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ON THE POLYCONVOLUTION OF HARTLEY INTEGRAL TRANSFORMS H_1, H_2, H_1 AND INTEGRAL EQUATIONS

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ABSTRACT. In this paper, we construct and study a new polyconvolution $*\,(f,g,h)\,(x)$ of functions $f,\ g,\ h$. We will show that the polyconvolution satisfy the following factorization equality

$$H_1\left[*\,(f,g,h)\right](y) = (H_2 f)(y)\,(H_1 g)(y)\,(H_1 h)(y), \forall y \in \mathbb{R}.$$

We prove the existence of this polyconvolution in the space $L(\mathbb{R})$. As examples, applications to solve an integral equation of polyconvolution type and two systems of integral equations of polyconvolution type are presented.

Key words and phrases: Integral equation; Convolution; Polyconvolution; Hartley transforms.

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1. INTRODUCTION

The concept of polyconvolution was first proposed by Kakichev in 1997 ([5]). Accoding to this definition, the polyconvolution for $n+1$ arbitrary integral transforms $K, K_1, K_2, ..., K_n$ with the weight function $\gamma(x)$ of functions $f_1, f_2, ..., f_n$ for which the factorization property holds.

$$K[*\left(f_1, f_2, ..., f_n\right)](y) = \left(K_1 f_1\right)(y)\left(K_2 f_2\right)(y) ... \left(K_n f_n\right)(y)$$

In this paper, we construct and study a new polyconvolution for Hartley integral transforms. We note that from the above factorization equality, the general definition of polyconvolution has the form

$$*\left(f_1, f_2, ..., f_n\right)(x) = K^{-1}\left[\left(K_1 f_1\right)(.)\left(K_2 f_2\right)(.) ... \left(K_n f_n\right)(.)\right](x)$$

With K^{-1} being the inverse operator of K.
Although it looks quite simple, it is not easy to have an explicit form of polyconvolution when applied to concrete integral transforms.

In this paper, we also apply the new polyconvolution to solve a class of integral equations and system of integral equations. We note that for such integral equation and system of integral equations, a representation of their solution in a closed form is an interesting and open problem ([4], [8]).

In this section, we recall some known convolutions, generalized convolutions.
The Hartley integral transform$H_1,\ H_2$ was introduced in [3]

$$(H_{\binom{1}{2}} f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \operatorname{cas}(\pm xy) dy, x \in \mathbb{R}.$$

Here $\operatorname{cas}(\pm x) = \cos x \pm \sin x$. The convolution for the Hartley integral transform H_1([6], [7])

$$\left(f \underset{H_1}{*} g\right)(x) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\left[g(x+u) + g(x-u) + g(u-x) - g(-x-u)\right] du, \tag{1.1}$$

satisfies the factorization property

$$H_1\left(f \underset{H_1}{*} g\right)(y) = \left(H_1 f\right)(y)\left(H_1 g\right)(y). \tag{1.2}$$

The generalized convolution for the Hartley integral transform ([2])

$$\left(f \underset{H_1, H_1, H_2}{*} g\right)(x) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[f(x+y) + f(x-y) - f(-x+y) + f(-x-y)\right] g(y) dy, \tag{1.3}$$

satisfies the factorization property

$$H_1\left(f \underset{H_1, H_1, H_2}{*} g\right)(y) = \left(H_1 f\right)(y)\left(H_2 g\right)(y). \tag{1.4}$$

2. POLYCONVOLUTION OF HARTLEY INTEGRAL TRANSFORMS $H_1,\ H_2,\ H_1$

Definition 2.1. *The polyconvolution for the Hartley integral transforms of the functions f, g and h is defined by*

$$(2.1)\qquad \begin{aligned} *\left(f,g,h\right)(x) =& \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[f\left(x-v-w\right)-f\left(-x-v+w\right)\right.\\ &\left.-f\left(-x+v-w\right)-f\left(x+v+w\right)\right]g\left(v\right)h\left(w\right)dvdw, x\in\mathbb{R}.\end{aligned}$$

Theorem 2.1. *Let f, g and h be functions in $L(\mathbb{R})$, then the polyconvolution (2.1) for the Hartley integral transforms H_1, H_2, H_1 of the functions f, g and h belongs to $L(\mathbb{R})$ and the factorization property holds*

$$(2.2)\qquad H_1\left[*\left(f,g,h\right)\right](y) = \left(H_2 f\right)(y)\left(H_1 g\right)(y)\left(H_1 h\right)(y), \forall y\in\mathbb{R}.$$

Proof. First, we prove that $*(f,g,h)(x)\in L(\mathbb{R})$. Indeed,

$$\begin{aligned}\int_{-\infty}^{\infty}\left|*\left(f,g,h\right)(x)\right|dx \leq& \frac{1}{4\pi}\int_{-\infty}^{\infty}\left|g\left(v\right)\right|dv\int_{-\infty}^{\infty}\left|h\left(w\right)\right|dv\int_{-\infty}^{\infty}\left[\left|f\left(x-v-w\right)\right|\right.\\ &\left.+\left|f\left(-x-v+w\right)\right|+\left|f\left(-x-v+w\right)\right|+\left|f\left(x+v+w\right)\right|\right]dx\end{aligned}$$

It is easy to see that

$$\begin{aligned}&\int_{-\infty}^{\infty}\left[\left|f\left(x-v-w\right)\right|+\left|f\left(-x-v+w\right)\right|+\left|f\left(-x+v-w\right)\right|\right.\\ &\left.+\left|f\left(x+v+w\right)\right|\right]dx = 4\int_{-\infty}^{\infty}\left|f\left(t\right)\right|dt.\end{aligned}$$

Hence,

$$\int_{-\infty}^{\infty}\left|*\left(f,g,h\right)(x)\right|dx \leq \frac{1}{\pi}\int_{-\infty}^{\infty}\left|g\left(v\right)\right|dv\int_{-\infty}^{\infty}\left|h\left(w\right)\right|dw\int_{-\infty}^{\infty}\left|f\left(t\right)\right|dt < +\infty.$$

So $*(f,g,h)(x)$ belongs to $L(\mathbb{R})$.

Now we prove the factorization property (2.2).

Since

$$\begin{aligned}&\left(H_2 f\right)(y)\left(H_1 g\right)(y)\left(H_1 h\right)(y) =\\ &= \frac{1}{2\pi}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\operatorname{cas}\left(-yu\right).\operatorname{cas}\left(yv\right).\operatorname{cas}\left(yw\right).f\left(u\right)g\left(v\right)h\left(w\right)dudvdw,\end{aligned}$$

and

$$\begin{aligned}\operatorname{cas}\left(-yu\right).\operatorname{cas}\left(yv\right).\operatorname{cas}\left(yw\right) =& \frac{1}{2}\left[\operatorname{cas}[y\left(u+v+w\right)]+\operatorname{cas}-y\left(u+v-w\right)\right.\\ &\left.+\operatorname{cas}-y\left(u-v+w\right)-\operatorname{cas}y\left(u-v-w\right)\right].\end{aligned}$$

Thus

$$\begin{aligned}
&(H_2 f)(y)(H_1 g)(y)(H_1 h)(y) = \\
=&\frac{1}{4\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\operatorname{cas} y(u+v+w)] + \operatorname{cas} -y(u+v-w) \\
&+\operatorname{cas} -y(u-v+w) - \operatorname{cas} y(u-v-w)]\, f(u)\, g(v)\, h(w)\, dudvdw \\
=&\frac{1}{4\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\operatorname{cas}(yt)\,[f(t-v-w) - f(-t-v+w) + \\
&-f(-t+v-w) - f(t+v+w)]\, g(v)\, h(w)\, dtdvdw \\
=&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} *(f,g,h)(y)\operatorname{cas} ytdt \\
=&H_1[*(f,g,h)](y), \forall y \in \mathbb{R}.
\end{aligned}$$

The proof is completed.

Theorem 2.2. *(Titchmarch-type Theorem)*
Let $f, g, h \in L(\mathbb{R})$. If $\forall x \in \mathbb{R}, *(f,g,h)(x) \equiv 0$, then either $f(x) = 0$, or $g(x) = 0$, or $h(x) = 0, \forall x \in \mathbb{R}$.

Proof. The hypothesis $*(f,g,h)(x) \equiv 0$ implies that

$$H[*(f,g,h)](y) = 0, \forall y \in \mathbb{R}.$$

Due to Theorem 2.1 we have

$$(H_2 f)(y)(H_1 g)(y)(H_1 h)(y) = 0, \forall y \in \mathbb{R}. \tag{2.3}$$

As $(H_2 f)(y), (H_1 g)(y), (H_1 h)(y)$ are analytic $\forall y \in \mathbb{R}$ so from (2.3) we have $(H_2 f) = 0, \forall y \in \mathbb{R}$, or $(H_1 g) = 0, \forall y \in \mathbb{R}$, or $(H_1 h) = 0, \forall y \in \mathbb{R}$.
It follows that either $f(x) = 0, \forall x \in \mathbb{R}$, or $g(x) = 0, \forall x \in \mathbb{R}$, or $h(x) = 0, \forall x \in \mathbb{R}$.
The theorem is proved.

In the sequel, for simplicity, we define the norm in the space $L(\mathbb{R})$ by

$$\|f\| = \frac{1}{\sqrt[3]{\pi}}\int_{-\infty}^{\infty}|f(x)|\, dx$$

Theorem 2.3. *If f, g, h belong to $L(\mathbb{R})$, then the following inequality holds*

$$\|*(f,g,h)\| \leq \|f\| \,.\, \|g\| \,.\, \|h\| \,.$$

Proof. From the proof of Theorem 2.1, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty}|*(f,g,h)(x)|\, dx &\leq \frac{1}{\pi}\int_{-\infty}^{\infty}|f(t)|\, dt\int_{-\infty}^{\infty}|g(v)|\, dv\int_{-\infty}^{\infty}|h(w)|\, dw \\
&= \frac{1}{\sqrt[3]{\pi}}\int_{-\infty}^{\infty}|f(t)|\, dt.\frac{1}{\sqrt[3]{\pi}}\int_{-\infty}^{\infty}|g(v)|\, dv.\frac{1}{\sqrt[3]{\pi}}\int_{-\infty}^{\infty}|h(w)|\, dw.
\end{aligned}$$

Thus

$$\|*(f,g,h)\| \leq \|f\| \,.\, \|g\| \,.\, \|h\| \,.$$

The proof is completed.

Corollary 2.1. *In the space $L(\mathbb{R})$ the polyconvolution (2.1) has following inequality*

$$*(f,g,h)(x) = *(f,h,g)(x).$$

Proof. Indeed,

$$H_1\left[*(f,g,h)\right](y) = (H_2 f)(y)(H_1 g)(y)(H_1 h)(y) = (H_2 g)(y)(H_1 h)(y)(H_1 g)(y),$$

implies that

$$*(f,g,h)(x) = *(f,h,g)(x).$$

3. APPLICATION TO SOLVING AN INTEGRAL EQUATION AND A SYSTEM OF INTEGRAL EQUATIONS

3.1. Consider the integral equations.

$$\begin{aligned} f(x) + \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &\left[g(x-v-w) - g(-x-v+w) - g(-x+v-w)\right. \\ &\left.-g(x+v+w)\right] f(v)\, h(w)\, dvdw = k(x), x\in\mathbb{R}. \end{aligned} \tag{3.1}$$

Here g, h and k are functions of $L(\mathbb{R})$, f is unknown function.

Theorem 3.1. *Let $f,\ g,\ h$ be given. Equation (3.1) has an unique solution in $L(\mathbb{R})$ if $1 + H_1\left(h \underset{H_1,H_1,H_2}{*} g\right)(y) \neq 0, \forall y \in \mathbb{R}$. The solution of $f(x)$ is defined by*

$$f(x) = k(x) - \left(k \underset{H_1}{*} l\right)(x).$$

Here, $l \in L(\mathbb{R})$ and it is determined by the equation

$$(H_1 l)(y) = \frac{H_1\left(h \underset{H_1,H_1,H_2}{*} g\right)(y)}{1 + H_1\left(h \underset{H_1,H_1,H_2}{*} g\right)(y)}.$$

Proof. The equation (3.1) can be rewriten in the form

$$f(x) + \left[*(g,f,h)\right](x) = k(x).$$

Due to Theorem 2.1

$$(H_1 f)(y) + (H_2 g)(y)(H_1 f)(y)(H_1 h)(y) = (H_1 k)(y), \forall y \in \mathbb{R}.$$

It follows that

$$(H_1 f)(y)\left[1 + (H_2 g)(y)(H_1 h)(y)\right] = (H_1 k)(y). \tag{3.2}$$

From (1.3) and (1.4) we get

$$(H_1 f)(y)\left[1 + H_1\left(h \underset{H_1,H_1,H_2}{*} g\right)(y)\right] = (H_1 k)(y).$$

With the condition $1 + H_1\left(h \underset{H_1,H_1,H_2}{*} g\right)(y) \neq 0$, we obtain

$$(H_1 f)(y) = (H_1 k)(y)\left[1 - \frac{H_1\left(h \underset{H_1,H_1,H_2}{*} g\right)(y)}{1 + H_1\left(h \underset{H_1,H_1,H_2}{*} g\right)(y)}\right].$$

Furthermore, according to the Wiener-Levy's theorem in [1], [9], there exists a function $l \in L(\mathbb{R})$ such that

$$(H_1 l)(y) = \frac{H_1\left(h \underset{H_1,H_1,H_2}{*} g\right)(y)}{1 + H_1\left(h \underset{H_1,H_1,H_2}{*} g\right)(y)}.$$

It follows that

$$(H_1 f)(y) = (H_1 k)(y) - H_1\left(k \underset{H_1}{*} l\right)(y).$$

Thus

$$f(x) = k(x) - \left(k \underset{H_1}{*} l\right)(x) \in L(\mathbb{R}).$$

The theorem is proved.

3.2. **Consider the system of integral equations with only one polyconvolution.**

$$\text{(3.3)} \quad \begin{cases} f(x) + \dfrac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\varphi(x-v-w) - \varphi(-x-v+w) - \varphi(-x+v-w)\right. \\ \qquad\qquad \left. -\varphi(x+v+w)\right] g(v)\,\psi(w)\,dvdw = h(x) \\ \dfrac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\left[p(x+u) + p(x-v) + p(-x+v) - p(-x-v)\right] dv \\ \qquad\qquad +g(x) = k(x),\, x \in \mathbb{R}. \end{cases}$$

Where φ, ψ, p, h, k are given functions in $L(\mathbb{R})$, and f, g are the unknown functions.

Theorem 3.2. *With the condition* $1 - H_1[*(\varphi,\psi,p)](y) \neq 0, \forall y \in \mathbb{R}$, *there exists a unique solution in* $L(\mathbb{R})$ *of (3.3) which is defined by*

$$\begin{cases} f(x) = h(x) + \left(h \underset{H_1}{*} l\right)(x) - [*(\varphi,\psi,k)](x) - \left[*(\varphi,\psi,k) \underset{H_1}{*} l\right](x) \in L(\mathbb{R}), \\ g(x) = k(x) + \left(k \underset{H_1}{*} l\right)(x) - \left(h \underset{H_1}{*} p\right)(x) - \left[\left(h \underset{H_1}{*} p\right) \underset{H_1}{*} l\right](x), \in L(\mathbb{R}). \end{cases}$$

Here $l \in L(\mathbb{R})$ *and defined by the equation*

$$(H_1 l)(y) = \frac{H_1[*(\varphi,\psi,p)](y)}{1 - H_1[*(p,\varphi,\psi)](y)}$$

Proof. System (3.3) can be written in the form

$$\begin{cases} f(x) + *(\varphi,g,\psi)(x) = h(x) \\ \left(f \underset{H_1}{*} p\right)(x) + g(x) = k(x), x \in \mathbb{R}. \end{cases}$$

Using the factorization property of the polyconvolution (2.1) and the convolution (1.1) we obtain the linear system of algebraic equations with respectively to $(H_1 f)(y)$ and $(H_1 g)(y)$:

$$\begin{cases} (H_1 f)(y) + (H_2\varphi)(y)(H_1 g)(y)(H_1\psi)(y) = (H_1 h)(y), \\ (H_1 f)(y)(H_1 p)(y) + (H_1 g)(y) = (H_1 k)(y), y \in \mathbb{R}. \end{cases}$$

Formally, we have

$$\Delta = \begin{vmatrix} 1 & (H_2\varphi)(y)(H_1\psi)(y) \\ (H_1 p)(y) & 1 \end{vmatrix} = 1 - H_1[*(\varphi,\psi,p)](y).$$

$$\Delta_1 = \begin{vmatrix} (H_1 h)(y) & (H_2\varphi)(y)\,(H_1\psi)(y) \\ (H_1 k)(y) & 1 \end{vmatrix} = (H_1 h)(y) - H_1\left[*\left(\varphi,\psi,k\right)\right](y).$$

$$\Delta_2 = \begin{vmatrix} 1 & (H_1 h)(y) \\ (H_1 p)(y) & (H_1 k)(y) \end{vmatrix} = (H_1 k)(y) - H_1\left(h \underset{H_1}{*} p\right)(y).$$

Since $1 - H_1\left[*\left(\varphi,\psi,p\right)\right](y) \neq 0$,

$$(H_1 f)(y) = \left\{(H_1 h)(y) - H_1\left[*\left(\varphi,\psi,k\right)\right](y)\right\}\left\{1 + \frac{H_1\left[*\left(\varphi,\psi,p\right)\right](y)}{1 - H_1\left[*\left(\varphi,\psi,p\right)\right](y)}\right\}.$$

Due to Wiener-Levy's theorem [1], [9] there exists a function $l \in L(\mathbb{R})$ such that

$$(H_1 l)(y) = \frac{H_1\left[*\left(\varphi,\psi,p\right)\right](y)}{1 - H_1\left[*\left(\varphi,\psi,p\right)\right](y)}.$$

It follows that

$$\begin{aligned}(H_1 f)(y) = & \left\{(H_1 h)(y) - H_1\left[*\left(\varphi,\psi,k\right)\right](y)\right\}.\left\{1 + (H_1 l)(y)\right\} \\ = & H_1\left(h \underset{H_1}{*} l\right)(y) - H_1\left[*\left(\varphi,\psi,k\right)\right](y) - H_1\left\{\left[*\left(\varphi,\psi,k\right)\right] \underset{H_1}{*} l\right\}(y) \\ & + (H_1 h)(y)\end{aligned}$$

Thus,

$$f(x) = h(x) + \left(h \underset{H_1}{*} l\right)(x) - \left[*\left(\varphi,\psi,k\right)\right](x) - \left[*\left(\varphi,\psi,k\right) \underset{H_1}{*} l\right](x) \in L(\mathbb{R}).$$

Similarly we obtain

$$\begin{aligned}(H_1 g)(y) = & \left\{(H_1 k)(y) - H_1\left(h \underset{H_1}{*} p\right)(y)\right\}\left\{1 + (H_1 l)(y)\right\} \\ = & (H_1 k)(y) + H_1\left(k \underset{H_1}{*} l\right)(y) - H_1\left[\left(h \underset{H_1}{*} p\right) \underset{H_1}{*} l\right](y) \\ & - H_1\left(h \underset{H_1}{*} p\right)(y).\end{aligned}$$

It follows that

$$g(x) = k(x) + \left(k \underset{H_1}{*} l\right)(x) - \left(h \underset{H_1}{*} p\right)(x) - \left[\left(h \underset{H_1}{*} p\right) \underset{H_1}{*} l\right](x), \in L(\mathbb{R}).$$

The proof is completed.

3.3. **Consider the system of integral equations with polyconvolutions.**

Lemma 3.1. *The polyconvolution for the Hartley integral transforms H_1, H_1, H_2 of the functions f, g and h is defined by*

$$\begin{aligned}\underset{1}{*}(f,g,h)(x) = & \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-f(-x+v-w) + f(x+v+w)\right. \\ & \left.+ f(x-v-w) + f(-x-v+w)\right] g(v)\,h(w)\,dv dw, x \in \mathbb{R}.\end{aligned} \tag{3.4}$$

Let f, g and h be functions in $L(\mathbb{R})$, then the polyconvolution (3.4) for the Hartley integral transforms H_1, H_1, H_2 of the functions f, g and h belongs to $L(\mathbb{R})$ and the factorization property holds

$$H_1\left[\underset{1}{*}(f,g,h)\right](y) = (H_1 f)(y)\,(H_1 g)(y)\,(H_2 h)(y), \forall y \in \mathbb{R}. \tag{3.5}$$

Now, we consider the following systems

$$\begin{cases} f(x) + \dfrac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\varphi(x-v-w) - \varphi(-x-v+w) - \varphi(-x+v-w)\right. \\ \hfill \left. -\varphi(x+v+w)\right] g(v)\,\psi(w)\,dvdw = h(x) \\ \dfrac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\left[-f(-+-v-w) + f(x+v+w) + f(x-v-w)\right. \\ \hfill \left. +f(-x-v+w)\right] p(v)\,q(w)dvdw + g(x) = k(x), x \in \mathbb{R}. \end{cases} \tag{3.6}$$

Where $p, q, \varphi, \psi, p, h$, k are given functions in $L(\mathbb{R})$, and f, g are the unknown functions.

Theorem 3.3. *With the condition* $1 - H_1\left\{[*(p,\varphi,\psi)] \underset{H_1,H_1,H_2}{*} m\right\}(y) \neq 0, \forall y \in \mathbb{R}$*, there exists a unique solution in $L(\mathbb{R})$ of (3.3) which is defined by*

$$\begin{cases} f(x) = h(x) + \left(h \underset{H_1}{*} l\right)(x) - [*(k,\varphi,\psi)](x) - \left\{[*(k,\varphi,\psi)] \underset{H_1}{*} l\right\}(x) \in L(\mathbb{R}), \\ g(x) = k(x) + \left(k \underset{H_1}{*} l\right)(x) - [*(p,q,h)](x) - \left\{[*(p,q,h)] \underset{H_1}{*} l\right\}(x) \in L(\mathbb{R}). \end{cases}$$

Here $l \in L(\mathbb{R})$ and defined by the equation

$$(H_1 l)(y) = \frac{H_1\left\{[*(p,\varphi,\psi)] \underset{H_1,H_1,H_2}{*} m\right\}(y)}{1 - H_1\left\{[*(p,\varphi,\psi)] \underset{H_1,H_1,H_2}{*} m\right\}(y)}.$$

Proof. System (3.6) can be written in the form

$$\begin{cases} f(x) + *(g,\varphi,\psi)(x) = h(x) \\ \underset{1}{*}(f,p,q)(x) + g(x) = k(x), x \in \mathbb{R}. \end{cases}$$

Using the factorization property of the polyconvolutions (2.1) and (3.4) we get

$$\begin{cases} (H_1 f)(y) + (H_1 g)(y).\,(H_2\varphi)(y)\,(H_1\psi)(y) = (H_1 h)(y), \\ (H_1 f)(y)\,(H_1 p)(y)\,(H_2 q)(y) + (H_1 g)(y) = (H_1 k)(y), y \in \mathbb{R}. \end{cases}$$

Formally, we have

$$\Delta = \begin{vmatrix} 1 & (H_2\varphi)(y)\,(H_1\psi)(y) \\ (H_1 p)(y)\,(H_2 q)(y) & 1 \end{vmatrix} = 1 - H_1\left[*(p,\varphi,\psi)\right](y)\,(H_2 q)(y)\,.$$

So,

$$\Delta = 1 - H_1\left\{[*(p,\varphi,\psi)] \underset{H_1,H_1,H_2}{*} q\right\}(y).$$

$$\Delta_1 = \begin{vmatrix} (H_1 h)(y) & (H_2\varphi)(y)\,(H_1\psi)(y) \\ (H_1 k)(y) & 1 \end{vmatrix} = (H_1 h)(y) - H_1\left[*(\varphi,k,\psi)\right](y)\,.$$

$$\Delta_2 = \begin{vmatrix} 1 & (H_1h)(y) \\ (H_1p)(y)(H_2q)(y) & (H_1k)(y) \end{vmatrix} = (H_1k)(y) - H_1[*(q,p,h)](y).$$

Since $1 - H_1\left\{[*(p,\varphi,\psi)] \underset{H_1,H_1,H_2}{*} q\right\}(y) \neq 0, \forall y \in \mathbb{R}$ we have

$$\begin{aligned}&(H_1f)(y)\\ &= \{(H_1h)(y) - H_1[*(k,\varphi,\psi)](y)\}\left\{1 + \frac{H_1\left\{[*(p,\varphi,\psi)] \underset{H_1,H_1,H_2}{*} q\right\}(y)}{1 - H_1\left\{[*(p,\varphi,\psi)] \underset{H_1,H_1,H_2}{*} q\right\}(y)}\right\}.\end{aligned}$$

Due to Wiener-Levy's theorem ([1], [9]), there exists a function $l \in L(\mathbb{R})$ such that

$$(H_1l)(y) = \frac{H_1\left\{[*(p,\varphi,\psi)] \underset{H_1,H_1,H_2}{*} q\right\}(y)}{1 - H_1\left\{[*(p,\varphi,\psi)] \underset{H_1,H_1,H_2}{*} q\right\}(y)}.$$

It follows that

$$\begin{aligned}(H_1f)(y) &= \{(H_1h)(y) - H_1[*(k,\varphi,\psi)](y)\}\{1 + (H_1l)(y)\}\\ &= H_1\left(h \underset{H_1}{*} l\right)(y) - H_1[*(k,\varphi,\psi)](y) - H_1\left\{[*(k,\varphi,\psi)] \underset{H_1}{*} l\right\}(y)\\ &\quad + (H_1h)(y)\end{aligned}$$

Thus,

$$f(x) = h(x) + \left(h \underset{H_1}{*} l\right)(x) - [*(k,\varphi,\psi)](x) - \left\{[*(k,\varphi,\psi)] \underset{H_1}{*} l\right\}(x) \in L(\mathbb{R}).$$

Similarly we obtain

$$\begin{aligned}(H_1g)(y) &= \{(H_1k)(y) - H_1[*(q,p,h)](y)\}\{1 + (H_1l)(y)\}\\ &= H_1\left(k \underset{H_1}{*} l\right)(y) - H_1[*(q,p,h)](y) - H_1\left\{[*(q,p,h)] \underset{H_1}{*} l\right\}(y)\\ &\quad + (H_1k)(y)\end{aligned}$$

It follows that

$$g(x) = k(x) + \left(k \underset{H_1}{*} l\right)(x) - [*(p,q,h)](x) - \left\{[*(p,q,h)] \underset{H_1}{*} l\right\}(x) \in L(\mathbb{R}).$$

The proof is completed.

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