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SOLVING TWO POINT BOUNDARY VALUE PROBLEMS BY MODIFIED SUMUDU TRANSFORM HOMOTOPY PERTURBATION METHOD

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ABSTRACT. This paper considers a combined form of the Sumudu transform with the modified homotopy perturbation method (MHPM) to find approximate and analytical solutions for nonlinear two point boundary value problems. This method is called the modified Sumudu transform homotopy perturbation method (MSTHPM). The suggested technique avoids the round-off errors and finds the solution without any restrictive assumptions or discretization. We will introduce an appropriate initial approximation and furthermore, the residual error will be canceled in some points of the interval (RECP). Only a first order approximation of MSTHPM will be required, as compared to STHPM, which needs more iterations for the same cases of study. After comparing figures between approximate, MSTHPM, STHPM and numerical solutions, it is found through the solutions we have obtained that they are highly accurate, indicating that the MSTHPM is very effective, simple and can be used to solve other types of nonlinear boundary value problems (BVPs).

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1. INTRODUCTION

In the last two decades, many analytical approximate methods have been presented to solve two point boundary value problems. Most of these problems generally occur commonly in many areas of Physics and Chemistry. Recently, many researchers have introduced various methods to obtain approximate solutions for nonlinear differential equations (NDEs), such as variational iteration method[1, 2, 3, 4, 5], homotopy analysis method [6, 7, 8], adomian decomposition method [9, 10, 11], and homotopy perturbation method [12, 13, 14, 15, 16]. Amongst all these methods, the HPM has been considered as one of the most popular one, due to its simplicity and its wide range of applications.

The HPM was suggested by He in [17, 18] and had been proven by many authors [19, 20, 21, 22, 23, 24, 25, 26, 27, 28] to be a powerful mathematical tool for solving various types of nonlinear problems, which represent a large number of modern science branches. In some applications when series solution is searched for, the HPM method has some drawbacks which reduce the efficiency of the method due to repeated calculations and calculations of massive unneeded terms. Hence, many authors had improved this scheme by integrating it with other methods to avoid these drawbacks; one of them is the Sumudu transform homotopy perturbation method (STHPM) which is a combination of HPM and Sumudu transform to obtain high accuracy numerical results when solving nonlinear equations. The advantage of this method is that it is an elegant combination of two powerful methods which produces an easy to implement approach that is simple, efficient and reliable which also reduces the volume computational work. Moreover, the proposed method gives approximate solutions without any limitations. Singh and Devendra [29] created STHPM with He's polynomials to find the solution of nonlinear partial equations with the initial conditions, the results appeared that the approach is very efficient and simple to apply. After that, the coupled method has been presented by many authors to be a powerful mathematical tool for solving a wide range of nonlinear operator equations [30, 32, 33, 31, 34, 35, 36, 37, 38, 39, 40, 41, 42]. In spite of the aforementioned advantages, STHPM do not provide accurate results in the region closer to the endpoints of the interest interval, as in the solution of nonlinear differential equations coupled with mixed boundary conditions where the endpoint at y(0) is unknown. However, the work which will be introduced in this paper is quite different from the techniques used to solve these types of problems; we will modify STHPM to improve the accuracy of the solution, particularly at unknown endpoints of the interval. The methodology used in the proposed method is based primarily on the exploitation of the freedom of the HPM in the choice of an arbitrary linear function as a suitable initial approximation; furthermore, the residual error will be canceled in some points of the interval. Implementing these steps will further accelerate the convergence of the approximate solutions, as it will be shown, the MSTHPM present more accurate results in a first order approximation than the fourth order STHPM approximation.

In this present study, the main goal is to employ the modified Sumudu transform homotopy perturbation method (MSTHPM) in solving boundary value problems with mixed and Neumann conditions. The proposed method produces the solutions in a rapid convergent series, where the advantage lies in its applicability and effectiveness for obtaining approximate solutions for nonlinear equations.

In sections two and three of this article, we provide the main concepts of HPM method and Sumudu transform (ST), respectively. The MSTHPM has been included in section four as a combination of ST and MHPM. In section five, we verified the effectiveness of the proposed method by applying it to two physical systems based on nonlinear differential equations, the conclusions are presented in the last section.

2. HOMOTOPY PERTURBATION METHOD (HPM)

To explain the fundamental idea of the HPM, consider the following nonlinear differential equation:

$$L(u) + N(u) = f(t), \tag{2.1}$$

with boundary conditions

$$\beta(u, \partial u/\partial t), \quad t \in \Gamma, \tag{2.2}$$

where L and N is a linear and nonlinear operators respectively, $f(t)$ is a known analytical function, β is a boundary operator and Γ is the domain boundary for Ω.

We construct the HPM [18] as $u(p,t) \times [0,1] \longrightarrow \Re$ which satisfies:

$$H(u,p) = (1-p)[L(u) - L(u_0)] + p(L(u) + N(u) - f(t)] = 0, \quad p \in [0,1], \quad t \in \Omega \tag{2.3}$$

or

$$H(u,p) = L(u) - L(u_0) + p(L(u_0) + N(u) - f(t)) = 0, \quad p \in [0,1], \quad t \in \Omega \tag{2.4}$$

where p is an embedding parameter, its values are varied from 0 to 1, u_0 is the initial approximate solution for Eq.(2.1) which satisfies the boundary conditions.

Suppose the solution for Eq.(2.3) or Eq.(2.4) can be expressed as a power series of p as

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots . \tag{2.5}$$

The values for the sequence $u_0, u_1, u_2, \dots$ can be found by substituting Eq.(2.5) into Eq.(2.4) and equating coefficients of p with the same power. When $p \longrightarrow 1$, it gives the approximate solution for Eq.(2.1) as

$$u(t) = \lim_{p\to 1} u = u_0 + u_1 + u_2 + u_3 + \dots, \tag{2.6}$$

the Series (2.6) has been proved its convergence in [17].

3. SUMUDU TRANSFORM (ST)

Watugula [43] introduced Sumudu transform as a new integral and is defined as:

$$F(\eta) = S\{f(t)\} = \int_0^\infty \frac{1}{\eta} e^{\frac{-t}{\eta}} f(t)dt, \tag{3.1}$$

or

$$F(\eta) = S\{f(t)\} = \int_0^\infty e^{-t} f(\eta t)dt. \tag{3.2}$$

The linearity is an essential property of ST, that is,

$$S\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 F_1(\eta) + c_2 F_2(\eta), \tag{3.3}$$

where c_1 and c_2 are two constants and $S\{f_1(t)\} = F_1(\eta)$, $S\{f_2(t)\} = F_2(\eta)$.

In this work we used the following properties of ST:

$$S\{t^n\} = n!\eta^n, \tag{3.4}$$

$$S\left\{f^{(n)}(t)\right\} = \frac{1}{\eta^n}F(\eta) - \frac{1}{\eta^n}\sum_{k=0}^{n-1}\eta^k f^(k)(0), \tag{3.5}$$

where $f^{(0)}(0) = f(0)$, $f^{(k)}(t)$, $k = 1, 2, 3, ..., n-1$ are the k^{th} derivatives of the function $f(t)$, and $S\left\{f^{(n)}(t)\right\} = F(\eta)$. If $F(\eta)$ is the Sumudu transform of $f(t)$, then $f(t)$ is called the inverse Sumudu transform of $F(\eta)$ and is expressed by $f(t) = S^{-1}\{F(\eta)\}$, where the inverse Sumudu transform operator is S^{-1}.

From Eq.(3.4) we get:

$$t^n = S^{-1}\{n!\eta^n\}. \tag{3.6}$$

Applying S^{-1} on Eq.(3.3), we obtain the following linearity property:

$$S^{-1}\{c_1F_1(\eta) + c_2F_2(\eta)\} = c_1f_1(t) + c_2f_2(t). \tag{3.7}$$

4. MODIFIED SUMUDU TRANSFORM HOMOTOPY PERTURBATION METHOD (MSTHPM)

In order to illustrate the basic idea of this method, we employ MSTPM to give analytical and approximate solutions for a nonlinear BVPs, as Eq.(2.1). Thus, the same steps of HPM follows until Step (2.4), after which $L(u_0)$ was substituted by an arbitrary function $Z(t)$, where the freedom of HPM was exploited. To solve the problems in this work, it is sufficient to select a polynomial trial function with unknown parameters, A, B, C ..., to be determined.

Taking the Sumudu transform of both sides of Eq.(2.4), we get:

$$S\{L(u) - Z(t) + p(Z(t) + N(u) - f(t))\} = 0. \tag{4.1}$$

Employing the differential property of ST, we have:

$$\frac{1}{\eta^n}S(u) - \frac{1}{\eta^n}u(0) - \frac{1}{\eta^{n-1}}u'(0) - ... - \frac{1}{\eta}u^{(n-1)}(0) = S\{Z(t) + p(-Z(t) - N(u) + f(t))\}, \tag{4.2}$$

or

$$S(u) = \eta^n\left\{\frac{1}{\eta^n}u(0) + \frac{1}{\eta^{n-1}}u'(0) + ... + \frac{1}{\eta}u^{(n-1)}(0)\right\} + \eta^n S\{Z(t) + p(-Z(t) - N(u) + f(t))\}. \tag{4.3}$$

Next, taking S^{-1} to both sides of Eq.(4.3), we have:

$$u(t) = S^{-1}\left\{\eta^n(\frac{1}{\eta^n}u(0) + \frac{1}{\eta^{n-1}}u'(0) + ... + \frac{1}{\eta}u^{(n-1)}(0))\right\} + S^{-1}\{\eta^n S\{Z(t) + p(-Z(t) - N(u) + f(t))\}\}. \tag{4.4}$$

Suppose that:

$$u(t) = \sum_{n=0}^{\infty}p^n u_n, \tag{4.5}$$

is a power series solution of Eq.(2.1).
Next, substituting Eq.(4.5) into Eq.(4.4), we obtain

$$\sum_{n=0}^{\infty} p^n u_n = S^{-1}\left\{\eta^n\left(\frac{1}{\eta^n}u(0)+\frac{1}{\eta^{n-1}}u'(0)+...+\frac{1}{\eta}u^{(n-1)}(0)\right)\right\}+ S^{-1}\left\{\eta^n S\left\{Z(t)+p(-Z(t)-N(\sum_{n=0}^{\infty}p^n u_n)+f(t))\right\}\right\}. \tag{4.6}$$

Equating the identical power terms of p , we obtain:

$$p^0 : u_0 = S^{-1}\left\{\eta^n(\left(\frac{1}{\eta^n}u(0)+\frac{1}{\eta^{n-1}}u'(0)+...+\frac{1}{\eta}u^{(n-1)}(0)\right)+S\left\{Z(t)\right\})\right\} \tag{4.7}$$

$$p^1 : u_1 = S^{-1}\left\{\eta^n S\left\{-N(u_0)-Z(t)+f(t)\right\}\right\} \tag{4.8}$$

$$p^2 : u_2 = S^{-1}\left\{\eta^n S\left\{-N(u_0,u_1)\right\}\right\} \tag{4.9}$$

$$p^3 : u_3 = S^{-1}\left\{\eta^n S\left\{-N(u_0,u_1,u_2)\right\}\right\} \tag{4.10}$$

$$...$$

$$p^j : u_j = S^{-1}\left\{\eta^n S\left\{-N(u_0,u_1,...,u_j)\right\}\right\} \tag{4.11}$$

$$...$$

The approximate solution will be:

$$u(t) = \lim_{p\longrightarrow 1} = u_0+u_1+u_2+u_3+... \tag{4.12}$$

The values of $A, B, C, ...$ are adequately calculated by solving the algebraic system, which is derived as follows:
1. Equation (4.12) should satisfy the boundary conditions at the end points of the interval.
2. In order to determine the values of all the parameters, we need to solve more algebraic equations by adding to those mentioned in Eq.(2.1), until we get the same number of equations and parameters to be determine. By substituting Eq.(4.12) into Eq.(2.1) where the residual is defined and becomes zero, the process will be illustrated by the following test examples.

5. CASE STUDIES

In this portion, we will compare STHPM and MSTHPM to illustrate the solution procedures of both methods in solving two nonlinear ordinary differential equations: first, with mixed boundary conditions and second, with Neumann boundary conditions. The numerical method (RKF45) was used to verify the efficiency of the proposed method, where the square residual error (S.R.E) was used to quantify the accuracy of the proposed solutions. The (S.R.E), which represents the total error made by the proposed approximate solution, is considered to be reliable, where it is equal to zero when the approximate solution turns out to be the exact solution of the differential equation studied.

MIXED BOUNDARY CONDITIONS

Consider the following nonlinear differential equation [41]:

$$u''(t) - \varepsilon u^4(t) = 0, \quad 0 \leq t \leq 1, \tag{5.1}$$

with the boundary conditions

$$u'(0) = 0, \quad u(1) = 1, \tag{5.2}$$

which shows the temperature distribution in a rectangular fin of uniform thickness due to radiation of free space with nonlinearity of high order.

We are going to solve the Eq.(5.1) in detail by using the STHPM and MSTHPM methods to compare them.

THE STHPM METHOD

By applying the fourth order approximation of STHPM method to find a solution for Eq.(5.1), we will proceed as follows:

Identifying terms:

$$L(u) = u''(t), \tag{5.3}$$

$$N(u) = -\varepsilon u^4(t). \tag{5.4}$$

Next, we make a homotopy in accordance with Eq.(2.4), thus we get an approximate analytical solution:

$$(1 - p)(u'' - u_0'') + \rho(u'' - \varepsilon u^4) = 0, \tag{5.5}$$

or

$$u'' = u_0'' + p(-u_0'' + \varepsilon u^4). \tag{5.6}$$

Applying the Sumudu transform, we obtain:

$$S\{u''\} = S\left\{u_0'' + \rho(-u_0'' + \varepsilon u^4)\right\}. \tag{5.7}$$

Employing the differential property of ST for $n = 2$, we get:

$$\frac{1}{\eta^2}U(\eta) - \frac{1}{\eta^2}\eta(0) - \frac{1}{\eta}u'(0) = S\left\{u_0'' + p(-u_0'' + \varepsilon u^4)\right\}. \tag{5.8}$$

that gives

$$u(t) = S^{-1}\left\{\eta^2(\frac{A}{\eta^2} + S\left\{u_0'' + \rho(-u_0'' + \varepsilon u^4)\right\})\right\}. \tag{5.9}$$

obtained upon Solving for $U(\eta)$ and applying the inverse of the Sumudu transforms S^{-1} , where we define $A = u(0)$, and using $u'(0) = 0$.

Then, we suppose that the series solution for $u(t)$ is given by

$$u(t) = \sum_{n=0}^{\infty} p^n u_n. \tag{5.10}$$

Consider the first approximation solution of Eq.(5.1) is:

$$\nu_0 = A, \tag{5.11}$$

which satisfies $u'(0) = 0$.
Substituting Eq.(5.10) and Eq.(5.11) into Eq.(5.9), we have:

$$\sum_{n=0}^{\infty} p^n u_n = S^{-1}\left\{\eta^2(\frac{A}{\eta^2} + \varepsilon S p(\sum_{n=0}^{\infty} p^n u_n)^4)\right\}. \tag{5.12}$$

Comparing the coefficients of like powers of p, we obtain:

$$p^0 : u_0 = S^{-1}\{A\}. \tag{5.13}$$

$$p^1 : u_1 = \varepsilon S^{-1}\left\{\eta^2 S\left\{u_0^4\right\}\right\}. \tag{5.14}$$

$$p^2 : u_2 = \varepsilon S^{-1}\left\{\eta^2 S\left\{4u_0^3 u_1\right\}\right\}. \tag{5.15}$$

$$p^3 : u_3 = \varepsilon S^{-1}\left\{\eta^2 S\left\{6u_0^2 u_1^2 + 4u_0^3 u_2\right\}\right\}. \tag{5.16}$$

$$p^4 : u_4 = \varepsilon S^{-1}\left\{\eta^2 S\left\{4u_1^3 u_0 + 12u_0^2 u_1 u_2 + 4u_0^3 u_3\right\}\right\}. \tag{5.17}$$

Solving equations (5.13) - (5.17) for $u_0(t), u_1(t), u_2(t), ...$, we get:

$$p^0 : u_0(t) = A. \tag{5.18}$$

$$p^1 : u_1(t) = \frac{\varepsilon A^4}{2} t^2. \tag{5.19}$$

$$p^2 : u_2(t) = \frac{\varepsilon^2 A^7}{6} t^4. \tag{5.20}$$

$$p^3 : u_3(t) = \frac{13\varepsilon^3 A^{10}}{180} t^6. \tag{5.21}$$

$$p^4 : u_4 = \frac{161\varepsilon^4 A^{13}}{5040} t^8. \tag{5.22}$$

Substituting (5.18)-(5.22) into (5.10) then, evaluating the *limit* when $p \longrightarrow 1$, the solution is given by:

$$u(t) = A + \frac{\varepsilon A^4}{2} t^2 + \frac{\varepsilon^2 A^7}{6} t^4 + \frac{13\varepsilon^3 A^{10}}{180} t^6 + \frac{161\varepsilon^4 A^{13}}{5040} t^8 + ... \tag{5.23}$$

By applying the boundary condition $u(1) = 1$ and taking $\varepsilon = 7$ as a case study, we can find the value of A as the following:

$$A = 0.535951596523. \tag{5.24}$$

Substituting (5.24) into (5.23), we have:

$$u(t) = 0.535951596523 + 0.288782133227\, t^2 + 0.103734641974\, t^4 + 0.048441842897\, t^6 + 0.023089753791\, t^8. \tag{5.25}$$

THE MSTHPM METHOD

We build a homotopy in accordance with (2.4), thus we get an approximate analytical solution:

$$u'' = Z(t) + p(-Z(t) + \varepsilon u^4), \tag{5.26}$$

where we have substituted $L(u_0)$ for a function of $Z(t)$, which will be defined later.
Applying ST, we obtain

$$S\{u''\} = S\left\{Z(t) + p(-Z(t) + \varepsilon u^4)\right\}. \tag{5.27}$$

Employing the differential property of ST for $n = 2$, we have:

$$\frac{1}{\eta^2}U(\eta) + \frac{1}{\eta^2}u(0) + \frac{1}{\eta}u'(0) = S\left\{Z(t) + p(-Z(t) + \varepsilon u^4)\right\}, \tag{5.28}$$

that gives

$$u(t) = S^{-1}\left\{u^2(\frac{A}{u^2} + S\left\{Z(t) + \rho(-Z(t) + \varepsilon u^4)\right\})\right\}, \tag{5.29}$$

obtained upon solving for $U(\eta)$ and applying the inverse Sumudu transform S^{-1}, where we define $A = u(0)$, and using $u'(0) = 0$. Then, we suppose that the series solution for $u(t)$ is given by

$$u(t) = \sum_{n=0}^{\infty} p^n u_n. \tag{5.30}$$

As well, to get a highly accurate approximation, it is sufficient to choose $Z(t)$, as a linear function

$$Z(t) = Ct + B. \tag{5.31}$$

Substituting (5.30) and (5.31) into (5.29), we have:

$$\sum_{n=0}^{\infty} p^n u_n = S^{-1}\left\{\eta^2(\frac{A}{\eta^2} + S\left\{Ct + B + p(-Ct - B + \varepsilon(\sum_{n=0}^{\infty} p^n u_n)^4)\right\})\right\}. \tag{5.32}$$

By comparing the coefficients of like powers of p, we obtain:

$$p^0 : u_0(t) = S^{-1}\left\{A + B\eta^2 + C\eta^3\right\}, \tag{5.33}$$

$$p^1 : u_1(t) = S^{-1}\left\{\eta^2 S\left\{-B - Ct + \varepsilon u_0^4\right\}\right\}. \tag{5.34}$$

By solving equations (5.33) - (5.34) for $u_0(t)$ and $u_1(t)$, we obtain:

$$p^0 : u_0(t) = A + \frac{B}{2}t^2 + \frac{C}{6}t^3. \tag{5.35}$$

$$\begin{aligned} p^1 : u_1(t) = -\frac{B}{2}t^2 - \frac{C}{6}t^3 + \varepsilon(\frac{A^4}{2}t^2 + \frac{A^3B}{6}t^4 + \frac{A^3C}{30}t^5 + \frac{A^2B^2}{20}t^6 + \frac{A^2BC}{42}t^7 + \\ \frac{336AB^3 + 122C^2A^2}{37632}t^8 + \frac{AB^2C}{144}t^9 + \frac{540B^4 + 1440C^2AB}{777600}t^{10} + \\ \frac{1320B^3C + 5940C^3A}{7840800}t^{11} + \frac{C^2B^2}{3168}t^{12} + \frac{BC^3}{16848}t^{13} + \frac{C^4}{234872}t^{14}). \end{aligned} \tag{5.36}$$

By substituting (5.35) and (5.36) into (5.30) and evaluating the lim when $p \longrightarrow 1$, the first

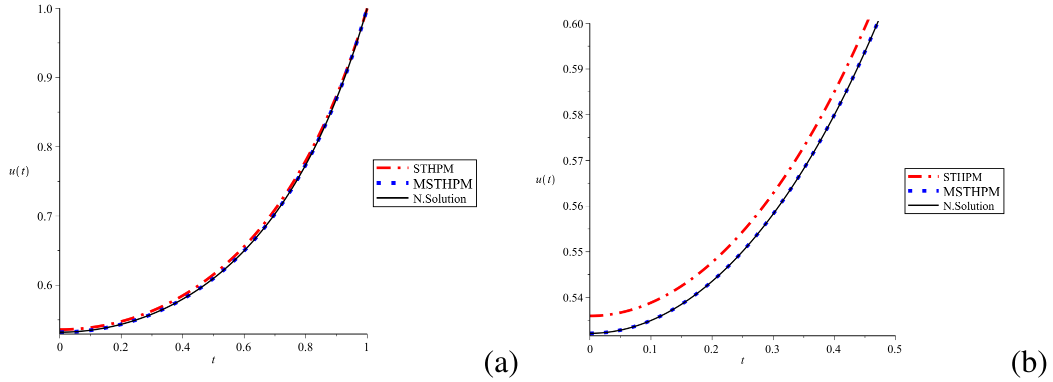

Figure 1: (a) and (b) show the comparison between STHPM (5.25), MSTHPM (5.39) approximations and numerical solution of (5.1).

order approximate solution is given by:

$$u(t) = A + \varepsilon(\frac{A^4}{2}t^2 + \frac{A^3B}{6}t^4 + \frac{A^3C}{30}t^5 + \frac{A^2B^2}{20}t^6 + \frac{A^2BC}{42}t^7 + \frac{336AB^3 + 122C^2A^2}{37632}t^8 + \frac{AB^2C}{144}t^9 + \frac{540B^4 + 1440C^2AB}{777600}t^{10} + \frac{1320B^3C + 5940C^3A}{7840800}t^{11} + \frac{C^2B^2}{3168}t^{12} + \frac{BC^3}{16848}t^{13} + \frac{C^4}{235872}t^{14}). \tag{5.37}$$

Applying the boundary condition $u(1) = 1$ on Eq.(5.37), to calculate the values of A, B, and C. In addition, following MSTHPM algorithm, we substitute (5.37) into (5.1) and evaluate the resultant expression for the values $t = 1 \times 10^{-4}$ and $t = 0.75$, which lies in [0,1], upon following the above procedure, we have a system of equations for A, B , and C . We take $\varepsilon = 7$ as a case study to obtain the values:

$$A = 0.5321388777 \quad B = 0.5612820355 \quad C = 0.6535691224. \tag{5.38}$$

Substituting (5.38) into (5.37), we obtain:

$$u(t) = 0.5321388777 + 0.28065190971\, t + 0.0986740378\, t^4 + 0.0229796431\, t^5 + 0.0312234083\, t^6 + 0.0173129628\, t^7 + 0.0084009121\, t^8 + 0.0053261617\, t^9 + 21362986 \times 10^{-3}\, t^{10} + 7.879266 \times 10^{-4}\, t^{11} + 2.973433 \times 10^{-4}\, t^{12} + 6.51037 \times 10^{-5}\, t^{13} + 5.4149 \times 10^{-6}\, t^{14}. \tag{5.39}$$

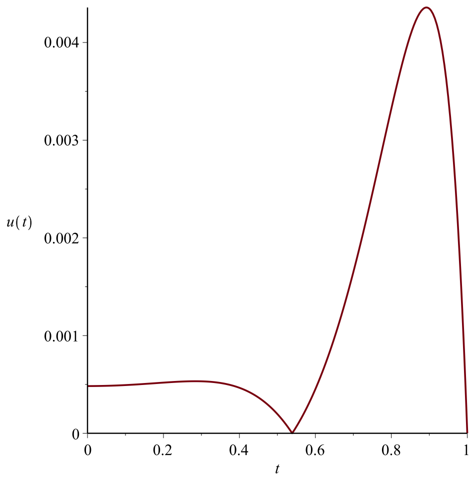

Figure 2: shows absolute error (AE) between MSTHPM approximation (5.39) and numerical solution of (5.1).

Figures 1(a) and (b) show that both schemes managed to accurately model the stationary temperature distribution, and they proved that the approximate solution of the proposed method is useful and adequate in some cases where the solution is difficult to find , such as, for the case where one of the boundary conditions is unknown where the region closer to the unknown endpoints of the interval is very hard to model; in our examples, $u(0)$, and the perturbation parameter, $\varepsilon = 7$ is large. In addition, upon calculating the (S.R.E) of Eq.(5.25) and Eq.(5.39), for the same big value of the parameter $\varepsilon = 7$, the resulting values obtained, were 0.2813957390937 and 0.0116413047163, respectively. In fact, the addition of the initial linear function led to an increase in accuracy of results and accelerate the convergence, while there was an augmentation in the number of the terms and a rise in the order of approximations. It is worth mentioning that Eq.(5.39) is just a first order approximation, whereas, Eq.(5.25) is a fourth order approximation. considering The above, it shows that MSTHPM is 28 times more accurate than STHPM. Therefore, it is demonstrated that the proposed method is not only limited to small parameters, but it can produce approximate solutions for difficult problems that contain large valued parameters. Actually, as shown in Figure 2, it is noteworthy that the biggest absolute error (AE) is only 0.0043 which is significantly precise, considering that (5.39) is just a first order approximate solution for (5.25).

NEUMANN BOUNDARY CONDITIONS

We will now extend our analysis to the following second order nonlinear differential equation [41]

$$u''(t) + u(t) - u^2(t) = 0, \quad 0 \le t \le 1, \tag{5.40}$$

with the boundary conditions

$$u'(0) = 0, \quad u'(1) = \frac{\pi}{4}. \tag{5.41}$$

THE STHPM METHOD

Applying the STHPM method to equation (5.40) to obtain the solution,
By constructing the homotopy, we obtain:

$$u'' = u_0'' + p(-u_0'' - u + u^2). \tag{5.42}$$

Applying the Sumudu transform to , we get:

$$S\{u''\} = S\left\{u_0'' + p(-u_0'' - u + u^2)\right\}. \tag{5.43}$$

Next, employing the differential property of ST for $n = 2$, we get:

$$\frac{1}{\eta^2}U(\eta) - \frac{1}{\eta^2}u(0) - \frac{1}{\eta}u'(0) = S\left\{u_0'' + p(-u_0'' - u + u^2)\right\}, \tag{5.44}$$

that gives

$$u(t) = S^{-1}\left\{\eta^2(\frac{A}{\eta^2} + S\left\{u_0'' + p(-u_0'' - u + u^2)\right\})\right\}, \tag{5.45}$$

obtained upon solving for $U(\eta)$ and applying the inverse of the Sumudu transforms S^{-1} , where we define $A = u(0)$, and using $u'(0) = 0$.
Employing the power series as the solution for $y(t)$:

$$u(t) = \sum_{n=0}^{\infty} p^n u_n. \tag{5.46}$$

Consider the first approximate solution of Eq.(5.40) is:

$$u_0 = A \tag{5.47}$$

which satisfies $u'(0) = 0$.
Substituting (5.46) and (5.47) into (5.45), we have:

(5.48)
$$u(t) = \sum_{n=0}^{\infty} p^n u_n = S^{-1}\left\{\eta^2(\frac{A}{\eta^2} + S\left\{u_0'' + \rho(-u_0'' - (\sum_{n=0}^{\infty} p^n u_n) + (\sum_{n=0}^{\infty} p^n u_n)^2)\right\})\right\}.$$

the identical coefficients of p can be readily identified as:

$$p^0 : u_0(t) = S^{-1}\{A\} \tag{5.49}$$

$$p^1 : u_1(t) = S^{-1}\left\{\eta^2 S\left\{-u_0 + u_0^2\right\}\right\} \tag{5.50}$$

$$p^2 : u_2(t) = S^{-1}\left\{\eta^2 S\{-u_1 + 2u_0u_1\}\right\} \tag{5.51}$$

$$p^3 : u_3(t) = S^{-1}\left\{\eta^2 S\left\{-u_2 + u_1^2 u_0 u_2\right\}\right\} \tag{5.52}$$

$$p^4 : u_4(t) = S^{-1}\left\{\eta^2 S\{-u_3 + 2u_0u_3 + u_1u_2\}\right\}. \tag{5.53}$$

By Solving equations (5.49) - (5.53) for $u_0(t),\ u_1(t),\ u_2(t),\ ...$, we obtain:

$$p^0 : u_0(t) = A \tag{5.54}$$

$$p^1 : u_1(t) = \frac{A^2 - A}{2}t^2 \tag{5.55}$$

$$p^2 : u_2(t) = \frac{A - 3A^2 + 2A^3}{4!}t^4 \tag{5.56}$$

$$p^3 : u_3(t) = \frac{(A^2 - A)(2A - 1)^2 + 6(A^2 - A)^2}{6!} t^6 \tag{5.57}$$

$$p^4 : u_4(t) = \frac{(A - 3A^2 + 2A^3)(36(A^2 - A)^2 + (2A - 1)^2)}{8!} t^8. \tag{5.58}$$

Substituting (5.54)-(5.58) into (5.46), we obtain:

$$u(t) = A + \frac{(A^2 - A)}{2} t^2 + \frac{A - 3A^2 + 2A^3}{4!} t^4 + \frac{10A^4 - 20A^3 + 11A^2 - A}{6!} t^6 + \frac{(A - 3A^2 + 2A^3)(36(A^2 - A)^2 + (2A - 1)^2)}{8!} t^8. \tag{5.59}$$

Applying the boundary condition $u'(1) = \frac{\pi}{4}$ on Eq.(5.59), to calculate the values of A as the following:

$$A = -0.6793160999. \tag{5.60}$$

Substituting (5.60) into (5.59), we obtain:

$$u(t) = -0.6793160999 + 0.5703932320\, t^2 - 0.1121123203\, t^4 + 0.01965933892\, t^6 - 0.003111881558\, t^8 \tag{5.61}$$

THE MSTHPM METHOD

By applying the MSTHPM method to (5.40), we build the homotopy as follows:

$$u''(t) = Z(t) + p(-Z(t) - u(t) + u^2(t)). \tag{5.62}$$

where we have substituted $L(u_0)$ for $Z(t)$.

Applying ST we obtain:

$$S\{u''\} = S\left\{Z(t) + p(-Z(t) - u + u^2)\right\}. \tag{5.63}$$

Employing the differential property of ST for $n = 2$, we obtain:

$$\frac{1}{\eta^2} U(\eta) - \frac{1}{\eta^2} u(0) - \frac{1}{\eta} u'(0) = S\left\{Z(t) + p(-Z(t) - u + u^2)\right\}, \tag{5.64}$$

that gives

$$u(t) = S^{-1}\left\{\eta^2\left(\frac{A}{\eta^2} + S\left\{Z(t) + p(-Z(t) - u + u^2)\right\}\right)\right\}, \tag{5.65}$$

obtained upon Solving for $U(\eta)$ and applying the inverse of the Sumudu transforms S^{-1}, where we define $A = u(0)$, and using $u'(0) = 0$.

Now, employing the power series as the solution for Eq.(5.65):

$$u(t) = \sum_{n=0}^{\infty} p^n u_n. \tag{5.66}$$

After substituting (5.66) and (5.31) into (5.65) we get:

$$\sum_{n=0}^{\infty} p^n u_n = S^{-1}\left\{\eta^2\left(\frac{A}{\eta^2} + S\left\{Ct + B + p\left(-Ct - B - \left(\sum_{n=0}^{\infty} p^n u_n\right) + \left(\sum_{n=0}^{\infty} p^n u_n\right)^2\right)\right\}\right)\right\}. \tag{5.67}$$

The identical coefficients of p can be readily identified as:

$$p^0 : u_0(t) = S^{-1}\left\{A + B\eta^2 + C\eta^3\right\}, \tag{5.68}$$

$$p^1 : u_1(t) = S^{-1}\left\{\eta^2 S\left\{-B - Ct + u_0 + u_0^2\right\}\right\}. \tag{5.69}$$

Solving equations (5.68) - (5.69) for $u_0(t)$ and $u_1(t)$, we obtain:

$$p^0 : u_0(t) = A + \frac{B}{2}t^2 + \frac{C}{6}t^3, \tag{5.70}$$

(5.71)
$$p^1 : u_1(t) = \frac{A - A^2 + B}{2}t^2 + \frac{C}{6}t^3 + \frac{B - 2AB}{24}t^4 + \frac{C - 2AC}{120}t^5 - \frac{B^2}{120}t^6 - \frac{AC}{252}t^7 - \frac{C^2}{2016}t^8.$$

By substituting (5.70) and (5.71) into (5.66) and evaluating the *limit* when $p \longrightarrow 1$, a first order approximation is given by:

(5.72)
$$u(t) = A + \frac{A^2 - A}{2}t^2 + \frac{2AB - B}{24}t^4 + \frac{2AC - C}{120}t^5 + \frac{B^2}{120}t^6 - \frac{B^2}{120}t^6 - \frac{BC}{252}t^7 + \frac{C^2}{2016}t^8.$$

Equation (5.72) should satisfy the boundary condition $u(1) = \frac{\pi}{4}$, hence, this fact is used to calculate the values of A, B, and C. In addition, following MSTHPM algorithm, we substitute (5.72) into (5.40) and evaluate the resultant expression for the values $t = 0.20$ and $t = 0.75$, which lies in [0,1]. After following the above procedure, we have a system of equations for A, B and C, to obtain the values:

$$A = -0.67731056268 \quad B = 1.16504731425 \quad C = -0.568773061101. \tag{5.73}$$

Substituting (5.73) into (5.72), we obtain:

(5.74)
$$u(t) = -0.67731056268 + 0.568030080497\,t^2 - 0.11430187576\,t^4 + 0.01116037554\,t^5 + 0.01131112704\,t^6 - 0.00262955368\,t^7 + 0.00016046766\,t^8.$$

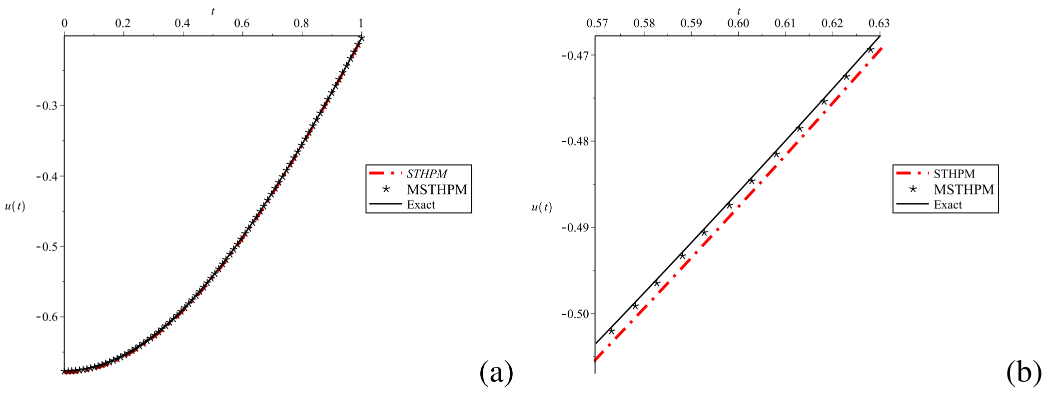

Figure 3: (a) and (b) show the comparison between STHPM (5.61), MSTHPM (5.74) approximations and numerical solution of (5.40).

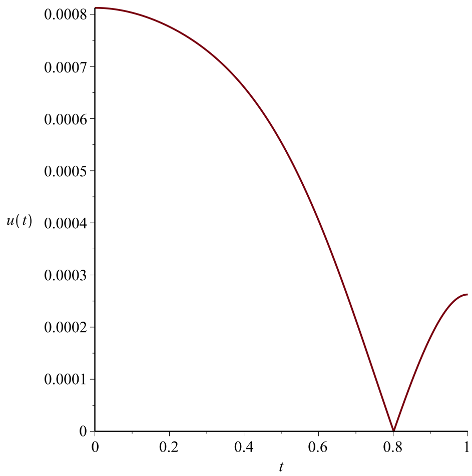

Figure 4: shows absolute error (AE) between MSTHPM approximation (5.74) and numerical solution of (5.40).

Based on Fig (3), it is clear that both the schemes are in good agreement with the numerical solution. Moreover, the (S.R.E) shows that the approximations of both schemes are 0.0000762972854 and 0.00002859606371 respectively, thus, the first order approximation of MSTHPM is better than the fourth order approximation of STHPM, while the biggest absolute error of the approximate solution of MSTHPM is 0.0008, which demonstrates that it is highly accurate.

From our study cases, we concluded that the MSTHPM is suitable and effective in dealing with the previous nonlinear BVPs and may be applied to other types of differential equations.

6. CONCLUSIONS

In this paper, MSTHPM has been efficiently used for solving the boundary value problems to obtain precise approximate solutions. Hence the solutions were compared with the STHPM and numerical solutions. Furthermore, the ordinary differential equations have been solved by the algebraic system to get the unknown conditions. The results showed that there is a possibility to accelerate the convergence of the solution by using MSTHPM for the given BVPs. Other advantages of this method is that it employs the first order approximation which can be used to solve other nonlinear problems.

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