The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 16, Issue 2, Article 8, pp. 1-16, 2019

A NEW ADOMIAN APPROACH TO SOLVING INTEGRAL EQUATIONS OF FREDHOLM AND VOLTERRA SECOND KIND

OUEDRAOGO SENY, NEBIE ABDOUL WASSIHA, YOUSSOUF PARE, BLAISE SOME

Received 29 January, 2019; accepted 7 August, 2019; published 7 October, 2019.

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ JOSEPH KI-ZERBO, BURKINA FASO.
oseny@yahoo.fr, nebwass@yahoo.fr
pareyoussouf@yahoo.fr, some@univ-ouaga.bf

ABSTRACT. In order to simplify the resolution of Fredholm and Volterra's second type integral equations, we propose a new approach based on the Adomian Decompositional Method (ADM). We test the new approach on several examples with success.

Key words and phrases: Adomian Decomposition Method(ADM), The modifie Adomian algorithm, Fredholm integral equations second type, Volterra integral equations second type, new approach of Adomian Decomposition Method (ADM)..

2000 *Mathematics Subject Classification.* Primary 345Axx, 45Bxx, 45Gxx, 47H30, 47Jxx, 34A12.

ISSN (electronic): 1449-5910

1. INTRODUCTION

In the literature, there are several methods of solving the integral equations, we put in place a new idea in order to simplify the problem, inspired by the method of Adomian. In this paper, we propose a new approach based on the Adomian decompositional method (ADM) [3, 4, 5, 6, 8, 9] to solve the linear integral equations of Fredholm and second-type Volterra [1, 2, 13, 14, 16]. In doing so, we obtain interesting results.

2. THE NEW ADOMIAN APPROACH FOR FREDHOML LINEAR INTEGRAL EQUATIONS SECOND TYPE

2.1. **Description of the new Adomian approach.** Let us consider the following Fredholm integral equation second kind:

$$\varphi(x) = f(x) + \lambda \int_{\alpha}^{\beta} K(x,t)\,\varphi(t)\,dt, \lambda > 0 \tag{2.1}$$

Where φ is the unknown function, $K(x,t)$ is the kernel of the integral equation and $x, t \in [\alpha, \beta] \subset \mathbb{R}, K(x,t) \in C(\Omega)$ whith $\Omega = [\alpha, \beta] \times [\alpha, \beta]$ .. Taking

$$h(x) = \int_{\alpha}^{\beta} K(x,t)\,\varphi(t)\,dt \tag{2.2}$$

we get:

$$\varphi(x) = f(x) + \lambda h(x) \tag{2.3}$$

Then

$$h(x) = \int_{\alpha}^{\beta} K(x,t)\,(f(t) + \lambda h(t))\,dt$$

$$\Leftrightarrow h(x) = \underbrace{\int_{\alpha}^{\beta} K(x,t)\,f(t)\,dt}_{l(x)} + \lambda \int_{\alpha}^{\beta} K(x,t)\,h(t)\,dt \tag{2.4}$$

(2.4) is the canonical form of Adomian. Let's look for the solution of the equation in the form of a convergent series $\left(\sum_{n=0}^{+\infty} h_n(x)\right)$, we obtain the relationship

$$\sum_{n=0}^{+\infty} h_n(x) = l(x) + \lambda \int_{\alpha}^{\beta} K(x,t) \left(\sum_{n=0}^{+\infty} h_n(t)\right) dt$$

then we get the following Adomian algorithm:

$$\begin{cases} h_0(x) = l(x) \\ h_{n+1}(x) = \lambda \displaystyle\int_{\alpha}^{\beta} K(x,t)\,h_n(t)\,dt, n \geq 0 \end{cases} \tag{2.5}$$

2.2. **Convergence of the algorithm.**

Theorem 2.1. *If* $f \in C([\alpha,\beta])$ *and* $K \in C(\Omega)$ *with* $\Omega = [\alpha,\beta]\times[\alpha,\beta]$ *then* $\forall t \in [\alpha,\beta], \exists m > 0$ *such that* $|f(t)| \leq m$ *and* $\forall (x,t) \in [\alpha,\beta]\times[\alpha,\beta], \exists M > 0$ *such that* $|K(x,t)| \leq M$, *and the series* $\left(\sum_{n=0}^{+\infty} h_n(x)\right)$ *is convergent and the solution of the equation*

$$\varphi(x) = f(x) + \lambda \int_{\alpha}^{\beta} K(x,t)\,\varphi(t)\,dt$$

is

$$\varphi(x) = f(x) + \lambda h(x)$$

exists and is unique.

Proof. The existence of the solution of the equation $\varphi(x) = f(x) + \lambda \int_{\alpha}^{\beta} K(x,t)\,\varphi(t)\,dt$ has been proved in [15]

Let's show that the series $\left(\sum_{n=0}^{+\infty} h_n(x)\right)$ is convervent, we obtain successively the following inequalities, where $m = \sup_{x\in[\alpha,\beta]} f(x)$ et $M = \sup_{(x,t)\in[\alpha,\beta]\times[\alpha,\beta]} K(x,t)$:

$$\begin{cases} |h_0(x)| \leq mM\,|\beta - a| \\ |h_1(x)| \leq \lambda \dfrac{(mM\,|\beta - a|)^2}{2!} \\ |h_2(x)| \leq \lambda^2 \dfrac{(mM\,|\beta - a|)^3}{3!} \\ |h_3(x)| \leq \lambda^3 \dfrac{(mM\,|\beta - a|)^4}{4!} \\ .... \\ |h_n(x)| \leq \lambda^n \dfrac{(mM\,|\beta - a|)^{n+1}}{(n+1)!} \end{cases}$$

$\Rightarrow$

$$\sum_{n=0}^{+\infty} |h_n(x)| \leq \frac{1}{\lambda} \sum_{n=0}^{+\infty} \frac{(mM\lambda\,|\beta - a|)^{n+1}}{(n+1)!} = \frac{1}{\lambda}\left(\exp(mM\lambda\,|\beta - a|) - 1\right)$$

then the series $\left(\sum_{n=0}^{+\infty} |h_n(x)|\right)$ is convergent and the serie the series $\left(\sum_{n=0}^{+\infty} h_n(x)\right)$ is convergent too.

Let us consider two different solutions of (2.1) $\phi(x)$ and $\varphi(x)$.

Let's apply the Adomian method to both $\phi(x)$ and $\varphi(x)$ two functions supposed different, then it becomes: $\phi(x) = \sum_{n=0}^{+\infty} \phi_n(x)$ and $\varphi(x) = \sum_{n=0}^{+\infty} \varphi_n(x)$ solutions of (2.1)

$\Rightarrow$

$$\begin{cases} \phi_0(x) = f(x) \\ \phi_{n+1}(x) = \lambda \displaystyle\int_{\alpha}^{\beta} K(x,t)\,\phi_n(t)\,dt, n \geq 0 \end{cases}$$

and

$$\begin{cases} \varphi_0(x) = f(x) \\ \varphi_{n+1}(x) = \lambda \int\limits_{\alpha}^{\beta} K(x,t)\,\varphi_n(t)\,dt, n \geq 0 \end{cases}$$

so by making the difference of the two series, we get:

$$\begin{cases} \phi_0(x) - \varphi_0(x) = f(x) - f(x) = 0 \\ \phi_{n+1}(x) - \varphi_{n+1}(x) = \lambda \int\limits_{\alpha}^{\beta} K(x,t)\,(\phi_n(t) - \varphi_n(t))\,dt, n \geq 0 \end{cases}$$

so we get:

$$\begin{cases} \phi_0(x) = \varphi_0(x) \\ \phi_n(x) = \varphi_n(x)\,, n \geq 1 \end{cases}$$

$\Rightarrow \ \forall x \in [\alpha,\beta]$, we obtain: $\phi(x) = \varphi(x)$,which is impossible because by hypothesis $\phi(x) \neq \varphi(x)$. So they are necessarily equal and we get $\phi(x) = \varphi(x)$. ∎

2.3. **Applications.**

2.3.1. *Example 1.* Let us consider the following linear integral of Fredholm second kind:

$$\varphi(x) = x + \int\limits_0^1 (xt\ln t)\,\varphi(t)\,dt \tag{2.6}$$

$\varphi(x) = f(x) + h(x)$
where

$$h(x) = \int\limits_0^1 (xt\ln t)\,\varphi(t)\,dt.$$

Then we get: $h(x) = \int\limits_0^1 (xt\ln t)\,f(t)\,dt + \int\limits_0^1 (xt\ln t)\,h(t)\,dt$. Let's look for the solution of the equation in the form of a convergent series $\left(\sum\limits_{n=0}^{+\infty} h_n(x)\right)$ and we get the following Adomian algoritm:

$$\begin{cases} h_0(x) = \int\limits_0^1 (xt\ln t)\,f(t)\,dt \\ h_{n+1}(x) = \int\limits_0^1 (xt\ln t)\,h_n(t)\,dt\ ; n \geq 0 \end{cases}$$

Let us calculate the following terms: $h_0(x), h_1(x), h_2(x), h_3(x), \ldots$

$$\begin{cases} h_0(x) = -\frac{1}{9}x \\ h_1(x) = \left(\frac{1}{9}\right)^2 x \\ h_2(x) = -\left(\frac{1}{9}\right)^3 x \\ \ldots \\ h_n(x) = \left(-\frac{1}{9}\right)^{n+1} x \end{cases}$$

We obtain:

$$h(x) = \sum_{n=0}^{+\infty} \left(-\frac{1}{9}\right)^{n+1} x = -\frac{1}{10}x.$$

Hence the exact solution of (2.6) is:

$$\varphi(x) = x - \frac{1}{10}x = \frac{9}{10}x.$$

2.3.2. *Example 2.* Let us consider the following linear integral of Fredholm second kind:

$$\varphi(x) = x + \frac{1}{2}\int_0^1 (x-t)\,\varphi(t)\,dt \tag{2.7}$$

We obtain:

$$\varphi(x) = f(x) + \frac{1}{2}h(x) \text{ with } h(x) = \int_0^1 (x-t)\,\varphi(t)\,dt$$

we get:

$$h(x) = \int_0^1 (x-t)\left(f(t) + \frac{1}{2}h(t)\right)dt$$

$\Rightarrow$

$$\begin{cases} h_0(x) = \frac{1}{2}x - \frac{1}{3} \\ h_{n+1}(x) = \frac{1}{2}\int_0^1 (x-t)\,h_n(t)\,dt\,;\, n \geq 0 \end{cases}$$

Let us calculate the following terms: $h_0(x), h_1(x), h_2(x), h_3(x), \ldots$

$$\begin{cases} h_0(x) = \frac{1}{2}x - \frac{1}{3} \\ h_1(x) = -\frac{1}{24}x \\ h_2(x) = \frac{1}{144} - \frac{1}{96}x \\ h_3(x) = \frac{1}{1152}x \\ h_4(x) = \frac{1}{4608}x - \frac{1}{6912} \\ h_5(x) = -\frac{1}{55296}x \\ h_6(x) = \frac{1}{331775} - \frac{1}{221184}x \\ \ldots \\ h_{2n}(x) = \left(-\frac{1}{48}\right)^n \left(\frac{1}{2}x - \frac{1}{3}\right); n \geq 0 \\ h_{2n+1}(x) = \left(-\frac{1}{48}\right)^n \left(-\frac{1}{24}\right) x; n \geq 0 \end{cases}$$

We obtain:

$$\Rightarrow \begin{cases} h(x) & = \left(\frac{1}{2}x - \frac{1}{3}\right) \sum_{n=0}^{+\infty} \left(-\frac{1}{48}\right)^n - \frac{1}{24} \sum_{n=0}^{+\infty} \left(-\frac{1}{48}\right)^n x \\ & = \frac{22}{49}x - \frac{16}{49} \end{cases}$$

Then, we obtain the exact solution of the equation (2.7):

$$\varphi(x) = \frac{60}{49}x - \frac{8}{49}.$$

2.3.3. *Example 3.* Let us consider the following linear integral of Fredholm second kind:

$$\varphi(x) = x + \frac{1}{2} \int_0^1 xt^2 \varphi(t)\, dt \tag{2.8}$$

$\varphi(x) = f(x) + \frac{1}{2} h(x)$
Where

$$h(x) = \int_0^1 xt^2 \varphi(t)\, dt$$

$$\Rightarrow h(x) = \int_0^1 xt^2 f(t)\, dt + \frac{1}{2} \int_0^1 xt^2 h(t)\, dt$$

$$\begin{cases} h_0(x) = \int_0^1 xt^3 dt \colon \frac{1}{4}x \\ h_{n+1}(x) = \frac{1}{2} \int_0^1 xt^2 h_n(t)\, dt \end{cases}$$

Let us calculate the following terms: $h_0(x), h_1(x), h_2(x), h_3(x), \ldots$

$$\begin{cases} h_0(x) = \left(\frac{1}{4}\right) x \\ h_1(x) = \frac{1}{32}x = \frac{1}{8}h_0(x) \\ h_2(x) = \frac{1}{256}x = \left(\frac{1}{8}\right)^2 h_0(x) \\ h_3(x) = \left(\frac{1}{8}\right)^3 h_0(x) \\ \ldots \\ h_n(x) = \left(\frac{1}{8}\right)^n h_0(x)\ ; n \geq 0 \end{cases}$$

We obtain:
$h(x) = h_0(x) \sum_{n=0}^{+\infty} \left(\frac{1}{8}\right)^n$
$\Rightarrow$

$$h(x) = \frac{2}{7}x$$

Hence the exact solution of (2.8) is:

$$\varphi(x) = \frac{8}{7}x.$$

2.3.4. *Example 4.* Let us consider the following linear integral of Fredholm second kind: $f(x)$

$$\varphi(x) = e^{2x} - \frac{1}{2}e^x(e-1) + \frac{1}{2}\int_0^1 e^{x-t}\varphi(t)\,dt \tag{2.9}$$

$\varphi(x) = f(x) + \frac{1}{2}h(x)$
Where $h(x) = \int_0^1 e^{x-t}\varphi(t)\,dt$
We obtain: $h(x) = \int_0^1 e^{x-t}f(t)\,dt + \frac{1}{2}\int_0^1 e^{x-t}h(t)\,dt$

$$\begin{cases} h_0(x) = \int_0^1 e^{x-t}f(t)\,dt \\ h_{n+1}(x) = \frac{1}{2}\int_0^1 e^{x-t}h_n(t)\,dt\ ; n \geq 0 \end{cases}$$

Calculating: $h_0(x), h_1(x), h_2(x), h_3(x), \ldots$

$$\begin{cases} h_0(x) = \int\limits_0^1 e^{x-t}\left(e^{2t} - \frac{1}{2}e^t(e-1)\right)dt \\ h_1(x) = \frac{1}{2}\int\limits_0^1 e^{x-t}\left(\frac{1}{2}e^t(e-1)\right)dt \\ h_2(x) = \frac{1}{2}\int\limits_0^1 e^{x-t}\left(\frac{1}{4}e^t(e-1)\right)dt \\ h_3(x) = \frac{1}{2}\int\limits_0^1 e^{x-t}\left(\frac{1}{8}e^t(e-1)\right)dt \\ ... \\ h_n(x) = \left(\frac{1}{2}\right)^{n+1} e^x(e-1) \qquad \forall\, n \geq 0 \end{cases}$$

We obtain:

$$h(x) = \frac{e^x(e-1)}{2}\sum_{n=0}^{+\infty}\left(\frac{1}{2}\right)^n = e^x(e-1)$$

Hence the exact solution exact of (2.9) is:

$$\Rightarrow \varphi(x) = e^{2x}.$$

3. THE NEW ADOMIAN APPROACH FOR VOLTERRA LINEAR INTEGRAL EQUATIONS SECOND TYPE

3.1. Description of the new approach. Let us consider the following Volterra integral equation second king:

$$\varphi(x) = f(x) + \lambda\int\limits_\alpha^x K(x,t)\,\varphi(t)\,dt, \lambda > 0, T < +\infty \tag{3.1}$$

Where $x, t \in [\alpha, T] \subset \mathbb{R}, K(x,t) \in C([\alpha, T] \times [\alpha, T])$ and $f \in C([\alpha, T])$. Taking

$$h(x) = \int\limits_\alpha^x K(x,t)\,\varphi(t)\,dt$$

we get

$$\varphi(x) = f(x) + \lambda h(x)$$

.

Then

$$h(x) = \int\limits_\alpha^x K(x,t)\left(f(t) + \lambda h(t)\right)dt$$

$\Rightarrow$

$$h(x) = \int_{\alpha}^{x} K(x,t) f(t) \, dt + \lambda \int_{\alpha}^{x} K(x,t) h(t) \, dt \tag{3.2}$$

(3.2) is the canonical form of Adomian. Let's look for the solution of (3.2) in the form of a convergent series $\left(\sum_{n=0}^{+\infty} h_n(x) \right)$, we obtain the relationship

$$\sum_{n=0}^{+\infty} h_n(x) = \int_{\alpha}^{x} K(x,t) f(t) \, dt + \lambda \int_{\alpha}^{x} K(x,t) \left(\sum_{n=0}^{+\infty} h_n(t) \right) dt$$

then (3.3) is the algorithm of Adomian:

$$\begin{cases} h_0(x) = \int_{\alpha}^{x} K(x,t) f(t) \, dt \\ h_{n+1}(x) = \lambda \int_{\alpha}^{x} K(x,t) h_n(t) \, dt; n \geq 0 \end{cases} \tag{3.3}$$

3.2. **Convergence of the algorithm.**

Theorem 3.1. *If* $f \in C([\alpha, T])$ *and* $K \in C(\Omega)$ *whith* $\Omega = [\alpha, T] \times [\alpha, T]$ *then* $\forall t \in [\alpha, T], \exists m > 0$ *such that* $|f(t)| \leq m$ *and* $\forall (x,t) \in [\alpha, T] \times [\alpha, T], \exists M > 0$ *such that* $|K(x,t)| \leq M,$ *and the series* $\left(\sum_{n=0}^{+\infty} h_n(x) \right)$ *is convergent and the solution of the equation*

$$\varphi(x) = f(x) + \lambda \int_{\alpha}^{x} K(x,t) \varphi(t) \, dt$$

is

$$\varphi(x) = f(x) + \lambda h(x)$$

exists and is unique.

Proof. The existence of the solution of the equation $\varphi(x) = f(x) + \lambda \int_{\alpha}^{x} K(x,t) \varphi(t) \, dt$ has been proved in [15]

Let's show that the series $\left(\sum_{n=0}^{+\infty} h_n(x)\right)$ is convervent, we obtain successively the following inequalities, where $m = \sup_{x\in[\alpha,T]} f(x)$ et $M = \sup_{(x,t)\in[\alpha,\beta]\times[\alpha,T]} K(x,t)$:

$$\begin{cases} |h_0(x)| \le mM|x-a| \\ |h_1(x)| \le \lambda \frac{(mM|x-a|)^2}{2!} \\ |h_2(x)| \le \lambda^2 \frac{(mM|x-a|)^3}{3!} \\ |h_3(x)| \le \lambda^3 \frac{(mM|x-a|)^4}{4!} \\ .... \\ |h_n(x)| \le \lambda^n \frac{(mM|x-a|)^{n+1}}{(n+1)!} \end{cases}$$

$\Rightarrow$

$$\sum_{n=0}^{+\infty} |h_n(x)| \le \frac{1}{\lambda} \sum_{n=0}^{+\infty} \frac{(mM\lambda|x-a|)^{n+1}}{(n+1)!} = \frac{1}{\lambda}\left(\exp(mM\lambda|x-a|) - 1\right)$$

then the series $\left(\sum_{n=0}^{+\infty} |h_n(x)|\right)$ is convergent and the series the series $\left(\sum_{n=0}^{+\infty} h_n(x)\right)$ is convergent too.

Let us consider two differents solutions of (2.1), $\phi(x)$ and $\varphi(x)$.

Let's apply the Adomian method to both $\phi(x)$ and $\varphi(x)$ two functions supposed different, then it comes:$\phi(x) = \sum_{n=0}^{+\infty} \phi_n(x)$ and $\varphi(x) = \sum_{n=0}^{+\infty} \varphi_n(x)$ solutions of (2.1).

$\Rightarrow$

$$\begin{cases} \phi_0(x) = f(x) \\ \phi_{n+1}(x) = \lambda \int_\alpha^\beta K(x,t)\,\phi_n(t)\,dt, n \ge 0 \end{cases}$$

and

$$\begin{cases} \varphi_0(x) = f(x) \\ \varphi_{n+1}(x) = \lambda \int_\alpha^\beta K(x,t)\,\varphi_n(t)\,dt, n \ge 0 \end{cases}$$

so by making the difference of the two series, we get:

$$\begin{cases} \phi_0(x) - \varphi_0(x) = f(x) - f(x) = 0 \\ \phi_{n+1}(x) - \varphi_{n+1}(x) = \lambda \int_\alpha^\beta K(x,t)\left(\phi_n(t) - \varphi_n(t)\right) dt, n \ge 0 \end{cases}$$

so we get:

$$\begin{cases} \phi_0(x) = \varphi_0(x) \\ \phi_n(x) = \varphi_n(x)\,, n \ge 1 \end{cases}$$

$\Rightarrow \forall x \in [\alpha, \beta]$, we obtain: $\phi(x) = \varphi(x)$,which is impossible because by hypothesis $\phi(x) \ne \varphi(x)$. So they are necessarily equal and we get $\phi(x) = \varphi(x)$. ∎

3.3. **Applications.**

3.3.1. *Example 1.* Let us consider the following linear integral equation of Volterra second type:

$$\varphi(x) = e^x + \int_0^x e^{x-t}\varphi(t)\,dt \tag{3.4}$$

Let us take $h(x) = \int_0^x e^{x-t}\varphi(t)\,dt$, we get $\varphi(x) = e^x + h(x)$

$\Rightarrow h(x) = \int_0^x e^{x-t}\left(e^t + h(t)\right)dt$

$\Rightarrow$

$$h(x) = \int_0^x e^x dt + \int_0^x e^{x-t}h(t)\,dt$$

And we get the following Adomian Algorithm:

$$\begin{cases} h_0(x) = \int_0^x e^x dt\colon xe^x \\ h_{n+1}(x) = \int_0^x e^{x-t}h_n(t)\,dt; n \geq 0 \end{cases}$$

Calculating some terms: $h_0(x), h_1(x), h_2(x), h_3(x), ...$

$$\begin{cases} h_0(x) = xe^x \\ h_1(x) = \frac{1}{2}x^2e^x \\ h_2(x) = \int_0^x e^{x-t}\frac{1}{2}t^2e^t dt\colon \frac{1}{6}x^3e^x \\ h_3(x) = \int_0^x e^{x-t}\frac{1}{6}t^3e^t dt\colon \frac{1}{24}x^4e^x \\ h_4(x) = \int_0^x e^{x-t}\frac{1}{24}t^4e^t dt : \frac{1}{120}x^5e^x \\ ... \\ h_n(x) = \frac{x^{n+1}}{(n+1)!}e^x \end{cases}$$

We obtain:

$$h(x) = \exp(x)\sum_{n=0}^{+\infty}\frac{x^{n+1}}{(n+1)!} = (\exp(x) - 1)\exp(x)$$

Hence the exact solution exact of (3.4) is:

$$\varphi(x) = \exp(2x).$$

3.3.2. *Example 2.* Let us consider the following linear integral equation of Volterra second type:

$$\varphi(x) = x + \frac{1}{2}\int_0^x (x-t)\,\varphi(t)\,dt \tag{3.5}$$

Taking $h(x) = \int_0^x (x-t)\,\varphi(t)\,dt$, we get $\varphi(x) = x + \frac{1}{2}h(x)$

Where $h(x) = \int_0^x (x-t)\left(t + \frac{1}{2}h(t)\right)dt$, we obtain:

$$h(x) = \int_0^x (x-t)\,t\,dt - \frac{1}{2}\int_0^x (x-t)\,h(t)\,dt$$

And we get the following Adomian Algorithm:

$$\begin{cases} h_0(x) = \int_0^x (x-t)\,t\,dt \\ h_{n+1}(x) = \frac{1}{2}\int_0^x (x-t)\,h_n(t)\,dt; n \geq 0 \end{cases}$$

Calculating some terms: $h_0(x), h_1(x), h_2(x), h_3(x), \ldots$

$$\begin{cases} h_0(x) = \int_0^x (x-t)\,t\,dt = \frac{1}{3!}x^3 \\ h_1(x) = \frac{1}{2}\int_0^x (x-t)\left(\frac{1}{6}t^3\right)dt\text{: } \left(\frac{1}{2}\right)\frac{x^5}{5!} \\ h_2(x) = \frac{1}{2}\int_0^x (x-t)\left(\frac{1}{240}t^5\right)dt =\text{: } \left(\frac{1}{2}\right)^2\frac{x^7}{7!} \\ h_3(x) = \frac{1}{2}\int_0^x (x-t)\left(\frac{1}{20\,160}t^7\right)dt\text{: } \left(\frac{1}{2}\right)^3\frac{x^9}{9!} \\ h_4(x) = \frac{1}{2}\int_0^x (x-t)\left(\frac{1}{2903\,040}t^9\right)dt\text{:}\left(\frac{1}{2}\right)^4\frac{x^{11}}{11!} \\ \ldots \\ h_n(x) = \left(\frac{1}{2}\right)^n\frac{x^{2n+3}}{(2n+3)!} \end{cases} \tag{3.6}$$

$$\Rightarrow \begin{cases} h(x) & = \sum_{n=0}^{+\infty} h_n(x) \\ & = \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n \frac{x^{2n+3}}{(2n+3)!} \\ & = \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \\ & = \sum_{n=0}^{+\infty} \frac{\left(\sqrt{\frac{1}{2}}\right)^{2(n+1)+1}}{\left(\sqrt{\frac{1}{2}}\right)^3} \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \\ & = \sum_{n=0}^{+\infty} \frac{\left(\sqrt{\frac{1}{2}}\right)^{2(n+1)+1}}{\frac{1}{2}\sqrt{\frac{1}{2}}} \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \\ & = 2\sqrt{2} \sum_{p=1}^{+\infty} \left(\sqrt{\frac{1}{2}}x\right)^{2p+1} \frac{1}{(2p+1)!} \\ & = 2\sqrt{2}\left(\sinh\left(\sqrt{\frac{1}{2}}x\right) - \sqrt{\frac{1}{2}}x\right) \\ & = 2\sqrt{2}\sinh\left(\sqrt{\frac{1}{2}}x\right) - 2x \end{cases}$$

We obtain:

$$h(x) = 2\sqrt{2}\sinh\left(\sqrt{\frac{1}{2}}x\right) - 2x$$

Hence the exact solution exact of (3.5) is:

$$\varphi(x) = \sqrt{2}\sinh\left(\frac{x}{\sqrt{2}}\right).$$

3.3.3. *Example 3.* Let us consider the following linear Volterra integral equation second kind:

$$\varphi(x) = x - \frac{1}{4}x^4 + \int_0^x \left(x^2 - t^2\right)\varphi(t)\,dt \tag{3.7}$$

By posing:

$h(x) = \int_0^x (x^2 - t^2)\varphi(t)\,dt$

we get:

$$\Rightarrow h(x) = \frac{1}{4}x^4 - \frac{1}{70}x^7 + \int_0^x \left(x^2 - t^2\right)h(t)\,dt$$

Let's look for the solution of (3.7), in the form of a convergent series : $h(x) = \sum_{n=0}^{+\infty} h_n(x)$

Let's calculate some of the terms of the series $\left(\sum_{n=0}^{+\infty} h_n(x)\right)$, for that use the modified Adomian Algorithm:$h_0(x), h_1(x), h_2(x), h_3(x), h_4(x), \ldots$

$$\begin{cases} h_0(x) = \frac{1}{4}x^4 \\ h_1(x) = -\frac{1}{70}x^7 + \int\limits_0^x (x^2 - t^2) h_0(t) \, dt \\ h_{n+1}(x) = \int\limits_0^x (x^2 - t^2) h_n(t) \, dt; n \geq 1: \end{cases}$$

$$\begin{cases} h_0(x) = \frac{1}{4}x^4 \\ h_1(x) = -\frac{1}{70}x^7 + \int\limits_0^x (x^2 - t^2) h_0(t) \, dt \\ h_{n+1}(x) = \int\limits_0^x (x^2 - t^2) h_n(t) \, dt; n \geq 1: \end{cases}$$

We have, after calculation:

$$\begin{cases} h_0(x) = \frac{1}{4}x^4 \\ h_1(x) = -\frac{1}{70}x^7 + \int\limits_0^x (x^2 - t^2) \left(\frac{1}{4}t^4\right) dt = 0 \\ h_{n+1}(x) = \int\limits_0^x (x^2 - t^2) h_n(t) \, dt = 0, n \geq 1: \end{cases}$$

So we get:

$$h(x) = h_0(x) = \frac{1}{4}x^4$$

Hence the exact solution of (3.7) is:

$$\varphi(x) = x - \frac{1}{4}x^4 + \frac{1}{4}x^4 = x.$$

.

3.3.4. *Example 4.* Let us consider the following integral equation:

$$\varphi(x) = \frac{4}{5}x + \frac{1}{x^4} \int\limits_0^x t^3 \varphi(t) \, dt \tag{3.8}$$

where $x > 0$.

By posing $h(x) = \dfrac{1}{x^4}\displaystyle\int_0^x t^3\varphi(t)\,dt$, we get:

$$h(x) = \frac{1}{x^4}\int_0^x t^3 f(t)\,dt + \int_0^x \frac{t^3}{x^4}h(t)\,dt \tag{3.9}$$

By applying the Adomian algorithm to (3.9), we obtain:

$$\begin{cases} h_0(x) = \displaystyle\int_0^x \frac{t^3}{x^4}f(t)\,dt \\ h_{n+1}(x) = \displaystyle\int_0^x \frac{t^3}{x^4}h_n(t)\,dt; n \geq 0 \end{cases}$$

Let's calculate some terms: $h_0(x), h_1(x), h_2(x), h_3(x), ...$

$$\begin{cases} h_0(x) = \displaystyle\int_0^x \frac{t^3}{x^4}\left(\tfrac{4}{5}t\right)dt\text{: } \tfrac{4}{25}x \\ h_1(x) = \displaystyle\int_0^x \frac{t^3}{x^4}\left(\tfrac{4}{25}t\right)dt\text{: } \tfrac{4}{125}x \\ h_2(x) = \displaystyle\int_0^x \frac{t^3}{x^4}\left(\tfrac{4}{125}t\right)dt = \tfrac{4}{625}x \\ h_3(x) = \displaystyle\int_0^x \frac{t^3}{x^4}\left(\tfrac{4}{625}t\right)dt\text{: } \tfrac{4}{3125}x \\ ... \\ h_n(x) = 4x\left(\dfrac{1}{5}\right)^{n+2}; n \geq 0 \end{cases}$$

We obtain:

$$h(x) = \sum_{n=0}^{+\infty} 4x\left(\frac{1}{5}\right)^{n+2} = \frac{1}{5}x$$

Hence the exact solution exact of (3.8) is:

$$\varphi(x) = \frac{4}{5}x + \frac{1}{5}x = x.$$

4. CONCLUSION

In this paper, we propose a new approach of the Adomian method for the integral equations of Fredholm and Volterra of the second kind, then prove the convergence of the algorithm associated with this new approach. Finally, we used this new approach to solve several examples successfully. Also, we can say that this new approach of the Adomian method is a good tool for these types of equations.

REFERENCES

[1] A. M. WAZWAZ, A new algorithm for solving differential equation of Lame-Emden type, *Appl. Math. Comput.*, **21**(7), (2011), pp. 387-395.

[2] A. M. WAZWAZ, The numerical soluton of sixth-order bundary value problem by the modifie decomposition method, *Appl. Math. Comput.*, **118**(2011), pp. 311-325.

[3] B. ABBO, N. NGARASTA, B. MAMPASSI, B. SOME and L. SOME, A new approach of the Adomian algorithm for solving nonlinear ordinary or partial differential equations, *Far East J. Appl. Math.*, **23**(3) (2006), pp. 299-312.

[4] BLAISE SOME, Convergence of the Adomian method applied to Fredholm integro-differential equations, *Africa Mat.*, **3**(145)(2001), pp. 71-88.

[5] B. MAMPASSI, B. SALEY and B. SOME, Solving some nonlinear reaction-diffusion equations using the new Adomian decomposition method, *Afr. Diaspora J. Math.*, **1**(1) (2004), pp. 1-9.

[6] G. ADOMIAN, *Nonlinear Stochastic Systems Theory and Application to Physics*, Kluwer Academic Pub (1989).

[7] G. BISSANGA, J. YINDOULA BONAZEBI, A new approach of contruction of the Adomian algorithm, application of the Adomian decomposition method (ADM) and the Some Blaise Abbo (SBA) method to solving the convection equation, *Pioneeer J. Math. Sci.*, **13**(2) (2015), pp. 127-138.

[8] K. ABBAOUI and Y. CHERUAULT, The decomposition method applied to the Cauchy problems, *Kybernetes*, **28**(1) (1999), pp. 68-74.

[9] K. ABBAOUI and Y. CHERUAULT, Convergence of the Adomian decomposition method applied to differential equations, *Math. Comput. Model.*, **28**(5)(1994), pp. 103-109.

[10] N. NGARASTA, B. SOME, K. ABBAOUI and Y. CHERUAULT, New numerical study of Adomian method applied to a diffusion model, *Kybernetes*, **31**(1) (2002), pp. 61-75.

[11] S. KHELIFA and Y. CHERUAULT, Biomedical and biocybernetics applications: the decomposition method for solving first order partial differential equations, *Kybernetes*, **31**(6) (2002), pp. 844-871.

[12] T. MAVOUNGOU and Y. CHERRUAULT, Numerical Study of Fisher's Equation by Adomian's Method, *Math. Comput. Modeling*, **19**(1)(1994), pp. 89-95.

[13] YOUSSOUF PARE, FRANCIS BASSONO and BLAISE SOME, A new technique for numerical resolution of non linear integral equations of Fredholm by SBA method, *Far East. J. Appl. Math.*, **70**(1) (2012), pp. 21-33.

[14] YOUSSOUF PARE, ABBO BAKARI, RASMANE YARO and BLAISE SOME, Solving first kind Abel integral equations using the SBA numerical method, *Nonl. Anal. Differ. Equ.*, **1**(3)(2013), pp. 115-128.

[15] YAYA MOUSSA, YOUSSOUF PARE, PIERRE CLOVIS NITIEMA and BLAISE SOME, New approach of the Adomian Decomposition Method, *Inter. J. Num. Methods. Appl.*, **16**(1)(2017), pp. 1-10.

[16] HERMANN BRUNNER, Theory and numerical solution of Volterra functional integral equations. *Hit Summer Seminar*, 5-16 july (2010).:browse confirm wa