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SOME INEQUALITIES OF THE HERMITE–HADAMARD TYPE FOR k-FRACTIONAL CONFORMABLE INTEGRALS

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ABSTRACT. In the paper, the authors deal with generalized k-fractional conformable integrals, establish some inequalities of the Hermite–Hadamard type for generalized k-fractional conformable integrals for convex functions, and generalize known inequalities of the Hermite–Hadamard type for conformable fractional integrals.

Key words and phrases: Gamma function; k-gamma function; Convex function; Inequality of the Hermite–Hadamard type; Riemann–Liouville fractional integral; Fractional conformable integral; Generalized k-fractional conformable integral.

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1. INTRODUCTION

The theory of fractional integral inequalities plays a vital role in the field of mathematical sciences. One of the most famous inequalities for convex functions, the Hermite–Hadamard integral inequality, reads that, if $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

There have been many mathematicians dedicated to generalizations and extensions of (1.1). For detailed information, please refer to [5, 11, 16, 29, 35] and closely related references therein.

Recall from [31, 32, 34] that the Riemann–Liouville fractional integrals $\mathfrak{I}_{a^+}^{\alpha}$ and $\mathfrak{I}_{b^-}^{\alpha}$ of order α can be defined respectively by

$$\mathfrak{I}_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}\,t, \quad x > a \tag{1.2}$$

and

$$\mathfrak{I}_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,\mathrm{d}\,t, \quad x < b, \tag{1.3}$$

where $\Re(\alpha) > 0$ and Γ is the classical Euler gamma function [17, 19, 33].

In [12], the Riemann–Liouville k-fractional integrals are respectively defined by

$$\mathfrak{I}_{k,a^+}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\alpha/k-1} f(t)\,\mathrm{d}\,t, \quad x > a \tag{1.4}$$

and

$$\mathfrak{I}_{k,b^-}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\alpha/k-1} f(t)\,\mathrm{d}\,t, \quad x < b \tag{1.5}$$

for $\Re(\alpha) > 0$. For more information on fractional integral operators (1.2), (1.3), (1.4), and (1.5), please refer to the papers [1, 2, 3, 6, 9, 10, 12, 13, 21, 23, 25, 28, 30, 31, 32, 34] and closely related references therein.

We now recall from [26] two inequalities of the Hermite–Hadamard type for the Riemann–Liouville fractional integrals as follows.

Theorem 1.1 ([26]). *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)}{2(b-a)^{\alpha}}\left[\mathfrak{I}_{a^+}^{\alpha} f(b) + \mathfrak{I}_{b^-}^{\alpha} f(a)\right] \le \frac{f(a)+f(b)}{2}. \tag{1.6}$$

Theorem 1.2 ([26]). *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping such that $a < b$ and $f' \in L[a, b]$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)}{2(b-a)^{\alpha}}\left[\mathfrak{I}_{a^+}^{\alpha} f(b) + \mathfrak{I}_{b^-}^{\alpha} f(a)\right]\right| \le \frac{b-a}{2(\alpha+1)}\left(1 - \frac{1}{2^{\alpha}}\right)\left(|f'(a)| + |f'(b)|\right). \tag{1.7}$$

If letting $\alpha = 1$, then the inequality (1.6) reduces to (1.1).

The left and right fractional conformable integral operators are respectively defined in [8] by

$${}^{\beta}\mathfrak{I}_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\beta)}\int_a^x \left[\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha}\right]^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}}\,\mathrm{d}\,t \tag{1.8}$$

and

$$(1.9)\qquad {}^{\beta}\mathfrak{J}_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left[\frac{(b-x)^\alpha - (b-t)^\alpha}{\alpha}\right]^{\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}} \,\mathrm{d}\, t$$

for $\alpha > 0$ and $\Re(\beta) > 0$. Obviously, if taking $a = 0$ and $\alpha = 1$, then (1.8) and (1.9) reduce to the Riemann–Liouville fractional integrals (1.2) and (1.3) respectively.

The generalized k-fractional conformable integrals are defined in [7] by

$${}^{\beta}_{k}\mathfrak{J}_{a+}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\beta)} \int_a^x \left[\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right]^{\beta/k-1} \frac{f(t)}{(t-a)^{1-\alpha}} \,\mathrm{d}\, t$$

and

$${}^{\beta}_{k}\mathfrak{J}_{b-}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\beta)} \int_a^x \left[\frac{(b-x)^\alpha - (b-t)^\alpha}{\alpha}\right]^{\beta/k-1} \frac{f(t)}{(b-t)^{1-\alpha}} \,\mathrm{d}\, t,$$

where $\alpha > 0$, $\Re(\beta) > 0$, and $\Gamma_k(x)$ is defined [4, 14, 15, 18, 20, 22] by

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{x/k-1}}{(x)_{n,k}}$$

in terms of

$$(\lambda)_{n,k} = \begin{cases} 1, & n = 0; \\ \lambda(\lambda+k)\cdots(\lambda+(n-1)k), & n \in \mathbb{N}. \end{cases}$$

In this paper, we will establish some inequality of the Hermite–Hadamard type for generalized k-fractional conformable integral operators and generalize several known inequalities of the Hermite–Hadamard type for k-fractional conformable integral operators.

2. A LEMMA

For proving our main results, we need the following lemma.

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that $a < b$ and $f' \in L[a, b]$. Then*

$$(2.1)\qquad \frac{f(a)+f(b)}{2} - \frac{k\Gamma_k(\beta+k)\alpha^{\beta/k}}{(b-a)^{\alpha\beta/k}} \left[{}^{\beta}_{k}\mathfrak{J}_{b-}^{\alpha} f(a) + {}^{\beta}_{k}\mathfrak{J}_{a+}^{\alpha} f(b)\right]$$
$$= \frac{(b-a)\alpha^{\beta/k}}{2} \int_0^1 \left[\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k} - \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}\right] f'(ta+(1-t)b) \,\mathrm{d}\, t$$

for $\alpha, \beta > 0$.

Proof. Denote

$$I_1 = \int_0^1 \left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k} f'(ta+(1-t)b) \,\mathrm{d}\, t$$

and

$$I_2 = \int_0^1 \left[\frac{1-(1-t)^\alpha}{\alpha}\right]^{\beta/k} f'(ta+(1-t)b) \,\mathrm{d}\, t.$$

Integrating by parts yields

$$I_1 = \int_0^1 \left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k} f'(ta+(1-t)b) \,\mathrm{d}\, t$$
$$= \left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k} \frac{f(ta+(1-t)b)}{a-b}\bigg|_0^1 - \int_0^1 \beta\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k} f'(ta+(1-t)b) \,\mathrm{d}\, t$$

$$= \frac{1}{\alpha^{\beta/k}} \frac{f(b)}{b-a} - \frac{\beta}{b-a} \frac{\Gamma_k(\beta)}{(b-a)^{\alpha\beta/k}} {}_k^{\beta}\mathfrak{J}_{b^-}^{\alpha} f(a)$$
$$= \frac{1}{b-a}\left[\frac{f(b)}{\alpha^{\beta/k}} - \frac{\Gamma_k(\beta+k)}{(b-a)^{\alpha\beta/k}} {}_k^{\beta}\mathfrak{J}_{b^-}^{\alpha} f(a)\right]$$

and

$$I_2 = -\frac{1}{b-a}\left[\frac{f(a)}{\alpha^{\beta/k}} - \frac{\Gamma_k(\beta+k)}{(b-a)^{\alpha\beta/k}} {}_k^{\beta}\mathfrak{J}_{a^+}^{\alpha} f(b)\right].$$

Adding I_1 and $-I_2$ and then multiplying on both sides by $\frac{b-a}{2}\alpha^{\beta/k}$ result in (2.1). The proof of Lemma 2.1 is complete. ∎

Remark 2.1. When $\alpha = 1$, the equality (2.1) in Lemma 2.1 reduces to

$$\frac{f(a)+f(b)}{2} - \frac{k\Gamma_k(\beta+k)}{(b-a)^{\beta/k}}\left[{}_k^{\beta}\mathfrak{J}_{b^-} f(a) + {}_k^{\beta}\mathfrak{J}_{a^+} f(b)\right]$$
$$= \frac{(b-a)}{2}\int_0^1 \left[(1-t)^{\beta/k} - t^{\beta/k}\right] f'(ta+(1-t)b)\,\mathrm{d}\,t$$

for $\beta > 0$.

When $k = 1$, the equality (2.1) in Lemma 2.1 becomes [27, Lemma 3.1].

When $\alpha = 1$ and $k = 1$, the equality (2.1) in Lemma 2.1 can be written as

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathfrak{J}_{a^+}^{\alpha} f(b) + \mathfrak{J}_{b^-}^{\alpha} f(a)\right]$$
$$= \frac{b-a}{2}\int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(ta+(1-t)b)\,\mathrm{d}\,t$$

which can be found in [26, Lemma 2].

3. MAIN RESULTS

We now in a position to establish some inequalities of the Hermite–Hadamard type for convex mappings for generalized k-fractional conformable integral operators.

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be such that $f \in L[a,b]$ and $a < b$. If f is convex on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{k\Gamma_k(\beta+k)\alpha^{\beta/k}}{2(b-a)^{\alpha\beta/k}}\left[{}_k^{\beta}\mathfrak{J}_{a^+}^{\alpha} f(b) + {}_k^{\beta}\mathfrak{J}_{b^-}^{\alpha} f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{3.1}$$

for $\alpha, \beta > 0$.

Proof. Since f is a convex function on $[a,b]$, we have

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}, \quad x, y \in [a,b].$$

Letting $x = ta + (1-t)b$ and $y = (1-t)a + tb$ gives

$$2f\left(\frac{a+b}{2}\right) \le f(ta+(1-t)b) + f((1-t)a+tb). \tag{3.2}$$

Multiplying on both sides of (3.2) by $\left(\frac{1-t^{\alpha}}{\alpha}\right)^{\beta/k-1} t^{\alpha-1}$ and then integrating with respect to t over $[0,1]$ lead to

$$2f\left(\frac{a+b}{2}\right)\int_0^1 \left(\frac{1-t^{\alpha}}{\alpha}\right)^{\beta/k-1} t^{\alpha-1}\,\mathrm{d}\,t$$

$$
\begin{aligned}
&\leq \int_0^1 \left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k-1} t^{\alpha-1} f(ta+(1-t)b)\,\mathrm{d}\,t \\
&\quad + \int_0^1 \left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k-1} t^{\alpha-1} f((1-t)a+tb)\,\mathrm{d}\,t \\
&= \frac{1}{b-a}\int_a^b \left[\frac{1-(\frac{b-u}{b-a})^\alpha}{\alpha}\right]^{\beta/k-1} \left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u)\,\mathrm{d}\,u \\
&\quad + \frac{1}{b-a}\int_a^b \left[\frac{1-(\frac{v-a}{b-a})^\alpha}{\alpha}\right]^{\beta/k-1} \left(\frac{v-a}{b-a}\right)^{\alpha-1} f(v)\,\mathrm{d}\,v \\
&= \frac{1}{(b-a)^{\alpha\beta/k}}\int_a^b \left[\frac{(b-a)^\alpha-(b-u)^\alpha}{\alpha}\right]^{\beta/k-1} \frac{f(u)}{(b-u)^{1-\alpha}}\,\mathrm{d}\,u \\
&\quad + \frac{1}{(b-a)^{\alpha\beta/k}}\int_a^b \left[\frac{(b-a)^\alpha-(v-a)^\alpha}{\alpha}\right]^{\beta/k-1} \frac{f(v)}{(v-a)^{1-\alpha}}\,\mathrm{d}\,v \\
&= \frac{k\Gamma_k(\beta)}{(b-a)^{\alpha\beta/k}}\left[{}_k^\beta\mathfrak{I}_{b^-}^\alpha f(a) + {}_k^\beta\mathfrak{I}_{a^+}^\alpha f(b)\right].
\end{aligned}
$$

From

$$\int_0^1 \left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k-1} t^{\alpha-1}\,\mathrm{d}\,t = \frac{k}{\beta\alpha^{\beta/k}},$$

it follows that

$$2f\left(\frac{a+b}{2}\right) \leq \frac{k\Gamma_k(\beta+k)\alpha^{\beta/k}}{(b-a)^{\alpha\beta/k}}\left[{}_k^\beta\mathfrak{I}_{b^-}^\alpha f(a) + {}_k^\beta\mathfrak{I}_{a^+}^\alpha f(b)\right]$$

which can be rewritten as the left hand side of the inequality (3.1).

Making use of the convexity of f arrives at

$$f(ta+(1-t)b) \leq tf(a)+(1-t)f(b)$$

and

$$f(tb+(1-t)a) \leq tf(b)+(1-t)f(a).$$

Adding the above two inequalities yields

$$f(ta+(1-t)b)+f(tb(1-t)a) \leq f(a)+f(b).$$

Multiplying on both sides of the above inequality by $\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k-1} t^{\alpha-1}$ and integrating with respect to t over $[0,1]$ reveal

$$\frac{k\Gamma_k(\beta+k)\alpha^{\beta/k}}{(b-a)^{\alpha\beta/k}}\left[{}_k^\beta\mathfrak{I}_{b^-}^\alpha f(a) + {}_k^\beta\mathfrak{I}_{a^+}^\alpha f(b)\right] \leq f(a)+f(b)$$

which can be rewritten as the right hand side of the inequality (3.1). The proof of Theorem 3.1 is complete. ∎

Remark 3.1. If $\alpha=1$, then the inequality (3.1) reduces to

$$f\left(\frac{a+b}{2}\right) \leq \frac{k\Gamma_k(\beta+k)}{2(b-a)^{\beta/k}}\left[{}_k^\beta\mathfrak{I}_{a^+} f(b) + {}_k^\beta\mathfrak{I}_{b^-} f(a)\right] \leq \frac{f(a)+f(b)}{2}$$

for $\beta>0$.

If $k=1$, then the inequality (3.1) in Theorem 3.1 becomes [27, Theorem 2.1].

If $\alpha=1$ and $k=1$, then the inequality (3.1) in Theorem 3.1 can be rearranged as (1.6).

Theorem 3.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that $a<b$ and $f'\in L[a,b]$. If $|f'|$ is a convex function on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{k\Gamma_k(\beta+k)\alpha^{\beta/k}}{(b-a)^{\alpha\beta/k}}\left[{}_k^\beta\mathfrak{J}_{b^-}^\alpha f(a)+{}_k^\beta\mathfrak{J}_{a^+}^\alpha f(b)\right]\right| \le \frac{(b-a)}{2\alpha}\left[2B_{1/(2^\alpha)}\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)-B\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)\right](|f'(a)|+|f'(b)|) \tag{3.3}$$

for $\alpha,\beta>0$.

Proof. By Lemma 2.1 and the convexity of $|f'|$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{k\Gamma_k(\beta+k)\alpha^{\beta/k}}{(b-a)^{\alpha\beta/k}}\left[{}_k^\beta\mathfrak{J}_{b^-}^\alpha f(a)+{}_k^\beta\mathfrak{J}_{a^+}^\alpha f(b)\right]\right|\\
&=\frac{(b-a)\alpha^{\beta/k}}{2}\left|\int_0^1\left[\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k}-\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}\right]f'(ta+(1-t)b)\,\mathrm{d}\,t\right|\\
&\le\frac{(b-a)\alpha^{\beta/k}}{2}\left|\int_0^1\left[\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k}-\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}\right]\right|(t|f'(a)|+(1-t)|f'(b)|\\
&\le\frac{(b-a)\alpha^{\beta/k}}{2}\left|\int_0^{1/2}\left[\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k}-\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}\right]\right|(t|f'(a)+(1-t)|f'(b)|\\
&+\frac{(b-a)\alpha^{\beta/k}}{2}\left|\int_{1/2}^1\left[\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}-\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k}\right]\right|(t|f'(a)+(1-t)|f'(b)|\\
&=\frac{(b-a)\alpha^{\beta/k}}{2}|f'(a)|\int_0^{1/2}\left[t\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k}-t\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}\right]\mathrm{d}\,t\\
&+\frac{(b-a)\alpha^{\beta/k}}{2}|f'(b)|\int_0^{1/2}\left[(1-t)\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k}-(1-t)\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}\right]\mathrm{d}\,t\\
&+\frac{(b-a)\alpha^{\beta/k}}{2}|f'(a)|\int_{1/2}^1\left[t\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}-t\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k}\right]\mathrm{d}\,t\\
&+\frac{(b-a)\alpha^{\beta/k}}{2}|f'(b)|\int_{1/2}^1\left[(1-t)\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}-(1-t)\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k}\right]\mathrm{d}\,t.
\end{aligned}\tag{3.4}$$

Changing variables by $x=t^\alpha$ and $y=(1-t)^\alpha$ results in

$$\int_0^{1/2}t\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta/k}\mathrm{d}\,t=\frac{1}{\alpha^{\beta/k+1}}B_{1/(2^\alpha)}\left(\frac{2}{\alpha},\frac{\beta}{k}+1\right),\tag{3.5}$$

$$\begin{aligned}\int_0^{1/2}t\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}\mathrm{d}\,t=\frac{1}{\alpha^{\beta/k+1}}\Bigg[B\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)\\
-B\left(\frac{2}{\alpha},\frac{\beta}{k}+1\right)+B_{1/(2^\alpha)}\left(\frac{2}{\alpha},\frac{\beta}{k}+1\right)-B_{1/(2^\alpha)}\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)\Bigg],\end{aligned}\tag{3.6}$$

$$\int_0^{1/2}t\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta/k}\mathrm{d}\,t=\frac{1}{\alpha^{\beta/k+1}}\left[B\left(\frac{2}{\alpha},\frac{\beta}{k}+1\right)-B\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)\right].\tag{3.7}$$

Substituting the equalities (3.5), (3.6) and (3.7) into the equality (3.4) leads to (3.3). The proof of Theorem 3.2 is complete. ∎

Remark 3.2. If $\alpha = 1$, then the inequality (3.3) in Theorem 3.2 reduces to

$$\left|\frac{f(a)+f(b)}{2} - \frac{k\Gamma_k(\beta+k)}{(b-a)^{\beta/k}}\left[{}_k^\beta\mathfrak{I}_{b-}f(a) + {}_k^\beta\mathfrak{I}_{a+}f(b)\right]\right| \le \frac{(b-a)}{2(\beta+1)}\left(1-\frac{1}{2^\beta}\right)(|f'(a)|+|f'(b)|)$$

for $\beta > 0$.

When $k = 1$, then the inequality (3.3) in Theorem 3.2 becomes [27, Theorem 3.1].

If $\alpha = 1$ and $k = 1$, then the inequality (3.3) in Theorem 3.2 can be reformulated as (1.7).

Remark 3.3. This paper is a slightly revised version of the preprint [24].

REFERENCES

[1] P. AGARWAL, F. QI, M. CHAND, and G. Singh, Some fractional differential equations involving generalized hypergeometric functions, *J. Appl. Anal.*, **25** (2019), in press.

[2] Z. DAHMANI, New inequalities in fractional integrals, *Int. J. Nonlinear Sci.*, **9** (2010), no. 4, pp. 493–497.

[3] Z. DAHMANI, L. TABHARIT, and S. Taf, New generalization of Grüss inequality using Riemann–Liouville fractional integrals, *Bull. Math. Anal. Appl.*, **2** (2010), no. 3, pp. 93–99.

[4] R. DIAZ and E. PARIGUAN, On hypergeometric functions and Pochhammer k-symbol, *Divulg. Mat.*, **15** (2007), no. 2, pp. 179–192.

[5] S. S. DRAGOMIR and C. E. M. PEARCE, *Selected Topics on Hermite-Hadamard Type Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000; Available online at `http://rgmia.org/monographs/hermite_hadamard.html`.

[6] R. GORENFLO and F. MAINARDI, Fractional calculus: integral and differential equations of fractional order, in *Fractals and Fractional Calculus in Continuum Mechanics* (Udine, 1996), pp. 223–276, CISM Courses and Lect., 378, Springer, Vienna, 1997.

[7] S. HABIB, S. MUBEEN, M. N. NAEEM, and F. QI, Generalized k-fractional conformable integrals and related inequalities, *HAL archives*, (2018), available online at `https://hal.archives-ouvertes.fr/hal-01788916`.

[8] F. JARAD, E. UĞURLU, T. ABDELJAWAD, and D. BĂLEANU, On a new class of fractional operators, *Adv. Difference Equ.*, **2017**, Paper No. 247, 16 pp. Available online at `https://doi.org/10.1186/s13662-017-1306-z`.

[9] A. A. KILBAS, Hadamard-type fractional calculus, *J. Korean Math. Soc.*, **38** (2001), no. 6, pp. 1191–1204.

[10] A. A. KILBAS, H. M. SRIVASTAVA, and J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[11] D. S. MITRINOVIĆ and I. B. LACKOVIĆ, Hermite and convexity, *Aequationes Math.*, **28** (1985), no. 3, pp. 229–232; Available online at `https://doi.org/10.1007/BF02189414`.

[12] S. MUBEEN and G. M. HABIBULLAH, k-fractional integrals and application, *Int. J. Contemp. Math. Sci.*, **7** (2012), no. 1-4, pp. 89–94.

[13] S. MUBEEN and S. IQBAL, Grüss type integral inequalities for generalized Riemann–Liouville k-fractional integrals, *J. Ineq. Appl.*, **2016**, Paper No. 109, 13 pp. Available online at `https://doi.org/10.1186/s13660-016-1052-x`.

[14] K. S. NISAR and F. QI, On solutions of fractional kinetic equations involving the generalized k-Bessel function, Note Mat. **37** (2017), no. 2, pp. 11–20; Available online at `https://doi.org/10.1285/i15900932v37n2p11`.

[15] K. S. NISAR, F. QI, G. RAHMAN, S. MUBEEN, and M. ARSHAD, Some inequalities involving the extended gamma function and the Kummer confluent hypergeometric k-function, *J. Inequal. Appl.*, (2018), **2018**:135, 12 pp; Available online at `https://doi.org/10.1186/s13660-018-1717-8`.

[16] M. E. ÖZDEMIR, C. YıLDıZ, A. O. AKDEMIR, and E. SET, On some inequalities for s-convex functions and applications, *J. Ineq. Appl.*, **2013**, 2013:333, 11 pp. Available online at `https://doi.org/10.1186/1029-242X-2013-333`.

[17] F. QI, Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities, *Filomat*, **27** (2013), no. 4, pp. 601–604; Available online at `http://dx.doi.org/10.2298/FIL1304601Q`.

[18] F. QI, A. AKKURT, and H. YILDIRIM, Catalan numbers, k-gamma and k-beta functions, and parametric integrals, *J. Comput. Anal. Appl.*, **25** (2018), no. 6, pp. 1036–1042.

[19] F. QI and W.-H. LI, Integral representations and properties of some functions involving the logarithmic function, *Filomat*, **30** (2016), no. 7, pp. 1659–1674; Available online at `https://doi.org/10.2298/FIL1607659Q`.

[20] F. QI and K. S. NISAR, Some integral transforms of the generalized k-Mittag-Leffler function, *Publ. Inst. Math. (Beograd) (N.S.)*, **105(119)** (2019), in press.

[21] F. QI, G. RAHMAN, S. M. HUSSAIN, W.-S. DU, and KOTTAKKARAN SOOPPY NISAR, Some inequalities of Čebyšev type for conformable k-fractional integral operators, *Symmetry*, **10** (2018), no. 11, Article 614, 8 pages; Available online at `https://doi.org/10.3390/sym10110614`.

[22] F. QI, G. RAHMAN, and K. S. NISAR, Convexity and inequalities related to extended beta and confluent hypergeometric functions, *HAL archives*, (2018), available online at `https://hal.archives-ouvertes.fr/hal-01703900`.

[23] G. RAHMAN, K. S. NISAR, and F. QI, Some new inequalities of the Grüss type for conformable fractional integrals, *AIMS Math.*, **3** (2018), no. 4, pp. 575–583; Available online at `https://doi.org/10.3934/Math.2018.4.575`.

[24] G. RAHMAN, F. QI, K. S. NISAR, and A. GHAFFAR, Some inequalities of the Hermite–Hadamard type concerning k-fractional conformable integrals, *ResearchGate Preprint*, (2018), available online at `https://doi.org/10.13140/RG.2.2.16226.63686`.

[25] S. G. SAMKO, A. A. KILBAS, and O. I. MARICHEV, *Fractional Integrals and Derivatives: Theory and Applications*, Edited and with a foreword by S. M. Nikol'skiĭ. Translated from the 1987 Russian original. Revised by the authors. Gordon and Breach Science Publishers, Yverdon, 1993.

[26] M. Z. SARıKAYA, E. SET, H. YALDIZ, and N. BAŞAK, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Modelling*, **57** (2013), no. 9-10, pp. 2403–2407; Available online at `https://doi.org/10.1016/j.mcm.2011.12.048`.

[27] E. SET, J. CHOI, and A. GÖZPINAR, Hermite–Hadamard type inequalities for new conformable fractional integral operator, *ResearchGate Preprint*, (2018), Available online at `https://www.researchgate.net/publication/322936389`.

[28] E. SET, İ. İŞCAN, M. Z. SARıKAYA, and M. E. ÖZDEMIR, On new inequalities of Hermite–Hadamard–Fejér type for convex functions via fractional integrals, *Appl. Math. Comput.*, **259** (2015), pp. 875–881; Online: `https://doi.org/10.1016/j.amc.2015.03.030`.

[29] E. SET, M. E. ÖZDEMIR, and M. Z. SARıKAYA, Inequalities of Hermite–Hadamard's type for functions whose derivatives absolute values are m-convex, *AIP Conference Proceedings*, **1309** (2010), no. 1, pp. 861–873; Available online at `https://doi.org/10.1063/1.3525219`.

[30] E. SET, M. Z. SARıKAYA, M. E. ÖZDEMIR, and H. YILDIRIM, The Hermite–Hadamard's inequality for some convex functions via fractional integrals and related results, *J. Appl. Math. Statis. Inform.*, **10** (2014), no. 2, pp. 69–83; Available online at `https://doi.org/10.2478/jamsi-2014-0014`.

[31] D.-P. SHI, B.-Y. XI, and F. QI, Hermite–Hadamard type inequalities for (m, h_1, h_2)-convex functions via Riemann–Liouville fractional integrals, *Turkish J. Anal. Number Theory*, **2** (2014), no. 1, 22–27; Available online at `https://doi.org/10.12691/tjant-2-1-6`.

[32] D.-P. SHI, B.-Y. XI, and F. QI, Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals of (α, m)-convex functions, *Fract. Diff. Calculus*, **4** (2014), no. 2, pp. 33–43; Available online at `https://doi.org/10.7153/fdc-04-02`.

[33] H. M. SRIVASTAVA and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier, Inc., Amsterdam, 2012; Available online at `http://dx.doi.org/10.1016/B978-0-12-385218-2.00001-3`.

[34] S.-H. WANG and F. QI, Hermite–Hadamard type inequalities for s-convex functions via Riemann–Liouville fractional integrals, *J. Comput. Anal. Appl.*, **22** (2017), no. 6, pp. 1124–1134.

[35] H.-P. YIN, J.-Y. WANG, and F. QI, Some integral inequalities of Hermite–Hadamard type for s-geometrically convex functions, *Miskolc Math. Notes*, **19** (2018), no. 1, pp. 699–705; Available online at `https://doi.org/10.18514/MMN.2018.2451`.