(X) := 0$.

By using the above s^k Choi et al showed their result for the cyclic cohomology. In addition, they used Connes-Tzygan lemma to conclude their result for the simplicial cohomology. In this section, we show, with two methods, how we manage $(s^{n-1}d^{n-1} + d^n s^n)$ to avoid cyclic cohomology and Connes-Tzygan lemma.

3.1. **First method.** In [1, Proposition 5.12] and [1, Remark 5.13] we have, for X, an elementary tensor without minimal element:

Proposition 3.2. *Let X an elementary tensor in $F_n^{j,h}$. Then*

$$(s^{n-1}d^{n-1}+d^n s^n)(X)=\sum_{i\in I_X}\Big[X+\bullet_{i-1}\otimes\langle x_ix_{i+1}]\otimes x_ix_{i+1}\otimes\bullet_{n-i}$$

$$-\quad \bullet_{i-1}\otimes\langle x_i]\otimes x_ix_{i+1}\otimes\bullet_{n-i}\Big]\ mod\ G_{n,j,h-1}.$$

Where $\bullet_{i-1}\otimes\langle x_ix_{i+1}]\otimes x_ix_{i+1}\otimes\bullet_{n-i}:=x_1\otimes\cdots\otimes x_{i-1}\otimes\langle x_ix_{i+1}]\otimes x_ix_{i+1}\otimes x_{i+2}\otimes\cdots\otimes x_{n+1}$ and $\bullet_{i-1}\otimes\langle x_i]\otimes x_ix_{i+1}\otimes\bullet_{n-i}:=x_1\otimes\cdots\otimes x_{i-1}\otimes\langle x_i]\otimes x_ix_{i+1}\otimes x_{i+2}\otimes\cdots\otimes x_{n+1}$. When $n+1\in I_X$, the corresponding term in square brackets should be interpreted as $X+(-1)^n x_2\otimes\cdots\otimes x_n\otimes\langle x_{n+1}x_1]\otimes x_{n+1}x_1+(-1)^{n+1}x_2\otimes\cdots\otimes x_n\otimes\langle x_{n+1}]\otimes x_{n+1}x_1$.

In order to analyse the above result, we need [1, Definition 5.14]:

Definition 3.3. *A left-block of length one is called a* one-block. *A one-block x_k in an elementary tensor X is called a* block-unit *if $x_k=\langle x_k]$ and $x_kx_{k+1}=x_{k+1}$.*

Given $X=x_1\otimes x_2\otimes\cdots\otimes x_{n+1}$, let

$$R_X=\{i\in I_X: x_i\ is\ a\ one-block\ but\ not\ block\ unit,\ and\ [x_i]\ \succ[x_{i+1}]\},$$

clearly R_X may be empty. By [1, Lemma 5.16] we have:

Lemma 3.4. *Let X be an elementary tensor in F_n^j, and let $i\in I_X$. Then precisely one of the following four cases can occur*

i) *x_i is not a one-block in X, in which case,*

$$\bullet_{i-1}\otimes\langle x_ix_{i+1}]\otimes x_ix_{i+1}\otimes\bullet_{n-i}=\bullet_{i-1}\otimes\langle x_i]\otimes x_ix_{i+1}\otimes\bullet_{n-i};$$

ii) *x_i is a one-block and $[x_i]\not\succeq[x_{i+1}]$, in which case, the tensor*

$$\bullet_{i-1}\otimes\langle x_i]\otimes x_ix_{i+1}\otimes\bullet_{n-i}$$

either has fewer left-blocks, or lower height (and the same number of left-blocks), than X;

iii) *x_i is a block-unit, in which case $\bullet_{i-1}\otimes\langle x_i]\otimes x_ix_{i+1}\otimes\bullet_{n-i}=X$;*

iv) *$i\in R_X$.*

In cases ii), iii), iv) the tensor $\bullet_{i-1}\otimes\langle x_ix_{i+1}]\otimes x_ix_{i+1}\otimes\bullet_{n-i}$ has fewer left-blocks than X.

Corollary 3.5. *Let X an elementary tensor in $F_n^{j,h}$. Then*

$$(s^{n-1}d^{n-1}+d^n s^n)(X)=(\mid m'_X\mid+\mid R'_X\mid)X-\sum_{\substack{1\le i\le n\\ i\in R_X}}V_i(X)$$

$$+\begin{cases} d_0^n s_{n+1}^n(X) & \text{if } n+1\in I_X \text{ and } [x_1]\prec[x_{n+1}],\\ 0 & \text{else,}\end{cases}\quad +\ mod\ G_{n,j,h-1}$$

where $m'_X=\big\{i\in I_X: i=n+1\ \ or\ \ [x_{i+1}]\not\prec[x_i]\big\}$, $R'_X=\{1\le i\le n: i\in R_X\}$, $V_i(X)=\bullet_{i-1}\otimes\langle x_i]\otimes x_ix_{i+1}\otimes\bullet_{n-i}$.

Remark If $\mid I_X \mid\neq 0$, it is simple to verify that $\mid m'_X \mid\neq 0$. Therefore, if X has no minimal element, then:

$$\begin{aligned}
P_{m_X}(X) &:= \big(I - \frac{1}{m_X}(s^{n-1}d^{n-1} + d^n s^n)\big)(X) \\
&= \frac{1}{m_X}\sum_{\substack{1\leq i\leq n\\ i\in R_X}} V_i(X) \\
&- \begin{cases} \frac{1}{m_X} d_0^n s_{n+1}^n(X) & \text{if } n+1\in I_X \text{ and } [x_1]\prec[x_{n+1}], \\ 0 & \text{else,}\end{cases} \\
&+ \ mod\, G_{n,j,h-1}
\end{aligned}$$

where $m_X =\mid m'_X \mid + \mid R'_X \mid$.

It is easy to see that $d_0^n s_{n+1}^n(X)$ has at most the same number of left-blocks of X, and at most the same height of X (if $[x_1]\prec[x_{n+1}]$ we have equalities). To eliminate $d_0^n s_{n+1}^n(X)$ we use an original idea. And to eliminate $V_i(X)$ we adapt some notions from [1].

Definition 3.6. *Let $X = x_1\otimes x_2\otimes\cdots\otimes x_{n+1}$ an elementary tensor without minimal element.*

(1) *Define a descending block in X to be a cyclic subtensor $x_k\otimes\cdots\otimes x_l$ with the property that $[x_k]\succ[x_{k+1}]\succ\cdots\succ[x_l]$, while $[x_{k-1}]\nsucc[x_k]$ and $[x_l]\nsucc[x_{l+1}]$.*
(2) *Let $1\leq i\leq n+1$, we define the descent of x_i in X, $desc_i(X)$, to be $l-i$, where x_l is the last element in the unique descending block that contains x_i (this is interpreted cyclically).*
(3) *We define $desc(X) = \sum_{i\in R_X} desc_i(X)$.*

By [1, Lemma 5.19] we have:

Lemma 3.7. *Let $X = x_1\otimes\cdots\otimes x_{n+1}$ be a tensor without minimal element, such that R_X is non-empty, and let $i\in R_X$. Then*

$$desc\big(V_i(X)\big) < desc(X).$$

For $X\in F_n^{j,h}$ let $D_k(X) = lin\{Y\in G_{n,j,h} : desc(Y)\leq desc(X)-k\}$. So, we have

$$\begin{aligned}
P_{m_X}(X) &= D+S \\
&- \begin{cases} \frac{1}{m_X} d_0^n s_{n+1}^n(X) & \text{if } n+1\in I_X \text{ and } [x_1]\prec[x_{n+1}], \\ 0 & \text{else,}\end{cases}
\end{aligned}$$

where $D\in D_1(X)$ and $S\in G_{n,j,h-1}$. For $m_X^1 = m_{d_0^n s_{n+1}^n(X)}$ we have, if $n+1\in I_X$ and $[x_1]\prec[x_{n+1}]$ then

$$\begin{aligned}
P_{m_X^1}(d_0^n s_{n+1}^n(X)) &= D+S \\
&- \begin{cases} \frac{1}{m_X^1} d_0^n s_{n+1}^n(d_0^n s_{n+1}^n(X)) & \text{if } n+1\in I_X \text{ and } [x_2]\prec[x_1]\prec[x_{n+1}], \\ 0 & \text{else,}\end{cases}
\end{aligned}$$

where $D\in D_1(d_0^n s_{n+1}^n(X))$ and $S\in G_{n,j,h-1}$. Let us prove that $D_1(d_0^n s_{n+1}^n(X))\subset D_1(X)$.

Lemma 3.8. *If $n+1\in I_X$ and $[x_1]\prec[x_{n+1}]$ then $desc(d_0^n s_{n+1}^n(X))\leq desc(X)$ and $D_1(d_0^n s_{n+1}^n(X))\subset D_1(X)$.*

Proof. As $[x_1] \prec [x_{n+1}]$ it is evident that $\mid I_{d_0^n s_{n+1}^n(X)} \mid = \mid I_X \mid$ and $h(d_0^n s_{n+1}^n(X)) = h(X)$. To get our result, we have to proof that $desc(d_0^n s_{n+1}^n(X)) \leq desc(X)$. We have $d_0^n s_{n+1}^n(X) = (-1)^{n+1} x_2 \otimes \cdots \otimes x_n \otimes \langle x_{n+1}] \otimes x_{n+1}x_1$. Let $Z = x_2 \otimes x_n \otimes x_{n+1} \otimes x_1$. It is clear that $desc(Z) = desc(X)$.

(1) If $n \in R_Z$ then, by Lemma 3.7, $desc(V_n(Z)) < desc(Z)$. As $desc(d_0^n s_{n+1}^n(X)) = desc(V_n(Z))$ and $desc(Z) = desc(X)$ we have our result.
(2) If $n \notin R_Z$ then x_{n+1} is a block-unit. Thus, $\langle x_{n+1}] = x_{n+1}$ and $x_{n+1}x_1 = x_1$. So, $desc(d_0^n s_{n+1}^n(X)) = desc(Z) = desc(X)$. And we have our result.

∎

Remark If X has no minimal element and $[x_1] \prec [x_{n+1}]$, then $d_0^n s_{n+1}^n(X)$ has no minimal element.

Corollary 3.9. *If X is an elementary tensor without minimal element, then*

$$P_{m_X^1} \circ \quad P_{m_X}(X) = D + S + \begin{cases} \frac{1}{m_X^1 m_X} d_0^n s_{n+1}^n(d_0^n s_{n+1}^n(X)) & \textit{if } n+1 \in I_X \textit{ and } [x_2] \prec [x_1] \prec [x_{n+1}], \\ 0 & \textit{else,} \end{cases}$$

where $D \in D_1(X)$ and $S \in G_{n,j,h-1}$.

Proof. We know that, if Y has no minimal element then $\mid m'_Y \mid \neq 0$, and so $m_Y \neq 0$.

$$\begin{aligned} P_{m_X^1} \circ P_{m_X}(X) &= P_{m_X^1}(D) + P_{m_X^1}(S) + D_1 + S_1 \\ &+ \begin{cases} \frac{1}{m_X^1 m_X} d_0^n s_{n+1}^n(d_0^n s_{n+1}^n(X)) & \text{if } n+1 \in I_X \text{ and } [x_2] \prec [x_1] \prec [x_{n+1}], \\ 0 & \text{else,} \end{cases} \end{aligned}$$

where $D, D_1 \in D_1(X)$ and $S, S_1 \in G_{n,j,h-1}$. To get our result, we have only to see that, $\forall m \geq 1$, $P_m(D)$ (respectively $P_m(S)$) $\in D_1(X) + G_{n,j,h-1}$. ∎

By iteration in the previous corollary we get, if X has no minimal element, then

$$\begin{aligned} &P_{m_X^{n-2}} \circ \cdots \circ P_{m_X}(X) = D + S \\ &+(-1)^{n-1} \begin{cases} \frac{1}{c_X}(d_0^n s_{n+1}^n)^{(n-1)}(X) & \text{if } n+1 \in I_X \text{ and } [x_{n-1}] \prec \cdots \prec [x_1] \prec [x_{n+1}], \\ 0 & \text{else,} \end{cases} \end{aligned}$$

where $m_X^k = m_{(d_0^n s_{n+1}^n)^{(k)}(X)}$, $c_X = m_X m_X^1 m_X^2 \ldots m_X^{n-2}$, $D \in D_1(X)$ and $S \in G_{n,j,h-1}$. This motivates the next corollary:

Corollary 3.10. *Let X an elementary tensor. Then $P^{(n)}(X) = D + S$, where $D \in D_1(X)$, $S \in G_{n,j,h-1}$ and $P = P_1 \circ P_2 \circ \cdots \circ P_{n+1}$.*

Remark By calculation, we verify that $P_m \circ P_{m'} = P_{m'} \circ P_m$, for all m, m'. The equality is independent with the definition of s^k. We use this remark in the other sections also.

Proof. If $m_X^k \neq 0$, for $k = 0, 1, \ldots, n-1$ then $P^{(n)}(X) = Q_{m_X^{n-1}} \circ \cdots \circ Q_{m_X} \circ P_{m_X^{n-1}} \circ \cdots \circ P_{m_X}(X) = D + S$, where $D \in D_1(X), S \in G_{n,j,h-1}$ and $Q_r = P_1 \circ \cdots \circ P_{r-1} \circ P_{r+1} \circ \cdots \circ P_{n+1}$. Note that, if $m_Y = 0$ then Y has a minimal element. So if $m_X^k = 0$ and $m_X^{k-1} \neq 0$ then $P_{m_X^{k-1}} \circ \cdots \circ P_{m_X}(X) = D + S$, where $D \in D_1(X), S \in G_{n,j,h-1}$. And the result is immediate. ∎

As $P^n(S) \in G_{n,j,h-1}$, if $S \in G_{n,j,h-1}$, and by the previous corollary we get the below result.

Corollary 3.11. *Let X an elementary tensor. Then, $P^{2n}(X) \in D_2(X) + G_{n,j,h-1}$. Moreover, as $0 \leq desc(X) \leq n(n-1)$ we get $P^{\left(n^2(n-1)\right)}(X) \in G_{n,j,h-1}$.*

Proposition 3.12. *Let X an elementary tensor. Then, $P^{\left(n^5(n-1)\right)}(X) \in F_{n,me}$.*

Proof. By Corollary 3.11, we have $P^{\left(n^2(n-1)\right)}(X) = U + V$, where $U \in F_n^{j-1}$ and $V \in F_n^{j,h-1}$. So $P^{\left(n^2(n-1)\right)}(V) = U_1 + V_1$, where $U_1 \in F_n^{j-1}$ and $V_1 \in F_n^{j,h-2}$. We note that $(n+1) \leq h(X) \leq n(n+1)$, if X has no minimal element. Then, $P^{\left(n^4(n-1)\right)}(X) \in F_n^{j-1}$. Moreover, as $2 \leq \mid I_X \mid \leq n+1$, if X has no minimal element, then we get $P^{\left(n^5(n-1)\right)}(X) \in F_{n,me}$. ∎

3.2. **Second method.** A part of working in the previous subsection was considered to eliminate $d_0^n s_{n+1}^n(X)$. To work around this problem, we use a new idea. We simply take $s_{n+1}^n = 0$. And we get $s^n = \sum_{\substack{i \in I_X \\ i \neq n+1}} s_i^n$. With this definition, and by adapting the proof of [1, Proposition 5.12], we get:

Proposition 3.13. *Let X an elementary tensor in $F_n^{j,h}$. Then*

$$\begin{aligned}(s^{n-1}d^{n-1} + d^n s^n)(X) &= \sum_{\substack{i \in I_X \\ 1 \leq i \leq n-1}} \Big[X + \bullet_{i-1} \otimes \langle x_i x_{i+1}] \otimes x_i x_{i+1} \otimes \bullet_{n-i} \\ &- \quad \bullet_{i-1} \otimes \langle x_i] \otimes x_i x_{i+1} \otimes \bullet_{n-i}\Big] \\ &+ \quad d_n^n s_n^n(X) + d_{n+1}^n s_n^n(X) \;\; mod\; G_{n,j,h-1}.\end{aligned}$$

In the next lemma, we illustrate the behaviour of $d_n^n s_n^n$ and $d_{n+1}^n s_n^n$.

Lemma 3.14. *Let X an elementary tensor. Then*

(1) $d_n^n s_n^n(X) = \begin{cases} X & \text{if } n \in I_X, \\ 0 & \text{else.}\end{cases}$

(2) $d_{n+1}^n s_n^n(X) = \begin{cases} 0 & \text{if } n \notin I_X, \\ Y & \text{if } n \in I_X \text{ and } [x_{n+1}] \npreceq [x_n], \\ d_{n+1}^n s_n^n(X) & \text{if } n \in I_X,\ [x_{n+1}] \preceq [x_n] \text{ and } d_{n+1}^n s_n^n(X) \neq -X, \\ -X & \text{else,}\end{cases}$

where $Y \in G_{n,j,h-1}$.

By the previous lemma we get

$$d_n^n s_n^n(X) + d_{n+1}^n s_n^n(X) = \begin{cases} 0 & \text{if } n \notin I_X, \\ X + Y & \text{if } n \in I_X \text{ and } [x_{n+1}] \npreceq [x_n], \\ X + d_{n+1}^n s_n^n(X) & \text{if } n \in I_X,\ [x_{n+1}] \preceq [x_n] \text{ and } d_{n+1}^n s_n^n(X) \neq -X, \\ 0 & \text{else.}\end{cases}$$

Corollary 3.15. *Let X an elementary tensor in $F_n^{j,h}$. Then*

$$(s^{n-1}d^{n-1} + d^n s^n)(X) = (\mid m''_X \mid + \mid R''_X \mid)X - \sum_{\substack{1 \leq i \leq n-1 \\ i \in R_X}} V_i(X)$$

$$+ \begin{cases} d_{n+1}^n s_n^n(X) & \text{if } n \in I_X,\ [x_{n+1}] \preceq [x_n] \text{ and } d_{n+1}^n s_n^n(X) \neq -X, \\ 0 & \text{else,}\end{cases} \quad + \; mod\; G_{n,j,h-1}$$

where $m''_X = \{i \in I_X : 1 \leq i \leq n-1, [x_{i+1}] \not\prec [x_i]\} \cup \{n \in I_X : \left([x_{n+1}] \npreceq [x_n]\right) \text{ or } \left([x_{n+1}] \preceq [x_n], d_{n+1}^n s_n^n(X) \neq -X\right)\}$, $R''_X = \{1 \leq i \leq n-1 : i \in R_X\}$, $V_i(X) = \bullet_{i-1} \otimes \langle x_i] \otimes x_i x_{i+1} \otimes \bullet_{n-i}$.

Lemma 3.16. *If $I_X \neq \emptyset$ then $m''_X \neq \emptyset$.*

Proof. Let $X = x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1}$ with no minimal element($I_X \neq \emptyset$). In this case, we have $|I_X| \geq 2$. Suppose that $m''_X = \emptyset$.
First case $I_X = \{n, n+1\}$. In this case we have, on one hand, $[x_{n+1}] \preceq [x_i]$, for $i \in \{1, 2, \ldots, n-1\}$. On other hand, as $n \in I_X$ and $n \notin m''_X$ we get $[x_{n+1}] \preceq [x_n]$. Then, we deduce that $[x_{n+1}]$ is a minimal element for X, impossible.
Second case $\{1, 2, \ldots, n-1\} \cap I_X \neq \emptyset$. Let $i_0 = min_{1 \leq i \leq n-1}\{i \in I_X\}$. So $\{x_1, x_2, \ldots, x_{i_0-1}\}$ are in the same left minimal block that contain x_{n+1} **(1)**. As $i_0 \in I_X$ and $i_0 \notin m''_X$ then $[x_{i_0+1}] \prec [x_{i_0}]$ and so $i_0 + 1 \in I_X$. If $i_0 + 1 \leq n-1$ we get also $[x_{i_0+2}] \prec [x_{i_0+1}]$ and so $i_0 + 2 \in I_X$. By iteration we get $[x_n] \prec [x_{n-1}] \prec \cdots \prec [x_{i_0}]$ **(2)** and $n \in I_X$. As $n \in I_X$ and $n \notin m''_X$ we get $[x_{n+1}] \preceq [x_n]$ **(3)**. By **(1), (2)** and **(3)** we have that x_{n+1} is a minimal element for X, impossible.
We have our result. ∎

In the Corollary 3.15 it is simple to see that if $n \in I_X$ then $V_n(X) = -d^n_{n+1}s^n_n(X)$.
Remark If $| I_X | \neq 0$ then

$$
\begin{aligned}
&P_{m_X}(X) = \big(I - \frac{1}{m_X}(s^{n-1}d^{n-1} + d^n s^n)\big)(X)\\
&= \frac{1}{m_X} \sum_{\substack{1 \leq i \leq n-1 \\ i \in R_X}} V_i(X)\\
+\quad & \frac{1}{m_X}\begin{cases} \cdots \bullet_{n-1} \otimes \langle x_n] \otimes x_n x_{n+1} & \text{if } n \in I_X,\ [x_{n+1}] \preceq [x_n] \text{ and } d^n_{n+1}s^n_n(X) \neq -X, \\ 0 & \text{else,} \end{cases}\\
+\quad & mod\, G_{n,j,h-1},
\end{aligned}
$$

where $m_X =| m''_X | + | R''_X |$.
By lemma 3.7 we get $P_{m_X}(X) = D + S$

$$
+\frac{1}{m_X}\begin{cases} \cdots \bullet_{n-1} \otimes \langle x_n] \otimes x_n x_{n+1} & \text{if } n \in I_X,\ [x_{n+1}] \preceq [x_n] \text{ and } d^n_{n+1}s^n_n(X) \neq -X, \\ 0 & \text{else,} \end{cases}
$$

where $D \in D_1(X)$ and $S \in G_{n,j,h-1}$. It is clear that if $n \in I_X$ and $[x_{n+1}] \preceq [x_n]$ then $I_{V_n(X)} \neq \emptyset$. So $m_{V_n(X)} \neq \emptyset$ and $P_{m_{V_n(X)}} \circ P_{m_X}(X) = D + S$ where $D \in D_1(X)$ and $S \in G_{n,j,h-1}$. We have the next corollary:

Corollary 3.17. *Let X an elementary tensor. Then $P^{(2)}(X) = D + S$, where $D \in D_1(X)$ and $S \in G_{n,j,h-1}$, where $P = P_1 \circ P_2 \circ \cdots \circ P_{n+1}$.*

With similar proofs to those of Corollary 3.11 and Proposition 3.12, we get the next corollary.

Corollary 3.18. *Let X an elementary tensor. Then $P^{\left(2n^4(n-1)\right)}(X) \in F_{n,me}$.*

4. WITH MINIMAL ELEMENT

4.1. **Non constant element.** In this section we will show that there are linear bounded maps $s^k : \mathcal{C}_k(\ell^1(S), \ell^1(S)) \longrightarrow \mathcal{C}_{k+1}(\ell^1(S), \ell^1(S))$, $k \in \{n-1, n\}$, such that for each elementary tensor $X \in F_{k,me}$ we have $\big(I - (s^{n-1}d^{n-1} + d^n s^n)\big)(X) \in F_{k,cons}$. By adapting the map s in [1, subsection 5.1], we have our goal.

Definition 4.1. *Let $X = x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1} \in \mathcal{C}_n\big(\ell^1(S), \ell^1(S)\big)$. We say that a cyclic subtensor $x_k \otimes \cdots \otimes x_l$ of X is a* minimal left block *if $[x_k] = \cdots = [x_l] = [X]$ for all i in the* *cyclic interval* $[k,l]$, *where* $[X]=[x_1x_2\dots x_{n+1}]$. *A cyclic subtensor is a* minimal-block *if it is a minimal left block, and it is not strictly included in another minimal left block.*

Notation If X is without minimal element (respectively, a constant tensor) we define: $I_X:=\emptyset$. If not, it is easy to see that X has a unique decomposition into minimal-blocks, and, therefore, we can define

$$I_X=\{i\in\{1,2,3,\dots,n+1\}: x_i \text{ is the first component of a minimal}-\text{block}\}.$$

Definition 4.2. *For* $k\in\{n-1,n\}$ *and* $1\le i\le k+1$ *let the linear bounded map* $s_i^k:\mathcal{C}_k(\ell^1(S),\ell^1(S))\longrightarrow\mathcal{C}_{k+1}(\ell^1(S),\ell^1(S))$ *defined by*

$$s_i^k(x_1\otimes x_2\otimes\cdots\otimes x_{k+1})=(-1)^i\,x_1\otimes\cdots\otimes\langle x_i]\otimes x_i\otimes\cdots\otimes x_{k+1},$$

and

$$s^k(X)=\sum_{i\in I_X}s_i^k(X).$$

If $I_X=\emptyset$ *we define* $s^k(X):=0$.

To simplify the study of $(s^{n-1}d^{n-1}+d^ns^n)$, we write it as:

$$\sum_{k=1}^{n-1}(s_k^{n-1}d_0^{n-1}+d_0^ns_{k+1}^n)+\sum_{1\le j<k\le n}(s_k^{n-1}d_j^{n-1}+d_j^ns_{k+1}^n)+\sum_{1\le k<j\le n}(s_k^{n-1}d_j^{n-1}+d_{j+1}^ns_k^n)$$

$$+(s_n^{n-1}d_0^{n-1}+d_{n+1}^ns_{n+1}^n+d_0^ns_1^n+d_0^ns_{n+1}^n)+\sum_{i=1}^{n}(s_i^{n-1}d_i^{n-1}+d_i^ns_i^n+d_i^ns_{i+1}^n+d_{i+1}^ns_i^n).$$

Lemma 4.3. *For* $X=x_1\otimes x_2\otimes\cdots\otimes x_{n+1}$ *we have the following:*

1. *If* $1\le k\le n+1$ *and* $k\in I_X$ *then* $[x_k]=[X]$, $k+1\notin I_X$ *and* $[x_{k-1}]\neq[X]$ *so* $k-1\notin I_X$. *Also* $k\in I_{d_k^{n-1}(X)}$.
2. *If* $1\le k\le n+1$ *and* $[x_kx_{k+1}]\neq[X]$ *then* $[x_k]\neq[X]$ *and* $[x_{k+1}]\neq[X]$.
3. *If* $0\le j\le n$ *then* $[d_j^{n-1}(X)]=[X]$.
4. *If* $2\le k\le n-1$ *and* $k+1\in I_X$ *then* $k\in I_{d_0^{n-1}(X)}$.
5. *If* $1\le k\le n-1$ *and* $k\in I_{d_0^{n-1}(X)}$ *then* $k+1\in I_X$.
6. *If* $1\le j<k\le n$ *and* $k\in I_{d_j^{n-1}(X)}$ *then* $k+1\in I_X$.
7. *If* $1\le j+1<k\le n$ *and* $k+1\in I_X$ *then* $k\in I_{d_j^{n-1}(X)}$.
8. *If* $1\le k<j\le n$ *and* $k\in I_{d_j^{n-1}(X)}$ *then* $k\in I_X$.
9. *If* $\big(1\le k<j\le n-1$ *or* $2\le k<j\le n\big)$ *and* $k\in I_X$ *then* $k\in I_{d_j^{n-1}(X)}$.

Proof.

1. Trivial.
2. Trivial.
3. Trivial.
4.

$$\begin{aligned}X&=x_1\otimes x_2\otimes\cdots\otimes x_k\otimes x_{k+1}\otimes x_{k+2}\otimes\cdots\otimes x_n\otimes x_{n+1},\\ d_0^{n-1}(X)&=\qquad x_2\otimes\cdots\otimes x_k\otimes x_{k+1}\otimes x_{k+2}\otimes\cdots\otimes x_n\otimes x_{n+1}x_1.\end{aligned}$$

   If $2\le k\le n-1$ and $k+1\in I_X$ then $[x_{k+1}]=[X]$ and $[x_k]\neq[X]$. It is easy to see that $[x_{k+1}]=[d_0^{n-1}(X)]$. As $2\le k\le n-1$ and $[x_k]\neq[d_0^{n-1}(X)]$ we deduce that $k\in I_{d_0^{n-1}(X)}$.

(5)

$$\begin{array}{rcl} X & = & x_1 \otimes x_2 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes x_{k+2} \otimes \cdots \otimes x_n \otimes x_{n+1}, \\ d_0^{n-1}(X) & = & x_2 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes x_{k+2} \otimes \cdots \otimes x_n \otimes x_{n+1}x_1. \end{array}$$

If $1 \leq k \leq n-1$ and $k \in I_{d_0^{n-1}(X)}$ then $[x_{k+1}] = [d_0^{n-1}(X)]$ and $[x_k] \neq [d_0^{n-1}(X)]$. As = $[d_0^{n-1}(X)] = [X]$ we deduce that $[x_{k+1}] = [X]$ and $[x_k] \neq [X]$, which means $k+1 \in I_X$.

(6)

$$\begin{array}{rcl} X & = & x_1 \otimes \cdots \otimes x_j \otimes x_{j+1} \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_{n+1}, \\ (-1)^j d_j^{n-1}(X) & = & x_1 \otimes \cdots \otimes x_j x_{j+1} \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_{n+1}. \end{array}$$

If $1 \leq j < k \leq n$ and $k \in I_{d_j^{n-1}(X)}$ then $[x_{k+1}] = [X]$ and $[x_k] \neq [X]$ (even if $k = j+1 \in I_{d_j^{n-1}(X)}$ we have $[x_{j+1}] \neq [X]$), which means that $k+1 \in I_X$.

(7)

$$\begin{array}{rcl} X & = & x_1 \otimes \cdots \otimes x_j \otimes x_{j+1} \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_{n+1}, \\ (-1)^j d_j^{n-1}(X) & = & x_1 \otimes \cdots \otimes x_j x_{j+1} \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_{n+1}. \end{array}$$

If $1 \leq j+1 < k \leq n$ and $k+1 \in I_X$ then $[x_{k+1}] = [X]$ and $[x_k] \neq [X]$, which means $[x_{k+1}] = [d_j^{n-1}(X)]$ and $[x_k] \neq [d_j^{n-1}(X)]$. So $k \in I_{d_j^{n-1}(X)}$.

(8)

$$\begin{array}{rcl} X & = & x_1 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_j \otimes x_{j+1} \otimes \cdots \otimes x_{n+1}, \\ (-1)^j d_j^{n-1}(X) & = & x_1 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_j x_{j+1} \otimes \cdots \otimes x_{n+1}. \end{array}$$

If $1 \leq k < j \leq n$ and $k \in I_{d_j^{n-1}(X)}$ then $[x_k] = [X]$ and $[x_{k-1}] \neq [X]$, which means $k \in I_X$.

(9)

$$\begin{array}{rcl} X & = & x_1 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_j \otimes x_{j+1} \otimes \cdots \otimes x_{n+1}, \\ (-1)^j d_j^{n-1}(X) & = & x_1 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_j x_{j+1} \otimes \cdots \otimes x_{n+1}. \end{array}$$

If $\big(1 \leq k < j \leq n-1$ or $2 \leq k < j \leq n\big)$ and $k \in I_X$ then $[x_k] = [X] = [d_j^{n-1}(X)]$ and $[x_{k-1}] \neq [X] = [d_j^{n-1}(X)]$, so $k \in I_{d_j^{n-1}(X)}$.

∎

Lemma 4.4.

(1) $\sum_{1\leq k\leq n-1} \big(s_k^{n-1}d_0^{n-1} + d_0^n s_{k+1}^n\big)(X) = \begin{cases} Y & \textit{or} \\ 0 \end{cases}$

where Y is an elementary tensor with more minimal elements than X.

(2) $\sum_{1\leq j<k\leq n} (s_k^{n-1}d_j^{n-1} + d_j^n s_{k+1}^n) \begin{cases} Y & \textit{or} \\ 0 \end{cases}$

where Y is an elementary tensor with more minimal elements than X.

(3) $\sum_{1\le k<j\le n}(s_k^{n-1}d_j^{n-1}+d_{j+1}^n s_k^n)(X)=\begin{cases} Y & \textit{or} \\ 0 \end{cases}$

where Y is an elementary tensor with more minimal elements than X.

Proof.

(1) By using [lemma 4.3, 4 and 5] it is easy to check that: $\sum_{k=1}^{n-1}\left(s_k^{n-1}d_0^{n-1}+d_0^n s_{k+1}^n\right)(X)=\left(s_1^{n-1}d_0^{n-1}+d_0^n s_2^n\right)(X)$. By [lemma 4.3, 5] we can see that it is enough to study the case $k+1=2\in I_X$.
If $k+1=2\in I_X$ then $[x_2]=[X]$ and $[x_1]\neq[X]$.
  (a) If $k=1\in I_{d_0^{n-1}(X)}$ then it is easy to see that $\left(s_1^{n-1}d_0^{n-1}+d_0^n s_2^n\right)(X)=0$.
  (b) If $k=1\notin I_{d_0^{n-1}(X)}$ then $\left(s_1^{n-1}d_0^{n-1}+d_0^n s_2^n\right)(X)=d_0^n s_2^n(X)$.
  $d_0^{n-1}(X)=x_2\otimes\cdots\otimes x_n\otimes x_{n+1}x_1$. As $[x_2]=[d_0^{n-1}(X)]=[X]$ and x_2 is not beginning of minimal-block for $d_0^{n-1}(X)$ (because $1\notin I_{d_0^{n-1}(X)}$) then $[x_{n+1}x_1]=[d_0^{n-1}(X)]=[X]$. We have $d_0^n s_2^n(X)=\langle x_2]\otimes x_2\otimes\cdots\otimes x_n\otimes x_{n+1}x_1$

$$\begin{aligned} d_0^n s_2^n(X) &= \langle x_2]\otimes x_2\otimes\cdots\otimes x_n\otimes x_{n+1}x_1, \\ X &= x_1\otimes x_2\otimes\cdots\otimes x_n\otimes x_{n+1}. \end{aligned}$$

  As $[x_1]\neq[X]$, $[\langle x_2]]=[x_2]=[X]$ and $[X]=[x_1x_{n+1}]\preceq[x_{n+1}]$ we can see that $d_0^n s_2^n(X)$ has more minimal elements than X.

(2) Using [lemma 4.3, 6 and 7], it is easy to check that: $\sum_{1\le j<k\le n}(s_k^{n-1}d_j^{n-1}+d_j^n s_{k+1}^n)=\sum_{1\le j\le n-1}(s_{j+1}^{n-1}d_j^{n-1}+d_j^n s_{j+2}^n)$. By [lemma 4.3, 6] we can see that it is enough to study the case $j+2\in I_X$, for $1\le j\le n-1$. Let us assume that $j+2\in I_X$ and $1\le j\le n-1$. Then we have $[x_{j+2}]=[X]$ and $[x_{j+1}]\neq[X]$.
  (a) If $j+1\in I_{d_j^{n-1}(X)}$ then it is easy to see that $\left(s_{j+1}^{n-1}d_j^{n-1}+d_j^n s_{j+2}^n\right)(X)=0$.
  (b) If $j+1\notin I_{d_j^{n-1}(X)}$ then $(s_{j+1}^{n-1}d_j^{n-1}+d_j^n s_{j+2}^n)(X)=d_j^n s_{j+2}^n(X)$.

$$\begin{aligned} d_j^n s_{j+2}^n(X) &= x_1\otimes\cdots\otimes x_jx_{j+1}\otimes\langle x_{j+2}]\otimes x_{j+2}\otimes\cdots\otimes x_{n+1}, \\ X &= x_1\otimes\cdots\otimes x_j\otimes\ \ x_{j+1}\otimes\ \ x_{j+2}\otimes\cdots\otimes x_{n+1}. \end{aligned}$$

  As $[x_{j+1}]\neq[X]$, $[\langle x_{j+2}]]=[x_{j+2}]=[X]$ and $[x_jx_{j+1}]\preceq[x_j]$ then $d_j^n s_{j+2}^n(X)$ has more minimal elements than X.

(3) Using [lemma 4.3, 8 and 9] it is easy to check that: $\sum_{1\le k<j\le n}(s_k^{n-1}d_j^{n-1}+d_{j+1}^n s_k^n)=(s_1^{n-1}d_n^{n-1}+d_{n+1}^n s_1^n)$. By [lemma 4.3, 8] we can see that it is enough to study the case $1\in I_X$. Let us assume that $1\in I_X$. Then $[x_1]=[X]$ and $[x_{n+1}]\neq[X]$.
  (a) If $1\in I_{d_n^{n-1}(X)}$ then it is easy to see that $\left(s_1^{n-1}d_n^{n-1}+d_{n+1}^n s_1^n\right)(X)=0$.
  (b) If $1\notin I_{d_n^{n-1}(X)}$ then $\left(s_1^{n-1}d_n^{n-1}+d_{n+1}^n s_1^n\right)(X)=d_{n+1}^n s_1^n(X)$.
  $d_n^{n-1}(X)=(-1)^{n-1}x_1\otimes x_2\otimes\cdots\otimes x_{n-1}\otimes x_nx_{n+1}$. As $[x_1]=[X]$ and x_1 is not the beginning of a minimal block for $d_n^{n-1}(X)$, because $1\notin I_{d_n^{n-1}(X)}$, we deduce that $[x_nx_{n+1}]=[X]$.

$$\begin{aligned} (-1)^{n+2}d_{n+1}^n s_1^n(X) &= \langle x_1]\otimes x_1\otimes x_2\otimes\cdots\otimes x_{n-1}\otimes x_nx_{n+1}, \\ X &= x_1\otimes x_2\otimes\cdots\otimes x_{n-1}\otimes x_n\otimes x_{n+1}. \end{aligned}$$

  As $[\langle x_1]]=[x_1x_{n+1}]=[X]$ and $[x_{n+1}]\neq[X]$ we deduce that $(-1)^{n+2}d_{n+1}^n s_1^n(X)$ has more minimal elements than X.

∎

Lemma 4.5. *For $1 \leq i \leq n$ we have*
$\big(s_i^{n-1}d_i^{n-1} + d_i^n s_i^n + d_i^n s_{i+1}^n + d_{i+1}^n s_i^n\big)(X) = \begin{cases} X+Y & \text{if } i \in I_X \\ Y & \text{else} \end{cases}$
where Y is zero or an elementary tensor with more minimal element than X.

Proof.

- $s_i^{n-1}d_i^{n-1}(X) = \begin{cases} x_1 \otimes \cdots \otimes \langle x_i x_{i+1}] \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1} & \text{if } i \in I_{d_i^{n-1}(X)} \\ 0 & \text{else} \end{cases}$
- $d_i^n s_i^n(X) = \begin{cases} X & \text{if } i \in I_X \\ 0 & \text{else} \end{cases}$
- $d_i^n s_{i+1}^n(X) = \begin{cases} -x_1 \otimes \cdots \otimes x_i \langle x_{i+1}] \otimes x_{i+1} \otimes \cdots \otimes x_{n+1} & \text{if } i+1 \in I_X \\ 0 & \text{else} \end{cases}$
- $d_{i+1}^n s_i^n(X) = \begin{cases} -x_1 \otimes \cdots \otimes \langle x_i] \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1} & \text{if } i \in I_X \\ 0 & \text{else} \end{cases}$

If $i+1 \in I_X$ then $[x_{i+1}] = [X]$ and $[x_i] \neq [X]$. So $[x_i \langle x_{i+1}]] = [X]$. It is easy to see that $d_i^n s_{i+1}^n(X)$ has more minimal elements than X. Note that if $i+1 \notin I_X$ then $d_i^n s_{i+1}^n(X) = 0$.

(1) If $i \in I_X$. Then $i \in I_{d_i^{n-1}(X)}$. It is clear that $\langle x_i x_{i+1}] = \langle x_i]$. We can see that $s_i^{n-1}d_i^{n-1}(X) + d_{i+1}^n s_i^n(X) = 0$.

(2) If $i \notin I_X$.
   (a) If $i \notin I_{d_i^{n-1}(X)}$. Then $s_i^{n-1}d_i^{n-1}(X) + d_{i+1}^n s_i^n(X) = 0$.
   (b) If $i \in I_{d_i^{n-1}(X)}$. Then $[x_i x_{i+1}] = [X]$ and $[x_{i-1}] \neq [X]$. Therefore $[x_i] \neq [X]$ $\big($because if $[x_i] = [X]$ and as $[x_{i-1}] \neq [X]$ we get $i \in I_X$, which is not the case$\big)$. As $[\langle x_i x_{i+1}]] = [X]$ and $[x_i] \neq [X]$ we can see that $s_i^{n-1}d_i^{n-1}(X)$ has more minimal elements that X.

∎

Lemma 4.6. $\big(s_n^{n-1}d_0^{n-1} + d_{n+1}^n s_{n+1}^n + d_0^n s_1^n + d_0^n s_{n+1}^n\big)(X) = \begin{cases} X+Y & \text{if } n+1 \in I_X \\ Y & \text{else} \end{cases}$
where Y is zero or an elementary tensor with more minimal elements than X.

Proof. The proof is similar to that of lemma 4.5.

∎

With lemmas 4.4, 4.5 and 4.6, we get:

Proposition 4.7. $(s^{n-1}d^{n-1} + d^n s^n)(X) = \sum\limits_{\substack{1 \leq k \leq n+1 \\ k \in I_X}} X + S$, *where S is zero or a finite sum of elementary tensors with more minimal elements than X.*

Lemma 4.8. *For an elementary tensor $X = x_1 \otimes \cdots \otimes x_{n+1}$, if $I_X \neq \emptyset$ then $1 \leq |I_X| \leq \frac{n+1}{2}$.*

Proof. For $X = x_1 \otimes \cdots \otimes x_{n+1}$ let us assume that $I_X \neq \emptyset$. Without loss of generality we can suppose that $1 \in I_X$. So $2 \notin I_X$ and $n+1 \notin I_X$. If $3 \in I_X$ then $4 \notin I_X$. By iteration, we get that $I_X \subseteq \{1, 3, 5, \dots\}$. As $n+1 \notin I_X$ we get our answer. ∎

Let
$$n_e = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{else.} \end{cases}$$
Let $Q_m = I - \frac{1}{m}(s^{n-1}d^{n-1} + d^n s^n)$ and $Q = Q_1 \circ Q_2 \circ \cdots \circ Q_n$.

Corollary 4.9. *If X is an elementary tensor with minimal element then $Q^{(n+1)_e-1}(X)$ is a linear sum of constant element.*

4.2. **Constant element.** In this section, we will show that there are linear bounded maps $s^k : \mathcal{C}_k(\ell^1(S), \ell^1(S)) \longrightarrow \mathcal{C}_{k+1}(\ell^1(S), \ell^1(S))$, $k \in \{n-1, n\}$, such that for each elementary tensor $X \in F_{k,cons}$ we have $\big(I - (s^{n-1}d^{n-1} + d^n s^n)\big)(X) = 0$.

Definition 4.10. *Let $n \geq 1$ and $s^k : \mathcal{C}_k(\mathcal{A}, \mathcal{A}) \longrightarrow \mathcal{C}_{k+1}(\mathcal{A}, \mathcal{A})$ the bounded linear map defined by*

$$s^n(x_1 \otimes \cdots \otimes x_{n+1}) = \begin{cases} (-1)^{n+1} x_1 \otimes x_2 \otimes \cdots \otimes \langle x_{n+1}] \otimes [x_{n+1}\rangle & \textit{if } [x_1] = \cdots = [x_{n+1}], \\ 0 & \textit{else.} \end{cases}$$

Proposition 4.11. *Let $X \in \mathcal{C}_n(\mathcal{A}, \mathcal{A})$ a constant element. Then*

$$\Big(I - (s^{n-1}d^{n-1} + d^n s^n)\Big)(X) = 0.$$

Proof. We verify easily that: $(s^{n-1}d^{n-1} + d^n s^n)(X) = \sum_{i=0}^{n} \big(s^{n-1}d_i^{n-1} + d_i^n s^n\big)(X) + d_{n+1}^n s^n(X) = X$. ∎

5. CONCLUSION

By Propositions 3.12, 4.11, and Corollary 4.9 we see that there exist $s^k : \mathcal{C}_k(\ell^1(S), \ell^1(S)) \longrightarrow \mathcal{C}_{k+1}(\ell^1(S), \ell^1(S))$, $k \in \{n-1, n\}$, bounded linear maps such that for all elementary tensors $X \in \mathcal{C}_k(\ell^1(S), \ell^1(S))$, we have $\big(I - (s^{n-1}d^{n-1} + d^n s^n)\big)(X) = 0$. This means that simplicial (co)-homology of $\ell^1(S)$ vanishes in all degrees.

REFERENCES

[1] Y. CHOI, F. GOURDEAU, and M. C. WHITE, Simplicial cohomology of band semigroup algebras, *Proc. Roy. Soc. Edinburgh Sect. A.*, **142** (2012), No. 4, pp. 715–744.

[2] Y. CHOI, Simplicial homology and Hochschild cohomology of Banach semilattice algebras, *Glasgow Math. J.*, **48** (2006), pp. 231–245.

[3] Y. CHOI, Simplicial homology of strong semilattices of Banach algebras, *Houston J. Math.*, **36** (2010), pp. 237–260.

[4] Y. FARHAT and F. GOURDEAU, Simplicial (co)homology of $\ell^1(\mathbb{Z}_+)$, *Can. Math. Bull.*, **62** (2019), No. 4, pp. 756–766.

[5] F. GOURDEAU, B. E. JOHNSON and M. C. WHITE, The cyclic and simplicial cohomology of $l^1(\mathbb{N})$, *Trans. Amer. Math. Soc.*, **357** (2005), No. 12, pp. 5097–5113.

[6] F. GOURDEAU and M. C. WHITE, The cyclic and simplicial cohomology of the bicyclic semigroup algebra, *Q. J. Math.*, **62** (2011), pp. 607–624.

[7] F. GOURDEAU and M. C. WHITE, The cyclic and simplicial cohomology of the Cuntz semigroup algebra, *J. Math. Anal. Appl.*, **386** (2012), No. 2, pp. 921–932.

[8] A. YA. HELEMSKII, Banach cyclic (co)homology and the Connes-Tzygan exact sequence, *J. Lond. Math. Soc.*, **s2-46** (1992), pp. 449–462.

[9] J. M. HOWIE, *Fundamentals of Semigroup Theory*, Oxford: Clarendon Press, (1995).