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# COMPOSITE VARIATIONAL-LIKE INEQUALITIES GIVEN BY WEAKLY RELAXED $\zeta$-SEMI-PSEUDOMONOTONE MULTI-VALUED MAPPING

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ABSTRACT. In this article, we introduce a composite variational-like inequalities with weakly relaxed $\zeta$-pseudomonotone multi-valued maping in reflexive Banach spaces. We obtain existence of solutions to the composite variational-like inequalities with weakly relaxed $\zeta$-pseudomon - otone multi-valued maps in reflexive Banach spaces by using KKM theorem. We have also checked the solvability of the composite variational-like inequalities with weakly relaxed $\zeta$-semi-pseudomonotone multi-valued maps in arbitrary Banach spaces using Kakutani-Fan-Glicksberg fixed point theorem.

*Key words and phrases:* Existence results; KKM-mapping; Weakly Relaxed $\zeta$-Semi-Pseudomonotone.

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## 1. INTRODUCTION

The theory of variational inequality provides us an elegant framework to study a wide class of linear and nonlinear problems arising in fluid flow through porous media, optimization, transportation, physical, applied and pure sciences, see [1, 2, 4, 8, 12, 14, 15, 16, 17, 18] and the references therein. Variational-like inequality, which is an extensions and the generalization of the variational inequality theory have been studied by several authors including Fang and Huang [7], Kang et al [10], Irfan and Khan [9] and references therein. By using the KKM technique, we study some existence results for variational-like inequalities with weakly-relaxed monotone mappings in reflexive Banach spaces. We also obtain the solvability of variational-like inequalities with weakly relaxed pseudomonotone mappings in reflexive Banach spaces using the Kakutani-Fan Glicksberg fixed point theorem. KKM theorem gives the foundation for many of the modern essential results in diverse areas of mathematical sciences.

Monotonicity is very important concept in nonlinear functional analysis. It plays an important role in variational inequality problems, equilibrium problems, fixed point theories, game theory and so on. In recent years a number of authors have also introduced many important generalizations of monotonicity such as quasimonotonicity, pseudomonotonicity, relaxed monotonicity, $p$-monotonicity, semimonotonicity etc.

In 1995 Chang et al [4] studied variational inequalities for monotone operators in non-reflexive Banach spaces. Kamaradian [11] showed that the problem of complementarity can be reduced to the variational inequality, while the relationship between mathematical programming and variational inequality was shown by Stampachia and Mancino [4, 5, 7, 12, 13, 15, 16].

Inspired and motivated by the recent research work, in this paper, we introduce and study a multi-valued generalized variational-like inequalities problem in real reflexive Banach vector space and then by using KKM technique, we proved some existence results. Our results improve and extend some corresponding results of [3].

## 2. PRELIMINARIES

In this paper, we suppose $\mathcal{B}$ is a reflexive Banach space and $\mathcal{B}^*$ is the topological dual of $\mathcal{B}$, $K$ a nonempty subset of $\mathcal{B}$.

**Definition 2.1.** Let $A, g : K \to K,\ \psi : K \times K \to \mathcal{B}$ be the mappings and $f : K \times K \to \mathbb{R}$ be a function. A multi-valued mapping $T : K \to 2^{\mathcal{B}^*}$ ($2^{\mathcal{B}^*}$ denotes the set of all the subset of $\mathcal{B}^*$) is said to be weakly relaxed $\zeta$-pseudomonotone if there exists a mapping $\alpha : E \to \mathbb{R}$ with $(\alpha(tz) = k(t)\alpha(z)$ for $z \in E$ and $t \in (0,1)$, where $k$ is a function from $(0,1)$ to $(0,1)$ with $\lim_{t\to 0} \frac{k(t)}{t} = 0$, such that for every pair of points $x, y \in K$ and for every $u \in T(x)$, we have

$$\langle Au, \psi(q, g(p))\rangle + f(q, g(p)) - f(g(p), g(p)) \geq 0,$$

implies

$$\langle Av, \psi(q, g(p))\rangle + f(q, g(p)) - f(g(p), g(p)) \geq \alpha(q - g(p)), \text{ for some } v \in T(q).$$

**Definition 2.2.** Let $K$ be a nonempty convex subset of $\mathcal{B}$, $g : K \to K,\ \psi : K \times K \to \mathcal{B}$ and $T : K \to 2^{\mathcal{B}^*}$ are mappings. $T$ is said to be $\psi$-hemicontinuous if for any fixed $p, q \in K$, the

map
$$t \longmapsto \langle T(g(p) + t(q - g(p)), \psi(q, g(p))\rangle \text{ for } 0 < t < 1,$$
is upper semicontinuous at $0^+$.

**Definition 2.3.** Let $K$ be a nonempty convex subset of $\mathcal{B}$, $A, g : K \to K$, $T : K \to 2^{\mathcal{B}^*}$ and $\psi : K \times K \to \mathcal{B}$ are mappings and $f : K \times K \to \mathbb{R} \cup \{+\infty\}$ be a proper function. $T$ is said to be $\psi$-coercive with respect to $f$ if there exists an $p_0 \in K$ such that
$$\frac{\langle Au - Au_0, \psi(g(p), g(p_0))\rangle + f(g(p), g(p_0))}{\psi(g(p), g(p_0))} \to \infty$$
whenever $\|p\| \to \infty$ for all $u \in T(g(p))$ and $u_0 \in T(g(p_0))$.

**Definition 2.4.** Let $K$ be a nonempty subset of a vector space $\mathcal{B}$. A multi-valued map $T : K \to 2^{\mathcal{B}}$ is said to be a KKM-map if for any finite subset $N$ of $K$, we have
$$co(N) \subset \bigcup_{p \in N} T(p).$$

**Theorem 2.1.** *(Fan-KKM Theorem*[6]*.) Let $\mathcal{B}$ be a topological vector space, $K \subset \mathcal{B}$ an arbitrary set, and $T : K \to 2^{\mathcal{B}}$ a KKM map. If all the sets $T(p)$ are closed in $\mathcal{B}$ and at least one of them is compact, then $\bigcap\{T(p) : p \in K)\}$ is nonempty.*

**Theorem 2.2.** *Let $K$ be a convex subset of $\mathcal{B}$, $A, g : K \to K$ are mappings, $T : K \to 2^{\mathcal{B}^*}$ be an $\psi$-hemicontinuous and weakly relaxed $\zeta$-pseudomonotone multi-valued map, and $f : K \times K \to \mathbb{R} \cup +\infty$ a proper function.*

(i) *$\psi(g(p), g(p)) = \overline{0}$ and $f(g(p), g(p)) = 0$ for $p \in K$;*

(ii) *for fixed $v \in \mathcal{B}^*$, $p \longmapsto \langle Av, \psi(g(p), .)\rangle + f(g(p), .)$ is convex.*

*Then the following variational inequalities (2.1) and (2.2) are equivalent.*

*Find $p \in K$ such that for each $q \in K$ , there exists $u \in T(p)$ satisfying,*
$$\langle Au, \psi(q, g(p))\rangle + f(q, g(p)) \geq 0. \tag{2.1}$$
*Find $p \in K$ such that for each $q \in K$, there exists $v \in T(q)$ satisfying,*
$$\langle Av, \psi(q, g(p))\rangle + f(q, g(p)) \geq \zeta(q - g(p)). \tag{2.2}$$

*Proof.* By the weakly relaxed $\zeta$-pseudomonotonicity of $T$, equation (2.1) implies (2.2).

Conversely, let $p$ be a point of $K$ such that for $q \in K$ with $f(q, g(p)) < \infty$ there exists $v \in T(q)$ satisfying,
$$\langle Av, \psi(q, g(p))\rangle + f(q, g(p)) \geq \zeta(q - g(p)).$$
Put $q_t = (1 - t)g(p) + tq, t \in (0, 1)$ then $q_t \in K$. It follows that,
$$\langle Av_t, \psi(q_t, g(p))\rangle + f(q_t, g(p)) \geq \zeta(q_t - g(p)), \text{ for some } v_t \in T(q_t).$$
Since, $\zeta(q_t - g(p)) = \zeta(t(q - g(p))) = k(t)\zeta(q - g(p))$, by the condition $(i)$ and $(ii)$, we have
$$t(\langle Av_t, \psi(q, g(p))\rangle + f(q, g(p)) \geq k(t)\zeta(q - g(p)).$$
Hence
$$\langle Av_t, \psi(q, g(p))\rangle + f(q, g(p)) \geq \frac{k(t)}{t}\zeta(q - g(p)).$$

Since, $T$ is $\psi$-hemicontinuous and weakly relaxed $\zeta$-pseudomonotone, by letting $t \to 0$, we get

$$\langle Au, \psi(q, g(p))\rangle + f(q, g(p)) \geq 0,$$

for some $Au \in T(p)$ and for all $q \in K$ with $f(q, g(p)) < \infty$. When $f(q, g(p)) = +\infty$, the inequality $\langle Au, \psi(q, g(p))\rangle + f(q, g(p)) \geq 0$ is trivial. Therefore, $p \in K$ is a solution of equation (2.1).

∎

**Theorem 2.3.** *Let $K$ be a nonempty bounded closed convex subset of a real reflexive Banach space $\mathcal{B}$. Let $A, g : K \to K$, $\psi : K \times K \to \mathcal{B}$ are mappings, $T : K \to 2^{\mathcal{B}^*}$ be an $\psi$-hemicontinuous and weakly relaxed $\zeta$-pseudomonotone map with nonempty compact values, and $f : K \times K \to \mathbb{R} \cup +\infty$ a proper function. Assume that*

(i) $\psi(g(p), q) + \psi(q, g(p)) = \overline{0}$ *and* $f(g(p), q) + f(q, g(p)) = 0$ *for* $p, q \in K$,
(ii) *for fixed* $v \in \mathcal{B}^*$, $p \longmapsto \langle Av, \psi(g(p), .)\rangle + f(g(p, .))$ *is convex, weakly lower semicontinuous,*
(iii) $\psi(p, g(p)) = 0$ *and* $f(p, g(p)) = 0$ *for all* $p \in K$,
(iv) $\zeta : \mathcal{B} \to \mathbb{R}$ *is weakly lower semicontinuous.*

*Then problem (2.1) is solvable.*

*Proof.* Define a multi-valued map, $F : K \to 2^{\mathcal{B}}$ as,

$$F(q) = \{p \in K | \langle Au, \psi(q, g(p))\rangle + f(q, g(p))| \geq 0 \text{ for some } u \in T(p) for\ q \in K\}.$$

Then $F$ is a KKM-map. In fact, suppose that there exist $\{g(q_1), g(q_2), ...., g(q_n)\}$ in $K$ and $t_i \geq 0$ with $\sum_{i=1}^n t_i = 1$ such that

$$g(q) = \sum_{i=1}^n t_i g(q_i) \notin \bigcup_{i=1}^n F(q_i).$$

Then for all $v \in T(q)$

$$\langle Av, \psi(q, g(q_i))\rangle + f(q, g(q_i)) < 0,\ i = 1, 2, ..., n.$$

It follows that

$$\begin{aligned}
0 &= \langle Av, \psi(q, g(q)) + f(q, g(q)) \\
&= \left\langle Av, \psi\left(q, \sum_{i=1}^n t_i g(q_i)\right)\right\rangle + f\left(q, \sum_{i=1}^n t_i g(q_i)\right) \\
&\leq \sum_{i=1}^n t_i \langle Av, \psi(q, g(q_i))\rangle + \sum_{i=1}^n t_i f(q, g(q_i)) \\
&< 0,
\end{aligned}$$

which is a contradiction. ∎

Define another multi-valued map, $G : K \to 2^{\mathcal{B}}$ by

$$G(q) = \{p \in K : \text{ for some } v \in T(q), \langle Av, \psi(q, g(p))\rangle + f(q, g(p)) \geq \zeta(q - g(p))\}$$

for $q \in K$. Then by the relaxed $\zeta$-pseudomonotonicity of $T$ it follows that $G$ is also a KKM-map. On the other hand, let $\{p_\beta\}$ be a net in $G(q)$ converging weakly to $p$. Then for some $v_\beta \in T(q)$,

$$\langle Av_\beta, \psi(q, g(p_\beta))\rangle + f(q, g(p_\beta)) \geq \zeta(q - g(p_\beta)),\ \text{for } \beta \in I.$$

Hence, $T(q)$ is compact, so that we may assume that $\{v_\beta\}$ converges to some $v \in T(q)$. Since, $p \longmapsto \langle Av, \psi(., g(p))\rangle + f(., g(p))$ is weakly upper semicontinuous, and $\zeta$ is weakly lower semicontinuous, and $\zeta$ is weakly lower semicontinuous, we have

$$\begin{aligned}
\langle Av, \psi(q, g(p))\rangle + f(q, g(p)) &\geq \overline{\lim}_\beta(\langle Av_\beta, \psi(q, g(p_\beta))\rangle + f(q, g(p_\beta)) \\
&\geq \frac{\lim}{\beta}(\langle Av_\beta, \psi(q, g(p_\beta))\rangle + f(q, g(p_\beta)) \\
&\geq \frac{\lim}{\beta}\alpha(q - g(p_\beta)) \\
&\geq \alpha(q - g(p)).
\end{aligned}$$

It follows that $p \in G(q)$ and $G(q)$ is weakly closed for all $q \in K$. Since $K$ is bounded closed and convex, $K$ is weakly compact, so $G(q)$ is weakly compact in $K$ for each $q \in K$. It follows from Theorem 2.1 and Fan-KKM theorem, that

$$\bigcup_{q\in K} F(q) = \bigcup_{q\in K} G(q) \neq \phi.$$

Hence, there exists an $p \in K$ such that for each $q \in K$, there exists $u \in T(p)$ satisfying

$$\langle Au, \psi(q, g(p))\rangle + f(q, g(p)) \geq 0.$$

**Theorem 2.4.** *Let $K$ be a nonempty unbounded closed convex subset of a real reflexive Banach space $\mathcal{B}$. Let $A, g : K \to K$, $\psi : K \times K \to \mathcal{B}$ are mappings, $T : K \to 2^{\mathcal{B}^*}$ be an $\psi$-hemicontinuous and relaxed $\zeta$-pseudomonotone multi-valued map, and $f : K \times K \to \mathbb{R} \cup +\infty$ a proper function. Assume that*

(i) *$\zeta : \mathcal{B} \to \mathbb{R}$ is weakly lower semicontinuous,*
(ii) *for fixed $v \in \mathcal{B}^*$, $p \longmapsto \langle Av, \psi(g(p), .)\rangle + f(g(p, .))$ is convex, weakly lower semicontinuous and*
(iii) *$\psi(g(p), q) + \psi(q, g(p)) = \overline{0}$ and $f(g(p), q) + f(q, g(p)) = 0$ for $p, q \in K$.*

*If $T$ is $\psi$-coercive with respect to $f$,then problem (2.1) is also solvable.*

*Proof.* Suppose $B_r = q \in \mathcal{B} : \|q\| \leq r$ and consider the following problem;
Find $p_r \in K \cap B_r$ such that for each $q \in K \cap B_r$, there exists $u_r \in T(p_r)$ satisfying

$$\langle Au_r, \psi(g(p_0), g(p_r)\rangle + f(q, p_r) \geq 0. \tag{2.3}$$

By Theorem 2.3, problem (2.3) has a solution $0p_r \in K \cap B_r$. Choose, $r \geq \|p_0\|$ with $p_0$ in the coercivity condition of Definition 2.3. Since, $p_0 \in K \cap B_r$, we have $u_r \in T(p_r)$ satisfying

$$\langle Au_r, \psi(g(p_0, g(p_r))\rangle + f(g((p_0), g(p_r)) \geq 0.$$

Moreover

$$\langle Au_r, \psi(g(p_0), g(p_r))\rangle + f(g(p_0), g(p_r))$$

$$\begin{aligned}
&= -\langle Au_r - Au_0, \psi(g(p_r), g(p_0))\rangle + f(g(p_0), g(p_r)) + \langle Au_0, \psi(g(p_0), g(p_r))\rangle \\
&\leq -\langle Au_r - Au_0, \psi(g(p_r), g(p_0))\rangle + f(g(p_0), g(p_r)) + \|Au_0\|\|\psi(g(p_r), g(p_0))\| \\
&\leq \|\psi(g(p_r), g(p_0))\| \left(-\frac{\langle Au_r - Au_0, \psi(g(p_r), g(p_0))\rangle + f(g(p_r), g(p_0))}{\|\psi(g(p_r), g(p_0))\|} + \|u_0\|\right),
\end{aligned}$$

for $u_0 \in T(p_0)$.

Now if $\|p_r\| = r$ for all $r$, we may choose $r$ large enough so that the above inequality and the $\psi$-coercivity of $T$ with respect to $f$ implies that,

$$\langle Au_r, \psi(g(p_0), g(p_r))\rangle + f(g(p_0), g(p_r)) < 0,$$

which contradicts

$$\langle Au_r, \psi(g(p_0), g(p_r))\rangle + f(g(p_0), g(p_r)) \geq 0.$$

Hence there exist $r$ such that $\|p_r\| < r$. For any $q \in K$, we can choose $\nu \in (0,1)$ small enough such that $g(p_r) + \nu(q - g(p_r)) \in K \cap B_r$. It follows from (2.3) that

$$\langle Au_r, \psi(g(p_r)) + \nu(q - g(p_r), g(p_r))\rangle + f(g(p_r)) + \nu(q - g(p_r), g(p_r)) \geq 0,$$

for some $u_r \in T(p_r)$. By conditions (i) and (ii), we have

$$\langle Au_r, \psi(q, g(p_r))\rangle + f(q, g(p_r)) \geq 0.$$

Thus $p_r \in K$ is a solution of (2.1). ∎

## 3. WEAKLY RELAXED $\zeta$-SEMI-PSEUDOMONOTONE COMPOSITE MULTI-VALUED VARIATIONAL-LIKE INEQUALITIES

In this section, we introduce and prove the existence of solutions for variational-like inequalities problems with a wakly relaxed $\zeta$-semi-pseudomonotone multi-valued mapping $B : K \times K \to 2^{\mathcal{B}^*}$, where $K$ is a nonempty closed convex subset of $\mathcal{B}^{**}$.

Find $p \in K$ such that for each $q \in K$ there exists $u \in B(g(p), g(p))$ satisfying

$$\langle Au, \psi(q, g(p))\rangle + f(q, g(p)) \geq 0. \tag{3.1}$$

**Definition 3.1.** Let $g : K \to K, \psi : K \times K \to \mathcal{B}^{**}$ are the mappings and $f : K \times K \to \mathbb{R}$ be a function. Let $K$ be a nonempty subset of $\mathcal{B}^{**}$. A multi-valued map $B : K \times K \to 2^{\mathcal{B}^*}$ is said to be weakly relaxed $\zeta$-semi-pseudomonotone if the following conditions hold:

(a) for each fixed $w \in K, B(w, .) : K \to 2^{\mathcal{B}^*}$ is weakly relaxed $\zeta$-pseudomonotone, i.e., there exists a function $\alpha : \mathcal{B}^{**} \to \mathbb{R}$ with $\alpha(tz) = k(t)\alpha(z)$ for $z \in \mathcal{B}^{**}$, where $k : (0,1) \to (0,1)$ is a function with $\lim_{t\to 0} \frac{k(t)}{t} = 0$, such that for every pair of points $p, q \in K$ and for every $u \in B(w, p)$, we have

$$\langle Au, \psi(q, g(p))\rangle + f(q, g(p)) \geq 0$$

implies

$$\langle Av, \psi(q, g(p))\rangle + f(q, g(p))\alpha(q - g(p)) \text{ for some } v \in T(q).$$

(b) for each fixed $q \in K, B(., q)$ is completely continuous, i.e., for any net $\{p_\beta\}$ in $\mathcal{B}^*$ such that $p_\beta \rightharpoonup^* p_0$, for every net $\{v\beta\}$ in $\mathcal{B}^{**}$ with $v_\beta \in B(p_\beta, q)$ has a convergent subnet, of each limit belongs to $B(p_0, q)$ in the norm topology of $\mathcal{B}^*$, where $\rightharpoonup$ denotes the weak$^*$ convergence in $\mathcal{B}^{**}$.

**Theorem 3.1.** *Let $\mathcal{B}$ be a real Banach space and $K$ be a nonempty bounded closed convex subset of $\mathcal{B}^{**}$. Let $B : K \times K \to 2^{\mathcal{B}^{(*)}}$ be a weakly relaxed $\zeta$-semi-pseudomonotone multi-valued map, $G : K \to K$ be a map and $f : K \times K \to \mathbb{R} \cup +\infty$ a proper function such that*

(i) *for fixed $v \in \mathcal{B}^*$, $x \longmapsto \langle Av, \psi(g(p), .)\rangle + f(g(p), .)$ is linear, weakly lower semicontinuous;*
(ii) *$\psi(g(p), q) + \psi(q, g(p)) = \overline{0}$ and $f(g(p), q) + f(q, g(p)) = 0$ for $p, q \in K$;*
(iii) *$\zeta : \mathcal{B}^{**} \to \mathbb{R}$ is convex, weakly lower semicontinuous;*

(iv) *for each* $p \in K$ , $A(g(p),.) : K \to 2^{\mathcal{B}^*}$ *is finite dimensional continuous.*
*Then problem (3.1) is solvable.*

*Proof.* Let $F \subset \mathcal{B}^{**}$ be a finite dimensional subspace with $K_F = K \bigcap F \neq \phi$. For each $w \in K$, we consider the following problem.

Find $p_0 \in K_F$ such that for $q \in K_F$, there exists $u_0 \in B(w, g(p_0))$ satisfying,

$$\langle Au_0, \psi(q, g(p_0))\rangle + f(q, g(p_0)) \geq 0. \tag{3.2}$$

For each $w \in K_F$, since $B(w,.)$ is weakly relaxed $\zeta$-pseudomonotone and continuous on a bounded closed convex subset $K_F$ of $F$. By Theorem 2.3, above problem has a solution $p_w \in K_F$. If we define a multi-valued map, $T : K_F \to 2^{K_F}$ as follows;

$$\begin{aligned} T(w) &= \{p \in K_F : \text{ for } q \in K_F, \text{ there exists } u \in B(w, g(p)) \text{ such that } \langle Au, \psi(q, g(p))\rangle \\ &\quad + f(q, g(p)) \geq 0\}, \end{aligned}$$

then $T(w)$ is nonempty, since $p_w \in T(w)$. By Theorem 2.1, $T(w)$ is equal to the set,

$$\begin{aligned} &\{p \in K_F : \text{ for } q \in K_F, \text{ there exists } v \in B(w, g(p)) \text{ such that } \langle Av, \psi(q, g(p))\rangle \\ &+ f(q, g(p)) \geq \alpha(q - g(p))\}. \end{aligned}$$

By conditions (i),(ii) and (iii), $T : K_F \to 2^{K_F}$ has a nonempty bounded closed and convex multi-values, and $T$ is upper semicontinuous by the complete continuity of $B(., q)$. By the Kakutani-Fan-Glickberg fixed point theorem, $T$ has a fixed point $w_0$ in $K_F$ i.e., for each $q \in K_F$, there exists $u \in B(w_0, w_0)$ satisfying,

$$\langle u, \psi(q, w_0)\rangle + f(q, w_0) \geq 0. \tag{3.3}$$

Let $\chi = \{F : F \text{ is a finite-dimensional subspace of } \mathcal{B}^{**} \text{ with } K_F \neq \phi\}$ and, for $F \in \chi$,

$$\begin{aligned} W_F &= \{p \in K : \langle Av, \psi(q, g(p))\rangle + f(q, g(p)) \geq \alpha(q - g(p)) \text{ for } q \in K_F, \\ &\qquad \text{and for some } v \in B(g(p), q))\}, \end{aligned}$$

By Theorem 2.1 and (3.3), we know that $W_F$ is nonempty and bounded. Since, the weak$^*$ closure $\overline{W_F}$ of $W_F$ in $\mathcal{B}^{**}$ is weak$^*$ compact in $\mathcal{B}^{**}$, for any $F_i \in \chi$, $(i = 1, 2, ....., n)$, we know that $W_{\cup F_i} \subset \bigcap_i W_{F_i}$ so $\{\overline{W_F} : F \in \chi\}$ has the finite intersection property. Therefore, it follows that,

$$\bigcap_{F \in \chi} \overline{W_F} \neq \phi.$$

Let $p \in \bigcap_{F \in \chi} \overline{W_F}$. We claim that for each $q \in K$, there exists $u \in B(g(p), g(p))$ satisfying $\langle Au, \psi(q, g(p))\rangle + f(q, g(p)) \geq 0$. Indeed, for each $w \in K$, let $F \in \chi$ such that $w \in K_F$ and $p \in K_F$, there exists a net $\{p_\beta\}$ in $W_F$ such that $p_\beta \rightharpoonup p$. It follows that

$$\langle Av_\beta, \psi(q, g(p_\beta))\rangle + f(q, g(p_\beta)) \geq \zeta(q - g(p_\beta))$$

for all $q \in K_F$ and for some $v_\beta \in B(g(p_\beta), q)$. Hence, we get

$$\langle Av, \psi(q, g(p))\rangle + f(q, g(p)) \geq \zeta(q - g(p))$$

for all $q \in K$ and for some $v \in B(g(p), q)$ by using the complete continuity of $B(., q)$ and the assumptions on $f, \psi$ and $\zeta$. From Theorem 2.1, for each $q \in K$ , there exists $u \in B(g(p), g(p))$ satisfying,

$$\langle u, \psi(q, g(p))\rangle + f(q, g(p)) \geq 0.$$

∎

**Theorem 3.2.** *Let $\mathcal{B}$ be a real Banach space and $K$ a nonempty unbounded closed convex subset of $\mathcal{B}^{**}$. Let $g : K \to K$ be a map, $B : K \times K \to 2^{\mathcal{B}^{(*)}}$ be a weakly relaxed $\zeta$-semi-pseudomonotone multi-valued map, and $f : K \times K \to \mathbb{R} \cup +\infty$ a proper function such that*

(i) *for each $p \in K$, $B(g(p), .) : K \to 2^{\mathcal{B}^*}$ is finite dimensional continuous;*
(ii) *$\zeta : \mathcal{B}^{**} \to \mathbb{R}$ is convex, weakly lower semicontinuous;*
(iii) *$\psi(g(p), q) + \psi(q, g(p)) = \bar{0}$ and $f(g(p), q) + f(q, g(p)) = 0$ for $p, q \in K$;*
(iv) *for fixed $v \in \mathcal{B}^*$, $p \to \langle Av, \psi(g(p), .)\rangle + f(g(p), .)$ is linear, lower semicontinuous;*
(v) *there exists an $p_0 \in K$ such that,*

$$\underline{lim}_{\|p\|\to\infty} \langle u, \psi(g(p), g(p_0)) + f(g(p), g(p_0)) \geq 0$$

*for all $p \in K$ and for all $u \in B(g(p), g(p))$. Then problem (3.1) is solvable.*

*Proof.* Denote the closed ball with radius $r$ and center at $0$ in $\mathcal{B}^{**}$ by $\mathbf{B}_r$. By Theorem 3.1, there exists a solution $p_r \in \mathbf{B}_r \bigcap K$ such that for each $q \in \mathbf{B}_r \bigcap K$ there exists, $u_r \in B(g(p_r), g(p_r))$ satisfying,

$$\langle Au_r, \psi(q, g(p_r))\rangle + f(q, g(p_r)) \geq 0.$$

Let $r$ be large enough so that $p_0 \in \mathbf{B}_r$, therefore there exists $u_r \in B(g(p_r), g(p_r))$ such that,

$$\langle Au_r, \psi(g(p_0), g(p_r))\rangle + f(g(p_0), g(p_r)) \geq 0.$$

By the condition $(v)$, we know that $\{p_r\}$ is bounded. So, we may suppose that, $p_r \rightharpoonup p$ as $r \rightharpoonup \infty$. It follows from Theorem 2.1 that for each $v_r \in B(g(p_r), y)$

$$\langle Av_r, \psi(q, g(p_r))\rangle + f(q, g(p_r)) \geq \zeta(q - g(p_r)) \text{ for all } q \in K.$$

Letting $r \to \infty$, we have for all $v \in B(g(p), q)$,

$$\langle Av, \psi(q, g(p))\rangle + f(q, g(p)) \geq \zeta(q - g(p)) \text{ for all } q \in K$$

Again by Theorem 2.1, we have for $q \in K$ , there exists, $u \in B(g(p), g(p))$ satisfying

$$\langle Au, \psi(q, g(p))\rangle + f(q, g(p)) \geq 0.$$

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