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DOUBLE DIFFERENCE OF COMPOSITION OPERATOR ON BLOCH SPACES

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ABSTRACT. In this paper we characterize the compactness of double difference of three non-compact composition operators on Bloch space induced by three holomorphic self maps on the unit disc.

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1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disc of the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. Consider the algebra $H(\mathbb{D})$ of all holomorphic functions with domain $\mathbb{D}$ and let $S(\mathbb{D})$ be the set of all analytic self maps of $\mathbb{D}$. Now if φ is an analytic self map of $\mathbb{D}$, then φ induces a composition operator C_φ defined by

$$C_\varphi f = f o \varphi,$$

for any analytic function $f : \mathbb{D} \to \mathbb{C}$. The study of composition operators on Bloch space $\mathcal{B}$ and little Bloch space $\mathcal{B}_o$ began with the paper [10] by Madigan and Matheson in which they have characterized the necessary and sufficient condition for compactness of composition operator on $\mathcal{B}$. In the last few decades the study of composition operator on various function spaces have been an active area of research for mathematician all across the globe. However in the last few years researchers have now started understanding the topological structure of the set of composition operators acting on certain function spaces specially the holomorphic function spaces. Berkson [1] in 1891 first introduced the idea of studying the topological structure of the sets of composition operator in the operator norm topology with his isolation results on Hardy space. Shapiro and Sunderg [14] continues the work of Berkson and went on further to raise a question on "when the difference of two composition operator on Hardy space H^2 is compact?" Since then the compact differences of composition operator on various settings has been an active area of research nowadays. While this question still remains unsolved for the Hardy space H^2, however various papers characterizing the compact differences of composition operators on Bloch spaces, little Bloch spaces and Bergman spaces has appeared in recent years. Moorhous [11] was the one who answered the question of when differences of two composition operator on Bergman spaces A^2_α, $\alpha > -1$ is compact. Hosokawa and Ohno ([5], [6]) initiated the study of the topological structures of the sets of composition operators on Bloch spaces $\mathcal{B}$ and little Bloch space $\mathcal{B}_o$ in the unit disc and gave a characterization on the question of compact difference of composition operator on these spaces. For more details on Bloch space and differences of composition operator on Bloch spaces one can refer to [3], [7],[8],[9], [12],[15] and [16].
Notations: Throughout the paper we shall use the notation $A \preceq B$ for nonnegative quantities A and B to mean that $A \leq KB$ for some constant $K > 0$ and the notation $A \approx B$ mean that both $A \preceq B$ and $B \preceq A$.

2. PREMILINARIES

We recall that the classical Bloch space denoted by $\mathcal{B}$ consists of all functions $f \in H(\mathbb{D})$ such that

$$sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) |f'(z)| < \infty. \tag{2.1}$$

and the little Bloch space $\mathcal{B}_o$ consists of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z|\to 1} \left(1 - |z|^2\right) |f'(z)| = 0 \tag{2.2}$$

The Bloch space $\mathcal{B}$ becomes a Banach space under the norm defined as

$$\|f\|_{\mathcal{B}} = |f(0)| + sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) |f'(z)| \tag{2.3}$$

and the little Bloch space $\mathcal{B}_o$ is a closed subspace of $\mathcal{B}$. We shall now review some basic properties of functions in the the Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_o$.
Let $z, w \in \mathbb{D}$, then the pseudo-hyperbolic distance between the two point is given by

$$\rho(w, z) = \left|\frac{w - z}{1 - \overline{w}z}\right| \tag{2.4}$$

and the hyperbolic metric which is also called the Bergman metric is then given by

$$\beta(w,z) = \frac{1}{2}\log\frac{1+\rho(w,z)}{1-\rho(w,z)}. \tag{2.5}$$

As Hosokawa and Ohno [5] has used the behavior of the Schwartz-Pick type derivative $\varphi^{\#}$ on the neighborhood of $\partial\mathbb{D}$ to study compact difference of composition operator on $\mathcal{B}$, we shall continue using this behavior of $\varphi^{\#}$ and results from Hosokawa and Ohno (see [5] and [6]) to study the double difference of composition operator on $\mathcal{B}$ and so we shall define the Schwartz-Pick type derivative as follows.
For $\varphi \in S(\mathbb{D})$, the Schwartz-Pick type derivative $\varphi^{\#}$ of φ is defined as

$$\varphi^{\#} = \frac{1-|z|^2}{1-|\varphi(z)|^2}\varphi'(z) \tag{2.6}$$

Further $\left|\varphi^{\#}(z)\right| \leq 1$ follows from the Schwartz-Pick lemma. We shall further use the estimate

$$|f(z)| \leq \frac{1}{\log 2}\,\|f\|_{\mathcal{B}}\,\frac{1}{1-|z|^2}\ \forall\ f \in \mathcal{B} \tag{2.7}$$

given in [13] in the proof of our main results. Note that we also write

$$\rho_{i,j}(z) = \rho\left(\varphi_i(z), \varphi_j(z)\right).$$

Let Δ denote the collection of all sequence $\{z_n\}$ in the unit disc $\mathbb{D}$ of $\mathbb{C}$ which converges to some point on the boundary of the unit disc denoted by $\partial\mathbb{D}$ such that the sequence $\{\varphi_i(z_n)\}$, $\{\varphi_i^*(z_n)\}$ and $\{\rho_{ij}(z_n)\}$ also converges for every i, j. Now we shall further state the following set which is defined in [4] as under

$$\begin{aligned}
I\{z_n\} &= \{i : \{z_n\} \in \Gamma_i\} \\
I_j\{z_n\} &= \left\{i : \{z_n\} \in \Gamma^{\star}_{i,j}\right\} \\
I_j^{\#}\{z_n\} &= \left\{i : \{z_n\} \in \Gamma^{\#}_{i,j}\right\} \\
I_j^{*}\{z_n\} &= \left\{i : \{z_n\} \in \Gamma^{*}_{i,j}\right\}
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_i &= \{\{z_n\} \in \Delta : |\varphi_i(z_n)| \to 1\} \\
\Gamma^{\star}_{i,j} &= \left\{\{z_n\} \in \Delta : \rho_{i,j}(z_n) \to 0\right\} \\
\Gamma^{\#}_{i,j} &= \left\{\{z_n\} \in \Gamma^{\star}_{i,j} : \varphi_i^{\#}(z_n) \nrightarrow 0\right\} \\
\Gamma^{*}_{i,j} &= \left\{\{z_n\} \in \Gamma^{\star}_{i,j} : lim_{n\to\infty}\varphi_i^{\#}(z_n) = lim_{n\to\infty}\varphi_j^{\#}(z_n)\right\}
\end{aligned}$$

Then it is clear that $\Gamma^{\#}_{i,j} \subseteq \Gamma^{\star}_{i,j} \subseteq \Gamma_i \subseteq \Delta$ and $\Gamma^{*}_{i} \subseteq \Gamma^{\star}_{i,j} \subseteq \Gamma_i \subseteq \Delta$. Now if $Card\,\Gamma^{\#}_{i} = \phi$, then by Madigan and Matheson [10] Compactness criterion, C_{φ_i} is compact on the Bloch space $\mathcal{B}$. The above construction induces a natural partition of the set $I\{z_n\}$ into disjoint sets of the form $I_j\{z_n\}$ defined above which can further be partitioned into disjoint sets of the form $I_j^*\{z_n\}$ as follows

$$I\{z_n\} = \cup_{s=1}^{t} I_{j_s}\{z_n\} \tag{2.8}$$

for some $j_1, j_2, \cdots, j_t \in I\{z_n\}$ and

$$I_j\{z_n\} = \cup_{q=1}^{p} I^*_{s_q}\{z_n\} \tag{2.9}$$

for some $s_1, s_2, \cdots, s_p \in I_j\{z_n\}$.
We have the following equivalent condition for an operator T to be compact on the Bloch Space $\mathcal{B}$ due to [4].

Theorem 2.1. *Let $\varphi_1, \varphi_2, \cdots \varphi_m$ be any analytic self-maps of the unit disc $\mathbb{D}$. Let*

$$T = \sum_{i=1}^{m} \lambda_i C_{\varphi_i} \tag{2.10}$$

be an operator on the Bloch space $\mathcal{B}$ where $\lambda_1, \lambda_2 \cdots, \lambda_m$ are nonzero complex scalars. Then the following condition are equivalent

(1) *T is compact.*

(2) *$\Sigma_{i \in I^{\#}_j\{z_n\}} \lambda_i = 0$ for all $\{z_n\} \in \Delta$, $j \in I\{z_n\}$.*

(3) *$\Sigma_{i \in I^*_j\{z_n\}} \lambda_i = 0$ for all $\{z_n\} \in \Delta$, $j \in I\{z_n\}$ with $\varphi^{\#}_j(z_n) \nrightarrow 0$.*

In [5] Hosokawa and Ohno further proved the following theorem.

Theorem 2.2. *Let φ and ψ be any analytic self maps of the unit disc $\mathbb{D}$. Suppose that neither C_φ nor C_ψ is compact on $\mathcal{B}$. Then the difference $C_\varphi - C_\psi$ is compact on Bloch space $\mathcal{B}$ if and only if*

(1) *$\varphi^{\#}(z_n)\rho(\varphi(z_n), \psi(z_n)) \to 0$ whenever $|\varphi(z_n)| \to 1$.*

(2) *$\psi^{\#}(z_n)\rho(\varphi(z_n), \psi(z_n)) \to 0$ whenever $|\psi(z_n)| \to 1$ and*

(3) *$|\varphi^{\#}(z_n) - \psi^{\#}(z_n)| \to 0$ whenever $|\varphi(z_n)| \to 1$ and $|\psi(z_n)| \to 1$.*

3. MAIN RESUT

We shall first prove for Bloch space a result similar to [2, Theorem 1.1] proved for Bergman spaces.

Theorem 3.1. *Let φ_1,φ_2 and φ_3 be an analytic self map of the unit disc and λ_1,λ_2 and λ_3 be three non zero complex number. Suppose that each of C_{φ_i} is not compact on the Bloch space $\mathcal{B}$ for $i = 1, 2, 3$. If $T = \lambda_1 C_{\varphi_1} + \lambda_2 C_{\varphi_2} + \lambda_3 C_{\varphi_3}$ is compact on $\mathcal{B}$, Then the following holds.*

(1) *$T = \lambda_i \left(C_{\varphi_i} - C_{\varphi_j} - C_{\varphi_k}\right)$ where (i, j, k) is some permutation of the set $\{1, 2, 3\}$.*

(2) *$T = \lambda_2 \left(C_{\varphi_2} - C_{\varphi_1}\right) + \lambda_3 \left(C_{\varphi_3} - C_{\varphi_1}\right)$*

Proof. Since T is compact implies that for each sequence $\{z_n\} \in \Delta$ and for all $j \in I\{z_n\}$, we have from Theorem 2.1, that

$$\Sigma_{i \in I^{\#}_j\{z_n\}} \lambda_i = 0. \tag{3.1}$$

Also we see from equation (3.1) that $Card\ I^{\#}_j\{z_n\} \neq 1$, since we have $\lambda_i \neq 0$ for all $i = 1, 2, 3$. Thus we see that for each sequence $\{z_n\} \in \Delta$ and for any $j \in I\{z_n\}$, we have

$Card\ I_j^{\#}\{z_n\} = \{0, 2, 3\}$. Thus we have the following cases

Case I: We first suppose that $Card\ I_j^{\#}\{z_n\} = 0$ for some sequence $\{z_n\} \in \Delta$ and some $j \in I\{z_n\}$. This implies that for each $i \in I\{z_n\}$, either $\rho_{i,j}(z_n) \nrightarrow 0$ or $\varphi_i^{\#}\{z_n\} \to 0$.

Subcase I: If $\varphi_i^{\#}(z_n) \to 0$ for $i \in I\{z_n\}$, then by Madigan and Matheson [10] compactness criterion, C_{φ_i} is compact which is not true.

Subcase II: If $\rho_{i,j}(z_n) \nrightarrow 0$ implies that $i \in I\{z_n\}$ but $i \notin I_j\{z_n\}$. However since $I\{z_n\} = \cup_{s=1}^t I_{j_s}\{z_n\}$ for some $j_1, j_2, \cdots j_t$ implies that $i \in I_{j_k}\{z_n\}$ for some $k = 1, 2, \cdots t$. Thus $i \in I_{j_k}^{\#}\{z_n\}$ which is a contradiction to the fact that $Card\ I_j^{\#}\{z_n\} = 0$ for all sequence $\{z_n\} \in \Delta$ and for each $j \in I\{z_n\}$. Thus $Card\ I_j^{\#}\{z_n\} = 0$ is not possible.

Case II: If $Card\ I_j^{\#}\{z_n\} = 2$. Suppose we assume that $I_j^{\#}\{z_n\} = \{1, 2\}$. Then we have

$$\lambda_1 + \lambda_2 = 0. \tag{3.2}$$

Since $3 \notin I_j^{\#}\{z_n\}$ implies that either $\varphi_3^{\#}(z_n) \to 0$ or $\rho_{3,j}(z_n) \nrightarrow 0$. If $\varphi_3^{\#}(z_n) \to 0$ as $|\varphi_3(z_n)| \to 1$ implies that C_{φ_3} is compact which is not true. Thus we must have $\rho_{3,j}(z_n) \nrightarrow 0$ which further implies that $3 \notin I_j\{z_n\}$. But since $3 \in I\{z_n\}$ and $I\{z_n\} = \cup_{s=1}^t I_{j_s}\{z_n\}$ for some $j_1, j_2 \cdots j_t \in I\{z_n\}$. Thus $3 \in I_{j_s}\{z_n\}$ for some $s = 1, 2 \cdots t$. Thus we have $3 \in I_{j_s}^{\#}\{z_n\}$ and since we are assuming that $Card\ I_j^{\#}\{z_n\} = 2$ for all sequence $\{z_n\} \in \Delta$ and for each $j \in I\{z_n\}$. Thus we must have either $I_j^{\#}\{z_n\} = \{3, 1\}$ or $I_j^{\#}\{z_n\} = \{3, 2\}$. Thus we have

$$\lambda_1 + \lambda_3 = 0 \text{ or } \lambda_2 + \lambda_3 = 0 \tag{3.3}$$

Solving equation (3.2) and (3.3) we get $T = \lambda_1(C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3})$ or $T = \lambda_2(C_{\varphi_2} - C_{\varphi_1} - C_{\varphi_3})$. Similarly we can get the other permutation in (1).

Case III: We suppose that $Card\ I_j^{\#}\{z_n\} = 3$ for some sequence $\{z_n\}$ and $j \in I\{z_n\}$. Then $I_j^{\#}\{z_n\} = \{1, 2, 3\}$ which further implies that $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Solving this we find the required result (2).

∎

In the next result we shall prove the necessary and sufficient condition for the double difference of composition operators to be compact constructed using three non compact composition operators.

Theorem 3.2. *Let $a, b \in \mathbb{C} \setminus \{0\}$ and $a + b \neq 0$. Assume that C_{φ_i} is not compact on $\mathcal{B}$ for each $i = 1, 2, 3$. Then $T = a(C_{\varphi_2} - C_{\varphi_1}) + b(C_{\varphi_3} - C_{\varphi_1})$ is compact on Bloch space $\mathcal{B}$ iff both $C_{\varphi_2} - C_{\varphi_1}$ and $C_{\varphi_3} - C_{\varphi_1}$ are compact on $\mathcal{B}$.*

Proof. The sufficiency condition is trivial. So we suppose that $T = a(C_{\varphi_2} - C_{\varphi_1}) + b(C_{\varphi_3} - C_{\varphi_1})$ is compact. We need to prove that $C_{\varphi_2} - C_{\varphi_1}$ and $C_{\varphi_3} - C_{\varphi_1}$ are both compact.
Suppose on the contrary that $C_{\varphi_2} - C_{\varphi_1}$ is not compact for instance. Then by Theorem 2.2, there exist some $\epsilon > 0$ and some sequence $\{z_n\} \in \Delta$ such that

$$\varphi_i^{\#}(z_n)\rho(\varphi_1(z_n), \varphi_2(z_n)) > \epsilon \text{ or } |\varphi_2^{\#}(z_n) - \varphi_1^{\#}(z_n)| > \epsilon \tag{3.4}$$

whenever $|\varphi_i(z_n)| \to 1$ for $i = 1, 2$.
We first suppose that $|\varphi_2^{\#}(z_n) - \varphi_1^{\#}(z_n)| > \epsilon$ whenever $|\varphi_i(z_n)| \to 1$ for $i = 1, 2$. Then it is

clear that $1, 2 \in I\{z_n\}$. Since T is compact implies that

$$\Sigma_{i \in I_j^*\{z_n\}} \lambda_i = 0 \tag{3.5}$$

for all $\{z_n\} \in \Delta$, $j \in I\{z_n\}$ with $\varphi_j^{\#}(z_n) \nrightarrow 0$ where $\lambda_1 = -(a+b), \lambda_2 = a$ and $\lambda_3 = b$. Now since $1 \in I\{z_n\}$ and clearly $\varphi_1^{\#}(z_n) \nrightarrow 0$ for otherwise C_{φ_1} is compact which is not true. Now consider the set $I_1^*\{z_n\}$. Then clearly $2 \notin I_1^*\{z_n\}$ because of our assumption. Thus $Card\, I_1^*\{z_n\} = 1\ or\ 2$, so that $I_1^*\{z_n\} = \{1, 3\}$ or $I_1^*\{z_n\} = \{1\}$ or $I_1^*\{z_n\} = \{3\}$.

Case I: If $I_1^*\{z_n\} = \{1, 3\}$. Then we have

$$\Sigma_{i \in I_1^*\{z_n\}} \lambda_i = 0$$

So that $-a = 0$ a contradiction.

Case I: If $I_1^*\{z_n\} = \{1\}$ implies that $a + b = 0$ a contradiction.

Case III If $I_1^*\{z_n\} = \{3\}$ implies that $b = 0$ again a contradiction.

Similarly we will again arrive at a contradiction when we consider $2 \in I\{z_n\}$. Next we suppose that $\varphi_i^{\#}(z_n)\rho(\varphi_1(z_n), \varphi_2(z_n)) \geq \epsilon$ say for $i = 1$. Then we must have

$$\frac{1 - |z_n|^2}{1 - |\varphi_1(z_n)|^2} \geq \epsilon_1 \text{ and } \rho_{1,2}(z_n) = \left|\frac{\varphi_1(z_n) - \varphi_2(z_n)}{1 - \overline{\varphi_1(z_n)}\varphi_2(z_n)}\right| \geq \epsilon_1 \tag{3.6}$$

for some ϵ_1 smaller then ϵ. This implies that $1 - |z_n|^2 \geq \epsilon_1 1 - |\varphi_1(z_n)|^2$. Thus we see that $1 - |z_n|^2 \approx 1 - |\varphi_1(z_n)|^2$. Now we shall consider the sequence of test function

$$f_n(z) = \frac{1}{(1 - z\overline{\varphi_1(z_n)})^k}$$

We have from our estimate that

$$|f_n(\varphi_1(z_n))| \preceq \|f_n\|_{\mathcal{B}} \log \frac{2}{(1 - |\varphi_1(z_n)|^2)}$$

So that

$$\frac{1}{(1 - |\varphi_1(z_n)|^2)^k} \preceq \|f_n\|_{\mathcal{B}} \log \frac{2}{(1 - |\varphi_1(z_n)|^2)} \tag{3.7}$$

Also

$$|Tf_n(z_n)| \preceq \|Tf_n\|_{\mathcal{B}} \log \frac{2}{(1 - |z_n|^2)} \tag{3.8}$$

From equation (3.7) and (3.8) we have

$$(1 - |\varphi_1(z_n)|^2)^k |Tf_n(z_n)| \preceq \frac{\|Tf_n\|_{\mathcal{B}}}{\|f_n\|_{\mathcal{B}}} \frac{\log \frac{2}{(1-|z_n|^2)}}{\log \frac{2}{(1-|\varphi_1(z_n)|^2)}} \tag{3.9}$$

Further since

$$\log \frac{2}{(1 - |\varphi_1(z_n)|^2)} \approx \log \frac{2}{(1 - |z_n|^2)}. \tag{3.10}$$

We have

$$(1 - |\varphi_1(z_n)|^2)^k |Tf_n(z_n)| \preceq \frac{\|Tf_n\|_{\mathcal{B}}}{\|f_n\|_{\mathcal{B}}} \tag{3.11}$$

Now let us construct a sequence as follows.

$$s_{j,n} = \frac{1}{1 + \overline{\varphi_1(z_n)}\alpha_j(z_n)} \tag{3.12}$$

where $\alpha_j(z_n) = \frac{\varphi_1(z_n)-\varphi_j(z_n)}{1-|\varphi_1(z_n)|^2}$ and $j = 1, 2, 3$. The it is clear from equation (3.12), that $\lim sup_{n\to\infty}|s_{2,n}| \lneq 1$, otherwise there is a subsequence say $\{z_{nk}\} \in \Delta$ such that $\alpha_j(z_{nk}) \to 0$ which implies that

$$\rho_{12}(z_{nk}) = |\alpha_j(z_{nk})| \left| \frac{1 - |\varphi_1(z_{nk})|}{1 - \overline{\varphi_1(z_{nk})}\varphi_2(z_{nk})} \right| \to 0$$

a contradiction to equation (3.6). Now from equation (3.11)

$$\begin{aligned}
\frac{\|Tf_n\|_{\mathcal{B}}}{\|f_n\|_{\mathcal{B}}} &\succeq (1 - |\varphi_1(z_n)|^2)^k |Tf_n(z_n)| \\
&\succeq \left| \lambda_1 + \lambda_2 \left(\frac{1 - |\varphi_1(z_n)|^2}{1 - \overline{\varphi_1(z_n)}\varphi_2(z_n)} \right)^k + \lambda_3 \left(\frac{1 - |\varphi_1(z_n)|^2}{1 - \overline{\varphi_1(z_n)}\varphi_3(z_n)} \right)^k \right| \\
&\succeq \left| 1 + \frac{\lambda_2}{\lambda_1} s_{2,n}^k + \frac{\lambda_3}{\lambda_1} s_{3,n}^k \right|
\end{aligned}$$

where $\lambda_1 = -(a + b), \lambda_2 = a$ and $\lambda_3 = b$. Since T is compact and so we have $\frac{\|Tf_n\|_{\mathcal{B}}}{\|f_n\|_{\mathcal{B}}} \to 0$ as $n \to \infty$ implies that

$$1 + \frac{\lambda_2}{\lambda_1} s_{2,n}^k + \frac{\lambda_3}{\lambda_1} s_{3,n}^k \to 0. \tag{3.13}$$

Also since $\lim sup_{n\to\infty}|s_{2,n}| \lneq 1$ and so equation (3.13) is possible only if $\frac{\lambda_3}{\lambda_1} = -1$, so that $0 = a$ a contradiction. Thus in each case we get a contradiction. Thus we must have both $C_{\varphi_2} - C_{\varphi_1}$ and $C_{\varphi_3} - C_{\varphi_1}$ compact. ∎

Corollary 3.3. *Let φ_i for $i = 1, 2, 3$ be three distinct holomorphic self maps of the unit disc and C_{φ_i} is not compact on $\mathcal{B}$ for each $i = 1, 2, 3$. Then $T = (C_{\varphi_2} - C_{\varphi_1}) + (C_{\varphi_3} - C_{\varphi_1})$ is compact on Bloch space $\mathcal{B}$ iff both $C_{\varphi_2} - C_{\varphi_1}$ and $C_{\varphi_3} - C_{\varphi_1}$ are compact on $\mathcal{B}$.*

4. CONCLUSION

Applying the behavior of the Schwartz Pick type derivative near the boundary of the unit disc $\mathbb{D}$,we have concluded the necessary and sufficient conditions for double difference of three non-compact composition operators on Bloch space $\mathcal{B}$ to be compact. These results will further help researchers in understanding the topological structure of the set of composition operators on Bloch space $\mathcal{B}$.

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