Aust. J. Math. Anal. Appl.
Vol. **17** (2020), No. 1, Art. 11, 16 pp.
AJMAA

ON RULED SURFACES ACCORDING TO QUASI-FRAME IN EUCLIDEAN 3-SPACE

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Received 30 January, 2020; accepted 14 May, 2020; published 26 May, 2020.

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ABSTRACT. This paper aims to study the skew ruled surfaces by using the quasi-frame of Smarandache curves in the Euclidean 3-space. Also, we reveal the relationship between Serret-Frenet and quasi-frames and give a parametric representation of a directional ruled surface using the quasi-frame. Besides, some comparative examples are given and plotted which support our method and main results.

Key words and phrases: Smarandache curves; Ruled surfaces; Serret-Frenet frame; Quasi-frame.

2010 *Mathematics Subject Classification.* 53A04, 53A05, 53A35.

ISSN (electronic): 1449-5910

This research was supported by Islamic University of Madinah. We would like to thank our colleagues from Deanship of Scientific Research who provided insight and expertise that greatly assisted the research.

1. INTRODUCTION

A ruled surface is a curved surface which can be generated by the continuous motion of a straight line in space along a space curve called a directrix. This straight line is called a generator, or ruling, of the surface. The importance of the ruled surface lies in the fact that it is used in many practical applications in computer aided geometric design. The ruled surface can be parameterized using the Serret-Frenet frame, however, this frame is undefined wherever the curvature vanishes, such as at points of inflection or along straight sections of the curve [8]. Thus, Bishop introduced a new frame along a space curve which is more suitable for applications [6]. But, it is well known that Bishop frame calculations are not an easy task, see [23, 24]. Therefore, various alternative methods have been proposed for computing the ruled surfaces, see [13, 14]. Klok defined the sweep surfaces using rotation-minimizing frames [17]. A robust computation of the rotation minimizing frame for sweep surfaces was introduced by Wang [23]. Inspired by the work of Coquillart [11], Mustafa introduced a new adapted frame along a space curve and denoted this the quasi-frame [20].

This paper is divided into three main parts. The first part is the introduction, which gives a brief idea about this article. The second part deals with the basic concepts of the theory of the classical differential geometry of curves in Euclidean 3-space. Also, the concepts of the quasi-frame and quasi-equations are highlighted. The main results of this paper, which deal with the study of skew ruled surfaces by using the quasi-frame of Smarandache curves are presented in third part of this paper. Moreover, some computational examples are illustrated to confirm the effectiveness of the proposed method.

2. PRELIMINARIES

Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a unit speed curve in Euclidean 3-space $\mathbb{E}^3$; by $\kappa(s)$ and $\tau(s)$ we denote the natural curvature and torsion of $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$, respectively. We assume $\boldsymbol{\alpha}''(s) \neq 0$ for all $s \in [0, L]$, since this would give us a straight line. Throughout the paper, $\boldsymbol{\alpha}'(s)$ denotes the derivative of $\boldsymbol{\alpha}$ with respect to arc length parameter s. For each point of $\boldsymbol{\alpha}(s)$, the set $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ is called the Serret-Frenet frame along $\boldsymbol{\alpha}(s)$, where $\mathbf{t}(s) = \boldsymbol{\alpha}'(s)$, $\mathbf{n}(s) = \boldsymbol{\alpha}''(s)/\left\|\boldsymbol{\alpha}''(s)\right\|$ and $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$ are the unit tangent, principal normal, and binormal vectors of the curve at the point $\boldsymbol{\alpha}(s)$, respectively. The arc-length derivative of the Serret-Frenet frame is governed by the following:

$$\begin{pmatrix} \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{b}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix}. \tag{2.1}$$

Although the Serret-Frenet frame can easily be computed, its rotation about the tangent of a general space curve often leads to undesirable twist in motion design or sweeping surface modeling. Also, the Serret-Frenet frame is not continuously defined for a C^1-continuous space curve, and even for a C^2-continuous space curve the Serret-Frenet frame becomes undefined at an inflection point (i.e., curvature $\kappa = 0$), this causing unacceptable discontinuity when used for surface modeling. In order to resolve the above problem of the Serret–Frenet frame, Coquillart and Mustafa et al. introduced a quasi-frame of a space curve as follows [11, 19, 20]:
Given a unit speed curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$, the quasi-frame (or simply Q-frame) along $\boldsymbol{\alpha}(s)$ is given by

$$\mathbf{e}_1(s) = \mathbf{t}(s), \ \mathbf{e}_2(s) = \frac{\mathbf{t} \times \boldsymbol{\zeta}}{\|\mathbf{t} \times \boldsymbol{\zeta}\|}, \ \mathbf{e}_3(s) = \mathbf{t} \times \mathbf{e}_2, \tag{2.2}$$

where $\boldsymbol{\zeta}$ is called the projection vector. The relation between Serret-Frenet frame and quasi-frame is given as follows [19]:

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix}. \tag{2.3}$$

By taking the derivative of Eq. (2.3) with respect to s and using the inverse transformation, we obtain

$$\begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \kappa_3 \\ -\kappa_2 & -\kappa_3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \tag{2.4}$$

The triple (κ_1, κ_2, κ_3) is called the Q-curvature functions of $\boldsymbol{\alpha}(s)$ and denoted by

$$\left.\begin{array}{l} \kappa_1 = \kappa\cos\varphi = <\mathbf{e}_1', \mathbf{e}_2>, \\ \kappa_2 = -\kappa\sin\varphi = <\mathbf{e}_1', \mathbf{e}_3>, \\ \kappa_3 = \tau + \varphi' = <\mathbf{e}_2', \mathbf{e}_3>. \end{array}\right\} \tag{2.5}$$

In terms of these quantities, the Q-geodesics, Q-line of curvatures, and Q-asymptotic lines on a smooth surface may be characterized as loci along which $\kappa_1 = 0$, $\kappa_3 = 0$, and $\kappa_2 = 0$, respectively. The Q-frame have many advantages over Serret-Frenet and Bishop frames. For instance, the Q-frame can be defined even along a line (i.e., curvature $\kappa = 0$). However, the Q-frame is singular in all cases where $\mathbf{t}$ and $\boldsymbol{\zeta}$ are parallel. Thus, in these cases, where $\mathbf{t}$ and $\boldsymbol{\zeta}$ are parallel, the projection vector $\boldsymbol{\zeta}$ can be chosen as $\boldsymbol{\zeta}$ = (0,1,0) or $\boldsymbol{\zeta}$ =$(0, 0, 1)$ (for details, see [19]).

Remark 1. If $\kappa_3 = 0$, then the Q-frame $\{\mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)\}$ turn out to the Bishop frame. In this case, one can show that:

$$\left.\begin{array}{l} \kappa_1 = \kappa\cos\varphi,\ \kappa_2 = \kappa\sin\varphi,\ \varphi = \tan^{-1}\left(\frac{\kappa_2}{\kappa_1}\right);\ \kappa_1 \neq 0 \\ \varphi - \varphi_0 = -\int\limits_{s_0}^{s} \tau ds. \end{array}\right\} \tag{2.6}$$

3. DIRECTIONAL RULED SURFACES

The parameterized surface

$$M : \mathbf{P}(s, v) = \boldsymbol{\alpha}(s) + v\mathbf{L}(s),\ v \in \mathbb{R}, \tag{3.1}$$

is called a ruled surface; $\boldsymbol{\alpha}(s)$ is called the base curve, and the line passing through $\boldsymbol{\alpha}(s)$ which is parallel to $\mathbf{L}(s)$ is called the ruling of the surface at $\boldsymbol{\alpha}(s)$. The surface $\mathbf{P}(s, v)$ is regular, if $\mathbf{P}_s \times \mathbf{P}_v \neq \mathbf{0}$ for all points, where $\mathbf{P}_s$ and $\mathbf{P}_v$ are the partial derivatives of $\mathbf{P}(s, v)$ with respect to s and v, respectively (see [2, 12, 15, 16, 18]).

3.1. **Directional Smarandache curves .** It is known that Smarandache geometry is a geometry which has at least one Smarandache denied axiom [1]. An axiom is said to be Smarandache denied, if it behaves in at least two different ways within the same space. Smarandache geometries are connected with the theory of relativity and the parallel universes. Smarandache curves are the objects of Smarandache geometry.

Consider a curve $\alpha = \alpha(s)$ in $\mathbb{E}^3$ and let $\{\mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)\}$ be a quasi-frame of α and κ_1, κ_2, and κ_3 the curvature functions in arc length parameter s of α. So, we have the following definition (for more details, see [3, 4, 5, 6, 7, 8, 9, 10, 13, 21, 22]).

Definition 3.1. A regular curve $\alpha(s)$ in $\mathbb{E}^3$, whose position vector is obtained by quasi-frame vectors of another regular curve $r(s)$ is called Smarandache curve.

In the following examples, we investigate some special Smarandache curves of a given curve according to its quasi-frame called $\mathbf{e}_1\mathbf{e}_2-$, $\mathbf{e}_1\mathbf{e}_3-$, $\mathbf{e}_2\mathbf{e}_3-$Smarandache curves and then obtain some of their differential geometric properties which represent the main results.

3.1.1. $\mathbf{e}_1\mathbf{e}_2$-*Smarandache curves* . Let $r = r(s)$ be a space curve in $\mathbb{E}^3$with Q- frame $\{\mathbf{e}_1(s)$, $\mathbf{e}_2(s)$, $\mathbf{e}_3(s)\}$. Then the e_1e_2- Smarandache curves of r are defined by

$$\alpha(\bar{s}) = \frac{1}{\sqrt{2}}\left(e_1 + e_2\right). \tag{3.2}$$

The curvature functions κ_α and τ_α of α can be computed as follows:

$$\begin{aligned}
\kappa_\alpha &= \frac{1}{\lambda_1}\left(\begin{array}{c} \kappa_1^2\left(2\kappa_1^2 + \left(\kappa_2+\kappa_3\right)^2\right)^2 \\ +\left(\kappa_3'\kappa_1 + \kappa_2'\kappa_1 - \kappa_1'\left(\kappa_2+\kappa_3\right) - \kappa_2\left(2\kappa_1^2 + \left(\kappa_2+\kappa_3\right)^2\right)\right)^2 \\ +\left(\kappa_3'\kappa_1 + \kappa_2'\kappa_1 - \kappa_1'\left(\kappa_2+\kappa_3\right) + \kappa_3\left(2\kappa_1^2 + \left(\kappa_2+\kappa_3\right)^2\right)\right)^2 \end{array}\right)^{\frac{1}{2}}; \\
\lambda_1 &= 2\left(\kappa_1^2 + \frac{1}{2}\left(\kappa_2+\kappa_3\right)^2\right)^{3/2}. \qquad (3.3)
\end{aligned}$$

$$\tau_\alpha = \frac{\sqrt{2}}{\lambda_2}\left(\begin{array}{c} \left(-\kappa_3'\kappa_1 - \kappa_2'\kappa_1 + \kappa_1'\left(\kappa_2+\kappa_3\right) + \kappa_2\left(2\kappa_1^2 + \left(\kappa_2+\kappa_3\right)^2\right)\right) \\ \left(\kappa_3'\kappa_2 + (3\kappa_3' + 2\kappa_2')\kappa_3 + \kappa_1\left(3\kappa_1' + \kappa_1^2 + \kappa_2^2 + \kappa_3^2\right) - \kappa_1''\right) \\ -\left(\kappa_3'\kappa_1 + \kappa_2'\kappa_1 - \kappa_1'\left(\kappa_2+\kappa_3\right) + \kappa_3\left(2\kappa_1^2 + \left(\kappa_2+\kappa_3\right)^2\right)\right) \\ \left((2\kappa_3' + 3\kappa_2')\kappa_2 + \kappa_2'\kappa_3 - \kappa_1\left(-3\kappa_1' + \kappa_1^2 + \kappa_2^2 + \kappa_3^2\right) + \kappa_1''\right) \\ -\kappa_1\left(2\kappa_1^2 + \left(\kappa_2+\kappa_3\right)^2\right) \\ \left(-\kappa_3'\kappa_1 + \kappa_2'\kappa_1 + 2\kappa_1'\left(\kappa_2-\kappa_3\right) + \left(\kappa_2+\kappa_3\right)\left(\kappa_1^2 + \kappa_2^2 + \kappa_3^2\right) - \kappa_3'' - \kappa_2''\right) \end{array}\right);$$

$$\lambda_2 = \left(\begin{array}{c} \kappa_1^2\left(2\kappa_1^2 + \left(\kappa_2+\kappa_3\right)^2\right)^2 \\ +\left(\kappa_3'\kappa_1 + \kappa_2'\kappa_1 - \kappa_1'\left(\kappa_2+\kappa_3\right) - \kappa_2\left(2\kappa_1^2 + \left(\kappa_2+\kappa_3\right)^2\right)\right)^2 \\ +\left(\kappa_3'\kappa_1 + \kappa_2'\kappa_1 - \kappa_1'\left(\kappa_2+\kappa_3\right) + \kappa_3\left(2\kappa_1^2 + \left(\kappa_2+\kappa_3\right)^2\right)\right)^2 \end{array}\right). \tag{3.4}$$

3.1.2. $\mathbf{e}_1\mathbf{e}_3$-*Smarandache curves* . Let $r = r(s)$ be a space curve in $\mathbb{E}^3$with Q- frame $\{\mathbf{e}_1(s)$, $\mathbf{e}_2(s)$, $\mathbf{e}_3(s)\}$. Then the e_1e_3- Smarandache curves of r are defined by

$$\beta(\bar{s}) = \frac{1}{\sqrt{2}}\left(e_1 + e_3\right). \tag{3.5}$$

The curvature functions κ_β and τ_β of β can be computed as follows:

$$\begin{aligned}
\kappa_\beta &= \frac{1}{\mu_1}\left(\begin{array}{c} \left((-\kappa_1' + \kappa_3')\kappa_2 + \kappa_1\left(2\kappa_2^2 + \left(\kappa_1-\kappa_3\right)^2\right) + \kappa_2'\left(\kappa_1-\kappa_3\right)\right)^2 \\ +\kappa_2^2\left(2\kappa_2^2 + \left(\kappa_1-\kappa_3\right)^2\right)^2 \\ +\left((-\kappa_1' + \kappa_3')\kappa_2 + \kappa_2'\left(\kappa_1-\kappa_3\right) + \left(2\kappa_2^2 + \left(\kappa_1-\kappa_3\right)^2\right)\kappa_3\right)^2 \end{array}\right)^{\frac{1}{2}}; \\
\mu_1 &= 2\left(\kappa_2^2 + \frac{1}{2}\left(\kappa_1-\kappa_3\right)^2\right)^{3/2}. \qquad (3.6)
\end{aligned}$$

$$\tau_\beta = \frac{\sqrt{2}}{\mu_2}\begin{pmatrix} \kappa_2\left(2\kappa_2^2+(\kappa_1-\kappa_3)^2\right) \\ \left((\kappa_1'+\kappa_3')\kappa_2+2\kappa_2'(\kappa_1+\kappa_3)+(\kappa_1-\kappa_3)(\kappa_1^2+\kappa_2^2+\kappa_3^2)-\kappa_1''+\kappa_3''\right) \\ -\left((-\kappa_1'+\kappa_3')\kappa_2+\kappa_1\left(2\kappa_2^2+(\kappa_1-\kappa_3)^2\right)+\kappa_2'(\kappa_1-\kappa_3)\right) \\ \left(-\kappa_3'(\kappa_1-3\kappa_3)-2\kappa_1'\kappa_3+\kappa_2(3\kappa_2'+\kappa_1^2+\kappa_2^2+\kappa_3^2)-\kappa_2''\right) \\ -\left((-\kappa_1'+\kappa_3')\kappa_2+\kappa_2'(\kappa_1-\kappa_3)+\left(2\kappa_2^2+(\kappa_1-\kappa_3)^2\right)\kappa_3\right) \\ \left(-2\kappa_3'\kappa_1+\kappa_1'(3\kappa_1-\kappa_3)-\kappa_2(-3\kappa_2'+\kappa_1^2+\kappa_2^2+\kappa_3^2)+\kappa_2''\right) \end{pmatrix};$$

(3.7)
$$\mu_2 = \begin{pmatrix} \left((-\kappa_1'+\kappa_3')\kappa_2+\kappa_1\left(2\kappa_2^2+(\kappa_1-\kappa_3)^2\right)+\kappa_2'(\kappa_1-\kappa_3)\right)^2 \\ +\kappa_2^2\left(2\kappa_2^2+(\kappa_1-\kappa_3)^2\right)^2 \\ +\left((-\kappa_1'+\kappa_3')\kappa_2+\kappa_2'(\kappa_1-\kappa_3)+\left(2\kappa_2^2+(\kappa_1-\kappa_3)^2\right)\kappa_3\right)^2 \end{pmatrix}.$$

3.1.3. $\mathbf{e}_2\mathbf{e}_3$-*Smarandache curves* . Let $r=r(s)$ be a space curve in $\mathbb{E}^3$ with Q- frame $\{\mathbf{e}_1(s)$, $\mathbf{e}_2(s)$, $\mathbf{e}_3(s)\}$. Then the e_2e_3- Smarandache curves of r are defined by

(3.8)
$$\gamma(\bar{s}) = \frac{1}{\sqrt{2}}(e_2+e_3).$$

The curvature functions κ_γ and τ_γ of γ can be computed as follows:

$$\kappa_\gamma = \frac{1}{\nu_1}\begin{pmatrix} \left((\kappa_1+\kappa_2)^2\kappa_3+2{\kappa_3}^3\right)^2 \\ +\left(\kappa_3'(\kappa_1+\kappa_2)-(\kappa_1'+\kappa_2')\kappa_3+\kappa_1\left((\kappa_1+\kappa_2)^2+2\kappa_3^2\right)\right)^2 \\ +\left(-\kappa_3'(\kappa_1+\kappa_2)+(\kappa_1'+\kappa_2')\kappa_3+\kappa_2\left((\kappa_1+\kappa_2)^2+2\kappa_3^2\right)\right)^2 \end{pmatrix}^{\frac{1}{2}};$$

(3.9)
$$\nu_1 = 2\left(\frac{1}{2}(\kappa_1+\kappa_2)^2+\kappa_3^2\right)^{3/2}.$$

$$\tau_\gamma = \frac{\sqrt{2}}{\nu_2}\begin{pmatrix} -\left(\kappa_3'(\kappa_1+\kappa_2)-(\kappa_1'+\kappa_2')\kappa_3+\kappa_1\left((\kappa_1+\kappa_2)^2+2\kappa_3^2\right)\right) \\ \left(2\kappa_1'\kappa_2+\kappa_2'(\kappa_1+3\kappa_2)+\kappa_3(3\kappa_3'+\kappa_1^2+\kappa_2^2+\kappa_3^2)-\kappa_3''\right) \\ +\left(-\kappa_3'(\kappa_1+\kappa_2)+(\kappa_1'+\kappa_2')\kappa_3+\kappa_2\left((\kappa_1+\kappa_2)^2+2\kappa_3^2\right)\right) \\ \left(2\kappa_2'\kappa_1+\kappa_1'(3\kappa_1+\kappa_2)-\kappa_3(-3\kappa_3'+\kappa_1^2+\kappa_2^2+\kappa_3^2)+\kappa_3''\right) \\ +\kappa_3\left((\kappa_1+\kappa_2)^2+2\kappa_3^2\right) \\ \left(2\kappa_3'(\kappa_1-\kappa_2)+(\kappa_1'-\kappa_2')\kappa_3+(\kappa_1+\kappa_2)(\kappa_1^2+\kappa_2^2+\kappa_3^2)-\kappa_1''-\kappa_2''\right) \end{pmatrix};$$

(3.10)
$$\nu_2 = \begin{pmatrix} \left((\kappa_1+\kappa_2)^2\kappa_3+2{\kappa_3}^3\right)^2 \\ +\left(\kappa_3'(\kappa_1+\kappa_2)-(\kappa_1'+\kappa_2')\kappa_3+\kappa_1\left((\kappa_1+\kappa_2)^2+2\kappa_3^2\right)\right)^2 \\ +\left(-\kappa_3'(\kappa_1+\kappa_2)+(\kappa_1'+\kappa_2')\kappa_3+\kappa_2\left((\kappa_1+\kappa_2)^2+2\kappa_3^2\right)\right)^2 \end{pmatrix}.$$

3.1.4. $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$-*Smarandache curves* . Let $r=r(s)$ be a space curve in $\mathbb{E}^3$ with Q-frame $\{\mathbf{e}_1(s)$, $\mathbf{e}_2(s)$, $\mathbf{e}_3(s)\}$. Then the $e_1e_2e_3-$ Smarandache curves of r are defined by

(3.11)
$$\delta(\bar{s}) = \frac{1}{\sqrt{3}}(e_1+e_2+e_3).$$

The curvature functions κ_δ and τ_δ of δ can be computed as follows:

$$\kappa_\delta = \frac{\sqrt{3}}{\sqrt{2}\xi_1}\left(\begin{array}{c}\left(\begin{array}{c}-\kappa_3'(\kappa_1+\kappa_2)+\kappa_2'(-\kappa_1+\kappa_3)+\kappa_1'\kappa_2+\kappa_3\\+2\kappa_2\left(\kappa_1^2+\kappa_2^2+\kappa_1(\kappa_2-\kappa_3)+\kappa_2\kappa_3+\kappa_3^2\right)\end{array}\right)^2\\+\left(\begin{array}{c}\kappa_3'(\kappa_1+\kappa_2)+\kappa_2'(\kappa_1-\kappa_3)-\kappa_1'(\kappa_2+\kappa_3)\\+2\kappa_3\left(\kappa_1^2+\kappa_2^2+\kappa_1(\kappa_2-\kappa_3)+\kappa_2\kappa_3+\kappa_3^2\right)\end{array}\right)^2\\+\left(\begin{array}{c}\kappa_3'(\kappa_1+\kappa_2)+\kappa_2'(\kappa_1-\kappa_3)-\kappa_1'(\kappa_2+\kappa_3)\\+2\kappa_1\left(\kappa_1^2+\kappa_1\kappa_2+\kappa_2^2+(-\kappa_1+\kappa_2)\kappa_3+\kappa_3^2\right)\end{array}\right)^2\end{array}\right)^{\frac{1}{2}};$$

$$\xi_1 = \left(2\left(\kappa_1^2+\kappa_2^2+\kappa_1(\kappa_2-\kappa_3)+\kappa_2\kappa_3+\kappa_3^2\right)^{3/2}\right). \tag{3.12}$$

$$\tau_\delta = \frac{\sqrt{3}}{\xi_2}\left(\begin{array}{c}-\left(\begin{array}{c}\kappa_3'(\kappa_1+\kappa_2)+\kappa_2'(\kappa_1-\kappa_3)-\kappa_1'(\kappa_2+\kappa_3)\\-2\kappa_2\left(\kappa_1^2+\kappa_2^2+\kappa_1(\kappa_2-\kappa_3)+\kappa_2\kappa_3+\kappa_3^2\right)\end{array}\right)\\\left(\begin{array}{c}\kappa_1'(3\kappa_1+\kappa_2)+2\kappa_2'(\kappa_1+\kappa_3)+\kappa_3'(\kappa_2+3\kappa_3)\\+(\kappa_1-\kappa_3)\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)-\kappa_1''+\kappa_3''\end{array}\right)\\+\left(\begin{array}{c}\kappa_3'(\kappa_1+\kappa_2)+\kappa_2'(\kappa_1-\kappa_3)-\kappa_1'(\kappa_2+\kappa_3)\\+2\kappa_3\left(\kappa_1^2+\kappa_2^2+\kappa_1(\kappa_2-\kappa_3)+\kappa_2\kappa_3+\kappa_3^2\right)\end{array}\right)\\\left(\begin{array}{c}2\kappa_3'(\kappa_1-\kappa_2)+\kappa_1'(-3\kappa_1+\kappa_3)-\kappa_2'(3\kappa_2+\kappa_3)\\+(\kappa_1+\kappa_2)\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)-\kappa_1''-\kappa_2''\end{array}\right)\\-\left(\begin{array}{c}\kappa_3'(\kappa_1+\kappa_2)+\kappa_2'(\kappa_1-\kappa_3)-\kappa_1'(\kappa_2+\kappa_3)\\+2\kappa_1\left(\kappa_1^2+\kappa_1\kappa_2+\kappa_2^2+(-\kappa_1+\kappa_2)\kappa_3+\kappa_3^2\right)\end{array}\right)\\\left(\begin{array}{c}\kappa_2'(\kappa_1+3\kappa_2)-\kappa_3'(\kappa_1-3\kappa_3)+2\kappa_1'(\kappa_2-\kappa_3)\\+(\kappa_2+\kappa_3)\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)-\kappa_3''-\kappa_2''\end{array}\right)\end{array}\right);$$

$$\xi_2 = \left(\begin{array}{c}\left(\begin{array}{c}-\kappa_3'(\kappa_1+\kappa_2)+\kappa_2'(-\kappa_1+\kappa_3)+\kappa_1'(\kappa_2+\kappa_3)\\+2\kappa_2\left(\kappa_1^2+\kappa_2^2+\kappa_1(\kappa_2-\kappa_3)+\kappa_2\kappa_3+\kappa_3^2\right)\end{array}\right)^2\\+\left(\begin{array}{c}\kappa_3'(\kappa_1+\kappa_2)+\kappa_2'(\kappa_1-\kappa_3)-\kappa_1'(\kappa_2+\kappa_3)\\+2\kappa_3\left(\kappa_1^2+\kappa_2^2+\kappa_1(\kappa_2-\kappa_3)+\kappa_2\kappa_3+\kappa_3^2\right)\end{array}\right)^2\\+\left(\begin{array}{c}\kappa_3'(\kappa_1+\kappa_2)+\kappa_2'(\kappa_1-\kappa_3)-\kappa_1'(\kappa_2+\kappa_3)\\+2\kappa_1\left(\kappa_1^2+\kappa_1\kappa_2+\kappa_2^2+(-\kappa_1+\kappa_2)\kappa_3+\kappa_3^2\right)\end{array}\right)^2\end{array}\right). \tag{3.13}$$

3.2. **Examples .**

Example 3.1. *Assume that the curve $r(s)$ is given by*

$$r = (\sin(e^s), \sin(e^s), e^s).$$

It is verify that the Serret–Frenet curvature and torsion of this curve are obtained by

$$\kappa = \frac{\sqrt{2}\sin(e^s)}{(2+\cos(2e^s))^{3/2}} \ \ and \ \ \tau = 0,$$

respectively.
Hence $\tau = 0$; the angle between the rotation minimizing frame (Bishop frame) and the Serret-Frenet frame is constant, therefore the Bishop frame is also not suitable for this example. The quasi-frame can be calculated by

$$\mathbf{e}_1 = \left(-\frac{1}{\sqrt{1+\cos^2(e^s)}}, 0, \frac{\cos(e^s)}{\sqrt{1+\cos^2(e^s)}}\right),$$

$$\mathbf{e}_2 = \left(-\frac{1}{\sqrt{1+\cos^2(e^s)}}, 0, \frac{\cos(e^s)}{\sqrt{1+\cos^2(e^s)}}\right),$$

$$\mathbf{e}_3 = \left(\frac{\cos^2(e^s)}{\sqrt{1+\cos^2(e^s)}\sqrt{2+\cos(2e^s)}}, -\frac{1}{\sqrt{2-\frac{2}{3+\cos(2e^s)}}}, \frac{\cos(e^s)}{\sqrt{1+\cos^2(e^s)}\sqrt{2+\cos(2e^s)}} \right).$$

Also, the quasi-curvature functions are obtained as follows

$$\kappa_1 = \frac{e^s(\sin(e^s)+\cos(e^s)\sin(2e^s))}{\sqrt{1+\cos^2(e^s)}\,(2+\cos(2e^s))^{3/2}},$$

$$\begin{aligned}\kappa_2 \;=\;& -\frac{e^s\cos^2(e^s)\sin(e^s)}{\sqrt{1+\cos^2(e^s)}\,(2+\cos(2e^s))^2} + \frac{e^s\sin(e^s)}{(2+\cos(2e^s))^{3/2}\sqrt{2-\frac{2}{3+\cos(2e^s)}}} \\ & + \frac{e^s\cos(e^s)\sin(2e^s)}{\sqrt{1+\cos^2(e^s)}\,(2+\cos(2e^s))^2},\end{aligned}$$

$$\kappa_3 = -\frac{e^s\sin(2e^s)}{\sqrt{2+\cos(2e^s)}\,(3+\cos(2e^s))}.$$

Now, we can construct the following Smarandache curves according to quasi-frame vectors as follows:

1. $\mathbf{e}_1\mathbf{e}_2-$Smarandache curve

The $\mathbf{e}_1\mathbf{e}_2-$Smarandache curve α is given by

$$\alpha = \frac{\mathbf{e}_1+\mathbf{e}_2}{\sqrt{2}},$$

and can be calculated as follows

$$\alpha = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{1+\cos^2(e^s)}} + \frac{\cos(e^s)}{\sqrt{2+\cos(2e^s)}}\right) \\ \frac{\cos(e^s)}{\sqrt{2}\sqrt{2+\cos(2e^s)}} \\ \frac{1}{\sqrt{2}}\left(\frac{\cos(e^s)}{\sqrt{1+\cos^2(e^s)}} + \frac{1}{\sqrt{2+\cos(2e^s)}}\right) \end{pmatrix}.$$

Using Eqs. (3.3) and (3.4), the curvatures of α are computed as follows

$$\begin{aligned}\kappa \;=\;& \frac{\sqrt{\dfrac{\begin{array}{c} e^{6s}(12+9\cos(2e^s)+\cos(4e^s)) \\ \epsilon_1^2\sin^6(e^s)\end{array}}{(6+5\cos(2e^s)+\cos^2(2e^s))^6}}}{4\left(\frac{e^{2s}\left(37+28\cos(2e^s)-4\sqrt{2}\cos(e^s)\sqrt{2+\cos(2e^s)}\sqrt{3+\cos(2e^s)}+3\cos(4e^s)\right)\sin^2(e^s)}{(13+10\cos(2e^s)+\cos(4e^s))^2}\right)^{3/2}}; \\ \epsilon_1 \;=\;& \begin{pmatrix} 37+28\cos(2e^s)-4\sqrt{2}\cos(e^s)\sqrt{2+\cos(2e^s)}\sqrt{3+\cos(2e^s)} \\ +3\cos(4e^s)\end{pmatrix},\end{aligned}$$

$$\tau = 0.$$

The ruled surface generated by the base curve r and generator α is given by

$$M_1 : \mathbf{\Psi}_1(s,t) = \beta + t\,\alpha; t \in \mathbb{R},$$

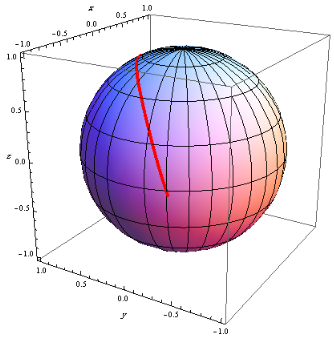
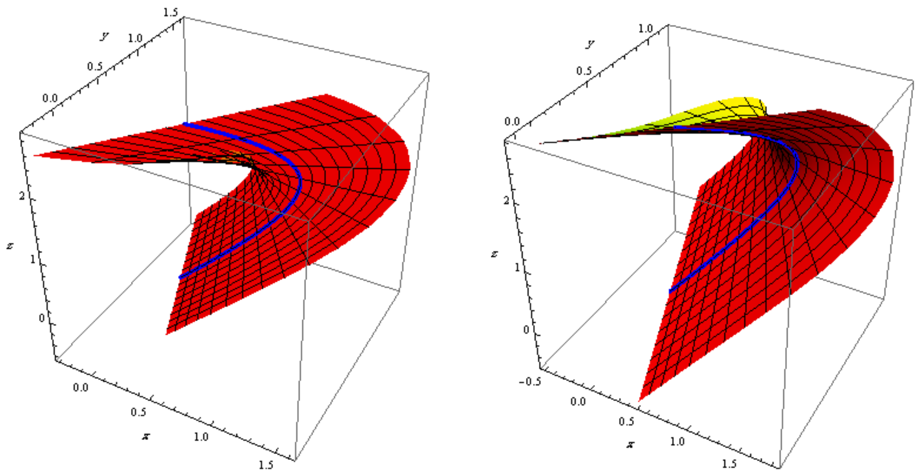


Figure 1: The quasi-spherical curve α.

which is calculated as follows

$$\mathbf{\Psi}_1(s,t) = \begin{pmatrix} \frac{t\cos(e^s)}{\sqrt{2}\sqrt{2+\cos(2e^s)}} - \frac{t}{\sqrt{3+\cos(2e^s)}} + \sin\left(e^s\right) \\ \frac{t\cos(e^s)}{\sqrt{2}\sqrt{2+\cos(2e^s)}} + \sin\left(e^s\right) \\ e^s + \frac{t}{\sqrt{2}\sqrt{2+\cos(2e^s)}} + \frac{t\cos(e^s)}{\sqrt{3+\cos(2e^s)}} \end{pmatrix}.$$

The ruled surfaces according to Serret-Frenet and quasi frames are shown in Figures 2a and 2b, respectively.



(A) (B)

Figure 2: (A) $\mathbf{\Psi}_1$ with Frenet vectors. (B) $\mathbf{\Psi}_1$ with quasi-normal vectors.

2. $\mathbf{e}_1\mathbf{e}_3-$Smarandache curves
The $\mathbf{e}_1\mathbf{e}_3-$Smarandache curve β is given by

$$\beta = \frac{\mathbf{e}_1 + \mathbf{e}_3}{\sqrt{2}},$$

and can be calculated as follows

$$\beta = \begin{pmatrix} \frac{\cos(e^s)\left(\cos(e^s)+\sqrt{1+\cos^2(e^s)}\right)}{\sqrt{2+\cos(2e^s)}\sqrt{3+\cos(2e^s)}} \\ \frac{1}{\sqrt{2}}\left(\frac{\cos(e^s)}{\sqrt{2+\cos(2e^s)}} - \frac{1}{\sqrt{2-\frac{2}{3+\cos(2e^s)}}}\right) \\ \frac{\cos(e^s)+\sqrt{1+\cos^2(e^s)}}{\sqrt{2+\cos(2e^s)}\sqrt{3+\cos(2e^s)}} \end{pmatrix}.$$

From Eqs. (3.6) and (3.7), the curvatures of β are computed as

$$\begin{aligned} \kappa &= \frac{\sqrt{\frac{e^{6s}\epsilon_2^2}{\frac{(81+79\cos(2e^s)+15\cos(4e^s)+\cos(6e^s))\sin^6(e^s)}{(2+\cos(2e^s))^6(3+\cos(2e^s))^7}}}}{\left(\frac{e^{2s}\left(15+10\cos(2e^s)+\sqrt{2}\sqrt{3+\cos(2e^s)}(5\cos(e^s)+\cos(3e^s))+\cos(4e^s)\right)\sin^2(e^s)}{(6+5\cos(2e^s)+\cos^2(2e^s))^2}\right)^{3/2}}; \\ \epsilon_2 &= \left(15+10\cos(2e^s)+\sqrt{2}\sqrt{3+\cos(2e^s)}\,(5\cos(e^s)+\cos(3e^s))+\cos(4e^s)\right), \end{aligned}$$

$$\tau = 0.$$

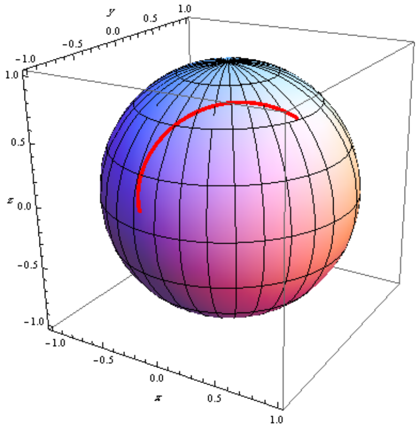


Figure 3: The quasi-spherical curve β.

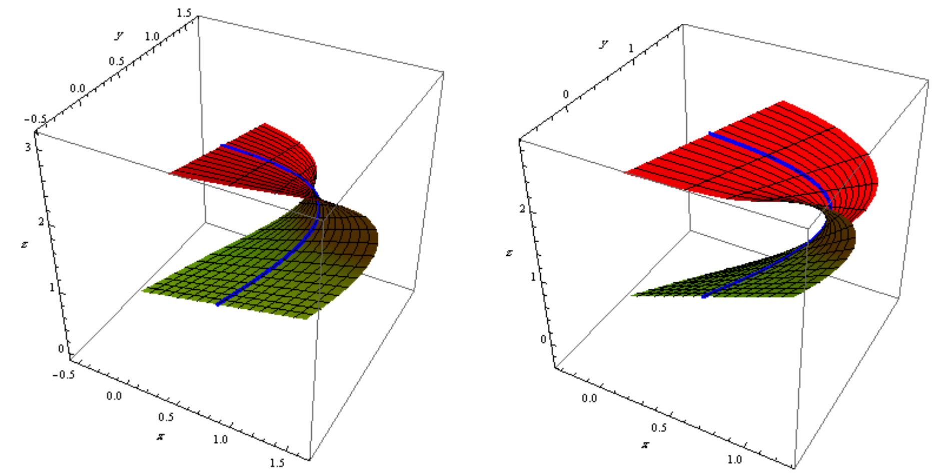
The ruled surface generated by the base curve r and generator β is given by

$$M_2 : \mathbf{\Psi}_2(s,t) = r + t\,\beta; t \in \mathbb{R},$$

which is calculated as follows

$$\mathbf{\Psi}_2(s,t) = \begin{pmatrix} \frac{t\cos(e^s)\left(\cos(e^s)+\sqrt{1+\cos^2(e^s)}\right)}{\sqrt{2+\cos(2e^s)}\sqrt{3+\cos(2e^s)}} + \sin(e^s) \\ \frac{t\left(\sqrt{2}\cos(e^s)-\sqrt{3+\cos(2e^s)}\right)}{2\sqrt{2+\cos(2e^s)}} + \sin(e^s) \\ e^s + \frac{t\left(\cos(e^s)+\sqrt{1+\cos^2(e^s)}\right)}{\sqrt{2+\cos(2e^s)}\sqrt{3+\cos(2e^s)}} \end{pmatrix}.$$

The ruled surfaces according to Serret–Frenet and quasi frames are shown in Figures 4a and 4b, respectively.



(A) (B)

Figure 4: (A) $\mathbf{\Psi}_2$ with Frenet vectors. (B) $\mathbf{\Psi}_2$ with quasi-normal vectors.

3. $\mathbf{e}_2\mathbf{e}_3-$Smarandache curves

The $\mathbf{e}_2\mathbf{e}_3-$Smarandache curve γ is given by

$$\gamma = \frac{\mathbf{e}_2 + \mathbf{e}_3}{\sqrt{2}},$$

and can be calculated as follows

$$\gamma = \begin{pmatrix} \frac{\cos^2(e^s)-\sqrt{2+\cos(2e^s)}}{\sqrt{2+\cos(2e^s)}\sqrt{3+\cos(2e^s)}} \\ -\frac{1}{2\sqrt{\frac{2+\cos(2e^s)}{3+\cos(2e^s)}}} \\ \frac{\cos(e^s)\left(1+\sqrt{2+\cos(2e^s)}\right)}{\sqrt{2+\cos(2e^s)}\sqrt{3+\cos(2e^s)}} \end{pmatrix}.$$

According to Eqs. (3.9) and (3.10), the curvatures of γ are computed as

$$\begin{aligned} \kappa &= \frac{\sqrt{2}\sqrt{\frac{e^{6s}(12+9\cos(2e^s)+\cos(4e^s))\ \epsilon_3 \sin^6(e^s)}{(13+10\cos(2e^s)+\cos(4e^s))^6}}}{\left(\frac{e^{2s}\left(29+12\sqrt{2+\cos(2e^s)}+4\cos(2e^s)\left(6+\sqrt{2+\cos(2e^s)}\right)+3\cos(4e^s)\right)\sin^2(e^s)}{(13+10\cos(2e^s)+\cos(4e^s))^2}\right)^{3/2}}; \\ \epsilon_3 &= \begin{pmatrix} 2971+1584\sqrt{2+\cos(2e^s)}+8\cos(2e^s) \\ \left(453+205\sqrt{2+\cos(2e^s)}\right) \\ +4\left(263+84\sqrt{2+\cos(2e^s)}\right)\cos(4e^s) \\ +8\left(19+3\sqrt{2+\cos(2e^s)}\right)\cos(6e^s)+9\cos(8e^s) \end{pmatrix}, \end{aligned}$$

$$\tau = 0.$$

The ruled surface generated by the base curve r and generator γ is given by

$$M_3 : \mathbf{\Psi}_3(s,t) = r + t\,\gamma; t \in \mathbb{R},$$

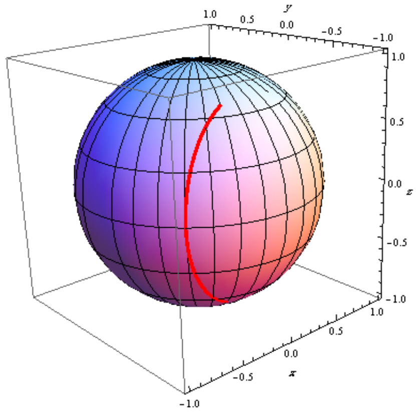
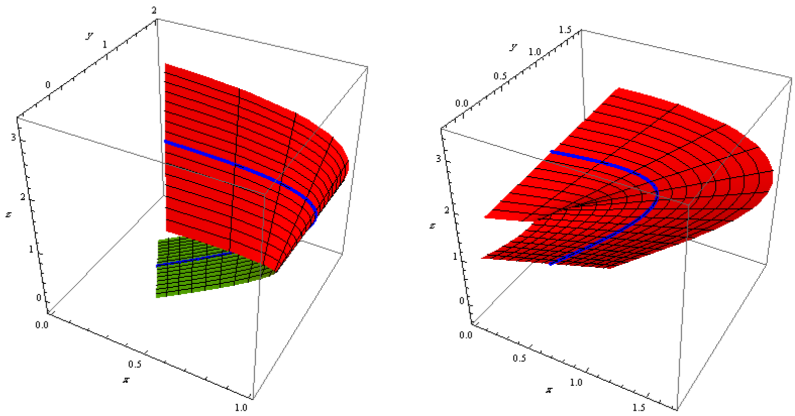


Figure 5: The quasi-spherical curve γ.

which is calculated as follows

$$\mathbf{\Psi}_3(s,t) = \begin{pmatrix} \frac{t\left(\cos^2(e^s)-\sqrt{2+\cos(2e^s)}\right)}{\sqrt{2+\cos(2e^s)}\sqrt{3+\cos(2e^s)}} + \sin\left(e^s\right) \\ -\frac{t}{2\sqrt{1-\frac{1}{3+\cos(2e^s)}}} + \sin\left(e^s\right) \\ e^s + \frac{t\cos(e^s)\left(1+\sqrt{2+\cos(2e^s)}\right)}{\sqrt{2+\cos(2e^s)}\sqrt{3+\cos(2e^s)}} \end{pmatrix}.$$

The ruled surfaces according to Serret-Frenet and quasi frames are shown in Figures 6a and 6b, respectively.



(A) (B)

Figure 6: (A) $\mathbf{\Psi}_3$ with Frenet vectors. (B) $\mathbf{\Psi}_3$ with quasi-normal vectors.

4. $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3-$ Smarandache curves

The $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3-$Smarandache curve δ is given by

$$\delta = \frac{\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3}{\sqrt{3}},$$

and can be calculated as follows

$$\delta=\begin{pmatrix}\frac{\cos^2(e^s)+\cos(e^s)\sqrt{1+\cos^2(e^s)}-\sqrt{2+\cos(2e^s)}}{\sqrt{3}\sqrt{1+\cos^2(e^s)}\sqrt{2+\cos(2e^s)}}\\ \frac{1}{\sqrt{3}}\left(\frac{\cos(e^s)}{\sqrt{2+\cos(2e^s)}}-\frac{1}{\sqrt{2-\frac{2}{3+\cos(2e^s)}}}\right)\\ \frac{\sqrt{1+\cos^2(e^s)}+\cos(e^s)\left(1+\sqrt{2+\cos(2e^s)}\right)}{\sqrt{3}\sqrt{1+\cos^2(e^s)}\sqrt{2+\cos(2e^s)}}\end{pmatrix}.$$

In the light of Eqs. (3.12) and (3.13) The curvature and torsion of δ are computed as

$$\begin{aligned}\kappa &= \frac{\sqrt{\frac{3}{2}}\sqrt{\frac{e^{6s}(12+9\cos(2e^s)+\cos(4e^s))\ \epsilon_4^2\sin^6(e^s)}{(6+5\cos(2e^s)+\cos^2(2e^s))^6}}}{\left(\frac{e^{2s}\ \epsilon_5\sin^2(e^s)}{(6+5\cos(2e^s)+\cos^2(2e^s))^2}\right)^{3/2}};\\ \epsilon_4 &= \begin{pmatrix}12+3\sqrt{2+\cos\left(2e^s\right)}+\sqrt{2}\cos\left(e^s\right)\sqrt{3+\cos\left(2e^s\right)}\\ \left(2+\cos\left(2e^s\right)-\sqrt{2+\cos\left(2e^s\right)}\right)\\ +\cos\left(2e^s\right)\left(9+\sqrt{2+\cos\left(2e^s\right)}\right)+\cos\left(4e^s\right)\end{pmatrix},\\ \epsilon_5 &= \begin{pmatrix}12+3\sqrt{2+\cos\left(2e^s\right)}\\ +\sqrt{2}\cos\left(e^s\right)\sqrt{3+\cos\left(2e^s\right)}\left(2+\cos\left(2e^s\right)-\sqrt{2+\cos\left(2e^s\right)}\right)\\ +\cos\left(2e^s\right)\left(9+\sqrt{2+\cos\left(2e^s\right)}\right)+\cos\left(4e^s\right)\end{pmatrix},\end{aligned}$$

$$\tau=0.$$

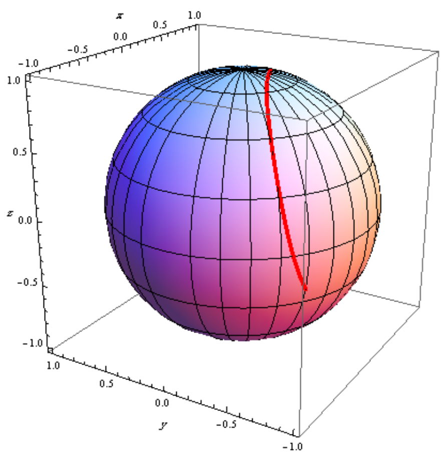


Figure 7: The quasi-spherical curve δ.

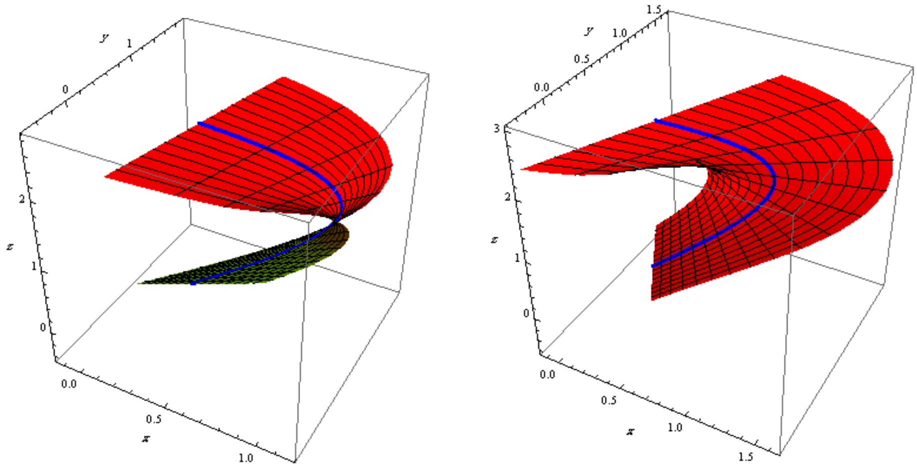
The ruled surface generated by the base curve r and generator δ is given by

$$M_4:\mathbf{\Psi}_4(s,t)=r+t\,\delta;t\in\mathbb{R},$$

which is calculated as follows

$$\mathbf{\Psi}_4(s,t)=\begin{pmatrix}\frac{t\left(\cos(e^s)\left(\cos(e^s)+\sqrt{1+\cos^2(e^s)}\right)-\sqrt{2+\cos(2e^s)}\right)}{\sqrt{3}\sqrt{1+\cos^2(e^s)}\sqrt{2+\cos(2e^s)}}+\sin\left(e^s\right)\\ \frac{t}{\sqrt{3}}\left(\frac{\cos(e^s)}{\sqrt{2+\cos(2e^s)}}-\frac{1}{\sqrt{2-\frac{2}{3+\cos(2e^s)}}}\right)+\sin\left(e^s\right)\\ e^s+\frac{t\left(\sqrt{1+\cos^2(e^s)}+\cos(e^s)\left(1+\sqrt{2+\cos(2e^s)}\right)\right)}{\sqrt{3}\sqrt{1+\cos^2(e^s)}\sqrt{2+\cos(2e^s)}}\end{pmatrix}.$$

The ruled surfaces according to Serret-Frenet and quasi frames are shown in Figures 8a and 8b, respectively.



(A) (B)

Figure 8: (A) $\mathbf{\Psi}_4$ with Frenet vectors. (B) $\mathbf{\Psi}_4$ with quasi-normal vectors.

Since the Serret-Frenet frame vectors (normal and binormal vectors) are not defined at points where the curvature of the curve is zero. Hence, the analytical expression of the ruled surface around a straight line is not obtainable. Therefore, in the following example, we obtain the analytical expression of the ruled surface generated by a line with a quasi-frame vectors.

Example 3.2. *Consider the curve $r(s)$, given by*

$$r=\left(e^s,e^s,0\right).$$

It is easy to see that this curve is a straight line and Serret-Frenet curvature vanishes. The quasi-frame vectors can be calculated by

$$\mathbf{e}_1=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right),$$

$$\mathbf{e}_2=(0,0,1),$$

$$\mathbf{e}_3=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right).$$

Smarandache curves of r are constructed as follows

$$\alpha=\left(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}\right),\ \beta=(1,0,0),\ \ \gamma=\left(\frac{1}{2},-\frac{1}{2},\frac{1}{\sqrt{2}}\right)$$

The ruled surfaces generated by the line with the quasi-frame vectors are, respectively given by

$$\begin{aligned}
\mathbf{\Psi}_1(s,t) &= \left(e^s+\frac{t}{2}, e^s+\frac{t}{2}, \frac{t}{\sqrt{2}}\right), \\
\mathbf{\Psi}_2(s,t) &= \left(e^s+t, e^s, 0\right), \\
\mathbf{\Psi}_3(s,t) &= \left(e^s+\frac{t}{2}, e^s-\frac{t}{2}, \frac{t}{\sqrt{2}}\right).
\end{aligned}$$

These ruled surfaces are shown in Figure 9.

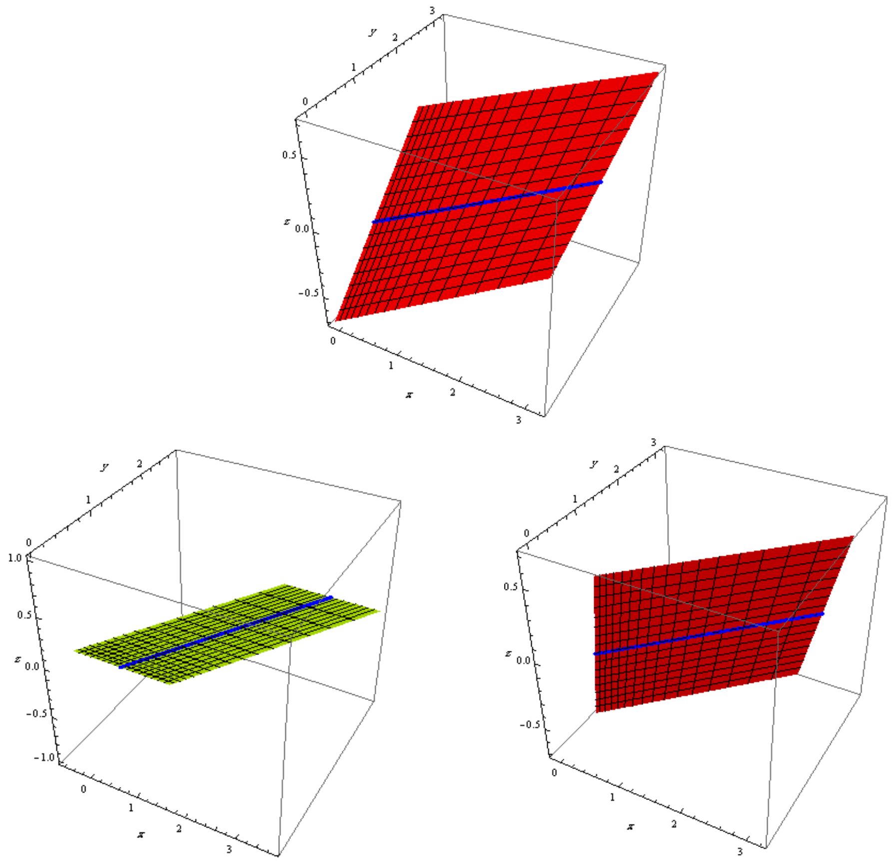


Figure 9: Ruled surfaces generated by straight lines with quasi-frame vectors.

4. Conclusion

In this study, ruled surfaces constructed by Smarandache curves according to quasi-frame are discussed. Also, some special Smarandache curves and their differential geometric properties are investigated. Finally, computational examples to confirm our main results are given and plotted. All calculations and figures in this paper have been done by using Wolfram Mathematica 7.0.

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