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STRUCTURAL AND SPECTRAL PROPERTIES OF $k-$QUASI CLASS Q OPERATORS

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ABSTRACT. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be $k-$quasi class Q if $\|T^{k+1}x\|^2 \leq \frac{1}{2}\left(\|T^{k+2}x\|^2 + \|T^k x\|^2\right)$, for all $x \in \mathcal{H}$, where k is a natural number. In this paper, first we will prove some results for the matrix representation of $k-$quasi class Q operators. Then, we will give the inclusion of approximate point spectrum of $k-$quasi class Q operators. Also, we will give the equivalence between Aluthge transformation and $*-$Aluthge transformation of $k-$quasi class Q operators.

Key words and phrases: $k-$quasi class Q operator; Approximate point spectrum; Contraction; Aluthge transformation.

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1. INTRODUCTION

In this paper let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and let $\mathcal{L}(\mathcal{H})$ denote the C^* algebra of all bounded operators on $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, we denote by $\ker T$ the null space, by $T(\mathcal{H})$ the range of T. By $\sigma(T)$ we write the spectrum of T, the $r(T)$ is the spectral radius of operator T which is defined by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. The $\sigma_a(T)$ is the approximate point spectrum of operator T and it is proved that if $\lambda \in \sigma_a(T)$, then there exist the sequence (x_n), such as $\|x_n\| = 1$ and $\|(T-\lambda I)x_n\| \to 0, n \to \infty$. A complex number λ is said to be in the point spectrum $\sigma_p(T)$ if there is a nonzero $x \in \mathcal{H}$ such that $(T-\lambda)x = 0$.

The null operator and the identity on $\mathcal{H}$ will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint, and $\|T\| = \|T^*\|$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is a positive operator, $T \geq O$, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. If two operator $T \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$ are positive operators and $TS = ST$ then TS is also positive operator.

The operator T is an isometry if $\|Tx\| = \|x\|$, for all $x \in \mathcal{H}$. The operator T is called unitary operator if $T^*T = TT^* = I$. The operator T is normaloid if $r(T) = \|T\|$ and it is quasinilpotent if $r(T) = 0$.

An operator $T \in \mathcal{L}(\mathcal{H})$ belongs to class Q if $T^{*2}T^2 - 2T^*T + I \geq O$. It is proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of class Q if $\|Tx\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2)$ (see [2]).

Definition 1.1. ([4]) An operator $T \in \mathcal{L}(\mathcal{H})$ belongs to $k-$quasi class Q if

$$\|T^{k+1}x\|^2 \leq \frac{1}{2}\left(\|T^{k+2}x\|^2 + \|T^kx\|^2\right),$$

for all $x \in \mathcal{H}$, where k is a natural number.

Equivalently operator $T \in \mathcal{L}(\mathcal{H})$ belongs to $k-$quasi class Q if

$$T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \geq O,$$

where k is a natural number.

For $k = 1$, then $1-$quasi class Q coincides with the quasi class Q (see [3]).

2. MATRIX REPRESENTATION OF $k-$QUASI CLASS Q OPERATORS

In this section we give some results for the matrix representation of $k-$quasi class Q operators.

Theorem 2.1. *Suppose that T^k does not have a dense range, then the following statements are equivalent:*

(1) *Operator T is a $k-$quasi class Q operator, for a positive integer k;*

(2) $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ *on* $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}$, *where A is an operator of the class Q on $\overline{T^k(\mathcal{H})}$, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.*

Proof. $(1) \Rightarrow (2)$ It has been proved in [4].

$(2) \Rightarrow (1)$ Suppose that $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ on $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}$, where A is an operator of the class Q on $\overline{T^k(\mathcal{H})}$, and $C^k = O$.

A simple calculation shows that:

$$T^* = \begin{pmatrix} A^* & 0 \\ B^* & C^* \end{pmatrix},$$

$$T^*T = \begin{pmatrix} A^*A & A^*B \\ B^*A & B^*B + C^*C \end{pmatrix},$$

$$T^{*2}T^2 = \begin{pmatrix} A^{*2}A^2 & A^{*2}AB + A^{*2}BC \\ B^*A^*A^2 + C^*B^*A^2 & |AB + BC|^2 + |C^2|^2 \end{pmatrix},$$

$$T^{*k} = \begin{pmatrix} A^{*k} & 0 \\ (\sum_{j=0}^{k-1} A^jBC^{k-1-j})^* & 0 \end{pmatrix},$$

$$T^k = \begin{pmatrix} A^k & (\sum_{j=0}^{k-1} A^jBC^{k-1-j}) \\ 0 & 0 \end{pmatrix},$$

Then, we have

$$\begin{aligned} & T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \\ &= \begin{pmatrix} A^{*k} & 0 \\ (\sum_{j=0}^{k-1} A^jBC^{k-1-j})^* & 0 \end{pmatrix} \\ &\times \begin{pmatrix} D & A^{*2}AB + A^{*2}BC \\ B^*A^*A^2 + C^*B^*A^2 - 2B^*A & |AB + BC|^2 + |C^2|^2 - 2(B^*B + C^*C) + I \end{pmatrix} \\ &\times \begin{pmatrix} A^k & \sum_{j=0}^{k-1} A^jBC^{k-1-j} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A^{*k}DA^k & A^{*k}D\sum_{j=0}^{k-1} A^jBC^{k-1-j} \\ (\sum_{j=0}^{k-1} A^jBC^{k-1-j})^*DA^k & M \end{pmatrix}, \end{aligned}$$

where $D = A^{*2}A^2 - 2A^*A + I$, $M = (\sum_{j=0}^{k-1} A^jBC^{k-1-j})^*D\sum_{j=0}^{k-1} A^jBC^{k-1-j}$. Let $v = x \oplus y$ be a vector in $\mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}$, where $x \in \overline{T^k(\mathcal{H})}$ and $y \in \ker T^{*k}$. Then,

$$\begin{aligned} & \left\langle T^{*k}(T^{*2}T^2 - 2T^*T + I)T^kv, v \right\rangle \\ &= \left\langle A^{*k}DA^kx, x \right\rangle \\ &+ \left\langle A^{*k}D\sum_{j=0}^{k-1} A^jBC^{k-1-j}y, x \right\rangle \\ &+ \left\langle (\sum_{j=0}^{k-1} A^jBC^{k-1-j})^*DA^kx, y \right\rangle \\ &+ \left\langle (\sum_{j=0}^{k-1} A^jBC^{k-1-j})^*D\sum_{j=0}^{k-1} A^jBC^{k-1-j}y, y \right\rangle \\ &= \left\langle D(A^kx + \sum_{j=0}^{k-1} A^jBC^{k-1-j}y), A^kx + \sum_{j=0}^{k-1} A^jBC^{k-1-j}y \right\rangle. \end{aligned}$$

Since A is an operator of the class Q, we have that $D = A^{*2}A^2 - 2A^*A + I \geq 0$. Therefore,

$$\left\langle T^{*k}(T^{*2}T^2 - 2T^*T + I)T^kv, v \right\rangle \geq 0$$

for all $v \in \mathcal{H}$. Hence,

$$T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \geq 0$$

So we have that T is a $k-$quasi class Q operator. ∎

Theorem 2.2. *Let T be an operator of $k-$quasi class Q, $0 \neq \lambda \in \sigma_p(T)$ and*

$$T = \begin{pmatrix} \lambda & A \\ O & B \end{pmatrix} \quad \textit{on } \mathcal{H} = \ker(T-\lambda) \oplus (\ker(T-\lambda))^{\perp}.$$

Then

$$A(B-\lambda)B^k = 0.$$

Proof. Without loss the generality, we may assume that $\lambda = 1$. For each natural number k,

$$T^k = \begin{pmatrix} 1 & A(I+B+B^2+...+B^{k-1}) \\ O & B^k \end{pmatrix},$$
$$T^{*k} = \begin{pmatrix} 1 & 0 \\ (I+B^*+B^{*2}+...+B^{*(k-1)})A^* & B^{*k} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ [A(I+B+B^2+...+B^{k-1})]^* & B^{*k} \end{pmatrix},$$
$$T^{*(k+2)}T^{k+2} = \begin{pmatrix} 1 & A(I+B+...+B^{k+1}) \\ [A(I+B+...+B^{k+1})]^* & |\, A(I+B+...+B^{k+1})|^2 + |\, B|^{2(k+2)} \end{pmatrix}$$

Since operator T is a $k-$quasi class Q operator, then,

$$T^{*(k+2)}T^{k+2} - 2T^{*(k+1)}T^{k+1} + T^{*k}T^k \geq 0,$$
$$\begin{pmatrix} 1 & A(I+B+...+B^{k+1}) \\ [A(I+B+...+B^{k+1})]^* & |\, A(I+B+...+B^{k+1})|^2 + |\, B|^{2(k+2)} \end{pmatrix}$$
$$-2\begin{pmatrix} 1 & A(I+B+...+B^{k}) \\ [A(I+B+...+B^{k})]^* & |\, A(I+B+...+B^{k})|^2 + |\, B|^{2(k+1)} \end{pmatrix}$$
$$+\begin{pmatrix} 1 & A(I+B+...+B^{k-1}) \\ [A(I+B+...+B^{k-1})]^* & |\, A(I+B+...+B^{k-1})|^2 + |\, B|^{2k} \end{pmatrix}$$

So,

$$T^{*(k+2)}T^{k+2} - 2T^{*(k+1)}T^{k+1} + T^{*k}T^k$$
$$= \begin{pmatrix} 0 & AB^{k+1} - AB^k \\ (AB^{k+1} - AB^k)^* & C \end{pmatrix} \geq 0,$$

for any positive operator C.
Hence, $AB^{k+1} - AB^k = A(B-I)B^k = 0$.
∎

Theorem 2.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be the operator defined as*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.$$

If A is operator of class Q, then T is an operator of $k-$quasi class Q.

Proof. Let be $D = A^{*2}A^2 - 2A^*A + I$. A simple calculation shows that:

$$T^* = \begin{pmatrix} A^* & 0 \\ B^* & 0 \end{pmatrix},$$

$$T^{*(k+2)} = \begin{pmatrix} A^{*(k+2)} & 0 \\ B^*A^{*(k+1)} & 0 \end{pmatrix},$$

$$T^{(k+2)} = \begin{pmatrix} A^{(k+2)} & A^{(k+1)}B \\ 0 & 0 \end{pmatrix},$$

$$T^{*(k+2)}T^{(k+2)} = \begin{pmatrix} A^{*(k+2)}A^{(k+2)} & A^{*(k+2)}A^{(k+1)}B \\ B^*A^{*(k+1)}A^{(k+2)} & B^*A^{*(k+1)}A^{(k+1)}B \end{pmatrix}.$$

$$\begin{aligned}
&T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \\
=&T^{*(k+2)}T^{(k+2)} - 2T^{*(k+1)}T^{(k+1)} + T^{*k}T^k \\
=&\begin{pmatrix} A^{*k}DA^k & A^{*k}DA^{(k-1)}B \\ B^*A^{*(k-1)}DA^k & B^*A^{*(k-1)}DA^{(k-1)}B \end{pmatrix}
\end{aligned}$$

Let $u = x \oplus y \in \mathcal{H} \oplus \mathcal{H}$. Then,

$$\begin{aligned}
&\langle (T^{*(k+2)}T^{(k+2)} - 2T^{*(k+1)}T^{(k+1)} + T^{*k}T^k)u, u\rangle \\
&= \langle A^{*k}DA^kx, x\rangle + \langle A^{*k}DA^{(k-1)}By, x\rangle \\
&+ \langle B^*A^{*(k-1)}DA^kx, y\rangle + \langle B^*A^{*(k-1)}DA^{(k-1)}By, y\rangle \\
&= \langle DA^kx, A^kx\rangle + \langle DA^{(k-1)}By, A^kx\rangle \\
&+ \langle DA^kx, A^{(k-1)}By\rangle + \langle DA^{(k-1)}By, A^{(k-1)}By\rangle \\
&= \langle D(A^kx + A^{(k-1)}By), (A^kx + A^{(k-1)}By)\rangle \geq 0
\end{aligned}$$

because A is operator of class Q then, $D = A^{*2}A^2 - 2A^*A + I \geq O$, so this proves the result. ∎

3. SOME PROPERTIES OF $k-$QUASI CLASS Q OPERATORS

In this section we prove some properties of $k-$quasi class Q operators. First, we give the inclusion of approximate point spectrum of $k-$quasi class Q operators.

Theorem 3.1. *Let $T \in L(H)$ be a regular $k-$quasi class Q operator. Then the approximate point spectrum of operator T lies in the disc*

$$\sigma_a(T) \subseteq \{\lambda \in C : \frac{\sqrt{2}}{\|T^{-k-1}\| \cdot \sqrt{\|T^{k+1}\|^2 + \|T^{k-1}\|^2}} \leq |\lambda| \leq \|T\|\}.$$

Proof. Let T be a regular $k-$quasi class Q operator. For every unit vector x in Hilbert space $\mathcal{H}$, we have:

$$\begin{aligned}
&\|x\|^2 = \|(T^{k+1})^{-1} \cdot (T^{k+1})x\|^2 \\
\leq&\|(T^{k+1})^{-1}\|^2 \cdot \|T^{k+1}x\|^2 \\
\leq&\|(T^{k+1})^{-1}\|^2 \cdot \frac{1}{2} \cdot (\|T^{k+2}x\|^2 + \|T^kx\|^2) \\
\leq&\frac{1}{2} \cdot \|(T^{k+1})^{-1}\|^2 \cdot (\|T^{k+1}\|^2 \cdot \|Tx\|^2 + \|T^{k-1}\|^2 \cdot \|Tx\|^2).
\end{aligned}$$

So,

$$2 \le \|Tx\|^2 \cdot \|(T^{k+1})^{-1}\|^2 \cdot (\|T^{k+1}\|^2 + \|T^{k-1}\|^2),$$

where we have

$$\|Tx\| \ge \frac{\sqrt{2}}{\|T^{-k-1}\| \cdot \sqrt{\|T^{k+1}\|^2 + \|T^{k-1}\|^2}}.$$

Now, assume that $\lambda \in \sigma_a(T)$, then there exists a sequence (x_n), such as $\|x_n\| = 1$ and $\|(T - \lambda I)x_n\| \to 0, n \to \infty$.

From the last inequation we have:

$$\|Tx_n - \lambda x_n\| \ge \|Tx_n\| - |\lambda| \cdot \|x_n\| \ge \frac{\sqrt{2}}{\|T^{-k-1}\| \cdot \sqrt{\|T^{k+1}\|^2 + \|T^{k-1}\|^2}} - |\lambda|.$$

Now, when $n \to \infty$ we have

$$|\lambda| \ge \frac{\sqrt{2}}{\|T^{-k-1}\| \cdot \sqrt{\|T^{k+1}\|^2 + \|T^{k-1}\|^2}}.$$

So, we have

$$\sigma_a(T) \subseteq \{\lambda \in C : \frac{\sqrt{2}}{\|T^{-k-1}\| \cdot \sqrt{\|T^{k+1}\|^2 + \|T^{k-1}\|^2}} \le |\lambda| \le \|T\|\}.$$

Therefore the proof is completed. ∎

A contraction is an operator T such that $\|T\| \le 1$, or $\|Tx\| \le \|x\|$, for every x in $\mathcal{H}$ (equivalently $T^*T \le I$) (see [2]).

Theorem 3.2. *Let T be an operator of $k-$quasi class Q. If operator T^2 is a contraction then so is operator T.*

Proof. Let T be a $k-$quasi class Q operator.

Then,

$$T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \ge O,$$
$$T^{*(k+2)}T^{k+2} - 2T^{*(k+1)}T^{k+1} + T^{*k}T^k \ge 0.$$

From this we have:

$$\begin{aligned}
&2T^{*(k+1)}T^{k+1} - 2T^{*k}T^k \le T^{*(k+2)}T^{k+2} - T^{*k}T^k \\
\le &2T^{*k}(T^*T - I)T^k \le T^{*k}(T^{*2}T^2 - I)T^k \\
\le &2(T^*T - I) \le T^{*2}T^2 - I.
\end{aligned}$$

Since operator T^2 is a contraction then

$$T^{*2}T^2 \le I.$$

From the last inequation we have that, $T^*T \le I$. So this implies that operator T is a contraction whenever T^2 is. ∎

Theorem 3.3. *Let be T an invertible operator and N be an operator such that N commutes with T^*T. Then N is $k-$quasi class Q if and only if TNT^{-1} is of $k-$quasi class Q.*

Proof. Let N be a $k-$quasi class Q operator. Then,

$$N^{*k}(N^{*2}N^2 - 2N^*N + I)N^k \geq 0.$$

From this we have that:

$$TN^{*k}[N^{*2}N^2 - 2N^*N + I]N^kT^* \geq 0.$$

Consider,

$$\begin{aligned} &TN^{*k}[N^{*2}N^2 - 2N^*N + I]N^kT^*[TT^*] \\ =&TN^{*k}[N^{*2}N^2 - 2N^*N + I]N^k[T^*T]T^* \\ =&T[T^*T]N^{*k}[N^{*2}N^2 - 2N^*N + I]N^kT^* \\ =&[TT^*]TN^{*k}[N^{*2}N^2 - 2N^*N + I]N^kT^*. \end{aligned}$$

So, we have that operator TT^* commutes with operator

$$TN^{*k}[N^{*2}N^2 - 2N^*N + I]N^kT^*.$$

Then operator $[TT^*]^{-1}$ also commutes with operator

$$TN^{*k}[N^{*2}N^2 - 2N^*N + I]N^kT^*.$$

Since the operators $[TT^*]^{-1}$ and $TN^{*k}[N^{*2}N^2 - 2N^*N + I]N^kT^*$ are positive and since they commute with each other we have that their product is also positive operator:

$$TN^{*k}[N^{*2}N^2 - 2N^*N + I]N^kT^*[TT^*]^{-1} \geq 0.$$

Now, since operator N commutes with operator T^*T, we get,

$$(TNT^{-1})^{*k} = (TNT^{-1})^*(TNT^{-1})^* \cdots (TNT^{-1})^*$$

$$= T^{*-1}N^*T^*T^{*-1}N^*T^* \cdots T^{*-1}N^*T^* = T^{*-1}N^{*k}T^* \tag{3.1}$$

$$(TNT^{-1})^k = TNT^{-1}TNT^{-1} \cdots TNT^{-1} = TN^kT^{-1} \tag{3.2}$$

$$(TNT^{-1})^{*2}(TNT^{-1})^2 = TN^{*2}N^2T^{-1} \tag{3.3}$$

$$(TNT^{-1})^*(TNT^{-1}) = T^{*-1}N^*T^*TNT^{-1} = TN^*NT^{-1} \tag{3.4}$$

To prove that TNT^{-1} is $k-$quasi class Q operator, the equation (3.1), (3.2), (3.3) and (3.4) we substitute in above expression:

$$(TNT^{-1})^{*k}[(TNT^{-1})^{*2}(TNT^{-1})^2 - 2(TNT^{-1})^*(TNT^{-1}) + I](TNT^{-1})^k$$

and we have

$$\begin{aligned} &T^{*-1}N^{*k}T^*[TN^{*2}N^2T^{-1} - 2TN^*NT^{-1} + I]TN^kT^{-1} \\ =&T^{*-1}N^{*k}T^*T[N^{*2}N^2 - 2N^*N + I]T^{-1}TN^kT^{-1} \\ =&T^{*-1}N^{*k}T^*T[N^{*2}N^2 - 2N^*N + I]N^kT^{-1} \\ =&T^{*-1}T^*TN^{*k}[N^{*2}N^2 - 2N^*N + I]N^kT^{-1} \\ =&TN^{*k}[N^{*2}N^2 - 2N^*N + I]N^kT^{-1} \end{aligned}$$

Now we have to prove that that the last expression is positive. From the fact that we prove before, that

$$TN^{*k}[N^{*2}N^2 - 2N^*N + I]N^kT^*[TT^*]^{-1} \geq 0$$

we have:

$$TN^{*k}[N^{*2}N^2 - 2N^*N + I]N^kT^*T^{*-1}T^{-1} \geq 0$$
$$\Rightarrow TN^{*k}[N^{*2}N^2 - 2N^*N + I]N^kT^{-1} \geq 0$$

Hence, TNT^{-1} is $k-$quasi class Q operator.
Conversely, let TNT^{-1} be a $k-$quasi class Q operator.

$$(TNT^{-1})^{*k}[(TNT^{-1})^{*2}(TNT^{-1})^2 - 2(TNT^{-1})^*(TNT^{-1}) + I](TNT^{-1})^k \geq 0.$$

Then similar as before, after substituting the equation (3.1), (3.2), (3.3) and (3.4) we have:

$$TN^{*k}[N^{*2}N^2 - 2N^*N + I]N^kT^{-1} \geq 0$$
$$\Rightarrow T^*TN^{*k}[N^{*2}N^2 - 2N^*N + I]N^kT^{-1}T \geq 0$$
$$\Rightarrow [T^*T]N^{*k}[N^{*2}N^2 - 2N^*N + I]N^k \geq 0.$$

Since operator $[T^*T]$ commutes with operator N and hence with operator

$$[T^*T]N^{*k}[N^{*2}N^2 - 2N^*N + I]N^k.$$

Then also operator $[T^*T]^{-1}$ commutes with operator

$$[T^*T]N^{*k}[N^{*2}N^2 - 2N^*N + I]N^k.$$

Since the operators $[T^*T]^{-1}$ and $[T^*T]N^{*k}[N^{*2}N^2 - 2N^*N + I]N^k$ are positive and since they commute with each other we have:

$$[T^*T]^{-1}[T^*T]N^{*k}[N^{*2}N^2 - 2N^*N + I]N^k \geq 0.$$

Therefore,

$$N^{*k}[N^{*2}N^2 - 2N^*N + I]N^k \geq 0.$$

Hence, N is $k-$quasi class Q operator. ∎

Theorem 3.4. *If $k-$quasi class Q operator T commutes with an isometric operator S, then TS is $k-$quasi class Q operator.*

Proof. Let T be a $k-$quasi class Q operator. Then,

$$T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \geq O.$$

$$\begin{aligned}
&(TS)^{*k}((TS)^{*2}(TS)^2 - 2(TS)^*(TS) + I)(TS)^k \\
=&(TS)^*(TS)^* \cdots (TS)^*((TS)^*(TS)^*(TS)(TS) - 2(TS)^*(TS) + I) \\
&(TS)(TS) \cdots (TS) \\
=&S^*T^*S^*T^* \cdots S^*T^*(S^*T^*S^*T^*TSTS - 2S^*T^*TS + I) \\
&TSTS \cdots TS \\
=&S^{*k}T^{*k}(T^{*2}T^2 - 2T^*T + I)T^kS^k \geq 0.
\end{aligned}$$

Hence, TS is an operator of the $k-$quasi class Q. ∎

In the following we give the necessary and sufficient conditions for a weighted shift operator T with decreasing weighted sequence (α_n) to be an operator of $k-$quasi class Q.

Theorem 3.5. *A weighted shift T with decreasing weighted sequence (α_n) is $k-$quasi class Q operator if*

$$\alpha_{n+k}^2\alpha_{n+k+1}^2 - 2\alpha_{n+k}^2 + 1 \geq 0.$$

Proof. Since T is a weighted shift, its adjoin T^* is also a weighted shift and defined by

$$T(e_n) = \alpha_n e_{n+1}$$

we have:

$$\begin{aligned} &T^*(e_n) = \alpha_{n-1}e_{n-1}, \\ &(T^*T)(e_n) = \alpha_n^2 e_n, \\ &(T^{*2}T^2)(e_n) = \alpha_n^2\alpha_{n+1}^2 e_n. \end{aligned}$$

Now, since T is $k-$quasi class Q operator then,

$$\begin{aligned} &T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \geq 0 \\ \Rightarrow &\alpha_n^2\alpha_{n+1}^2 \cdots \alpha_{n+k-1}^2(\alpha_{n+k}^2\alpha_{n+k+1}^2 - 2\alpha_{n+k}^2 + 1) \geq 0 \\ \Rightarrow &\alpha_{n+k}^2\alpha_{n+k+1}^2 - 2\alpha_{n+k}^2 + 1 \geq 0. \end{aligned}$$

∎

The following example shows that there exists quasi nilpotent operator that is $k-$quasi class Q.

Example 1. Consider the operator $T : l^2 \to l^2$ defined by

$$T(x) = (0, \alpha_1 x_1, \alpha_2 x_2, \ldots)$$

where $\alpha_n = \dfrac{n}{2^n}$ for $n \geq 1$. Operator T is of $k-$quasi class Q and quasi nilpotent.
Given $T(x) = (0, \alpha_1 x_1, \alpha_2 x_2, \ldots)$. Then, $T^*(x) = (\alpha_1 x_1, \alpha_2 x_2, \ldots)$,

$$\begin{aligned} &T^2(x) = (0, 0, \alpha_1\alpha_2 x_1, \alpha_2\alpha_3 x_2, \ldots), \\ &T^k(x) = (\overbrace{0, 0, 0, \ldots, 0}^{k-time}, \alpha_1\alpha_2 \ldots \alpha_k x_1, \alpha_2\alpha_3 \ldots \alpha_{k+1} x_2, \ldots), \\ &T^*T^k(x) = (\overbrace{0, 0, 0, \ldots, 0}^{(k-1)-time}, \alpha_1\alpha_2 \ldots \alpha_k^2 x_1, \alpha_2\alpha_3 \ldots \alpha_{k+1}^2 x_2, \ldots), \\ &T^{*k}T^k(x) = (\alpha_1^2\alpha_2^2 \ldots \alpha_k^2 x_1, \alpha_2^2\alpha_3^2 \ldots \alpha_{k+1}^2 x_2, \ldots), \\ &T^{*k}T^{k+1}(x) = (0, \alpha_1^2\alpha_2^2 \ldots \alpha_k^2\alpha_{k+1} x_1, \alpha_2^2\alpha_3^2 \ldots \alpha_{k+1}^2\alpha_{k+2} x_2, \ldots), \\ &T^{*(k+1)}T^{k+1}(x) = (\alpha_1^2\alpha_2^2 \ldots \alpha_k^2\alpha_{k+1}^2 x_1, \alpha_2^2\alpha_3^2 \ldots \alpha_{k+1}^2\alpha_{k+2}^2 x_2, \ldots), \\ &T^{*(k+1)}T^{k+2}(x) = (0, \alpha_1^2\alpha_2^2 \ldots \alpha_{k+1}^2\alpha_{k+2} x_1, \alpha_2^2\alpha_3^2 \ldots \alpha_{k+2}^2\alpha_{k+3} x_2, \ldots), \\ &T^{*(k+2)}T^{k+2}(x) = (\alpha_1^2\alpha_2^2 \ldots \alpha_{k+1}^2\alpha_{k+2}^2 x_1, \alpha_2^2\alpha_3^2 \ldots \alpha_{k+2}^2\alpha_{k+3}^2 x_2, \ldots). \end{aligned}$$

Now consider,

$$\begin{aligned}
&\langle T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k x, x\rangle \\
&=\langle (T^{*(k+2)}T^{k+2} - 2T^{*(k+1)}T^{k+1} + T^{*(k)}T^k)x, x\rangle \\
&=\langle (\alpha_1^2\alpha_2^2\ldots\alpha_{k+1}^2\alpha_{k+2}^2 x_1, \alpha_2^2\alpha_3^2\ldots\alpha_{k+2}^2\alpha_{k+3}^2 x_2, \ldots) \\
&-2(\alpha_1^2\alpha_2^2\ldots\alpha_k^2\alpha_{k+1}^2 x_1, \alpha_2^2\alpha_3^2\ldots\alpha_{k+1}^2\alpha_{k+2}^2 x_2, \ldots) \\
&+(\alpha_1^2\alpha_2^2\ldots\alpha_k^2 x_1, \alpha_2^2\alpha_3^2\ldots\alpha_{k+1}^2 x_2, \ldots), (x_1, x_2 \ldots)\rangle \\
&=\langle (\alpha_1^2\alpha_2^2\ldots\alpha_{k+1}^2\alpha_{k+2}^2 - 2\alpha_1^2\alpha_2^2\ldots\alpha_k^2\alpha_{k+1}^2 + \alpha_1^2\alpha_2^2\ldots\alpha_k^2)x_1, x_1\rangle + \\
&\quad \langle (\alpha_2^2\alpha_3^2\ldots\alpha_{k+3}^2 - 2\alpha_2^2\alpha_3^2\ldots\alpha_{k+2}^2 + \alpha_2^2\alpha_3^2\ldots\alpha_{k+1}^2)x_2, x_2\rangle + \ldots \\
&=(\alpha_1^2\alpha_2^2\ldots\alpha_{k+1}^2\alpha_{k+2}^2 - 2\alpha_1^2\alpha_2^2\ldots\alpha_k^2\alpha_{k+1}^2 + \alpha_1^2\alpha_2^2\ldots\alpha_k^2)\|x_1\|^2 \\
&+(\alpha_2^2\alpha_3^2\ldots\alpha_{k+2}^2\alpha_{k+3}^2 - 2\alpha_2^2\alpha_3^2\ldots\alpha_{k+1}^2\alpha_{k+2}^2 + \alpha_2^2\alpha_3^2\ldots\alpha_{k+1}^2) \\
&\quad \|x_2\|^2 + \ldots \\
&=\alpha_1^2\alpha_2^2\ldots\alpha_k^2(\alpha_{k+1}^2\alpha_{k+2}^2 - 2\alpha_{k+1}^2 + 1)\|x_1\|^2 \\
&+\alpha_2^2\alpha_3^2\ldots\alpha_{k+1}^2(\alpha_{k+2}^2\alpha_{k+3}^2 - 2\alpha_{k+2}^2 + 1)\|x_2\|^2 + \ldots \geq 0.
\end{aligned}$$

Because

$$\begin{aligned}
&\alpha_{n+k}^2\alpha_{n+k+1}^2 - 2\alpha_{n+k}^2 + 1 \\
=&\left(\frac{n+k}{2^{n+k}}\right)^2 \cdot \left(\frac{n+k+1}{2^{n+k+1}}\right)^2 - 2\left(\frac{n+k}{2^{n+k}}\right)^2 + 1 \geq 0, k \geq 1, n \geq 1.
\end{aligned}$$

From $T(e_k) = \dfrac{k}{2^k}e_{k+1}$ we have

$$\begin{aligned}
&T^2(e_k) = \frac{k}{2^k} \cdot \frac{k+1}{2^{k+1}} e_{k+2} \\
&. \\
&. \\
&. \\
&T^n(e_k) = \frac{k}{2^k} \cdot \frac{k+1}{2^{k+1}} \cdot \ldots \cdot \frac{k+n-1}{2^{k+n-1}} e_{k+n}.
\end{aligned}$$

Since, $\|T^n\| = \sup_k \| \dfrac{k}{2^k} \cdot \dfrac{k+1}{2^{k+1}} \ldots \dfrac{k+n-1}{2^{k+n-1}} \| = \dfrac{1}{2} \cdot \dfrac{2}{2^2} \ldots \dfrac{n}{2^n}$,

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1 \cdot 2 \cdot \ldots \cdot n}{2^{\frac{n\cdot(n+1)}{2}}} \right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{2^{\frac{n+1}{2}}}$$

$$< \lim_{n\to\infty} \frac{\sqrt[n]{(\frac{n+1}{2})^n}}{2^{\frac{n+1}{2}}} = \lim_{n\to\infty} \frac{\frac{n+1}{2}}{2^{\frac{n+1}{2}}} = 0.$$

Hence, operator T is quasi nilpotent.

4. ALUTHGE TRANSFORMATION OF $k-$QUASI CLASS Q OPERATORS

In this section we give the equivalence between Aluthge transformation and $*-$Aluthge transformation of $k-$quasi class Q operators.

Aluthge defined a transformation $\tilde{T}$ of operator T by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of operator T. $\tilde{T}$ is called Aluthge transformation (see [1]).

Yamazaki defined the $^*-$Aluthge transformation of operator T. The $*-$Aluthge transformation is defined by $\tilde{T}^{(*)} \overset{def}{=} (\tilde{T^*})^* = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$ (see [6]).

It is proved that $U^*|T^*|^{\frac{1}{2}} = |T|^{\frac{1}{2}}U^*, U^*|T^*| = |T|U^*, U|T|^{\frac{1}{2}} = |T^*|^{\frac{1}{2}}U, U|T| = |T^*|U.$

Theorem 4.1. *Let $T \in L(H)$. Then $\tilde{T}$ is $k-$quasi class Q operator if and only if $\tilde{T}^{(*)}$ is $k-$quasi class Q operator.*

Proof. Assume that $\tilde{T}$ is $k-$quasi class Q then,

$$\tilde{T}^{*k}(\tilde{T}^{*2}\tilde{T}^2 - 2\tilde{T}^*\tilde{T} + I)\tilde{T}^k \geq 0.$$

We need to prove that $\tilde{T}^{(*)}$ is $k-$quasi -class Q operator.

$$\begin{aligned}
&\tilde{T}^{(*)*k}(\tilde{T}^{(*)*2}\tilde{T}^{(*)2} - 2\tilde{T}^{(*)*}\tilde{T}^{(*)} + I)\tilde{T}^{(*)k}\\
=&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^{*k}\\
&((|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^{*2}(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^2\\
&-2(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}) + I)\\
&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^k\\
=&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*\cdots(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*\\
&[(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\\
&-2(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}) + I]\\
&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\cdots(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\\
=&(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})\cdots(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})\\
&[(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\\
&-2(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}) + I]\\
&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\cdots(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\\
=&UU^*(|T^*|^{\frac{1}{2}}U^*|T^*|U^*\cdots U^*|T^*|^{\frac{1}{2}})UU^*\\
&[|T^*|^{\frac{1}{2}}U^*|T^*|U^*|T^*U|T^*|U|T^*|^{\frac{1}{2}} - 2|T^*|^{\frac{1}{2}}U^*|T^*|U|T^*|^{\frac{1}{2}} + I]\\
&UU^*(|T^*|^{\frac{1}{2}}U|T^*|U\cdots U|T^*|^{\frac{1}{2}})UU^*
\end{aligned}$$

$$
\begin{aligned}
&= U(U^*|T^*|^{\frac{1}{2}}U^*|T^*|U^*\cdots U^*|T^*|^{\frac{1}{2}}U)\\
&\quad [U^*|T^*|^{\frac{1}{2}}U^*|T^*|U^*|T^*U|T^*|U|T^*|^{\frac{1}{2}}U\\
&\quad -2U^*|T^*|^{\frac{1}{2}}U^*|T^*|U|T^*|^{\frac{1}{2}}U+I](U^*|T^*|^{\frac{1}{2}}U|T^*|U\cdots U|T^*|^{\frac{1}{2}}U)U^*\\
&= U(|T|^{\frac{1}{2}}U^*|T|U^*\cdots U^*|T|^{\frac{1}{2}}U)[|T|^{\frac{1}{2}}U^*|T|U^*|T|U|T|U|T|^{\frac{1}{2}}\\
&\quad -2|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}}+I](|T|^{\frac{1}{2}}U|T|U\cdots U|T|^{\frac{1}{2}})U^*\\
&= U(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})\cdots(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})\\
&\quad [(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})\\
&\quad -2(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})+I]\\
&\quad (|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})\cdots(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})U^*\\
&= U\tilde{T}^{*k}(\tilde{T}^{*2}\tilde{T}^2-2\tilde{T}^*\tilde{T}+I)\tilde{T}^kU^*\geq 0.
\end{aligned}
$$

Therefore,

$$\tilde{T}^{*k}(\tilde{T}^{*2}\tilde{T}^2-2\tilde{T}^*\tilde{T}+I)\tilde{T}^k\geq 0.$$

Hence, $\tilde{T}^{(*)}$ is $k-$quasi class Q operator.
Conversely, assume that $\tilde{T}^{(*)}$ is $k-$quasi class Q operator, then

$$\tilde{T}^{(*)*k}(\tilde{T}^{(*)*2}\tilde{T}^{(*)2}-2\tilde{T}^{(*)*}\tilde{T}^{(*)}+I)\tilde{T}^{(*)k}\geq 0.$$

We need to prove that $\tilde{T}$ is $k-$quasi class Q.
Consider,

$$
\begin{aligned}
&\tilde{T}^{*k}(\tilde{T}^{*2}\tilde{T}^2-2\tilde{T}^*\tilde{T}+I)\tilde{T}^k\\
&= U^*U(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})\cdots(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})U^*U\\
&\quad [(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})\\
&\quad -2(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})+I]\\
&\quad U^*U(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})\cdots(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})U^*U\\
&= U^*(U|T|^{\frac{1}{2}}U^*|T|U^*\cdots U^*|T|^{\frac{1}{2}}U^*)\\
&\quad [U|T|^{\frac{1}{2}}U^*|T|U^*|T|U|T|U|T|^{\frac{1}{2}}U^*-2U|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}}U^*+I]\\
&\quad (U|T|^{\frac{1}{2}}U|T|U\cdots U|T|^{\frac{1}{2}}U^*)U\\
&= U^*(|T^*|^{\frac{1}{2}}U^*|T^*|U^*\cdots U^*|T^*|^{\frac{1}{2}})\\
&\quad [|T^*|^{\frac{1}{2}}U^*|T^*|U^*|T^*|U|T^*|U|T^*|^{\frac{1}{2}}-2|T^*|^{\frac{1}{2}}U^*|T^*|U|T^*|^{\frac{1}{2}}+I]\\
&\quad (|T^*|^{\frac{1}{2}}U|T^*|U\cdots U|T^*|^{\frac{1}{2}})U
\end{aligned}
$$

$$
\begin{aligned}
=&U^*(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})\cdots(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})\\
&[(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\\
&-2(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})+I]\\
&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\cdots(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})U\\
=&U^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^{*k}\\
&[(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^{*2}(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^2\\
&-2(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})+I](|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^kU\\
=&U^*\tilde{T}^{(*)*k}(\tilde{T}^{(*)*2}\tilde{T}^{(*)2}-2\tilde{T}^{(*)*}\tilde{T}^{(*)}+I)\tilde{T}^{(*)k}U
\end{aligned}
$$

Therefore,

$$\tilde{T}^{*k}(\tilde{T}^{*2}\tilde{T}^2-2\tilde{T}^*\tilde{T}+I)\tilde{T}^k)\geq 0.$$

Hence, $\tilde{T}$ is $k-$quasi class Q operator. ∎

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