The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 16, Issue 2, Article 12, pp. 1-11, 2019

COUNTABLE ORDINAL SPACES AND COMPACT COUNTABLE SUBSETS OF A METRIC SPACE

BORYS ÁLVAREZ-SAMANIEGO AND ANDRÉS MERINO

Received 8 March, 2019; accepted 2 October, 2019; published 11 November, 2019.

NÚCLEO DE INVESTIGADORES CIENTÍFICOS, FACULTAD DE CIENCIAS,
UNIVERSIDAD CENTRAL DEL ECUADOR (UCE), QUITO, ECUADOR.
borys_yamil@yahoo.com
balvarez@uce.edu.ec

ESCUELA DE CIENCIAS FÍSICAS Y MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS Y NATURALES,
PONTIFICIA UNIVERSIDAD CATÓLICA DEL ECUADOR, APARTADO: 17-01-2184, QUITO, ECUADOR.
aemerinot@puce.edu.ec

ABSTRACT. We show in detail that every compact countable subset of a metric space is homeomorphic to a countable ordinal number, which extends a result given by Mazurkiewicz and Sierpinski for finite-dimensional Euclidean spaces. In order to achieve this goal, we use Transfinite Induction to construct a specific homeomorphism. In addition, we prove that for all metric space, the cardinality of the set of all the equivalence classes, up to homeomorphisms, of compact countable subsets of this metric space is less than or equal to aleph-one. We also show that for all cardinal number smaller than or equal to aleph-one, there exists a metric space with cardinality equals the aforementioned cardinal number.

Key words and phrases: Cantor-Bendixson's derivative; Ordinal numbers; Ordinal topology.

2000 *Mathematics Subject Classification.* Primary 54A25; 03E15.

ISSN (electronic): 1449-5910

The first author would like to dedicate this paper to the memory of his beloved mother, Mrs. María Esther Samaniego Rodríguez.

1. INTRODUCTION

The study of homeomorphisms between compact countable subsets of a topological space and countable ordinal numbers was began by S. Mazurkiewicz and W. Sierpinski in [10]. More precisely, they showed that for every compact countable subset of an n-dimensional Euclidean space, there exists a homeomorphism between this subset and some countable ordinal number. Moreover, a detailed proof of this last result when the Euclidean space, under consideration, is the real line, was given by the authors in [1]. Some related propositions can also be found in [3, 4, 9, 7]. The main result of Section 2 is Theorem 2.5 below, which extends Lemma 3.6 in [1] for an arbitrary metric space. It is worth mentioning that Theorem 2 in [2] considers compact, dispersed topological spaces with some additional properties, while Theorem 2.5 below regards the case of a metric space. Furthermore, it is stated in [8], without proof, that it is a known fact which can be proved by induction that Y is a countable locally compact space if and only if Y is homeomorphic to some countable ordinal number (with the order topology). In this way, Lemmas 2.1, 2.3 and 2.4, proved in Section 2, are the comprehensive induction steps required in the Transfinite Induction used in the proof of Theorem 2.5. Section 3 is devoted to study the cardinality of the set of all the equivalence classes $\mathscr{K}_M$, up to homeomorphisms, of compact countable subsets of a metric space (M, d). Propositions 3.1 and 3.2 are used in the proof of Theorem 3.3, where it is shown that for all metric space (E, d), the cardinality of $\mathscr{K}_E$ is less than or equal to $\aleph_1$. Propositions 3.4 to 3.6 shows that for all cardinal number $\kappa \leq \aleph_1$, there exists a metric space (E_κ, d_κ) such that the cardinality of the set $\mathscr{K}_{E_\kappa}$ is equal to κ. Proposition 3.7 says that there exists a countable metric space (F, d_F) such that $|\mathscr{K}_F| = \aleph_1$. Finally, Proposition 3.8 asserts that there is an uncountable metric space (G, d_G) such that $|\mathscr{K}_G| = \aleph_0$.

We denote by $\mathbf{OR}$, the class of all ordinal numbers. In addition, ω represents the set of all natural numbers and ω_1 is the set of all countable ordinal numbers. Further, we consider any ordinal number as being a topological space, endowed with its natural order topology. In order to describe this last topology, for all $\alpha, \beta \in \mathbf{OR}$ such that $\alpha \leq \beta$, we write

$$(\alpha, \beta) := \{\gamma \in \mathbf{OR} : \alpha < \gamma < \beta\},$$
$$[\alpha, \beta) := \{\gamma \in \mathbf{OR} : \alpha \leq \gamma < \beta\}.$$

Thus, for any $\delta \in \mathbf{OR}$, the natural order topology for δ is given by the following topological basis

$$\{(\beta, \gamma) : \beta, \gamma \in \mathbf{OR},\ \beta < \gamma \leq \delta\} \cup \{[0, \beta) : \beta \in \mathbf{OR},\ \beta \leq \delta\}.$$

Next definition was first introduced by G. Cantor in [5].

Definition 1.1 (Cantor-Bendixson's derivative)**.** Let A be a subset of a topological space. For a given ordinal number $\alpha \in \mathbf{OR}$, we define, using Transfinite Recursion, the *α-th derivative* of A, written $A^{(\alpha)}$, as follows:

- $A^{(0)} = A$,
- $A^{(\beta+1)} = (A^{(\beta)})'$, for all ordinal number β,
- $A^{(\lambda)} = \bigcap_{\gamma<\lambda} A^{(\gamma)}$, for all limit ordinal number $\lambda \neq 0$,

where B' denotes the derived set of B, i.e., the set of all limit points (or accumulation points) of the subset B.

Remark 1.1. Given any subset of a T_1 topological space, its derived set is closed. As a consequence of this last result, we have that if F is a closed subset of a T_1 topological space, then $(F^{(\alpha)})_{\alpha\in\mathbf{OR}}$ is a decreasing family of closed subsets.

Moreover, $\mathcal{P}(C)$ and $|C|$ denote, respectively, the power set and the cardinality of the set C. We also write $A \sim B$ if there is a homeomorphism between the topological spaces A and B. If (X, τ) is a topological space, then $\mathcal{K}_X$ represents the set of all compact countable subsets of X, where a countable set is either a finite set or a countably infinite set, and $\mathscr{K}_X := \mathcal{K}_X / \sim$ denotes the set of all the equivalence classes, up to homeomorphisms, of elements of $\mathcal{K}_X$. If (E, d) is a metric space, $x \in E$ and $r > 0$, we denote by $B(x, r)$ and $B[x, r]$ the open and closed balls, centered at x with radius $r > 0$, respectively. Furthermore, for all $Y \neq \varnothing$, ρ_Y is used to designate the discrete metric on the set Y. We now give the following definition.

Definition 1.2 (Cantor-Bendixson's characteristic)**.** Let D be a subset of a topological space such that there exists an ordinal number $\beta \in \mathbf{OR}$ with the property that $D^{(\beta)}$ is finite. We say that $(\alpha, p) \in \mathbf{OR} \times \omega$ is the *Cantor-Bendixson characteristic* of D if α is the smallest ordinal number such that $D^{(\alpha)}$ is finite and $|D^{(\alpha)}| = p$. In this case, we write $\mathcal{CB}(D) = (\alpha, p)$.

For the sake of completeness, we give here the proof of the following theorem, which was first introduced by G. Cantor in [6] for an n-dimensional Euclidean space. It deserves to point out that there are some known extensions, considering topological spaces, of the next result.

Theorem 1.1. *Let (X, τ) be a Hausdorff space. For all $K \in \mathcal{K}_X$, there exists $\alpha \in \omega_1$ such that $K^{(\alpha)}$ is a finite set.*

Proof. Let $K \in \mathcal{K}_X$. We suppose, for a contradiction, that for all countable ordinal number γ, $K^{(\gamma)}$ is an infinite set. Let $\alpha \in \omega_1$. Since $\alpha + 1 \in \omega_1$, we have that $K^{(\alpha+1)}$ is an infinite set, and thus it is a nonempty set. By Remark 1.1, $K^{(\alpha+1)} \subseteq K$. Then, $K^{(\alpha+1)}$ is a countable set. Using the fact that every nonempty, compact, perfect, Hausdorff space is uncountable, we obtain that $K^{(\alpha+2)} \neq K^{(\alpha+1)}$. Thus, by using again Remark 1.1, we get $K^{(\alpha+2)} \subsetneq K^{(\alpha+1)}$. We now define

$$K_\alpha := K^{(\alpha+1)} \smallsetminus K^{(\alpha+2)} \neq \varnothing.$$

Then, $\{K_\gamma : \gamma \in \omega_1\}$ is a family of nonempty sets. By the Axiom of Choice, there exists a function

$$f \colon \omega_1 \to \bigcup_{\gamma \in \omega_1} K_\gamma$$

such that for all $\gamma \in \omega_1$, $f(\gamma) \in K_\gamma$. We claim that f is injective. In fact, let $\beta, \delta \in \omega_1$ be such that $\beta < \delta$. Thus, $\beta + 2 \leq \delta + 1$. By Remark 1.1,

$$K^{(\delta+1)} \subseteq K^{(\beta+2)}.$$

Then,

$$K_\beta \cap K_\delta = (K^{(\beta+1)} \smallsetminus K^{(\beta+2)}) \cap (K^{(\delta+1)} \smallsetminus K^{(\delta+2)}) = \varnothing.$$

Since $f(\beta) \in K_\beta$ and $f(\delta) \in K_\delta$, it follows that $f(\beta) \neq f(\delta)$. Hence, f is a one-to-one function. Therefore,

$$\aleph_1 := |\omega_1| \leq \left| \bigcup_{\gamma \in \omega_1} K_\gamma \right| \leq |K| \leq \aleph_0,$$

giving a contradiction. This finishes the proof of the theorem. ∎

Remark 1.2. Last theorem implies that if (X, τ) is a Hausdorff space and $K \in \mathcal{K}_X$, then $\mathcal{CB}(K)$ is well-defined and furthermore $\mathcal{CB}(K) \in \omega_1 \times \omega$.

2. EXISTENCE OF HOMEOMORPHISMS

Lemma 2.1. *If (E,d) is a metric space, $K \in \mathcal{K}_E$, and $\mathcal{CB}(K) = (1,1)$, then*

$$K \sim \omega + 1.$$

Proof. Since $\mathcal{CB}(K) = (1,1)$, there exists $x \in E$ such that $K' = \{x\}$. Moreover, we see that $K = K^{(0)}$ is infinite. Then, $K \setminus K'$ is a countably infinite set. Thus, there is a bijection g from $K \setminus K'$ onto ω. We now define the following function

$$\begin{aligned} f \colon K &\longrightarrow \omega + 1 \\ z &\longmapsto f(z) = \begin{cases} g(z), & \text{if } z \neq x, \\ \omega, & \text{if } z = x. \end{cases} \end{aligned}$$

From the definition of f, we obtain directly that f is a bijective function. We will now show that f is continuous. Since every point belonging to $K \setminus K'$ is an isolated point of K, it follows that f is continuous at every point of $K \setminus K'$. Thus, it remains to show the continuity of f at the point x. We take an open basic neighborhood V of $f(x) = \omega$ with regard to the order topology of $\omega + 1$. We will now show that $f^{-1}(V)$ is a neighborhood of x. If $V = [0, \beta)$, we have that $\beta = \omega + 1$. Thus, $V = \omega + 1$ and $f^{-1}(V) = K$ is a neighborhood of x. On the other hand, if $V = (n, \alpha)$, then

$$n < \omega < \alpha \leq \omega + 1.$$

Therefore, $n \in \omega$ and $\alpha = \omega + 1$. We now define the following set

$$A := \{z \in K : f(z) \leq n\}.$$

Thus, $x \notin A$. Moreover, since f is an injective function, we see that A is a finite set. Let us take $r := \min\{d(z,x) : z \in A\} > 0$. Then,

$$K \cap B(x,r) \subseteq f^{-1}((n, \omega + 1)).$$

In fact, if $z \in K$ satisfies $d(z,x) < r$, then $z \notin A$. Hence, $f(z) > n$. In addition, using the definition of function f, we see directly that $f(z) < \omega + 1$. Thus, $f(z) \in (n, \omega + 1) = V$. Consequently, $f^{-1}((n, \omega + 1))$ is a neighborhood of x. Therefore, f is continuous at the point x. We thus conclude that f is continuous at every point of its domain. Then, f is a continuous function. Finally, since f is a continuous bijective function, K is compact and $\omega + 1$ is a Hausdorff space, it follows that f is a homeomorphism. In conclusion, $K \sim \omega + 1$. ∎

The next lemma extends Lemma 3.4 in [1] to the case of an arbitrary T_1 topological space.

Lemma 2.2. *Let K and F be closed subsets of a T_1 topological space such that $K \cap F = K \cap \operatorname{int}(F)$, where $\operatorname{int}(F)$ is the set of all interior points of F. Then, for all $\alpha \in \mathbf{OR}$, we have that*

$$(K \cap F)^{(\alpha)} = K^{(\alpha)} \cap F. \tag{2.1}$$

Proof. We will use Transfinite Induction.

- The case $\alpha = 0$ follows directly.
- We assume that the result holds for a given $\alpha \in \mathbf{OR}$, i.e., $(K \cap F)^{(\alpha)} = K^{(\alpha)} \cap F$. Then,

  $$(K \cap F)^{(\alpha+1)} = \left((K \cap F)^{(\alpha)}\right)' = (K^{(\alpha)} \cap F)' \subseteq (K^{(\alpha)})' \cap F' \subseteq K^{(\alpha+1)} \cap F,$$

  where in the last expression we have used the fact that F is closed. To show the other inclusion, let $x \in K^{(\alpha+1)} \cap F$. Since K is a closed subset of a T_1 topological space, using Remark 1.1, it follows that $x \in K \cap F = K \cap \operatorname{int}(F)$. Therefore, there exists a neighborhood U of x such that $U \subseteq F$. Let V be a neighborhood of x. We now take $W := U \cap V$. We see that W is also a neighborhood of x. Then,

$$\begin{aligned}\varnothing &\neq \big(W \smallsetminus \{x\}\big) \cap K^{(\alpha)}\\ &= \big(W \smallsetminus \{x\}\big) \cap K^{(\alpha)} \cap F\\ &= \big(W \smallsetminus \{x\}\big) \cap (K \cap F)^{(\alpha)}\\ &\subseteq \big(V \smallsetminus \{x\}\big) \cap (K \cap F)^{(\alpha)}.\end{aligned}$$

Hence, $x \in (K \cap F)^{(\alpha+1)}$. Thus, $K^{(\alpha+1)} \cap F \subseteq (K \cap F)^{(\alpha+1)}$. Therefore, $(K \cap F)^{(\alpha+1)} = K^{(\alpha+1)} \cap F$.

- Lastly, let $\lambda \neq 0$ be a limit ordinal number. We assume that for all $\beta \in \mathbf{OR}$ such that $\beta < \lambda$, $(K \cap F)^{(\beta)} = K^{(\beta)} \cap F$. Hence,

$$(K \cap F)^{(\lambda)} = \bigcap_{\beta<\lambda} (K \cap F)^{(\beta)} = \bigcap_{\beta<\lambda} (K^{(\beta)} \cap F) = \bigcap_{\beta<\lambda} K^{(\beta)} \cap F = K^{(\lambda)} \cap F.$$

This finishes the proof. ∎

Lemma 2.3. *Let (E, d) be a metric space and let $\alpha > 1$ be a countable ordinal number. Suppose that for all ordinal number $\beta \in \mathbf{OR}$ such that $0 < \beta < \alpha$ and for all $\widetilde{K} \in \mathcal{K}_E$ with $\mathcal{CB}(\widetilde{K}) = (\beta, p) \in \mathbf{OR} \times \omega$, we have that $\widetilde{K} \sim \omega^\beta \cdot p + 1$. Then, for all $K \in \mathcal{K}_E$ such that $\mathcal{CB}(K) = (\alpha, 1)$, we get*

$$K \sim \omega^\alpha + 1.$$

Proof. Let $K \in \mathcal{K}_E$ be such that $\mathcal{CB}(K) = (\alpha, 1)$ with $\alpha > 1$. Then, there exists $x \in K$ with $K^{(\alpha)} = \{x\}$. We see that $x \in K^{(\alpha)} \subseteq K''$. Thus, x is an accumulation point of K'. Then, there is a sequence $(x_n)_{n\in\omega}$ in $K' \smallsetminus \{x\}$ such that $(d(x_n, x))_{n\in\omega}$ is a strictly decreasing sequence converging to 0. Moreover, since $\{d(z, x) \in \mathbb{R} : z \in K\}$ is a countable set, it follows that for all $n \in \omega$,

$$A_n := \{d(z, x) \in \mathbb{R} : z \in K\}^c \cap (d(x_{n+1}, x), d(x_n, x))$$

is a nonempty set. Therefore, $\{A_n : n \in \omega\}$ is a nonempty family of nonempty sets. By the Axiom of Countable Choice, there exists a sequence $(r_n)_{n\in\omega}$ of real numbers such that for all $n \in \omega$,

$$d(x_{n+1}, x) < r_n < d(x_n, x)$$

and for all $z \in K$ we have that $d(z, x) \neq r_n$. Thus, for all $n \in \omega$, we define the following sets

$$\begin{aligned}F_0 &:= B(x, r_0)^c,\\ F_{n+1} &:= B[x, r_n] \smallsetminus B(x, r_{n+1})\end{aligned}$$

and

$$K_n := K \cap F_n.$$

We claim that for all $n \in \omega$,

$$K \cap F_n = K \cap \operatorname{int}(F_n).$$

In fact, let $n \in \omega$. We see immediately that $K \cap \operatorname{int}(F_n) \subseteq K \cap F_n$. Reciprocally, let $z \in K \cap F_n$. We first consider the case when $n = 0$. We obtain that $z \in K$ and $d(z, x) \geq r_0$. Since $z \in K$, we have that $d(z, x) \neq r_0$. Thus, $\varepsilon_0 := d(z, x) - r_0 > 0$. It is not difficult to see now that $B(z, \varepsilon_0) \subseteq F_0$. Then, $z \in \operatorname{int}(F_0)$. Hence, $K \cap F_0 \subseteq K \cap \operatorname{int}(F_0)$. We now consider the case $n \in \omega \smallsetminus \{0\}$. We have that $z \in K$ and

$$r_n \leq d(z, x) \leq r_{n-1}.$$

Since $z \in K$, we obtain that $d(z,x) \neq r_n$ and $d(z,x) \neq r_{n-1}$. We now take $\varepsilon_n := \min\{d(z,x)-r_n, r_{n-1} - d(z,x)\} > 0$. We get $B(z,\varepsilon_n) \subseteq F_n$. Hence, $z \in \operatorname{int}(F_n)$. Therefore, $K \cap F_n \subseteq K \cap \operatorname{int}(F_n)$.

We can now see that the family $\{K_n : n \in \omega\}$ has the following properties:

- Since K is a closed subset of E, we obtain that for all $n \in \omega$, $x_n \in K_n$.
- For all $n \in \omega$, $K_n \subseteq K$.
- Since the intersection of two closed subsets is also closed, we see that for all $n \in \omega$, K_n is a closed subset.
- Since every closed subset of a compact space is compact, we have that for all $n \in \omega$, K_n is compact.
- Since for all $n \in \omega$, K_n is a countable set, we obtain that for all $n \in \omega$, $K_n \in \mathcal{K}_E$.
- For all $n \in \omega$, $K'_n \neq \varnothing$. In fact, let $n \in \omega$. By Lemma 2.2, we have that $K'_n = (K \cap F_n)' = K' \cap F_n$. Moreover, since $x_n \in K' \cap F_n$, we see that $x_n \in K'_n$.
- Since $\{F_n : n \in \omega\}$ is a pairwise disjoint family of sets, we obtain that the family of sets $\{K_n : n \in \omega\}$ is also pairwise disjoint.
- We have that
$$K = \biguplus_{n\in\omega} K_n \uplus \{x\}.$$
In fact, since the sequence $(r_n)_{n\in\omega}$ converges to 0, we see that $\biguplus_{n\in\omega} F_n \uplus \{x\} = E$. Then,
$$\begin{aligned}\biguplus_{n\in\omega} K_n \uplus \{x\} &= \biguplus_{n\in\omega} (K \cap F_n) \uplus \{x\} \\ &= \left(K \cap \biguplus_{n\in\omega} F_n\right) \uplus \{x\} \\ &= K \cap \left(\biguplus_{n\in\omega} F_n \uplus \{x\}\right) \\ &= K \cap E = K.\end{aligned}$$
- For all $n \in \omega$, $K_n^{(\alpha)} = \varnothing$. In fact, by Lemma 2.2, we see that for all $n \in \omega$,
$$K_n^{(\alpha)} = (K \cap F_n)^{(\alpha)} = K^{(\alpha)} \cap F_n = \{x\} \cap F_n = \varnothing.$$
- Using the fact that an infinite subset of a compact subset of a topological space has at least a limit point in the compact subset, using also Remark 1.1 and the Cantor intersection theorem in a Hausdorff topological space, we see that the last assertion implies that for all $n \in \omega$, if $\mathcal{CB}(K_n) = (\beta_n, p_n) \in \mathbf{OR} \times \omega$, then $0 < \beta_n < \alpha$ and $p_n \in \omega \smallsetminus \{0\}$.

It follows from the hypothesis that for all $n \in \omega$, $K_n \sim \omega^{\beta_n} \cdot p_n + 1$. By the Axiom of Countable Choice, there is a sequence $(f_n)_{n\in\omega}$ of homeomorphisms such that for all $n \in \omega$, $f_n \colon K_n \to \omega^{\beta_n} \cdot p_n + 1$ is a homeomorphism of the topological space K_n onto $\omega^{\beta_n} \cdot p_n + 1$. We now define the following function

$$\begin{aligned} f \colon K &\longrightarrow \tau + 1 \\ z &\longmapsto f(z) = \begin{cases} f_0(z), & \text{if } z \in K_0, \\ \displaystyle\sum_{k=0}^{n-1} \omega^{\beta_k} \cdot p_k + 1 + f_n(z), & \text{if } z \in K_n, \text{ for some } n \in \omega \smallsetminus \{0\}, \\ \tau, & \text{if } z = x, \end{cases}\end{aligned}$$

where

$$\tau := \sum_{k\in\omega} \omega^{\beta_k}\cdot p_k := \sup\left\{\sum_{k=0}^{n} \omega^{\beta_k}\cdot p_k : n\in\omega\right\}.$$

Proceeding in a similar way to the proof of Lemma 3.3 in [1], it is possible to show that $\tau = \omega^\alpha$ and f is a homeomorphism of K onto $\tau+1$. Hence, $K \sim \omega^\alpha + 1$. ∎

Lemma 2.4. *Let (E,d) be a metric space. Let α be a countable ordinal number such that $\alpha > 0$ and let $p\in\omega$. We assume that for all $\widetilde{K}\in\mathcal{K}_E$ such that $\mathcal{CB}(\widetilde{K}) = (\alpha, 1)$, we have that $\widetilde{K}\sim\omega^\alpha+1$. Then, for all $K\in\mathcal{K}_E$ with $\mathcal{CB}(K) = (\alpha,p)$, we get*

$$K\sim\omega^\alpha\cdot p+1.$$

Proof. Let $K\in\mathcal{K}_E$ be such that $\mathcal{CB}(K)=(\alpha,p)$ with $\alpha>0$. As mentioned in the proof of the previous lemma, we can show that $p\in\omega\smallsetminus\{0\}$. Thus,

$$K^{(\alpha)} = \{x_0, x_1, \ldots, x_{p-1}\},$$

where for all $i,j\in\{0,\ldots,p-1\}$ such that $i\neq j$, $x_i\neq x_j$. For all $m\in\{1,\ldots,p-1\}$, we define

$$d_m := \min\{d(x_m,x_j)\in\mathbb{R} : 0\le j\le p-1 \quad\text{and}\quad j\neq m\} > 0.$$

Let $m\in\{1,\ldots,p-1\}$. Since

$$\{d(z,x_m)\in\mathbb{R} : z\in K\}$$

is a countable set, there exists $r_m>0$ such that

$$r_m\in\{d(z,x_m)\in\mathbb{R} : z\in K\}^c\cap(0,d_m).$$

Thus, for all $z\in K$, we have that $d(z,x_m)\neq r_m$. We now define

$$F_m := B[x_m, r_m]$$

and

$$F_0 := E\smallsetminus\bigcup_{j=1}^{p-1} B(x_j,r_j).$$

Moreover, for all $n\in\{0,\ldots,p-1\}$, we also define

$$K_n := K\cap F_n.$$

We observe that for all $n\in\{0,\ldots,p-1\}$,

$$K\cap F_n = K\cap\operatorname{int}(F_n).$$

In fact, let $n\in\{0,\ldots,p-1\}$. Since $\operatorname{int}(F_n)\subseteq F_n$, we see that $K\cap\operatorname{int}(F_n)\subseteq K\cap F_n$. Reciprocally, given $z\in K\cap F_n$, we see that $z\in K$ and we consider the following two cases:

- We first examine the situation when $n\in\{1,\ldots,p-1\}$. Since $F_n := B[x_n,r_n]$, we have that

  $$d(z,x_n)\le r_n.$$

  In addition, $z\in K$ implies that $d(z,x_n)\neq r_n$. By taking $\varepsilon_n := r_n - d(z,x_n) > 0$, we obtain that $B(z,\varepsilon_n)\subseteq F_n$. Hence, $z\in\operatorname{int}(F_n)$.
- We now assume that $n=0$. Using the fact that $z\in F_0 := E\smallsetminus\bigcup_{j=1}^{p-1}B(x_j,r_j)$, we conclude that for all $j\in\{1,\ldots,p-1\}$,

  $$d(z,x_j)\ge r_j.$$

Furthermore, since $z \in K$, we see that for all $j \in \{1, \ldots, p-1\}$, $d(z, x_j) \neq r_j$. We now take $\varepsilon_0 := \min\{d(z, x_j) - r_j : 1 \leq j \leq p-1\} > 0$. Then, $B(z, \varepsilon_0) \subseteq F_0$. In order to prove this last assertion, let $w \in B(z, \varepsilon_0)$. We now suppose, to derive a contradiction, that there exists $i \in \{1, \ldots, p-1\}$ such that $d(w, x_i) < r_i$. Then, $\varepsilon_0 + r_i \leq d(z, x_i) \leq d(z, w) + d(w, x_i) < \varepsilon_0 + r_i$, which is a contradiction. Thus, $z \in \operatorname{int}(F_0)$.

Therefore, $K \cap F_n \subseteq K \cap \operatorname{int}(F_n)$.
Proceeding in a similar manner as in the proof of Lemma 2.3, we can show that the family $\{K_n : 0 \leq n \leq p-1\}$ satisfies the next properties:

- Using Remark 1.1, we have that for all $n \in \{0, \ldots, p-1\}$, $x_n \in K_n$.
- For all $n \in \{0, \ldots, p-1\}$, $K_n \subseteq K$.
- For all $n \in \{0, \ldots, p-1\}$, K_n is a closed subset of E.
- Since every closed subset of a compact space is compact, we obtain that for all $n \in \{0, \ldots, p-1\}$, K_n is compact.
- For all $n \in \{0, \ldots, p-1\}$, $K_n \in \mathcal{K}_E$.
- $\{K_n : 0 \leq n \leq p-1\}$ is a pairwise disjoint family of sets.
- $K = \biguplus_{n=0}^{p-1} K_n$.
- Using Lemma 2.2, we conclude that for all $n \in \{0, \ldots, p-1\}$,
$$K_n^{(\alpha)} = (K \cap F_n)^{(\alpha)} = K^{(\alpha)} \cap F_n = \{x_n\}.$$
- It follows from the last assertion that for all $n \in \{0, \ldots, p-1\}$, $\mathcal{CB}(K_n) = (\alpha, 1)$.

By using the hypothesis, we see that for all $n \in \{0, \ldots, p-1\}$, there exists a homeomorphism $g_n \colon K_n \to \omega^\alpha + 1$ from the topological space K_n onto $\omega^\alpha + 1$. We now consider the following function

$$\begin{aligned} g\colon K &\longrightarrow \tau + 1 \\ z &\longmapsto g(z) = \begin{cases} g_0(z), & \text{if } z \in K_0, \\ \omega^\alpha \cdot n + 1 + g_n(z), & \text{if } z \in K_n, \text{ for some } n \in \{1, \ldots, p-1\}, \end{cases} \end{aligned}$$

where $\tau := \omega^\alpha \cdot p$. By a similar argument to the one used in the proof of Lemma 2.3 above, we obtain that function g is a homeomorphism from K onto $\tau + 1$. Hence, $K \sim \omega^\alpha \cdot p + 1$. ∎

Theorem 2.5. *Suppose that (E, d) is a metric space. Let α be a countable ordinal number such that $\alpha > 0$ and let $p \in \omega$. If $K \in \mathcal{K}_E$ satisfies $\mathcal{CB}(K) = (\alpha, p)$, then*

$$K \sim \omega^\alpha \cdot p + 1.$$

Proof. We proceed by Strong Transfinite Induction on the ordinal number $\alpha > 0$. By Lemmas 2.1 and 2.4, the result holds for $\alpha = 1$. Now, let $\alpha \in \omega_1$ be such that $\alpha > 1$. We suppose that the conclusion is true for all ordinal number β such that $0 < \beta < \alpha$. By Lemmas 2.3 and 2.4, the result is also valid for α. Thus, the theorem holds for all countable ordinal number greater than zero. ∎

Remark 2.1. The hypothesis about the countable cardinality of the ordinal number α in Lemma 2.3, Lemma 2.4 and Theorem 2.5 can be omitted. In fact, if (E, d) is a metric space, $K \in \mathcal{K}_E$, $(\alpha, p) \in \mathbf{OR} \times (\omega \setminus \{0\})$ and $K \sim \omega^\alpha \cdot p + 1$, then $\alpha \in \omega_1$.

3. CARDINALITY OF THE PARTITION

Let (X,τ) be a topological space. We consider the set $\mathcal{K}_X$ of all compact countable subsets of X. The set $\mathscr{K}_X := \mathcal{K}_X/\sim$ provides a partition of the set $\mathcal{K}_X$ into disjoint equivalence classes, more precisely,

$$\mathscr{K}_X = \{[K] \in \mathcal{P}(\mathcal{K}_X) : K \in \mathcal{K}_X\},$$

where, for all $K \in \mathcal{K}_X$

$$[K] := \{K_1 \in \mathcal{K}_X : K_1 \sim K\}.$$

The following two propositions will be used in the proof of Theorem 3.3 below.

Proposition 3.1. *Let (X,τ) be a T_1 topological space. For all $K_1, K_2 \in \mathcal{K}_X$ such that $K_1 \sim K_2$, we have that $\mathcal{CB}(K_1) = \mathcal{CB}(K_2)$.*

Proof. Let $K_1, K_2 \in \mathcal{K}_X$ be such that $K_1 \sim K_2$ and let $f\colon K_1 \to K_2$ be a homeomorphism from K_1 onto K_2. We will first show that for all ordinal number $\alpha \in \mathbf{OR}$, $K_1^{(\alpha)} \sim K_2^{(\alpha)}$, where $f|_{K_1^{(\alpha)}}$ is a homeomorphism between these two sets. In order to prove this last assertion, we use below Transfinite Induction.

- In the case when $\alpha = 0$, we see that $K_1^{(0)} = K_1 \sim K_2 = K_2^{(0)}$ and $f = f|_{K_1^{(0)}}\colon K_1^{(0)} \to K_2^{(0)}$ is a homeomorphism from $K_1^{(0)}$ onto $K_2^{(0)}$.
- We now suppose that the result holds for a given ordinal number α, i.e., $K_1^{(\alpha)} \sim K_2^{(\alpha)}$ and $f|_{K_1^{(\alpha)}}$ is a homeomorphism between $K_1^{(\alpha)}$ and $K_2^{(\alpha)}$. By Remark 1.1 above, we have that $K_1^{(\alpha+1)} \subseteq K_1^{(\alpha)}$. Thus,
$$f(K_1^{(\alpha+1)}) = f|_{K_1^{(\alpha)}}(K_1^{(\alpha+1)}) = f|_{K_1^{(\alpha)}}((K_1^{(\alpha)})') = (K_2^{(\alpha)})' = K_2^{(\alpha+1)}.$$
Then, $f|_{K_1^{(\alpha+1)}}\colon K_1^{(\alpha+1)} \to K_2^{(\alpha+1)}$ is a homeomorphism from $K_1^{(\alpha+1)}$ onto $K_2^{(\alpha+1)}$. Therefore, $K_1^{(\alpha+1)} \sim K_2^{(\alpha+1)}$.
- Finally, let $\lambda \neq 0$ be a limit ordinal number. We presume that for all $\beta \in \mathbf{OR}$ such that $\beta < \lambda$, $K_1^{(\beta)} \sim K_2^{(\beta)}$, where $f|_{K_1^{(\beta)}}$ is a homeomorphism from $K_1^{(\beta)}$ onto $K_2^{(\beta)}$. Since f is an injection, we have that
$$f(K_1^{(\lambda)}) = f\left(\bigcap_{\beta<\lambda} K_1^{(\beta)}\right) = \bigcap_{\beta<\lambda} f(K_1^{(\beta)}) = \bigcap_{\beta<\lambda} K_2^{(\beta)} = K_2^{(\lambda)}.$$
Therefore, $f|_{K_1^{(\lambda)}}$ is a homeomorphism between $K_1^{(\lambda)}$ and $K_2^{(\lambda)}$, i.e., $K_1^{(\lambda)} \sim K_2^{(\lambda)}$.

Then, for all $\alpha \in \mathbf{OR}$, $|K_1^{(\alpha)}| = |K_2^{(\alpha)}|$. We suppose that $\mathcal{CB}(K_1) = (\beta, p) \in \mathbf{OR} \times \omega$. Thus, β is the smallest ordinal number such that $K_1^{(\beta)}$ is finite. Furthermore, since $|K_1^{(\beta)}| = p$, we obtain that $|K_2^{(\beta)}| = |K_1^{(\beta)}| = p$. With this, we conclude that $\mathcal{CB}(K_2) = (\beta, p) = \mathcal{CB}(K_1)$. ∎

Proposition 3.2. *Let (E,d) be a metric space and let $K_1, K_2 \in \mathcal{K}_E$. If $\mathcal{CB}(K_1) = \mathcal{CB}(K_2)$, then $K_1 \sim K_2$.*

Proof. Let $\mathcal{CB}(K_1) = \mathcal{CB}(K_2) = (\alpha, p)$, for some ordinal number α and some $p \in \omega$.

- If $\alpha = 0$, we have that K_1 and K_2 are both finite sets with p elements, therefore $K_1 \sim K_2$.
- If $\alpha > 0$ and $p \in \omega$, by Theorem 2.5, we have that $K_1 \sim \omega^\alpha \cdot p + 1$ and $K_2 \sim \omega^\alpha \cdot p + 1$, thus $K_1 \sim K_2$. ∎

Propositions 3.1 and 3.2 imply that for any metric space (E, d), the partition of $\mathcal{K}_E$ is fully characterized by the Cantor-Bendixson characteristic.

Theorem 3.3. *Let (E, d) be a metric space. The cardinality of $\mathscr{K}_E$ is less than or equal to $\aleph_1$.*

Proof. We define the function

$$\begin{aligned} \widetilde{\mathcal{CB}}\colon \mathscr{K}_E &\longrightarrow \omega_1 \times \omega \\ [K] &\longmapsto \widetilde{\mathcal{CB}}([K]) = \mathcal{CB}(K). \end{aligned}$$

By Theorem 1.1 and Proposition 3.1, we see that function $\widetilde{\mathcal{CB}}$ is well-defined. Moreover, Proposition 3.2 shows that $\widetilde{\mathcal{CB}}$ is an injective function. Thus,

$$|\mathscr{K}_E| \leq |\omega_1 \times \omega| = |\omega_1| =: \aleph_1. \quad \blacksquare$$

In general, we cannot strengthen the last result. To show this, we give the following three propositions.

Proposition 3.4. *For all $n \in \omega$, there exists a metric space (E_n, d_n) such that $|\mathscr{K}_{E_n}| = n$.*

Proof. Let $n \in \omega \setminus \{0\}$. We now consider the space (n, ρ_n), where ρ_n is the discrete metric on the set $n := \{0, \ldots, n-1\}$. Since n is a finite set, we have that every subset of n is compact, i.e.,

$$\mathcal{K}_n = \mathcal{P}(n).$$

Moreover, for all $K \in \mathcal{K}_n$, we see that $K^{(0)} = K$ is a finite set. Therefore, $\mathcal{CB}(K) = (0, |K|)$. Thus,

$$\widetilde{\mathcal{CB}}(\mathscr{K}_n) \subseteq \{0\} \times n.$$

On the other hand, for all $k \in \{0, \ldots n-1\}$, there exists $F \subseteq n$ such that $|F| = k$, i.e., $\mathcal{CB}(F) = (0, k)$. Thus,

$$\widetilde{\mathcal{CB}}(\mathscr{K}_n) = \{0\} \times n.$$

Hence,

$$|\mathscr{K}_n| = |\widetilde{\mathcal{CB}}(\mathscr{K}_n)| = n. \quad \blacksquare$$

Proposition 3.5. *There exists a metric space (E, d_E) such that $|\mathscr{K}_E| = \aleph_0$.*

Proof. We consider the metric space (ω, ρ_ω). Since ρ_ω is the discrete metric on the set ω, we see that a subset of ω is compact if and only if it is a finite set. Then, for all $K \in \mathcal{K}_\omega$, $K^{(0)} = K$ is a finite set. Hence, for all $K \in \mathcal{K}_\omega$, $\mathcal{CB}(K) = (0, |K|)$. Thus, $\widetilde{\mathcal{CB}}(\mathscr{K}_\omega) \subseteq \{0\} \times \omega$. On the other hand, since for all $k \in \omega$, there exists $K \subseteq \omega$ such that $|K| = k$, it follows that $\widetilde{\mathcal{CB}}(\mathscr{K}_\omega) = \{0\} \times \omega$. Therefore,

$$|\mathscr{K}_\omega| = |\widetilde{\mathcal{CB}}(\mathscr{K}_\omega)| = \aleph_0. \quad \blacksquare$$

Proposition 3.6. *There exists a metric space (F, d_F) such that $|\mathscr{K}_F| = \aleph_1$.*

Proof. We take the metric space $(\mathbb{R}, d)$, where d is the usual metric on the set $\mathbb{R}$. By Theorem 3.4 in [1], we obtain that

$$|\mathscr{K}_\mathbb{R}| = \aleph_1. \quad \blacksquare$$

Finally, it is worth mentioning that the last two results do not depend on the cardinality of the underlying metric spaces considered there, as it can be seen in the next two propositions.

Proposition 3.7. *There exists a countable metric space (G, d_G) such that $|\mathscr{K}_G| = \aleph_1$.*

Proof. Proceeding in a similar way as in the proof of Theorem 2.1 in [1] and considering the density of the rational numbers, $\mathbb{Q}$, in $\mathbb{R}$, we can see that for all countable ordinal number $\alpha \in \omega_1$, and for all $a, b \in \mathbb{Q}$ such that $a < b$, there exists a set $K \in \mathcal{K}_{\mathbb{Q}}$ such that $K \subseteq (a, b]$ and $K^{(\alpha)} = \{b\}$. By using this last statement, we can prove an analogous result to Corollary 2.1 in [1], more precisely, we have that for all countable ordinal number $\alpha \in \omega_1$, and for all $p \in \omega$, there is a set $K \in \mathcal{K}_{\mathbb{Q}}$ such that $|K^{(\alpha)}| = p$. Then,

$$\widetilde{\mathcal{CB}}(\mathscr{K}_{\mathbb{Q}}) = \big(\omega_1 \times (\omega \smallsetminus \{0\})\big) \cup \{(0,0)\}.$$

Hence,

$$|\mathscr{K}_{\mathbb{Q}}| = |\widetilde{\mathcal{CB}}(\mathscr{K}_{\mathbb{Q}})| = |\omega_1 \times \omega| = |\omega_1| =: \aleph_1. \quad \blacksquare$$

Proposition 3.8. *There exists an uncountable metric space* (H, d_H) *such that* $|\mathscr{K}_H| = \aleph_0$.

Proof. We take the uncountable metric space $(\mathbb{R}, \rho_{\mathbb{R}})$, where $\rho_{\mathbb{R}}$ is the discrete metric on the real line. Proceeding in a similar fashion as in the proof of Proposition 3.5, we obtain

$$|\mathscr{K}_{\mathbb{R}}| = \aleph_0. \quad \blacksquare$$

REFERENCES

[1] B. ÁLVAREZ-SAMANIEGO and A. MERINO, A primitive associated to the Cantor-Bendixson derivative on the real line, *J. Math. Sci. Adv. Appl.*, **41** (2016), pp. 1–33.

[2] J. BAKER, Compact spaces homeomorphic to a ray of ordinals, *Fund. Math.*, **76** (1972), pp. 19–27.

[3] J. BAKER, Dispersed images of topological spaces and uncomplemented subspaces of $C(X)$, *Proc. Amer. Math. Soc.*, **41** (1973), pp. 309–314.

[4] J. BAKER, Ordinal subspaces of topological spaces, *General Topology Appl.*, **3** (1973), pp. 85–91.

[5] G. CANTOR, Ueber unendliche, lineare Punktmannichfaltigkeiten I, *Math. Ann.*, **15** (1879), pp. 1–7.

[6] G. CANTOR, Sur divers théorèmes de la théorie des ensembles de points situés dans un espace continu à n dimensions, *Acta Math.*, **2** (1883), pp. 409–414.

[7] M. FRÉCHET, Les dimensions d'un ensemble abstrait, *Math. Ann.*, **68** (1910), pp. 145–168.

[8] W. D. GILLAM, Embeddability properties of countable metric spaces, *Topology Appl.*, **148** (2005), pp. 63–82.

[9] G. KNASTER and K. URBANIK, Sur les espaces complets séparables de dimension 0, *Fund. Math.*, **40** (1953), pp. 194–202.

[10] S. MAZURKIEWICZ and W. SIERPINSKI, Contribution à la topologie des ensembles dénombrables, *Fund. Math.*, **1** (1920), pp. 17–27.