The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 16, Issue 2, Article 15, pp. 1-17, 2019

A NEW APPROACH TO THE STUDY OF FIXED POINT FOR SIMULATION FUNCTIONS WITH APPLICATION IN G-METRIC SPACES

KOMI AFASSINOU [1] AND OJEN KUMAR NARAIN [2]

Received 18 April, 2019; accepted 15 November, 2019; published 16 December, 2019.

[1] DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF ZULULAND, KWADLANGEZWA, SOUTH AFRICA
komia@aims.ac.za

[2] SCHOOL OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE, UNIVERSITY OF KWAZULU-NATAL, DURBAN, SOUTH AFRICA.
naraino@ukzn.ac.za

ABSTRACT. The purpose of this work is to generalize the fixed point results of Kumar et al. [11] by introducing the concept of (α, β)-$\mathcal{Z}$-contraction mapping, Suzuki generalized (α, β)-$\mathcal{Z}$-contraction mapping, (α, β)-admissible mapping and triangular (α, β)-admissible mapping in the framework of G-metric spaces. Fixed point theorems for these class of mappings are established in the framework of a complete G-metric spaces and we establish a generalization of the fixed point result of Kumar et al. [11] and a host of others in the literature. Finally, we apply our fixed point result to solve an integral equation.

Key words and phrases: (α, β)-$\mathcal{Z}$-contraction; Suzuki generalized (α, β)-$\mathcal{Z}$ contraction mappings; fixed point; G-metric space.

2010 *Mathematics Subject Classification.* Primary 47H09, 47H10. Secondary 49J20, 49J40.

ISSN (electronic): 1449-5910

1. INTRODUCTION AND PREMILINARIES

It is well-known that one of the most important notion in fixed point theory is to introduce new contractive conditions and new iterative algorithm that generalizes new and existing contractive mappings and iterative algorithms in the literature (see [1, 2, 3, 4, 6, 8, 10] and the reference therein). In 2008, Suzuki [19] introduced the concept of mappings satisfying condition (C). This is also known as Suzuki-type generalized nonexpansive mapping and he proved some fixed point theorems for such class of mappings.

Definition 1.1. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to satisfy condition (C) if for all $x, y \in X$,

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq d(x,y).$$

Theorem 1.1. *Let (X, d) be a compact metric space and $T : X \to X$ be a mapping satisfying condition (C) for all $x, y \in X$. Then T has a unique fixed point.*

Samet et al. [15] introduced the notion of α-admissible mapping and obtained some fixed point results for this class of mappings.

Definition 1.2. [15] Let $\alpha : X \times X \to [0,\infty)$ be a function. We say that a self mapping $T : X \to X$ is α-admissible if for all $x, y \in X$,

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1.$$

Definition 1.3. [15] Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be mappings. We say that T is a triangular α-admissible if

(1) T is α-admissible and
(2) $\alpha(x,y) \geq 1$ and $\alpha(y,z) \geq 1 \Rightarrow \alpha(x,z) \geq 1$ for all $x, y, z \in X$.

Theorem 1.2. [15] *Let (X, d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Suppose that the following conditions hold:*

(1) *for all $x, y \in X$, we have $\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y))$, where $\psi : [0,\infty) \to [0,\infty)$ is a nondecreasing function such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$;*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(3) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$.*

Then T has a fixed point.

In [5] Chandok extended and improved the concept of α-admissible by introducing the notion of (α,β)-admissible mapping and obtained some fixed point theorems.

Definition 1.4. [5] Let X be a nonempty set and $\alpha, \beta : X \times X \to [0,\infty)$ be functions. We say that a self mapping $T : X \to X$ is (α,β)-admissible if for all $x, y \in X$, $\alpha(x,y) \geq 1$ and $\beta(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1$ and $\beta(Tx,Ty) \geq 1$.

In 2015, Khojasteh et al. [9] introduced the notion of $\mathcal{Z}$-contraction which generalizes the well-known Banach contraction and a host of other contractive conditions. They gave the following definition for $\mathcal{Z}$ as follows.

Definition 1.5. Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a mapping, then ζ is called a simulation function if it satisfies the following conditions:
$\zeta(i)$ $\zeta(0,0) = 0$;
$\zeta(ii)$ $\zeta(t,s) < s - t$, for all $t, s > 0$;
$\zeta(iii)$ if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

We denote the set of all simulation functions by $\mathcal{Z}$.

Definition 1.6. Let (X, d) be a metric space, $T : X \to X$ a mapping and $\zeta \in \mathcal{Z}$. Then T is called a $\mathcal{Z}$-contraction with respect to ζ, if the following condition is satisfied

$$\zeta(d(Tx, Ty), d(x, y)) > 0,$$

for all distinct $x, y \in X$.

Example 1.1. *Suppose $\zeta_i : [0, \infty)^2 \to [0, \infty), i = 1, 2, 3, 4$ defined as*

(1) $\zeta_1(t, s) = s - \phi(s) - t$ *for all* $t, s \in [0, \infty)$ *where* $\phi : [0, \infty) \to [0, \infty)$ *is a continuous function such that* $\phi(t) = 0$ *if and only if* $t = 0$.
(2) $\zeta_2(t, s) = \eta(s) - t$ *for all* $t, s \in [0, \infty)$ *where* $\eta : [0, \infty) \to [0, \infty)$ *be an upper semicontinuous mapping such that* $\eta(t) < t$ *for all* $t > 0$ $\eta(t) = 0$ *if and only if* $t = 0$.
(3) $\zeta_3(t, s) = \lambda s - t$ *for all* $t, s \in [0, \infty)$ *where* $0 < \lambda < 1$.
(4) $\zeta_4(t, s) = \frac{s}{s+1} - t$ *for all* $t, s \in [0, \infty)$.

Theorem 1.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to a simulation function $\zeta \in \mathcal{Z}$. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$, the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point of T.*

Recently, Kumam et al. [10] introduced the notion of Suzuki type $\mathcal{Z}$-contraction with respect to ζ in the framework of metric spaces. They established some fixed point results for this class of mapping and also show that the Suzuki type $\mathcal{Z}$-contraction with respect to ζ is a generalization of $\mathcal{Z}$-contraction mapping with respect to ζ. They gave the following definition and result.

Definition 1.7. Let (X, d) be a metric space, $T : X \to X$ a mapping and $\zeta \in \mathcal{Z}$. Then T is called a Suzuki type $\mathcal{Z}$-contraction with respect to ζ, if the following condition is satisfied

$$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow \zeta(d(Tx, Ty), d(x, y)) > 0,$$

for all distinct $x, y \in X$.

Theorem 1.4. *Let (X, d) be a complete metric space and $T : X \to X$ be a Suzuki $\mathcal{Z}$-contraction with respect to a simulation function $\zeta \in \mathcal{Z}$. Then T has a unique fixed point.*

Mustafa and Sims [13] introduced the concept of generalized metric space (G-metric) and they established some fixed point theorems in the framework of complete G-metric spaces.

Definition 1.8. Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties

(1) $G(x, y, z) = 0$ if and only if $x = y = z$,
(2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
(4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all the three variables),
(5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

The function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 1.9. A G-metric space is said to be symmetric if $G_b(x, y, y) = G_b(y, x, x)$ for all $x, y \in X$.

Proposition 1.5. *Let X be a G-metric space. Then for each $x, y, z, a \in X$, it follows that*

(1) $G(x, y, z) = 0$ *then* $x = y = z$,
(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(3) $G(x, y, y) \leq 2G(y, x, x)$,
(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$.

Definition 1.10. Let X be a G-metric space. A sequence $\{x_n\}$ in X is said to be:

(1) G-Cauchy if for each $\epsilon > 0$ there exists a positive integer n_0 such that for all $m, n, l \geq n_0, G(x_n, x_m, x_l) < \epsilon$;
(2) G-convergent to a point $x \in X$, if for $\epsilon > 0$ there exists a positive integer n_0 such that for all $m, n \geq n_0, G(x_n, x_m, x) < \epsilon$. That is $\lim_{n,m\to\infty} G(x_n, x_m, x) = 0$. We call x the limit of the sequence $\{x_n\}$ and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Definition 1.11. A G-metric space is called G-complete, if every G-Cauchy sequence is G-convergent in X.

Proposition 1.6. *Let (X, G) be a G-metric space. The following statements are equivalent:*

(1) x_n *is G-convergent to* x;
(2) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$;
(3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$;
(4) $G(x_n, x_m, x) \to 0$ *as* $m, n \to \infty$.

Proposition 1.7. *Let (X, G) be a G-metric space. The following statements are equivalent:*

(1) $\{x_n\}$ *is G-Cauchy sequence.*
(2) $G(x_m, x_n, x_n) \to 0$ *as* $n, m \to \infty$.

Very recently, Kumar et al. [11] introduced the concept of $\mathcal{Z}$-contraction with respect to ζ in the framework of G-metric spaces. They establish some fixed point results and gave an example to support their main result.

Definition 1.12. Let (X, G) be a G-metric space, $T : X \to X$ a mapping and $\zeta \in \mathcal{Z}$. Then T is called a $\mathcal{Z}$-contraction with respect to ζ, if the following condition is satisfied

$$\zeta(G(Tx, Ty, Tz), G(x, y, z)) > 0,$$

for all distinct $x, y, z \in X$.

Theorem 1.8. *Let (X, G) be a complete G-metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to a simulation function $\zeta \in \mathcal{Z}$. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$, the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point of T.*

Motivated by the research works of Khojasteh et al. [9], Kuman et al. [10], Kumar et al. [11] and the research work in this direction, our purpose in this paper is to introduce the notion of (α, β)-$\mathcal{Z}$-contraction mapping and Suzuki generalized (α, β)-$\mathcal{Z}$-contraction mapping with respect to ζ in the framework G-metric spaces. We prove some fixed point results for these types of mappings and then give some examples to support our main results.

2. MAIN RESULT

In this section, we introduce the notion of (α, β)-admissible mapping, triangular (α, β)-admissible mapping, (α, β)-$\mathcal{Z}$-contraction mapping and Suzuki generalized (α, β)-$\mathcal{Z}$-contraction mapping with respect to ζ in the framework G-metric spaces and established the existence and uniqueness results of the fixed point for this class of mappings.

Definition 2.1. Let X be a nonempty set, $T : X \to X$ and $\alpha, \beta : X \times X \times X \to [0, \infty)$ be mappings. Then T is called (α, β)-admissible if for all $x, y, z \in X$ with $\alpha(x, y, z) \geq 1$ and $\beta(x, y, z) \geq 1$ implies $\alpha(Tx, Ty, Tz) \geq 1$ and $\beta(Tx, Ty, Tz) \geq 1$.

Definition 2.2. Let X be a nonempty set, $T : X \to X$ and $\alpha, \beta : X \times X \times X \to [0, \infty)$ be mappings. Then T is called triangular (α, β)-admissible if

(1) T is (α, β)-admissible,
(2) $\alpha(x, a, a) \geq 1, \alpha(a, y, z) \geq 1$ and $\beta(x, a, a) \geq 1, \beta(a, y, z) \geq 1$ implies $\alpha(x, y, z) \geq 1$ and $\beta(x, y, z) \geq 1$,

for all $x, y, z, a \in X$.

Lemma 2.1. *Let X be a nonempty set and T be a triangular (α, β)-admissible mapping and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0, Tx_0) \geq 1$. Suppose that the sequence $\{x_n\}$ is defined by $x_{n+1} = Tx_n$, then $\alpha(x_m, x_n, x_n) \geq 1$ and $\beta(x_m, x_n, x_n) \geq 1$ for all $n, m \in \mathbb{N} \cup \{0\}$, with $m < n$.*

Proof. Suppose that T is triangular (α, β)-admissible mapping and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0, Tx_0) \geq 1$, we then have that $\alpha(x_0, Tx_0, Tx_0) = \alpha(x_0, x_1, x_1) \geq 1$ and $\beta(x_0, Tx_0, Tx_0) = \beta(x_0, x_1, x_1) \geq 1$, which implies that $\alpha(Tx_0, Tx_1, Tx_1) = \alpha(x_1, x_2, x_2) \geq 1$ and $\beta(Tx_0, Tx_1, Tx_1) = \beta(x_1, x_2, x_2) \geq 1$. Continuing the process, we obtain that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ and $\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$. For all $n, m \in \mathbb{N} \cup \{0\}$ with $m < n$, now observed that since $\alpha(x_m, x_{m+1}, x_{m+1}) \geq 1, \beta(x_m, x_{m+1}, x_{m+1}) \geq 1$ and $\alpha(x_{m+1}, x_{m+2}, x_{m+2}) \geq 1, \beta(x_{m+1}, x_{m+2}, x_{m+2}) \geq 1$, we obtain $\alpha(x_m, x_{m+2}, x_{m+2}) \geq 1, \beta(x_m, x_{m+2}, x_{m+2}) \geq 1$. Also, since $\alpha(x_m, x_{m+2}, x_{m+2}) \geq 1, \beta(x_m, x_{m+2}, x_{m+2}) \geq 1$ and $\alpha(x_{m+2}, x_{m+3}, x_{m+3}) \geq 1, \beta(x_{m+2}, x_{m+3}, x_{m+3}) \geq 1$, we obtain $\alpha(x_m, x_{m+3}, x_{m+3}) \geq 1, \beta(x_m, x_{m+3}, x_{m+3}) \geq 1$. Continuing the process, we have that

$$\alpha(x_m, x_n, x_n) \geq 1 \text{ and } \beta(x_m, x_n, x_n) \geq 1. \quad \blacksquare$$

Definition 2.3. Let (X, G) be a G-metric space, $\alpha, \beta \times X \times X \to [0, \infty)$ be a function and T be a self map on X. The mapping T is said to be (α, β)-$\mathcal{Z}$-contraction mapping with respect to ζ, if

$$\zeta(\alpha(x, y, z)\beta(x, y, z)G(Tx, Ty, Tz), G(x, y, z)) \geq 0 \tag{2.1}$$

for all distinct $x, y, z \in X$.

Remark 2.1. If we take $\alpha(x, y, z)\beta(x, y, z) = 1$, we obtain Definition 1.12.

Remark 2.2. It is easy to see from the definition of ζ that $\zeta(t, s) < 0$, for all $t \geq s > 0$. Hence, T is an (α, β)-$\mathcal{Z}$-contraction with respect to ζ, then

$$\alpha(x, y, z)\beta(x, y, z)G(Tx, Ty, Tz) < G(x, y, z)$$

for all distinct $x, y, z \in X$.

Theorem 2.2. *Let (X, G) be a G-complete metric space and $T : X \to X$ be an (α, β)-$\mathcal{Z}$-contraction mapping with respect to ζ. Suppose the following conditions hold:*

(1) *T is (α, β)-admissible mapping,*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0, Tx_0) \geq 1$,*
(3) *if for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1, \beta(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, x) \geq 1$ and $\beta(x_n, x, x) \geq 1$*

Then T has a fixed point.

Proof. To establish that T has a fixed point, we divide the proof into four steps.

Step 1: We will establish that $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$.

Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0, Tx_0) \geq 1$. We define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If we suppose that $x_{n+1} = x_n$, for some $n \in \mathbb{N} \cup \{0\}$, we obtain the desired result. Now, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is (α, β)-admissible mapping and $\alpha(x_0, x_1, x_1) = \alpha(x_0, Tx_1, Tx_1) \geq 1, \beta(x_0, x_1, x_1) = \beta(x_0, Tx_1, Tx_1) \geq 1$, we have that $\alpha(x_1, x_2, x_2) = \alpha(Tx_0, Tx_1, Tx_1) \geq 1$, and $\beta(x_1, x_2, x_2) = \beta(Tx_0, Tx_1, Tx_1) \geq 1$, continuing this process, we obtain that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ and $\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. As such we have that

$$\alpha(x_n, x_{n+1}, x_{n+1})\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$. We obtain from (2.1) and using $\zeta(ii)$ that

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n, x_{n+1}, x_{n+1})\beta(x_n, x_{n+1}, x_{n+1})G(Tx_n, Tx_{n+1}, Tx_{n+1}), G(x_n, x_{n+1}, x_{n+1})) \\ &= \zeta(\alpha(x_n, x_{n+1}, x_{n+1})\beta(x_n, x_{n+1}, x_{n+1})G(x_{n+1}, x_{n+2}, x_{n+2}), G(x_n, x_{n+1}, x_{n+1})) \\ &< G(x_n, x_{n+1}, x_{n+1}) - \alpha(x_n, x_{n+1}, x_{n+1})\beta(x_n, x_{n+1}, x_{n+1})G(x_{n+1}, x_{n+2}, x_{n+2}). \end{aligned} \tag{2.2}$$

From (2.2), we obtain

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+2}) &\leq \alpha(x_n, x_{n+1}, x_{n+1})\beta(x_n, x_{n+1}, x_{n+1})G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &< G(x_n, x_{n+1}, x_{n+1}). \end{aligned} \tag{2.3}$$

It is easy to see from (2.3) that the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}$ is a monotonically decreasing sequence of nonnegative real. Therefore, there exists $c \geq 0$ such that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = c.$$

Suppose that $c > 0$, clearly $\lim_{n\to\infty} G(x_{n+1}, x_{n+2}, x_{n+2}) = c$ and from (2.3), using the Sandwich Theorem we have that $\lim_{n\to\infty} \alpha(x_n, x_{n+1}, x_{n+1})\beta(x_n, x_{n+1}, x_{n+1})G(x_{n+1}, x_{n+2}, x_{n+2}) = c$. Since T is an (α, β)-$\mathcal{Z}$-contraction mapping with respect to $\zeta \in \mathcal{Z}$ and using $\zeta(iii)$, we have

$$0 \leq \limsup_{n\to\infty} \zeta(\alpha(x_n, x_{n+1}, x_{n+1})\beta(x_n, x_{n+1}, x_{n+1})G(x_{n+1}, x_{n+2}, x_{n+2}), G(x_n, x_{n+1}, x_{n+1})) < 0.$$

This is a contradiction, thus $c = 0$ and so we have that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{2.4}$$

Step 2: We will establish that $\{x_n\}$ is bounded.

Suppose that $\{x_n\}$ is not a bounded sequence, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}, n_{k+1}$ is the minimum integer such that

$$G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) > 1 \text{ and } G(x_m, x_{n_k}, x_{n_k}) \leq 1 \tag{2.5}$$

for $n_k \leq m \leq n_{k+1} - 1$. Using the triangular inequality, (2.5) and Proposition 1.5 (3), we have

$$\begin{aligned} 1 < G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) &\leq G(x_{n_{k+1}}, x_{n_{k+1}-1}, x_{n_{k+1}-1}) + G(x_{n_{k+1}-1}, x_{n_k}, x_{n_k}) \\ &\leq 2G(x_{n_{k+1}-1}, x_{n_{k+1}}, x_{n_{k+1}}) + 1. \end{aligned}$$

Letting $k \to \infty$ and using (2.4), we obtain

$$\lim_{k\to\infty} G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) = 1.$$

Using the definition of (α, β)-$\mathcal{Z}$-contraction with respect to ζ, we obtain

$$\alpha(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})\beta(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) \leq G(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1}),$$

and it follows that

$$\begin{aligned} G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) &\leq \alpha(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})\beta(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) \\ &\leq G(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1}), \end{aligned}$$

using the triangular inequality, (2.5) and Proposition 1.5 (3), we have that

$$\begin{aligned} 1 < G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) &\leq \alpha(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})\beta(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) \\ &\leq G(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1}) \\ &\leq G(x_{n_{k+1}-1}, x_{n_k}, x_{n_k}) + G(x_{n_k}, x_{n_k-1}, x_{n_k-1}) \\ &\leq 1 + 2G(x_{n_k-1}, x_{n_k}, x_{n_k}). \end{aligned}$$

Letting $k \to \infty$ and using (2.4), we obtain

$$\lim_{n\to\infty} \alpha(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})\beta(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) = 1$$

by definition of (α, β)-$\mathcal{Z}$-contraction with respect to ζ, and by $\zeta(iii)$, we obtain

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} \zeta(\alpha(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})\beta(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})G(Tx_{n_{k+1}-1}, Tx_{n_k-1}, Tx_{n_k-1}), \\ &\quad G(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})) \\ &\leq \limsup_{k\to\infty} \zeta(\alpha(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})\beta(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})G(x_{n_{k+1}}, x_{n_k}, x_{n_k}), \\ &\quad G(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})) < 0. \end{aligned}$$

This is a contradiction. Thus $\{x_n\}$ is bounded.

Step 3: We will establish that $\{x_n\}$ is Cauchy.

Suppose that $C_n = \sup\{G(x_i, x_j, x_j) : i, j \geq n\}, n \in \mathbb{N}$. Since $\{x_n\}$ is bounded, we have that $C_n < \infty$ for all $n \in \mathbb{N}$, as such C_n is a positive monotonically decreasing sequence which converges. That is $\lim_{n\to\infty} C_n = C \geq 0$. Suppose that $C > 0$, then by definition of C_n for every $k \in \mathbb{N}$, we can find n_k, m_k such that $m_k > n_k > k$ and

$$C_n - \frac{1}{K} < G(x_{m_k}, x_{n_k}, x_{n_k}) \leq C_k,$$

letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} G(x_{m_k}, x_{n_k}, x_{n_k}) = C. \tag{2.6}$$

Now, observe that

$$\begin{aligned} G(x_{m_k}, x_{n_k}, x_{n_k}) &\leq G(x_{m_k}, x_{m_{k-1}}, x_{m_{k-1}}) + G(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}) + G(x_{n_{k-1}}, x_{n_k}, x_{n_k}) \\ &\leq 2G(x_{m_{k-1}}, x_{m_k}, x_{m_k}) + G(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}) + G(x_{n_{k-1}}, x_{n_k}, x_{n_k}) \end{aligned}$$

and

$$\begin{aligned} G(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}) &\leq G(x_{m_{k-1}}, x_{m_k}, x_{m_k}) + G(x_{m_k}, x_{n_k}, x_{n_k}) + G(x_{n_k}, x_{n_{k-1}}, x_{n_{k-1}}) \\ &\leq G(x_{m_{k-1}}, x_{m_k}, x_{m_k}) + G(x_{m_k}, x_{n_k}, x_{n_k}) + 2G(x_{n_{k-1}}, x_{n_k}, x_{n_k}). \end{aligned}$$

Letting $k \to \infty$ and using (2.4) and (2.6), we obtain

$$\lim_{k\to\infty} G(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}) = C. \tag{2.7}$$

By definition of (α, β)-$\mathcal{Z}$-contraction with respect to ζ, we have that

$$\begin{aligned} G(x_{m_k}, x_{n_k}, x_{n_k}) &\leq \alpha(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})\beta(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})G(x_{m_k}, x_{n_k}, x_{n_k}) \\ &\leq G(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}), \end{aligned}$$

it is easy to see that

$$\lim_{n\to\infty} \alpha(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})\beta(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})G(x_{m_k}, x_{n_k}, x_{n_k}) = C \tag{2.8}$$

Then using (2.8), (2.7) and $\zeta(iii)$, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta(\alpha(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})\beta(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}), \\ &G(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})) \\ &\leq \limsup_{n\to\infty} \zeta(\alpha(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})\beta(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})G(x_{m_k}, x_{n_k}, x_{n_k}), \\ &G(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})) < 0. \end{aligned}$$

This is a contradiction, thus $C = 0$. Hence, $\{x_n\}$ is a Cauchy sequence.

Step 4: We will establish that T has a fixed point.

Since $\{x_n\}$ is a Cauchy sequence and X is a complete G-metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Using condition (3), since $\alpha(x_n, x, x) \geq 1$, $\beta(x_n, x, x) \geq 1$, we have that $\alpha(x_n, x, x)\beta(x_n, x, x) \geq 1$, and since T is (α, β)-$\mathcal{Z}$-contraction with respect to ζ and using $\zeta(iii)$, we obtain

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n, x, x)\beta(x_n, x, x)G(Tx_n, Tx, Tx), G(x_n, x, x)) \\ &< G(x_n, x, x) - \alpha(x_n, x, x)\beta(x_n, x, x)G(x_{n+1}, Tx, Tx,), \end{aligned}$$

it follows that

$$G(x_{n+1}, Tx, Tx,) \leq \alpha(x_n, x, x)\beta(x_n, x, x)G(x_{n+1}, Tx, Tx,) \leq G(x_n, x, x).$$

Letting $n \to \infty$, we obtaing

$$G(x, Tx, Tx,) \leq \alpha(x, x, x)\beta(x, x, x)G(x, Tx, Tx,) \leq G(x, x, x) = 0.$$

in the above inequality, we must have that $G(x, Tx, Tx) = 0$, that is, $x = Tx$. ∎

Example 2.1. *Let $X = [0, \infty)$ and $G : X \times X \times X \to [0, \infty)$ be defined as $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$ for all $x, y \in X$. It is clear that (X, G) is a G-metric space. We defined $T : X \to X$ by*

$$Tx = \begin{cases} \frac{x}{10} & \textit{if} \quad x \in [0, 1] \\ 2x - \frac{11}{8} & \textit{if} \quad x \in (1, \infty), \end{cases}$$

$\alpha, \beta : X \times X \times X \to [0, \infty)$ *by*

$$\alpha(x, y, z) = \begin{cases} 1 & \textit{if} \quad x, y, z \in [0, 1] \\ 0 & \textit{if} \quad x, y, z \in (1, \infty), \end{cases}$$

$$\beta(x, y, z) = \begin{cases} 2 & \textit{if} \quad x, y, z \in [0, 1] \\ 0 & \textit{if} \quad x, y, z \in (1, \infty), \end{cases}$$

and $\zeta(t,s) = \frac{1}{2}s - t$. Then T is an (α,β)-$\mathcal{Z}$-contraction mapping with respect to ζ and that that Theorem 1.8 is not applicable.

Proof. It is easy to see that T is triangular (α,β)-admissible mapping for any $x,y,z \in [0,1]$ and that for any $x_0 \in [0,1]$, we have $\alpha(x_0,Tx_0,Tx_0) = 1$ and $\beta(x_0,Tx_0,Tx_0) = 2$. Now suppose that $\{x_n\}$ is a sequence in X with $\alpha(x_n,x_{n+1},x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup 0$ and that $x_n \to x$ as $n \to \infty$, from the definition of α and β, it is clear that $\{x_n\} \subset [0,1]$, as such $x \in [0,1]$. Thus $\alpha(x_n,x,x) = 1$ and $\beta(x_n,x,x) = 2 > 1$ for all $n \in \mathbb{N} \cup 0$. Since $\alpha(x,y,z) = 1$ and $\beta(x,y,z) = 2 > 1$if $x,y,z \in [0,1]$, we need to show that

$$\zeta(\alpha(x,y,z)\beta(x,y,z)G(Tx,Ty,Tz),G(x,y,z)) \geq 0$$

for any $x,y,z \in [0,1]$. Without loss of generality, we suppose that $x \geq y \geq z$, so that

$$\begin{aligned}
&\zeta(\alpha(x,y,z)\beta(x,y,z)G(Tx,Ty,Tz),G(x,y,z)) = \zeta\Big(2G\big(\frac{x}{10},\frac{y}{10},\frac{z}{10}\big),G(x,y,z)\Big)\\
&= \zeta\left(2\max\left\{\left|\frac{x}{10}-\frac{y}{10}\right|,\left|\frac{y}{10}-\frac{z}{10}\right|,\left|\frac{z}{10}-\frac{x}{10}\right|\right\},\max\{|x-y|,|y-z|,|z-x|\}\right)\\
&= \zeta\Big(\frac{1}{5}|z-x|,|z-x|\Big) = \frac{1}{2}|z-x| - \frac{1}{5}|z-x| \geq 0.
\end{aligned}$$

Thus, T is an (α,β)-$\mathcal{Z}$-contraction with respect to ζ. However, to show that Theorem 1.8 is not applicable, suppose that $x = 10, y = 2$ and $z = 0$. Now, observe that

$$\begin{aligned}
&\zeta(G(Tx,Ty,Tz),G(x,y,z)) = \zeta\Big(G\big(\frac{21}{8},\frac{1}{10},0\big),G(2,1,0)\Big)\\
&= \zeta\left(\max\left\{\left|\frac{21}{8}-\frac{1}{10}\right|,\left|\frac{1}{10}-0\right|,\left|0-\frac{21}{8}\right|\right\},\max\{|2-1|,|1-0|,|0-1|\}\right)\\
&= \zeta\Big(\frac{21}{8},1\Big) = \frac{1}{2} - \frac{21}{8} < 0. \blacksquare
\end{aligned}$$

Remark 2.3. It is clear from the above example that T has two fixed points $x = 0$ and $x = \frac{31}{8}$. For the uniqueness of the fixed point, we need additional conditions.

Theorem 2.3. *Suppose that the hypothesis of Theorem 2.2 holds and in addition suppose $\alpha(x,y,y) \geq 1$ and $\beta(x,y,y) \geq 1$ for all $x,y \in F(T)$, where $F(T)$ is the set of fixed point of T. Then T has a unique fixed point.*

Proof. Let $x,y \in F(T)$, that is $Tx = x$ and $Ty = y$ such that $x \neq y$. Using our hypothesis that $\alpha(x,y,y) \geq 1$, $\zeta(iii)$, we obtain from 2.1 that

$$\begin{aligned}
0 &\leq \zeta(\alpha(x,y,y)\beta(x,y,y)G(Tx,Ty,Ty),G(x,y,y))\\
&< G(x,y,y) - \alpha(x,y,y)\beta(x,y,y)G(Tx,Ty,Ty)\\
&= G(x,y,y) - \alpha(x,y,y)\beta(x,y,y)G(x,y,y)\\
&\leq 0,
\end{aligned}$$

which is a contradiction, as such, we must have that $G(x,y,y) = 0 \Rightarrow x = y$. Hence, T has a unique fixed point. $\blacksquare$

Example 2.2. *Let $X = [0,\infty)$ and $G : X \times X \times X \to [0,\infty)$ be defined as $G(x,y,z) = \max\{|x-y|,|y-z|,|z-x|\}$ for all $x,y \in X$. It is clear that (X,G) is a G-metric space. We* *defined* $T : X \to X$ *by*

$$Tx = \begin{cases} \frac{x}{13} & \textit{if} \quad x \in [0,1] \\ 2x & \textit{if} \quad x \in (1,\infty), \end{cases}$$

$\alpha, \beta : X \times X \times X \to [0,\infty)$ *by*

$$\alpha(x,y,z) = \begin{cases} 2 & \textit{if} \quad x,y,z \in [0,1] \\ 0 & \textit{if} \quad x,y,z \in (1,\infty), \end{cases}$$

$$\beta(x,y,z) = \begin{cases} 3 & \textit{if} \quad x,y,z \in [0,1] \\ 0 & \textit{if} \quad x,y,z \in (1,\infty), \end{cases}$$

and $\zeta(t,s) = \frac{1}{2}s - t$. *Then* T *is an* (α,β)*-*$\mathcal{Z}$*-contraction mapping with respect to* ζ *but not an* $\mathcal{Z}$*-contraction mapping with respect to* ζ *as defined by Kumar et al.* [11].

Proof. It is easy to see that T is triangular (α,β)-admissible mapping for any $x,y,z \in [0,1]$ and that for any $x_0 \in [0,1]$, we have $\alpha(x_0,Tx_0,Tx_0) = 2 > 1$ and $\beta(x_0,Tx_0,Tx_0) = 3 > 1$. Now suppose that $\{x_n\}$ is a sequence in X with $\alpha(x_n,x_{n+1},x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup 0$ and that $x_n \to x$ as $n \to \infty$, from the definition of α and β, it is clear that $\{x_n\} \subset [0,1]$, as such $x \in [0,1]$. Thus $\alpha(x_n,x,x) = 2 > 1$ and $\beta(x_n,x,x) = 3 > 1$ for all $n \in \mathbb{N} \cup 0$. Since $\alpha(x,y,z) = 2 > 1$ and $\beta(x,y,z) = 3 > 1$ if $x,y,z \in [0,1]$, we need to show that

$$\zeta(\alpha(x,y,z)\beta(x,y,z)G(Tx,Ty,Tz),G(x,y,z)) \geq 0$$

for any $x,y,z \in [0,1]$. Without loss of generality, we suppose that $x \geq y \geq z$, so that

$$\begin{aligned}
&\zeta(\alpha(x,y,z)\beta(x,y,z)G(Tx,Ty,Tz),G(x,y,z)) = \zeta\left(6G\big(\frac{x}{2},\frac{y}{2},\ \frac{z}{2}\big),G(x,y,z)\right) \\
&= \zeta\left(6\max\left\{\left|\frac{x}{13}-\frac{y}{13}\right|,\left|\frac{y}{13}-\frac{z}{13}\right|,\left|\frac{z}{13}-\frac{x}{13}\right|\right\},\max\left\{|x-y|,|y-z|,|z-x|\right\}\right) \\
&= \zeta\left(\frac{6}{13}|z-x|,|z-x|\right) = \frac{1}{26}|z-x| \geq 0.
\end{aligned}$$

Thus, T is an (α,β)-$\mathcal{Z}$-contraction with respect to ζ and all the hypotheses of Theorem 2.3 are satisfied with $x = 0$ the unique fixed point of T.
However, to show that Theorem 1.8 is not applicable, let $x = 2, y = 1$ and $z = 0$ Now, observe that

$$\begin{aligned}
&\zeta(G(Tx,Ty,Tz),G(x,y,z)) = \zeta\left(G\big(4,\frac{1}{13},0\big),G(2,1,0)\right) \\
&= \zeta\left(\max\left\{\left|4-\frac{1}{13}\right|,\left|\frac{1}{13}-0\right|,|0-4|\right\},\max\left\{|2-1|,|1-0|,|0-2|\right\}\right) \\
&= \zeta(4,2) = 1-4 < 0. \quad \blacksquare
\end{aligned}$$

Remark 2.4. It is clear from the above result and example that our result generalizes the result of Kurmam et al. [11]. Also our result extends and improves the results of Khojasteh et al. [9], Kumam et al. [10] and a host of other results in the literature.

Definition 2.4. Let (X, G) be a G-metric space, $\alpha, \beta : X \times X \times X \to [0, \infty)$ be a function and T be a self map on X. The mapping T is said to be Suzuki generalized (α, β)-$\mathcal{Z}$-contraction mapping with respect to ζ, if

(2.9)

$$\alpha(x, y, z)\beta(x, y, z) \geq 1 \text{ and } \frac{1}{3}G(x, Tx, Tx) \leq G(x, y, z) \Rightarrow \zeta(G(Tx, Ty, Tz), M(x, y, z)) \geq 0,$$

where $M(x, y, z) = \max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Tz)\}$ for all distinct $x, y, z \in X$.

Remark 2.5. It is easy to see from the definition of ζ that $\zeta(t, s) < 0$, for all $t \geq s > 0$. Hence, T is a Suzuki generalized (α, β)-$\mathcal{Z}$-contraction with respect to ζ, then

$$\alpha(x, y, z)\beta(x, y, z) \geq 1 \text{ and } \frac{1}{3}G(x, Tx, Tx) \leq G(x, y, z) \Rightarrow G(Tx, Ty, Tz) < M(x, y, z)$$

for all distinct $x, y, z \in X$.

Theorem 2.4. *Let (X, G) be a G-complete metric space and $T : X \to X$ be a Suzuki generalized (α, β)-$\mathcal{Z}$-contraction mapping with respect to ζ. Suppose the following conditions hold:*

(1) *T is triangular (α, β)-admissible mapping,*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0, Tx_0) \geq 1$,*
(3) *if for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ and $\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, x) \geq 1$ and $\beta(x_n, x, x) \geq 1$.*

Then T has a fixed point.

Proof. To establish that T has a fixed point, we divide the proof into four steps.

Step 1: We will establish that $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$.

We define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If we suppose that $x_{n+1} = x_n$, for some $n \in \mathbb{N}$, we obtain the desired result. Now, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\frac{1}{3}G(x_n, Tx_n, Tx_n) = \frac{1}{3}G(x_n, x_{n+1}, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+1})$, and from Lemma 2.1, it is easy to see that $\alpha(x_n, x_{n+1}, x_{n+1})\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$, we obtain from (2.9)

$$\begin{aligned} 0 &\leq \zeta(G(Tx_n, Tx_{n+1}, Tx_{n+1}), M(x_n, x_{n+1}, x_{n+1})) \\ &= \zeta(G(x_{n+1}, x_{n+2}, x_{n+2}), M(x_n, x_{n+1}, x_{n+1})) \\ &< M(x_n, x_{n+1}, x_{n+1}) - G(x_{n+1}, x_{n+2}, x_{n+2}), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} M(x_n, x_{n+1}, x_{n+1}) &= \max\{G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\} \\ &= \max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\}. \end{aligned}$$

If we suppose that $\max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\} = G(x_{n+1}, x_{n+2}, x_{n+2})$, we obtain from (2.10) that

$$0 \leq \zeta(G(x_{n+1}, x_{n+2}, x_{n+2}), M(x_n, x_{n+1}, x_{n+1})) < G(x_{n+1}, x_{n+2}, x_{n+2}) - G(x_{n+1}, x_{n+2}, x_{n+2}) < 0,$$

which is a contradiction, as such

$$\max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\} = G(x_n, x_{n+1}, x_{n+1}),$$

that is,

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq G(x_n, x_{n+1}, x_{n+1}).$$

Thus, we obtain from (2.10) that

$$0 \leq \zeta(G(x_{n+1}, x_{n+2}, x_{n+2}), M(x_n, x_{n+1}, x_{n+1})) < G(x_n, x_{n+1}, x_{n+1}) - G(x_{n+1}, x_{n+2}, x_{n+2})$$

It is easy to see that the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}$ is a monotonically decreasing sequence of nonnegative real numbers. As such, there exists $c \geq 0$ such that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = c.$$

Now, suppose that $c > 0$, since T is a Suzuki generalized (α, β)-$\mathcal{Z}$-contraction mapping with respect to ζ and using $\zeta(iii)$, we have

$$0 \leq \limsup_{n\to\infty} \zeta(G(x_{n+1}, x_{n+2}, x_{n+2}), G(x_n, x_{n+1}, x_{n+1})) < 0.$$

This is a contradiction, thus $c = 0$ and so we have that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{2.11}$$

Step 2: We will establish that $\{x_n\}$ is bounded.

Suppose that $\{x_n\}$ is not a bounded sequence, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}, n_{k+1}$ is the minimum integer such that

$$G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) > 1 \text{ and } G(x_m, x_{n_k}, x_{n_k}) \leq 1 \tag{2.12}$$

for $n_k \leq m \leq n_{k+1} - 1$. Using the triangular inequality, (2.12) and Proposition 1.5 (3), we have

$$\begin{aligned} 1 < G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) &\leq G(x_{n_{k+1}}, x_{n_{k+1}-1}, x_{n_{k+1}-1}) + G(x_{n_{k+1}-1}, x_{n_k}, x_{n_k}) \\ &\leq 2G(x_{n_{k+1}-1}, x_{n_{k+1}}, x_{n_{k+1}}) + 1. \end{aligned}$$

Letting $k \to \infty$ and using (2.11), we obtain

$$\lim_{k\to\infty} G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) = 1.$$

Since $\frac{1}{3}G(x_{n_{k+1}-1}, Tx_{n_k-1}, Tx_{n_k-1}) = \frac{1}{3}G(x_{n_{k+1}-1}, x_{n_k}, x_{n_k}) < G(x_{n_{k+1}-1}, x_{n_k}, x_{n_k})$ and from Lemma 2.1, we obtain that $\alpha(x_{n_{k+1}-1}, x_{n_k}, x_{n_k})\beta(x_{n_{k+1}-1}, x_{n_k}, x_{n_k}) \geq 1$, by definition of Suzuki generalized (α, β)-$\mathcal{Z}$-contraction with respect to ζ, we obtain

$$G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) \leq M(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1}),$$

it follows that

$$\begin{aligned} 1 &< G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) \leq M(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1}) \\ &= \max\{G(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1}), G(x_{n_k-1}, x_{n_{k+1}}, x_{n_{k+1}}), G(x_{n_k-1}, x_{n_k}, x_{n_k})\} \\ &\leq \max\{G(x_{n_{k+1}-1}, x_{n_k}, x_{n_k}) + G(x_{n_k}, x_{n_k-1}, x_{n_k-1}), G(x_{n_k-1}, x_{n_k}, x_{n_k}) \\ &+ G(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}}), G(x_{n_k-1}, x_{n_k}, x_{n_k})\} \\ &\leq \max\{1 + 2G(x_{n_k-1}, x_{n_k}, x_{n_k}), 1 + G(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}}), G(x_{n_k-1}, x_{n_k}, x_{n_k})\}. \end{aligned}$$

Letting $k \to \infty$ and using (2.11), we obtain

$$\lim_{k\to\infty} M(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1}) = 1.$$

Furthermore, since

$$\frac{1}{3}G(x_{n_{k+1}-1}, Tx_{n_k-1}, Tx_{n_k-1}) = \frac{1}{3}G(x_{n_{k+1}-1}, x_{n_k}, x_{n_k}) < G(x_{n_{k+1}-1}, x_{n_k}, x_{n_k})$$

and from Lemma 2.1, we obtain that $\alpha(x_{n_{k+1}-1},x_{n_k},x_{n_k})\beta(x_{n_{k+1}-1},x_{n_k},x_{n_k})\geq 1$, by definition of Suzuki generalized (α,β)-$\mathcal{Z}$-contraction with respect to ζ and $\zeta(iii)$, we obtain

$$\begin{aligned}0&\leq \limsup_{k\to\infty}\zeta(G(Tx_{n_{k+1}-1},Tx_{n_k-1},Tx_{n_k-1}),M(x_{n_{k+1}-1},x_{n_k-1},x_{n_k-1}))\\&\leq \limsup_{k\to\infty}\zeta(G(x_{n_{k+1}},x_{n_k},x_{n_k}),M(x_{n_{k+1}-1},x_{n_k-1},x_{n_k-1}))<0.\end{aligned}$$

This is a contradiction. Thus $\{x_n\}$ is bounded.

Step 3: We will establish that $\{x_n\}$ is Cauchy.

Suppose that $C_n=\sup\{G(x_i,x_j,x_j):i,j\geq n\}, n\in\mathbb{N}$. Since $\{x_n\}$ is bounded, we have that $C_n<\infty$ for all $n\in\mathbb{N}$, as such C_n is a positive monotonically decreasing sequence which converges. That is $\lim_{n\to\infty}C_n=C\geq 0$. Suppose that $C>0$, then by definition of C_n for every $k\in\mathbb{N}$, we can find n_k,m_k such that $m_k>n_k>k$ and

$$C_n-\frac{1}{K}<G(x_{m_k},x_{n_k},x_{n_k})\leq C_k,$$

letting $k\to\infty$, we obtain

$$\lim_{k\to\infty}G(x_{m_k},x_{n_k},x_{n_k})=C. \tag{2.13}$$

Now, observe that

$$\begin{aligned}G(x_{m_k},x_{n_k},x_{n_k})&\leq G(x_{m_k},x_{m_{k-1}},x_{m_{k-1}})+G(x_{m_{k-1}},x_{n_{k-1}},x_{n_{k-1}})+G(x_{n_{k-1}},x_{n_k},x_{n_k})\\&\leq 2G(x_{m_{k-1}},x_{m_k},x_{m_k})+G(x_{m_{k-1}},x_{n_{k-1}},x_{n_{k-1}})+G(x_{n_{k-1}},x_{n_k},x_{n_k})\end{aligned}$$

and

$$\begin{aligned}G(x_{m_{k-1}},x_{n_{k-1}},x_{n_{k-1}})&\leq G(x_{m_{k-1}},x_{m_k},x_{m_k})+G(x_{m_k},x_{n_k},x_{n_k})+G(x_{n_k},x_{n_{k-1}},x_{n_{k-1}})\\&\leq G(x_{m_{k-1}},x_{m_k},x_{m_k})+G(x_{m_k},x_{n_k},x_{n_k})+2G(x_{n_{k-1}},x_{n_k},x_{n_k}).\end{aligned}$$

Letting $k\to\infty$ and using (2.11) and (2.13), we obtain

$$\lim_{k\to\infty}G(x_{m_{k-1}},x_{n_{k-1}},x_{n_{k-1}})=C. \tag{2.14}$$

Also, since

$$\frac{1}{3}G(x_{m_k-1},Tx_{m_k-1},Tx_{m_k-1})<\frac{1}{3}G(x_{m_k-1},x_{n_k-1},x_{n_k-1})<G(x_{m_k-1},x_{n_k-1},x_{n_k-1})$$

and from Lemma 2.1, we obatin that $\alpha(x_{n_{k+1}-1},x_{n_k},x_{n_k})\beta(x_{n_{k+1}-1},x_{n_k},x_{n_k})\geq 1$, and since T is Suzuki generalized (α,β)-$\mathcal{Z}$-contraction,
we have $G(Tx_{m_{k-1}},Tx_{n_k-1},Tx_{n_k-1})\leq M(x_{m_k-1},x_{n_k-1},x_{n_k-1}))$. It then follows that

$$\begin{aligned}G(x_{m_k},x_{n_k},x_{n_k})&=G(Tx_{m_k-1},Tx_{n_k-1},Tx_{n_k-1})\leq M(x_{m_k-1},x_{n_k-1},x_{n_k-1}))\\&=\max\{G(x_{m_k-1},x_{n_k-1},x_{n_k-1}),G(x_{m_k-1},x_{m_k},x_{m_k}),G(x_{n_k-1},x_{n_k},x_{n_k})\}.\end{aligned}$$

Letting $k\to\infty$, using (2.11), (2.13) and (2.14), we have that

$$\lim_{k\to\infty}M(x_{m_{k-1}},x_{n_{k-1}},x_{n_{k-1}})=C. \tag{2.15}$$

Using (2.13) and (2.15) we have

$$0\leq \limsup_{n\to\infty}\zeta(G(x_{m_k},x_{n_k},x_{n_k}),M(x_{m_k-1},x_{n_k-1},x_{n_k-1}))<0.$$

This is a contradiction, thus $C=0$. Hence, $\{x_n\}$ is a Cauchy sequence.

Step 4: We will establish that T has a fixed point.

Since $\{x_n\}$ is a Cauchy sequence and X is a complete G-metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

Claim: We claim that

$$G(x_n, x, x) < \frac{1}{3}G_b(x_n, x_{n+1}, x_{n+1})$$

or

$$G(x_{n+1}, x, x) < \frac{1}{3}G(x_{n+1}, x_{n+2}, x_{n+2}).$$

Proof of claim:

Using the fact that $G(x_{n+1}, x_{n+2}, x_{n+2}) \leq G(x_n, x_{n+1}, x_{n+1})$, we have

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) &\leq G(x_n, x, x) + G(x, x_{n+1}, x_{n+1}) \\
&\leq G(x_n, x, x) + 2G(x_{n+1}, x, x) \\
&< \frac{1}{3}G(x_n, x_{n+1}, x_{n+1}) + \frac{2}{3}G(x_{n+1}, x_{n+2}, x_{n+2}) \\
&\leq (\frac{1}{3} + \frac{2}{3})G(x_n, x_{n+1}, x_{n+1}) \\
&= G(x_n, x_{n+1}, x_{n+1})
\end{aligned}$$

The above inequality is a contradiction, thus we must have that

$$\frac{1}{3}G(x_n, x_{n+1}, x_{n+1}) \leq G(x_n, x, x) \quad \text{or} \quad \frac{1}{3}G(x_{n+1}, x_{n+2}, x_{n+2}) \leq G(x_{n+1}, x, x).$$

Now, supposing that $\frac{1}{3}G(x_n, x_{n+1}, x_{n+1}) \leq G(x_n, x, x)$ and $\alpha(x_n, x, x)\beta(x_n, x, x) \geq 1$, we have

$$0 \leq \zeta(G(Tx_n, Tx, Tx), M(x_n, x, x)).$$

Using $\zeta(ii)$, we obtain

$$\begin{aligned}
G(x_{n+1}, Tx, Tx) = G(Tx_n, Tx, Tx) &< M(x_n, x, x) \\
&= \max\{G(x_n, x, x), G(x_n, x_{n+1}, x_{n+1}), G(x, Tx, Tx)\}
\end{aligned}$$

letting $k \to \infty$, we have $\lim_{n\to\infty} M(x_n, x, x) = G(x, Tx, Tx)$, it then follows that

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty} \zeta(G(Tx_n Tx, Tx), M(x_n, x, x)) \\
&< \limsup_{n\to\infty} (M(x_n x, x) - G(Tx_n, Tx, Tx)) \\
&= G(x, Tx, Tx) - G(x, Tx, Tx) = 0.
\end{aligned}$$

This is a contradiction, as such we must have that $G(x, Tx, Tx) = 0 \Rightarrow x = Tx$. Using a similar approach, we can also show that T has a fixed point using

$$\frac{1}{3}G(x_{n+1}, x_{n+2}, x_{n+2}) \leq G(x_{n+1}, x, x). \quad \blacksquare$$

Theorem 2.5. *Suppose that the hypothesis of Theorem 2.4 holds and in addition suppose* $\alpha(x, y, y) \geq 1$ *and* $\beta(x, y, y) \geq 1$ *for all* $x, y \in F(T)$, *where* $F(T)$ *is the set of fixed point of* T. *Then* T *has a unique fixed point.*

Proof. Suppose that x and y are fixed points of T such that $x \neq y$.
Since $\frac{1}{3}G(x,Tx,Tx) = 0 < G(x,y,y)$ and $\alpha(x,y,y)\beta(x,y,y) \geq 1$, we obtain from (2.9)

$$\begin{aligned} 0 &\leq \zeta(G(Tx,Ty,Ty), M(x,y,y)) \\ &= \zeta(G(Tx,Ty,Ty), \max\{G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty)\}) \\ &= \zeta(G(Tx,Ty,Ty), G(x,y,y)) \\ &= \zeta(G(x,y,y), G(x,y,y)) \\ &< G(x,y,y) - G(x,y,y) = 0. \end{aligned}$$

This is a contradiction, as such we must have that $G(x,y,y) = 0 \Rightarrow x = y$. ∎

Corollary 2.6. *Let (X,G) be a G-complete metric space and $T : X \to X$ be mapping such that there exists and $k \in (0,1)$ satisfying*

$$\alpha(x,y,z)\beta(x,y,z) \geq 1 \Rightarrow G(Tx,Ty,Tz) \leq kG(x,y,z)$$

for all distinct $x,y,z \in X$. Suppose the following conditions hold:

(1) *T is a triangular (α,β)-admissible mapping,*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0,Tx_0) \geq 1, \beta(x_0,Tx_0,Tx_0) \geq 1$,*
(3) *if for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1},x_{n+1}) \geq 1$ and $\beta(x_n,x_{n+1},x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n,x,x) \geq 1$. and $\beta(x_n,x,x) \geq 1$.*

Then T has a fixed point.

Corollary 2.7. *Let (X,G) be a G-complete metric space and $T : X \to X$ be mapping such that there exists and $k \in (0,1)$ satisfying*

$$\frac{1}{3}G(x,Tx,Tx) \leq G(x,y,z) \Rightarrow G(Tx,Ty,Tz) \leq kM(x,y,z)$$

for all distinct $x,y,z \in X$, where $M(x,y,z) = \max\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Tz)\}$. Then T has a unique fixed point.

3. APPLICATION

In this section, we present an application of Corollary 2.7 to guarantee the existence and uniqueness problem of the solution to an integral equation of the form:

$$x(t) = f(t) + \int_0^1 H(t,s,u(s))ds, \quad t \in [0,1]. \tag{3.1}$$

Let $X = C([0,1])$ be the space of real continuous function defined on $[0,1]$. It is well-known that $C([0,1])$ endowed with the G-metric

$$G(x,y,z) = \sup_{t\in[0,1]} |x(t) - y(t)| + \sup_{t\in[0,1]} |y(t) - z(t)| + \sup_{t\in[0,1]} |z(t) - x(t)|$$

is a complete G-metric space. Define $T : X \to X$ by

$$Tx(t) = f(t) + \int_0^1 H(t,s,x(s))ds, \quad t \in [0,1].$$

Theorem 3.1. *Suppose that the following hypothesis hold:*

(1) *$H : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous,*

(2) *there exists* $K : [0,1] \times [0,1] \to [0,\infty)$ *such that* $\frac{1}{3}G(x,Tx,Tx) \leq G(x,y,y)$ *implies that*

$$|H(t,s,u) - H(t,s,v)| \leq K(t,s)|u-v|$$

for all distinct $x, y \in X, t, s \in [0,1]$ *and* $u, v \in \mathbb{R}$,

(3) $\sup_{t\in[0,1]} \int_0^1 K(t,s)ds < \tau$, *where* $\tau \in (0,1)$.

Then the integral equation (3.1) *has a solution* $x \in X$.

Proof. For $x, y \in X$, we have

$$\begin{aligned}
G(Tx,Ty,Ty) &= 2 \sup_{t\in[0,1]} |Tx(t) - Ty(t)| \\
&= 2 \sup_{t\in[0,1]} \left| \int_0^1 H(t,s,x(s)) - H(t,s,y(s))ds \right| \\
&\leq 2 \sup_{t\in[0,1]} \int_0^1 |H(t,s,x(s)) - H(t,s,y(s))|ds \\
&\leq 2 \sup_{t\in[0,1]} \int_0^1 K(t,s)|x(s) - y(s)|ds \\
&\leq 2 \sup_{t\in[0,1]} |x(t) - y(t)| \sup_{t\in[0,1]} \int_0^1 K(t,s)ds \\
&\leq \tau G(x,y,y) \\
&\leq G(x,y,y) \\
&\leq M(x,y,y).
\end{aligned}$$

Thus, Corollary 2.7 is applicable to T which guarantees the existence and the uniqueness of the fixed point $x \in X$. Thus, x is the unique solution of the integral equation 3.1. ∎

4. CONCLUSION

In this paper, we introduced the notion of (α,β)-$\mathcal{Z}$-contraction and Suzuki generalized (α,β)-$\mathcal{Z}$-contraction in the framework of complete G-metric space which improved, generalized and unified various comparable results [9, 10, 11] in the existing literature. In addition, we presented some examples to establish that this generalization was an important one. Finally, we applied our result to show the existence and uniqueness of the solution of an integral equation.

REFERENCES

[1] C. T. AAGE and J. N . SALUNKE, Fixed points for weak contractions in G-metric spaces, *Appl. Math. E-Notes.*, **12** (2012), pp. 23–28.

[2] A. AGHAJANI, M. ABBAS, and J. ROSHAN, Common fixed points of generalized weak contractive mappings in partially ordered G_b-metric spaces, *Filomat,* **28** (2014), pp. 1087–1101.

[3] A. H. ANSARI, Note on ϕ-ψ-contractive type mappings and related fixed point, *Proceedings of the 2nd Regional Conference on Mathematics and Applications,* Payame Noor University, Tonekabon, Iran, (2014), pp. 377–380.

[4] S. BANACH, Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales, *Fundamenta Mathematicae,* **3** (1922), pp. 133–181.

[5] S. CHANDOK, Some fixed point theorems for (α,β)-admissible Geraghty type contractive mappings and related results, *Mathematical Sciences,* **9** (2015), No. 3, pp. 127–135.

[6] D. DORIĆ, Common fixed point for generalized (ψ, ϕ)-weak contractions, *Appl. Math. Lett.*, 22 (2009), pp. 1896–1900.

[7] P. N. DUTTA and B. S. CHOUDHARY, A generalization of contraction principle in metric spaces, *Fixed Point Theory Appl.*, 2008 (2008) Article ID 406368, pp. 1–8.

[8] N. HUSSAIN, M. A. KUTBI and P. SALIMI, Fixed point theory in α-complete metric spaces with applications, *Abstr. Appl. Anal.*, (2014), Article ID 280817.

[9] F. KHOJASTEH, S. SHUKLA and S. RADENOVIĆ, A new approach to the study of fixed point theorems via simulation functions, *Filomat,* **9** (2015), pp. 1189–1194.

[10] P. KUMAM, D. GOPAL and L. BUDHIYI, A new fixed point theorem under Suzuki type $\mathcal{Z}$-contraction mappings, *J. Math. Anal.,* **8** (2017), No. 2, pp. 113–119.

[11] M. KUMAR and R. SHARMA, A New Approach to the Study of Fixed Point Theorems for Simulation Functions in G-Metric Spaces, *Bol. Soc. Paran. Mat.,* (2019), pp. 115–121.

[12] P. P. MURTHY, L. N. MISHRA and U. D. PATEL, n-tupled fixed point theorems for weak-contraction in partially ordered complete G-metric spaces, *New Trends Math. Sci.,* **3** (2015), No. 4, pp. 50–75.

[13] Z. MUSTAFA and B. SIMS, A new approach to generalized metric space, *J. Nonlinear Convex Anal.,* **7** (2006), pp. 289–297.

[14] P. SALINI and V. PASQUALE, A result of Suzuki type in partial G-metric spaces, *Acta Math. Sci. Ser. B (Engl. Ed.).*, **34** (2014), No. 2, pp. 274–284.

[15] B. SAMET, C. VETRO and P. VETRO, Fixed point theorem for α-ψ-contractive type mappings, *Nonlinear Anal.,* **75** (2012), pp. 2154–2165.

[16] W. SHATANAWI and M. POSTOLACHE, Some fixed-point results for a G-weak contraction in G-metric spaces, *Abstr. Appl. Anal.,* (2012), pp. 1–19.

[17] S. L. SINGH, R. KAMAL, M. DE LA SEN M and R. CHUGH, A fixed point theorem for generalized weak contractions, *Filomat,* **29** (2015), No. 7, pp. 1481–1490.

[18] W. SINTUNAVARAT, Nonlinear integral equations with new admissibility types in b-metric spaces, *J. Fixed Point Theory Appl.*, DOI 10.1007/s11784-015-0276-6, pp. 1–20.

[19] T. SUZUKI, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, *J. Math. Anal. Appl.,* **340** (2008), No. 2, pp. 1088–1095.

[20] T. SUZUKI, A new type of fixed point theorem in metric spaces, *Nonlinear Anal.,* **71** (2009), No. 11, pp. 5313–5317.

[21] U. MISHRA, H.K. NASHINE and R.KEWAT, Fixed point theory for generalized (ψ, ϕ)-weak contractions involving $f - g$ reciprocally continuity, *Indian J. Math.,* **56** (2014), No. 2, pp. 153–168.

[22] F. YAN, Y. SU and Q. FENG, A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations, *Fixed Point Theory Appl.,* **152** (2012), pp. 1–13.

[23] O. YAMAODA and W. SINTUNAVARAT, Fixed point theorems for $(\alpha, \beta) - (\psi, \varphi)$-contractive mappings in b-metric spaces with some numerical results and applications, *J. Nonlinear Sci. Appl.,* **9** (2016), pp. 22–33.