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WEYL'S THEOREM FOR CLASS Q AND k - QUASI CLASS Q OPERATORS

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ABSTRACT. In this paper, we give some properties of class Q operators. It is proved that every class Q operators satisfies Weyl's theorem under the condition that T^2 is isometry. Also we proved that every k quasi class Q operators is Polaroid and the spectral mapping theorem holds for this class of operator. It will be proved that single valued extension property, Weyl and generalized Weyl's theorem holds for every k quasi class Q operators.

Key words and phrases: class Q operator; k-quasi class Q operator; spectrum of an operator; SVEP .

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1. INTRODUCTION

Let H be an infinite dimensional separable complex Hilbert space with inner product $\langle ., . \rangle$ and $B(H)$ be $C*$ algebra of all bounded linear operators acting on H. For an operator $T \in B(H)$, we denote T^*, the adjoint of T. The spectrum of an operator $T \in B(H)$ is denoted by $\sigma(T)$. The range and kernel of T is denoted by ran(T) and ker(T) respectively. Here an operator $T \in B(H)$ is called p-hyponormal for $0 < p \leq 1$ if $(T^*T)^p - (TT^*)^p \geq 0$; when $p = 1$, T is called hyponormal; when $p = 1/2$, T is called semihyponormal. T is called log-hyponormal if T is invertible and $log(T^*T) \geq log(TT^*)$ and an operator T is called paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in H$ or equivalently $T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0$ for all $\lambda > 0$.

An operator T is called quasi paranormal if $\|T^2x\|^2 \leq \|T^3x\|\|Tx\|$ for all $x \in H$ and T is called k-quasi paranormal, if for every positive integer k, $\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|\|T^kx\|$ for all $x \in H$. In [10] B. P. Duggal, Kubrusly, Levan, introduced and studied some properties of class Q operators. An operator $T \in B(H)$ belongs to class Q if

$$T^{*2}T^2 - 2T^*T + I \geq 0.$$

It is proved that an operator $T \in B(H)$ is of class Q if

$$\|Tx\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2) \text{ for every } x \in H.$$

Devika, Suresh [9] introduced a new class of operators, the quasi class Q. An operator $T \in B(H)$ is said to belong to the quasi class Q if

$$T^{*3}T^3 - 2T^{*2}T^2 + T^*T \geq 0.$$

It is proved that an operator $T \in B(H)$ is of the quasi class Q if

$$\|T^2x\|^2 \leq \frac{1}{2}(\|T^3x\|^2 + \|Tx\|^2) \text{ for every } x \in H.$$

In [20], V. R. Hamiti introduced a new class of operator called k quasi class Q operator.

Definition 1.1. An operator T is said to be k quasi class Q operator if, for every positive integer k,

$$\|T^{k+1}x\|^2 \leq \frac{1}{2}(\|T^{k+2}x\|^2 + \|T^kx\|^2) \text{ for every } x \in H.$$

When $k = 1$, T is called quasi class Q operator.
In [20], it is also proved that, T is called quasi class Q operator if and only if

$$T^{*k+2}T^{k+2} - 2T^{*k+1}T^{k+1} + T^{*k}T^K \geq 0.$$

Also he showed that every k quasi paranormal operators is k quasi class Q operator, every quasi paranormal operator is quasi class Q operator.

2. PRELIMINARIES

Let $T \in B(H)$. $N(T)$ denotes the null space of T and let $\alpha(T) = dimN(T)$. For an operator T, ran T denotes the range of T and $\overline{ranT}$ denotes the closure of ran T. Let $\beta(T) = dimH/ranT$. T is called semi-Fredholm if it has closed range and either $\alpha(T) < \infty$ or $\beta(T) < \infty$. T is called Fredholm if it is semi-Fredholm and both $\alpha(T) < \infty,\ \beta(T) < \infty$. T is called Weyl if it is Fredholm of index zero, i.e., $i(T) = \alpha(T) - \beta(T) = 0$.

The Weyl spectrum of T is defined by $w(T) = \{\lambda \in C | T - \lambda \text{ is not Weyl }\}$. $\pi_{00}(T)$ denotes the set of all eigenvalues of T such that λ is an isolated point of $\sigma(T)$ and $0 < \alpha(T - \lambda) < \infty$. We write $\sigma_e(T)$ for the essential spectrum of T. The spectral picture SP(T) of T consists of $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$ and indices associated with these holes and pseudoholes. We say T to be isoloid if every isolated point in $\sigma(T)$ is an eigenvalue of T.

The essential approximate point spectrum $\sigma_{ea}(T)$ of T is defined by $\sigma_{ea}(T) = \{\sigma_a(T+K) :$ K is a compact operator$\}$, where $\sigma_a(T)$ denotes the approximate point spectrum of T.

We say that Weyl's theorem holds for T if $\sigma(T)/w(T) = \pi_{00}(T)$. An operator $T \in B(H)$ is said to have Bishop's property (β) if $(T-z)f_n(z) \to 0$ uniformly on every compact subset of D for analytic functions $f_n(z)$ on D, then $f_n(z) \to 0$ uniformly on every compact subset of D. T is said to have the single valued extension property, abbreviated, T has SVEP if $f(z)$ is an analytic vector valued function on some open set $D \subset C$ such that $(T-z)f(z) = 0$ for all $z \in D$, then $f(z) = 0$ for all $z \in D$. M. Berkani investigated generalized Weyl's theorem which extends Weyl's theorem, and proved that generalized Weyl's theorem holds for hyponormal operators [5]-[7].

S. Mecheri et all showed that generalized Weyl's theorem holds for (p,k) quasi hyponormal operators. X. Cao, M. Guo and B. Meng were proved Weyl type theorems for p hyponormal operators. M. Berkani investigated B Fredholm theory as follows [1], [5] - [7]. An operator T is called B Fredholm if there exists $n \in N$ such that $ran(T^n)$ is closed and the induced operator $T_{[n]} : ran(T^n) \ni x \to Tx \in ran(T^n)$ is Fredholm, i.e., $ran(T_{[n]}) = ran(T^{n+1})$ is closed, $\alpha(T_{[n]}) = dim\ N(T_{[n]}) < \infty$ and $\beta(T_{[n]}) = dim\ ran(T^n)/ran(T_{[n]}) < \infty$. Similarly, B Fredholm operator T is called B Weyl if $i(T_{[n]}) = 0$.

M. Berkani and M. Sarih [7] have proved that for $T \in B(H)$, If $ran(T^n)$ is closed and $T_{[n]}$ is Fredholm, then $R(T^m)$ is closed and $T_{[m]}$ is Fredholm for every $m \geq n$. Moreover, $indT_{[n]} = indT_{[n]}(= indT)$ and he also proved, An operator T is B Fredholm (B Weyl) if and only if there exist T invariant subspaces M and N such that $T = T|M \oplus T|N$ where $T|M$ is Fredholm (Weyl) and $T|N$ is nilpotent.

The B Weyl spectrum $\sigma_{BW}(T)$ are defined by

$$\sigma_{BW}(T) = \lambda \in C : T - \lambda\ is\ not\ B - Weyl\ \subset \sigma_W(T)$$

We say that generalized Weyl's theorem holds for T if $\sigma(T)|\sigma_{BW}(T) = E(T)$, where $E(T)$ denotes the set of all isolated points of the spectrum which are eigenvalues. Note that, if the generalized Weyl's theorem holds for T, then so does Weyl's theorem [6].

M. Berkani and A. Arroud showed that if T is hyponormal, then generalized Weyl's theorem holds for T. Salah Mecheri, et all have defined that, $T \in SF_+^-$ if $R(T)$ is closed, $dim\ ker(T) < \infty$ and $indT \leq 0$. Let $\pi_{00}^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < dim\ ker(T-\lambda) < \infty$. Let $\sigma_{SF_+^-}(T) = \lambda\, T - \lambda \notin SF_+^- \subset \sigma_W(T)$. a Weyl's theorem holds for T if $\sigma_a(T)\ \sigma_{SF_+^-}(T) = \pi_{00}^a(T)$. V. Rakocevic ([18], Corollary 2.5) proved that if a Weyl's theorem holds for T, then Weyl's theorem holds for T.

Also Salah Mecheri, et all $T \in SBF_+^-$ if there exists a positive integer n such that $ran(T^n)$ is closed, $T_{[n]} : ran(T^n) \ni x \to Tx \in ran(T^n)$ is upper semi-Fredholm and defined that $\sigma_{SBF_+^-}(T) = \lambda|T - \lambda \notin SBF_+^- \subset \sigma_{SF_+^-}$. Let $E^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < dim\ ker(T-\lambda)$. The quasinilpotent part of T is defined by $H_0(T) = x \in H : lim_{n\to\infty}\|T^n x\|^{1/n} = 0$. In general, $H_0(T)$ is not closed.

3. WEYL'S THEOREM FOR CLASS Q OPERATORS

In this section we prove some properties of class Q operators.

Theorem 3.1. *Let $T \in B(H)$ be class Q operator and T^2 is isometry then $N(T-\lambda I) \leq N(T^* - \lambda I)$ for each non zero complex number λ.*

Proof. Suppose T is class Q operator, then $T^{*2}T^2 - 2T^*T + I \geq 0$. This gives that $2\|Tx\|^2 \leq \|T^2x\| + \|x\|^2$. Let T^2 is isometry and $Tx = \lambda x$, Then
$2\langle Tx, Tx\rangle \leq \langle T^2x, T^2x\rangle + \langle x, x\rangle$.
$2\lambda\langle T^*x, x\rangle \leq 2|\lambda|^2\langle x, x\rangle$. Hence $\|T^*\| \leq |\lambda|$

Now consider $\|T^*x - \overline{\lambda}x\| = \|T^*x\|^2 - \langle\overline{\lambda}x, T^*x\rangle - \langle T^*x, \overline{\lambda}x\rangle + \|\overline{\lambda}x\|^2$
$= \|T^*x\|^2 - \langle\overline{\lambda}x, T^*x\rangle - \langle T^*x, \overline{\lambda}x\rangle + |\overline{\lambda}|^2\|x\|^2$
$= \|T^*x\|^2 - \overline{\lambda}\langle Tx, x\rangle - |\overline{\lambda}|^2\|x\|^2 + |\overline{\lambda}|^2\|x\|^2$
$= |\overline{\lambda}|^2\|x\|^2 - |\overline{\lambda}|^2\|x\|^2$
$= 0.$
$\Rightarrow T^*x = \overline{\lambda}x.$ ∎

Theorem 3.2. *Let A be an invertible operator and T be an operator such that T commutes with A^*A. Then T is class Q operator if and only if ATA^{-1} is class Q operator.*

Proof. Let T be a class Q operator. Then $T^{*2}T^2 - 2T^*T + I \geq 0$. Also T commutes with A^*A, so we have $(ATA^{-1})^{*2}(ATA^{-1})^2 = AT^{*2}T^2A^{-1}$ and $(ATA^{-1})^*(ATA^{-1}) = AT^*TA^{-1}$. So we have $(ATA^{-1})^{*2}(ATA^{-1})^2 - 2(ATA^{-1})^*(ATA^{-1}) + I = A(T^{*2}T^2 - 2T^*T + I)A^{-1}$. But $A(T^{*2}T^2 - 2T^*T + I)A^* \geq 0$.

Now consider $A(T^{*2}T^2 - 2T^*T + I)A^*(AA^*) = (AA^*)A(T^{*2}T^2 - 2T^*T + I)A^*$ Therefore AA^* commutes with $A(T^{*2}T^2 - 2T^*T + I)A^*$. Which gives that $(AA^*)^{-1}$ also commutes with $A(T^{*2}T^2 - 2T^*T + I)A^*$. Since $(AA^*)^{-1}$ and $A(T^{*2}T^2 - 2T^*T + I)A^*$ are positive $A(T^{*2}T^2 - 2T^*T + I)A^{-1} \geq 0$. Hence ATA^{-1} is class Q operator.
Conversely, let ATA^{-1} be class Q operator then,

$$\begin{aligned}(ATA^{-1})^{*2}(ATA^{-1})^2 - 2(ATA^{-1})^*(ATA^{-1}) + I &\geq 0\\ A(T^{*2}T^2 - 2T^*T + I)A^{-1} &\leq 0\\ [A^*A](T^{*2}T^2 - 2T^*T + I) &\geq 0.\end{aligned}$$

Also $(A^*A)^{-1}$ commutes with $(A^*A)(T^{*2}T^2 - 2T^*T + I)$. Since $(A^*A)^{-1}$ and $(A^*A)(T^{*2}T^2 - 2T^*T + I)$ are positive then we have $T^{*2}T^2 - 2T^*T + I \geq 0$. Hence T is class Q operator.
∎

Corollary 3.3. *Let T be class Q operator and A be any positive operator such that $A^{-1} = A^*$. Then $S = A^{-1}TA$ is class Q operator.*

By simple calculation we get the result.

Theorem 3.4. *If T is a class Q operator and M be a closed T-invariant subspace of H. Then the restriction $T|_M$ of class Q operator T to M is class Q operator.*

Proof. Let P be an orthogonal projection on M. Then $T_1 = TP = PTP$. Since T is class Q operator

$$T^{*2}T^2 - 2T^*T + I \geq 0$$
$$P(T^{*2}T^2 - 2T^*T + I)P \geq 0$$

Then $T_1^{*2}T_1^2 - 2T_1^*T_1 + I \geq 0 \therefore T_{|M}$ is class Q operator. ∎

Theorem 3.5. *Let $T \in B(H)$ be class Q operator, the range of T does not have dense range then T has the following 2×2 matrix representation $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T)} \oplus \ker T^*$, then T_1 is class Q operator on $\overline{ran(T)}$ and $T_3 = 0$. Further more $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

Proof. Let P be an orthogonal projection of H onto $\overline{ran(T)}$. Then $T_1 = TP = PTP$. By The definition of class Q operator we have that

$$T^{*2}T^2 - 2T^*T + IT^k \geq 0$$
$$\text{Which implies} \qquad P(T^{*2}T^2 - 2T^*T + I)P \geq 0$$

Then $T_1^{*2}T_1^2 - 2T_1^*T_1 + I \geq 0$ So T_1 is class Q operator on $\overline{ran(T)}$.
Also for any $x = (x_1, x_2) \in H$,

$$\begin{aligned}\langle T_3x_2, x_2\rangle &= \langle T(I-P)x, (I-P)x\rangle \\ &= \langle (I-P)x, T^*(I-P)x\rangle \\ &= 0\end{aligned}$$

This implies $T_3 = 0$

Since $\sigma(T) \cup \tau = \sigma(T_1) \cup \sigma(T_3)$ where τ is the union of certain holes in $\sigma(T)$, which happens to be a subset of $\sigma(T_1) \cap \sigma(T_3)$ [by corollary 7, [20]] and $\sigma(T_3) = 0$. $\sigma(T_1) \cap \sigma(T_3)$ has no interior points. So we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.

∎

Corollary 3.6. *Let $T \in B(H)$ be a class Q operator. If T_1 is invertible, then T is similar to a direct sum of a class Q and nilpotent operator.*

Theorem 3.7. *If $T \in B(H)$ is class Q with T^2 is isometry, then $asc(T - \lambda) \leq 1$ for all λ.*

By using the above theorem and corollary we get the proof.

Corollary 3.8. *If T is a class Q operator with T^2 is isometry, then T has SVEP and T satisfies Weyl's theorem.*

4. SPECTRAL PROPERTIES OF K-QUASI CLASS Q OPERATORS

We begin with the following theorem, this will be utilized to get the several important properties of k quasi class Q operators.

Theorem 4.1. *Let $T \in B(H)$ be k quasi class Q operator for any positive integer $k > 0$ and let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^k)} \oplus \ker T^{*k}$ be 2×2 matrix expression. Assume that, the range of T^k be not dense if and only if T_1 is class Q operator on $\overline{ran(T^k)}$ and $T_3^k = 0$. Further more $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

Proof. Suppose that $T \in B(H)$ is an operator of k quasi class Q. Let P be the projection of H onto $\overline{ran(T^k)}$. Then $T_1 = TP = PTP$.
Since T is k quasi class Q operator, we have

$$P(T^{*2}T^2 - 2T^*T + I)P \geq 0.$$

Then

$$\begin{aligned}P(T^{*2}T^2)P - 2P(T^*T)P + PIP &\geq 0 \\ T_1^{*2}T_1^2 - 2T_1^*T_1 + I &\geq 0\end{aligned}$$

For any $x = (x_1, x_2) \in H$

$$\begin{aligned}\langle T_3^k x_2, x_2\rangle &= \langle T^k(I-P)x, (I-P)x\rangle \\ &= \langle (I-P)x, T^{*k}(I-P)x\rangle \\ &= 0\end{aligned}$$

This implies $T_3^k = 0$
Since $\sigma(T) \cup M = \sigma(T_1) \cup \sigma(T_3)$ where M is the union of the holes in $\sigma(T)$, which happens

to be a subset of $\sigma(T_1) \cap \sigma(T_3)$. Then $\sigma(T_3) = 0$ and $\sigma(T_1) \cap \sigma(T_3)$ has no interior points we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^k)} \oplus \ker T^{*k}$ where $T_1^{*2}T_1^2 - 2T_1^*T_1 + I \geq 0$, and $T_3^k = 0$

$$T^k = \begin{pmatrix} T_1^k & \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned} T^kT^{*k} &= \begin{pmatrix} (T_1^kT_1^{*k}) + \sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j}(\sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j})^* & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Where $A = A^* = (T_1^kT_1^{*k}) + \sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j}(\sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j})^* \geq 0$.
Therefore

$$\begin{aligned} &T^kT^{*k}(T^{*2}T^2 - 2T^*T + I)T^kT^{*k} \\ &= \begin{pmatrix} (A(T_1^{*2}T_1^2 - 2T_1^*T_1 + I)A) & 0 \\ 0 & 0 \end{pmatrix} \geq 0 \end{aligned}$$

It follows that $T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \geq 0$ for $k > 0$ on $H = \overline{ran(T^k)} \oplus \ker T^{*k}$. Thus T is k quasi class Q operator. ∎

Corollary 4.2. *Let T be k quasi class Q operator. If T is quasi nilpotent, then it must be a nilpotent operator.*

Proof. Since T is quasi nilpotent operator, $\sigma(T) = \{0\}$. Since T is k quasi class Q operator and "by Theorem 4.1", we have $\sigma(T) = \sigma(T_1) \cup \{0\}$. Then $\sigma(T_1) = \{0\}$. This implies that $T_1 = 0$. But $T_3^k = 0$, So

$$T^{k+1} = \begin{pmatrix} 0 & T_2T_3^k \\ 0 & T_3^{k+1} \end{pmatrix} = 0.$$

Therefore T is nilpotent operator. ∎

Corollary 4.3. *If T is a k quasi class Q operator with $\sigma(T) \subseteq \{0, 1\}$, then $T^{k+1} = T^{k+2}$.*

Proof. By Theorem 4.1, we have $\sigma(T_1) \subseteq \{0, 1\}$. Since T_1 is class Q operator, then we say it is a projection. So $T_1^2 = T_1$.

By simple calculation we have $T^{k+1} = T^{k+2}$. ∎

Theorem 4.4. *Let T be an operator on $H \bigoplus K$, where K be an infinite dimensional separable Hilbert space and T is defined as $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$. If T is class Q operator, then T is quasi class Q operator.*

Proof. Calculate, $T^*(T^{*2}T^2 - 2T^*T + I)T$

$$\begin{aligned} &= \begin{pmatrix} T_1^* & 0 \\ T_2^* & 0 \end{pmatrix} \{ \begin{pmatrix} T_1^{*2} & 0 \\ T_2^*T_1^* & 0 \end{pmatrix} \begin{pmatrix} T_1^2 & 0 \\ T_2T_1 & 0 \end{pmatrix} - 2 \begin{pmatrix} T_1^* & 0 \\ T_2^* & 0 \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \} \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} T_1^* & 0 \\ T_2^* & 0 \end{pmatrix} \{ \begin{pmatrix} T_1^{*2}T_1^2 & 0 \\ T_2^*T_1^*T_1^2 & 0 \end{pmatrix} - 2 \begin{pmatrix} T_1^*T_1 & T_1^*T_2 \\ T_2^*T_1 & T_2^*T_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \} \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} T_1^*(T_1^{*2}T_1^2 - 2T_1^*T_1 + I)T_1 & T_1^*(T_1^{*2}T_1^2 - 2T_1^*T_1 + I)T_2 \\ T_2^*(T_1^{*2}T_1^2 - 2T_1^*T_1 + I)T_1 & T_2^*(T_1^{*2}T_1^2 - 2T_1^*T_1 + I)T_2 \end{pmatrix} \geq 0$$

Let $u = x \bigoplus y \in H \bigoplus K$.

Then $\langle (T^*(T^{*2}T^2 - 2T^*T + I)T)u, u \rangle$

$$= (T_1^*(T_1^{*2}T_1^2 - 2T_1^*T_1 + I)T_1)x, x\rangle + \langle (T_1^*(T_1^{*2}T_1^2 - 2T_1^*T_1 + I)T_2)y, x\rangle$$
$$+ \langle (T_2^*(T_1^{*2}T_1^2 - 2T_1^*T_1 + I)T_1)x, y\rangle + \langle (T_2^*(T_1^{*2}T_1^2 - 2T_1^*T_1 + I)T_2)y, y\rangle$$

$$= \langle (T_1^{*2}T_1^2 - 2T_1^*T_1 + I)(T_1x + T_2y), (T_1x + T_2y)\rangle \geq 0.$$

Since T_1 is class Q operator, $T_1^{*2}T_1^2 - 2T_1^*T_1 + I \geq 0$.

Therefore T is quasi class Q operator. ∎

Corollary 4.5. *Let T be an operator on $H \bigoplus K$, where K be an infinite dimensional separable Hilbert space and T is defined as $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$. If T_1 is class Q operator, then T is k quasi class Q operator.*

Proof. By 'Theorem 4.4', we have,

$\langle (T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k)u, u\rangle$

$$= \langle (T_1^{*2}T_1^2 - 2T_1^*T_1 + I)(T_1^kx + T_1^{k-1}T_2y), (T_1^kx + T_1^{k-1}T_2y)\rangle \geq 0.$$

Since T_1 is class Q operator. ∎

Theorem 4.6. *Let $T \in B(H)$ be an algebraically k quasi class Q operator and $\sigma(T) = \mu_0$, then $T - \mu_0$ is nilpotent.*

Proof. Assume $p(T)$ is k quasi class Q operator for some non constant polynomial $p(z)$. Since $\sigma(p(T)) = p(\sigma(T)) = p(\mu_0)$. This implies that $p(T) - p(\mu_0)$ is nilpotent (by Corollary 4.3).

Let $p(z) - p(\mu_0) = a(z - \mu_0)^{k_0}(z - \mu_1)^{k_1}.....(z - \mu_t)^{k_t}$ where $\mu_j \neq \mu_s$ for $j \neq s$. We have $0 = p(T) - p(\mu_0)$ gives $0 = (p(T) - p(\mu_0))^m = a^m(T - \mu_0)^{mk_0}(T - \mu_1)^{mk_1}.....(T - \mu_t)^{mk_t}$. This gives $(T - \mu_0)^{mk_0} = 0$. That is $(T - \mu_0)^n$. Therefore $T - \mu_0$ is nilpotent. ∎

Theorem 4.7. *Let T be k quasi class Q operator, $\lambda \in C$, and assume that $\sigma(T) = \lambda$. Then $T = \lambda$.*

Proof. Suppose that $\lambda = 0$. Since T is k quasi class Q operator, T is normaloid. Therefore $T = 0$. Suppose that $\lambda \neq 0$. Since T is invertible k quasi class Q, T^{-1} is also k quasi class Q. This implies that T is normaloid and $\sigma(T^{-1}) = \{1/\lambda\}$. Then $\|T\| \, \|T^{-1}\| = 1$. Hence T is convexoid. Therefore $W(T) = \{\lambda\}$. Which gives $T = \lambda$. ∎

Theorem 4.8. *let T be an algebraically k quasi class Q operator, then T is polaroid.*

Proof. If T is an algebraically k quasi class Q operator, then $p(T)$ is a k quasi class Q operator for some non constant polynomial $p(z)$. Let $\mu \in iso\sigma(T)$ and E_μ be the Riesz idempotent associated to μ defined by,

$$E_\mu := \frac{1}{2\pi i} \int_{\sigma D} (\lambda I - T)^{-1} d\lambda,$$

where D is the cl points of the osed disc centered at μ which contains no othersupremum of T. Then T can be represented as follows $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, where $\sigma(T_1) = \{\mu\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\mu\}$. Since T_1 is algebraically k quasi class Q operator and $\sigma(T_1) = \{\mu\}$, then by Theorem 4.7, $T_1 - \mu I$ is nilpotent. Therefore $T_1 - \mu I$ has finite ascent and descent. Hence μ is a pole of the resolvent of T. Now if $\mu \in iso\sigma(T)$ then $\mu \in \pi(T)$. Thus $iso\sigma(T) \in \pi(T)$, where $\pi(T)$ denote the set of poles of the resolvent of T. Hence T is polaroid. ∎

Theorem 4.9. *let T be k quasi class Q operator, then T is isoloid.*

Proof. Let $\lambda \in iso\sigma(T)$ and let $P = \dfrac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$ be the associated Riesz idempotent,where D is a closed disc centered at λ which contains no other points of $\sigma(T)$. Therefore Now $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$. Since T is algebraically k quasi class Q operator, then $p(T)$ is a k quasi class Q operator for some non constant polynomial p. Since $\sigma(T_1) = \{\lambda\}$ we must have $\sigma(p(T_1)) = p(\sigma(T_1)) = \{p\{\lambda\}\}$. Therefore $p(T_1) - p(\lambda)$ is quasi nilpotent by Theorem 4.6. Then we have $p(T_1) - p(\lambda) = 0$ put $q(z) = p(z) - p(\lambda)$ then $q(T_1) = 0$ and hence T_1 is algebraically class Q. By Theorem 4.7, $T_1 - (\lambda)$ is nilpotent. Therefore $\lambda \in \pi_0(T_1) \Longrightarrow \lambda \in \pi_0(T)$. Hence T is isoloid. ∎

Theorem 4.10. *Let $T \in B(H)$ be k quasi class Q operator , then T has SVEP.*

Proof. If the range of T^k is dense then T is class Q. By Theorem 4.1, the range of T^k is not dense we have $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ranT^k} \oplus kerT^{*k}$. Let D be an open subset of C and $f_n(z)$ be analytic function on D to H. Assume $(T - z)f_n(z) \to 0$ uniformly on every compact subset of D. Put $f_n(z) = f_{n_1}(z) \oplus f_{n_2}(z)$ on $H = \overline{ranT^k} \oplus kerT^{*k}$.
Then
$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix}\begin{pmatrix} f_{n_1}(z) \\ f_{n_1}(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_{n_1}(z) + T_2 f_{n_2}(z) \\ (T_3 - z)f_{n_2}(z) \end{pmatrix}.$$
Since T_3 is nilpotent, T_3 has bishop property β. Hence uniformly on every compact subset of D. Then $(T_1 - z)f_{n_1}(z) \to 0$. Since T_1 is class Q, T_1 has bishop property β. Hence T has SVEP. Since $f_{n_1}(z) \to 0$ uniformly on every compact subset of D. ∎

Corollary 4.11. *Let T be a k quasi class Q operator. Then the following assertions hold:*
(i) $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$, for every analytic function f on some open neighborhood of $\sigma(T)$.
(ii) T obeys a-Browder's theorem, that T is $\sigma_{ea}(T) = \sigma_{ab}(T)$
(where $\sigma_{ab}(T) = \bigcap \sigma_a(T + K) : TK = KT and\ K\ is\ \ a\ compact\ operator.$
(iii) a Browder's theorem holds for $f(T)$ for every analytic function f on some open neighborhood of $\sigma(T)$.

Proof. Note that above theorem implies that T has SVEP. By [3], (i) follows. Assertion (ii) is a consequence of ([7], Corollary 2.3). Since $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$, the rest of the argument follows as in ([7], Corollary 2.3). ∎

Theorem 4.12. *An operator quasi similar to a k quasi class Q operator has SVEP.*

Proof. Let T be k quasi class Q. Suppose S is an operator quasi similar to T. Then there exist an injective operator A with dense range such that $AS = TA$. Let U be an open set and $f : U \to H$ be an analytic function for which $(S - zI)f(z) = 0$ on U. Then $0 = A(S - zI)f(z) = (T - zI)Af(z)$ for all z in U. Since T has SVEP, we find $Af(z) = 0$. Since A is injective, it is immediate that $f(z) = 0$ for all z in U. ∎

Theorem 4.13. *Weyl's theorem holds for a k quasi class Q operator T.*

Proof. Let $\lambda \in \sigma(T)|w(T)$. Then $T - \lambda$ is weyl and not invertible. If λ is an interior point of $\sigma(T)$ there exists an open set G such that $\lambda \in G \subset \sigma(T)|w(T)$. Hence $dimN(T - \mu) > 0$ for all $\mu \in G$ and T does not have the single valued extension property. Which is a contradiction to

Theorem 4.10. Hence λ is a boundary point of $\sigma(T)$ and hence an isolated point of $\sigma(T)$. Thus $\lambda \in \pi_{00}(T)$.

Let $\lambda \in \pi_{00}(T)$ and E_λ be the Riesz idempotent for λ of T. Then $0 < dimN(T-\lambda) < \infty$, $T = T|_{E_\lambda H} \oplus T|_{(I-E_\lambda)H}$ $\sigma(T|_{E_\lambda H}) = \lambda$ and $\sigma(T|_{(I-E_\lambda)H}) = \sigma(T)|\lambda$. By Theorem 4.1, $T|_{E_\lambda H}$ is k quasi class Q operator.

If $\lambda \neq 0$ then $T|_{E_\lambda H} = \lambda$. Hence $E_\lambda H \subset N(T-\lambda)$ and E_λ is of finite rank. Since $(T-\lambda)|_{(I-E_\lambda)H}$ is invertible, $(T-\lambda) = 0|_{E_\lambda H} \oplus (T-\lambda)|_{(I-E_\lambda)H}$ is Weyl. Hence $\lambda \in \sigma(T)|w(T)$.

If $\lambda = 0$, then $(T|_{E_0 H})^k = 0$. Hence $E_0 H \subset N(T^k)$ and $dim\ E_0 H \leq dim\ N(T^k) \leq k\, dim\, N(T) < \infty$. Then $T|_{(I-E_\lambda)H}$ is compact. Since $T|_{(I-E_0)H}$ is invertible, $\lambda \in \sigma(T)|w(T)$ ∎

Corollary 4.14. *Weyl's theorem holds for every k quasi class Q operator T.*

Proof. Let $T \in B(H)$ is a k quasi class Q operator. Then by Theorem 4.1, T has the following matrix representation,

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \overline{ran(T^k)} \oplus \ker T^{*k}$$

where T_1 is class Q operator, T_3 is nilpotent operator. Therefore Weyl's theorem holds for because Weyl's theorem holds for class Q operator and nilpotent operator and both class Q operator and nilpotent operator are isoloid. Hence Weyl's theorem holds for $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ because $SP(T_3)$ has no pseudoholes. ∎

Theorem 4.15. *Generalized Weyl's theorem holds for k quasi class Q operator T.*

Proof. Let $\lambda \in \sigma(T)|\sigma_{BW}(T)$. Then $T-\lambda$ is B weyl and not invertible. Then

$$(T-\lambda) = (T-\lambda)|_M \oplus (T-\lambda)|_N$$

where $(T-\lambda)|_M$ is Weyl and $(T-\lambda)|_N$ is nilpotent.

The case $M = 0$ or $N = 0$ is easy to prove. For the case $M \neq 0$ and $N \neq 0$, we assume that $\lambda \in \sigma(T|_M)$. In this case $T|_M$ is k quasi class Q operator by Theorem 4.1, and $\lambda \in \sigma(T|_M)|\sigma_w(T|_M) = \pi_{00}(T|_M)$ Hence λ is an isolated point of $\sigma(T|_M)$ and an eigenvalue of $T|_M$. Hence λ is an eigenvalue of T. Also, $(T-\lambda)|_N$ is nilpotent, so λ is an isolated point of $\sigma(T)$. Hence $\lambda \in E(T)$.

Secondly we assume $\lambda \notin \sigma(T|_M)$. In this case, $(T-\lambda)|_N$ is nilpotent, and λ is an eigenvalue of $T|_N$ and T. Since $(T-\lambda)|_M$ is invertible, λ is an isolated point of $\sigma(T)$. Hence $\lambda \in E(T)$. Conversely, let $\lambda \in E(T)$. Since λ is an isolated point of $\sigma(T)$, $(T-\lambda) = (T-\lambda)|_{E_\lambda H} \oplus T|_{(I-E_\lambda)H}$ where E_λ denotes the Riesz idempotent for λ of T. Then $(T-\lambda)|_{E_\lambda H}$ is k quasi class Q by Theorem 1 and $\sigma(T|_{E_\lambda H}) = \lambda$.

If $\lambda \neq 0$ then $T|_{E_\lambda H} = \lambda$. Hence $(T-\lambda) = 0|_{E_\lambda H} \oplus (T-\lambda)|_{(I-E_\lambda)H}$ Since $(T-\lambda)|_{(I-E_\lambda)H}$ is invertible, $(T-\lambda)$ is B Weyl. Hence $\lambda \in \sigma(T)|_(\sigma_{BW}(T))$.

If $\lambda = 0$, then $(T|_{E_\lambda H})^k = 0$. Hence $\lambda \in \sigma(T)|_(\sigma_{BW}(T))$ ∎

Theorem 4.16. *Let m be a positive integer and $\lambda \in iso\sigma(T)$.*
(1) The following assertions are equivalent:
(a)$EH = ker(T-\lambda)^m$.
(b) $kerE = (T-\lambda)^m H$.
(2) If $\lambda \in p_0(T)$ and the order of λ is m, the following assertions are equivalent:
(a) E is self-adjoint.
(b) $ker(T-\lambda)^m = ker(T-\lambda)^{*m}$.
(c) $ker(T-\lambda)^m \subseteq ker(T-\lambda)^{*m}$.

Theorem 4.17. *Let T be a k quasi class Q operator and $\lambda \in C$.*

(1) $H_0(T) = kerT^{k+1}$, and if $\lambda \neq 0$, then $H_0(T-\lambda) = ker(T-\lambda)$.

(2) Let $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$. on $ker(T-\lambda) \oplus [(T-\lambda)^* H]$. if $0 \neq \lambda \in iso\sigma(T)$ and $ker(T_3)^* = 0$, then $E = E^*$

Proof. The local spectral subspace $X_T(F)$ of T is closed for every closed set $F \subseteq C$. Thus $H_0(T-\lambda) = X_{T-\lambda}(0)$ is closed and $\sigma(S) \subseteq \lambda$ where $S = T|_{H_0(T-\lambda)}$. Moreover, S is k quasi class Q operator. If $\sigma(S)$ is empty, then $H_0(T-\lambda) = 0$ and $ker(T-\lambda) = 0$. If $\sigma(S)$ is not empty, then $\sigma(S) = \lambda$. By Theorem 4.15, $S^{1+k} = 0$ when $\lambda = 0$, and $S = \lambda$ when $\lambda \neq 0$. Hence (i) is true.

By Theorem 4.15,Theorem 4.8 and Theorem 4.9, λ is a simple pole of the resolvent of T and it is sufficient to prove $ker(T-\lambda) \subseteq ker(T-\lambda)^*$, that is, $T_2 = 0$. In fact, $\lambda \in iso\sigma(T) \subset \rho(T_3) \cup iso\sigma(T_3)$. Since T_3 is k quasi class Q operator and isoloid by Theorem 4.15, Theorem4.8 and 4.9, this together with $ker(T_3-\lambda) = 0$ implies that $\lambda \in \rho(T_3)$. Hence $T_2 T_3^k = 0$ and $T_2 = 0$ by the assumption $ker(T_3)^* = 0$. Therefore $ker(T-\lambda) \subseteq ker(T-\lambda)^*$. ∎

Theorem 4.18. *If T^* is k quasi class Q, then Weyl's theorem holds for T.*

Proof. By Theorem 4.15, we have $\sigma(T^*) \setminus (\sigma_{BW}(T^*)) = E(T^*)$.

It is obvious that $(\sigma(T^*) \setminus \sigma_{BW}(T^*))^* = \sigma(T)|_(\sigma_{BW}(T))$. Hence we have to show that $(E(T^*)^*) = E(T)$

Let $\lambda^* \in E(T^*)$. Then λ is an isolated point of $\sigma(T)$. Let F_{λ^*} denotes the Riesz idempotent for λ^* of T^*.

If $\lambda^* \neq 0$, F_{λ^*} is self-adjoint, $0 \neq F_{\lambda^*} H = N((T-\lambda)^*) = N(T-\lambda)$. Hence $\lambda \in E(T)$. If $\lambda^* = 0$, then $T^* \setminus F_0$ is k quasi class Q operator by Theorem 4.1 and $(T^*|_{F_0 H})^k = 0$. Hence $T^{*k} F_0 = 0$. Let $E_0 = F_0^*$ be the Riesz idempotent for 0 of T. Then $T^k E_0 = (T^{*k} F_0)^* = 0$. Hence $T|_{E_0 H}$ is nilpotent. Thus $0 = \lambda \in E(T)$.

Conversely, Let $\lambda \in E(T)$. Then λ^* is an isolated point of $\sigma(T^*)$. Let F_{λ^*} be the Riesz idempotent for λ^* of T^*. if $\lambda \neq 0$, then F_{λ^*} is self adjoint and $0 \neq F_{\lambda^*} H = N((T-\lambda)^*) = N(T-\lambda)$. Hence $\lambda^* = E(T^*)$. Let $\lambda = 0$. Since $T^*|_{F_0 H}$ is k quasi class Q operator and $\sigma(T^*|_{F_0 H}) = 0$ we have $(T^*|_{F_0 H})^k = 0$. this implies that $(T^*|_{F_0 H})$ is nilpotent. Thus $0 = \lambda^* \in E(T^*)$.

∎

Corollary 4.19. *If T^* is k quasi class Q, then a Weyl's theorem holds for T.*

Corollary 4.20. *If T^* is k quasi class Q, then generalized a Weyl's theorem holds for T.*

5. CONCLUSION

Weyl's theorem plays an important role in operator theory. We proved that, "Weyl's theorem hold for class Q, quasi class Q and k quasi class Q operators with the condition that T^2 is isometry".

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