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ON FINDING INTEGRATING FACTORS AND FIRST INTEGRALS FOR A CLASS OF HIGHER ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. If the $n-th$ order differential equation is not exact, under certain conditions, an integrating factor exists which transforms the differential equation into an exact one. Thus, the order of differential equation can be reduced to the lower order. In this paper, we present a technique for finding integrating factors of the following class of differential equations:

$$\begin{aligned} F_n\left(t,y,y',y'',\dots,y^{(n-1)}\right)y^{(n)} + F_{n-1}\left(t,y,y',y'',\dots,y^{(n-1)}\right)y^{(n-1)} + \cdots + \\ + F_1\left(t,y,y',y'',\dots,y^{(n-1)}\right)y' + F_0\left(t,y,y',y''\dots,y^{(n-1)}\right) \\ = 0. \end{aligned}$$

Here, the functions $F_0, F_1, F_2, \cdots, F_n$ are assumed to be continuous functions with their first partial derivatives on some simply connected domain $\Omega \subset \mathbb{R}^{n+1}$. We also presented some demonstrative examples.

Key words and phrases: Higher order differential equation; Exact differential equations; Non-exact differential equations; Integrating factors; First integrals.

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1. INTRODUCTION

Differential equations play a major role in Applied Mathematics, Physics, and Engineering [5, 7, 9, 11, 15, 16]. To find the general solution of a differential equation is not an easy problem in the general case. In fact, a very specific class of differential equations can be solved by using special techniques and transformations. One of these techniques is to reduce the order of the differential equation by finding a proper integrating factor. Recently, many studies appear to deal with the problem of existence and finding integrating factors of differential equations. In [1, 3, 4, 6, 10, 13], the authors investigated the existence of integrating factors for some classes of second order differential equations. In [4], the authors investigated the existence of integrating factors of n-th order system of differential equations which has known symmetries of certain type.

In [8], the authors improve some symbolic algorithms to compute integrating factors for a class of third order differential equations. In this paper, we presented a technique to find integrating factors for the following class of differential equations:

$$\begin{aligned} F_n\left(t,y,y',y'',\ldots,y^{(n-1)}\right)y^{(n)} &+ F_{n-1}\left(t,y,y',y'',\ldots,y^{(n-1)}\right)y^{(n-1)}+\cdots+\\ &+F_1\left(t,y,y',y'',\ldots,y^{(n-1)}\right)y'+F_0\left(t,y,y',y''\ldots,y^{(n-1)}\right)\\ &=0 \end{aligned} \tag{1.1}$$

where $F_0, F_1, F_2, \cdots, F_n$ are assumed to be continuous functions with their first partial derivatives on some simply connected domain $\Omega \subset \mathbb{R}^{n+1}$. To demonstrate our technique, we present some illustrative examples. The paper layout: In section **2**, we prove the main result. In section **3** is devoted for concluding remarks.

2. INTEGRATING FACTORS AND FIRST INTEGRALS FOR A CLASS OF n-TH ORDER DIFFERENTIAL EQUATIONS

In this section, we investigate the existence of certain forms of integrating factors for equation (1.1) when it is a non exact differential equation. In general, the n-th order differential equation

$$f(t,y,y',\cdots,y^{(n-1)},y^{(n)})=0$$

is called exact if there exists a differentiable function $\Psi(t,y,y',\cdots,y^{(n-1)})=c$, such that $\frac{d}{dt}\Psi(t,y,y',\cdots,y^{(n-1)})=f(t,y,y',\cdots,y^{(n-1)},y^{(n)})=0$. In this case, $\Psi(t,y,y',\cdots,y^{(n-1)})=c$ is called the first integral of $f(t,y,y',\cdots,y^{(n-1)},y^{(n)})=0$, e.g., see, [12, 14]. In [2], the author gave the explicit conditions for (1.1) to be exact. He also gave an explicit formula for the first integral $\Psi\left(t,y,y',\cdots,y^{(n)}\right)=c$. Particularly, we have the following theorem:

Theorem 2.1. [2]. *Assume that $F_0, F_1, F_2, \ldots, F_n$ are continuous with their first partial derivatives on a simply connected domain Ω in $\mathbb{R}^{n+1}$. Then the differential equation* (1.1) *is exact if* $\dfrac{\partial F_i}{\partial y^{(j-1)}}=\dfrac{\partial F_j}{\partial y^{(i-1)}}$ *for all $i=2,3,\ldots,n$ and $j=1,2,\ldots,i-1$, and* $\dfrac{\partial F_i}{\partial t}=\dfrac{\partial F_0}{\partial y^{(i-1)}}$ *for all $i=1,2,3,\ldots,n$. Moreover, the first integral of* (1.1) *is explicitly given by*

$$\begin{aligned} \Psi\left(t,y,y',\cdots,y^{(n-1)}\right) &= \int_{t_0}^{t}F_0\left(\eta,y,y',\cdots,y^{(n-1)}\right)d\eta+\int_{y_0}^{y}F_1\left(t_0,\eta,y',\cdots,y^{(n-1)}\right)d\eta\\ &+ \cdots+\int_{y_0^{(n-1)}}^{y^{(n-1)}}F_n\left(t_0,y_0,y_0',\cdots,\eta\right)d\eta=c \end{aligned}$$

where c is an integrating constant. $\square$

Assume that (1.1) is a non exact differential equation. Then according to the above theorem, an integrating factor $\mu(t, y, y', \cdots, y^{(n-1)})$ of (1.1) exists if it solves the following system of $n!$ first order partial differential equations:

$$\left\{\begin{array}{c} \mu(\mathbf{y})\dfrac{\partial F_i(\mathbf{y})}{\partial t} + \dfrac{\partial \mu(\mathbf{y})}{\partial t}F_i(\mathbf{y}) = \mu(\mathbf{y})\dfrac{\partial F_0(\mathbf{y})}{\partial y^{(i-1)}} + \dfrac{\partial \mu(\mathbf{y})}{\partial y^{(i-1)}}F_0(\mathbf{y}), \\ i = 1, 2, \cdots, n, \\ \\ \mu(\mathbf{y})\dfrac{\partial F_i(\mathbf{y})}{\partial y^{(j-1)}} + \dfrac{\partial \mu(\mathbf{y})}{\partial y^{(j-1)}}F_i(\mathbf{y}) = \mu(\mathbf{y})\dfrac{\partial F_j(\mathbf{y})}{\partial y^{(i-1)}} + \dfrac{\partial \mu(\mathbf{y})}{\partial y^{(i-1)}}F_j(\mathbf{y}), \\ i = 2, \cdots, n,\ j = 1, 2, \cdots, i-1 \end{array}\right. \tag{2.1}$$

where $\mathbf{y} = (t, y, y', \cdots, y^{(n-1)})$. Generally, to solve such system of partial differential equations is not easy. Thus, we consider some special forms of the integrating factor $\mu(t, y, y', \cdots, y^{(n-1)})$. Particularly, we look for integrating factors $\mu(\xi)$ where

$$\xi := \xi(t, y, y', \cdots, y^{(n-1)}) = \alpha(t)\prod_{k=1}^{n}\alpha_k(y^{(k-1)}).$$

The functions $\alpha(t)$ and $\alpha_k\left(y^{(k-1)}\right)$, $k = 1, 2, \cdots, n$ are assumed to be differentiable functions. By substituting $\mu(\xi)$ in (2.1), we get

$$\left\{\begin{array}{c} \mu(\xi)\frac{\partial F_i(\mathbf{y})}{\partial t} + \mu'(\xi)\xi_t F_i(\mathbf{y}) = \mu(\xi)\frac{\partial F_0(\mathbf{y})}{\partial y^{(i-1)}} + \mu'(\xi)\xi_{y^{(i-1)}}F_0(\mathbf{y}), \\ i = 1, 2, \cdots, n, \\ \\ \mu(\xi)\frac{\partial F_i(\mathbf{y})}{\partial y^{(j-1)}} + \mu'(\xi)\xi_{y^{(j-1)}}F_i(\mathbf{y}) = \mu(\xi)\frac{\partial F_j(\mathbf{y})}{\partial y^{(i-1)}} + \mu'(\xi)\xi_{y^{(i-1)}}F_j(\mathbf{y}), \\ i = 2, \cdots, n, j = 1, 2, \cdots, i-1 \end{array}\right. \tag{2.2}$$

where $\mu'(\xi) = \frac{d\mu}{d\xi}$ and ξ_η denotes to $\frac{\partial \xi}{\partial \eta}$. Equivalently, we have

$$\left\{\begin{array}{l} \dfrac{\mu'(\xi)}{\mu(\xi)} = \dfrac{\frac{\partial F_0(\mathbf{y})}{\partial y^{(i-1)}} - \frac{\partial F_i(\mathbf{y})}{\partial t}}{\xi_t F_i(\mathbf{y}) - \xi_{y^{(i-1)}}F_0(\mathbf{y})},\ i = 1, 2, \cdots, n, \\ \\ \dfrac{\mu'(\xi)}{\mu(\xi)} = \dfrac{\frac{\partial F_j(\mathbf{y})}{\partial y^{(i-1)}} - \frac{\partial F_i(\mathbf{y})}{\partial y^{(j-1)}}}{\xi_{y^{(j-1)}}F_i(\mathbf{y}) - \xi_{y^{(i-1)}}F_j(\mathbf{y})},\ i = 2, \cdots, n,\ j = 1, 2, \cdots, i-1. \end{array}\right. \tag{2.3}$$

Hence, an integrating factor $\mu(\xi)$ of equation (1.1) exists if

$$\frac{\frac{\partial F_0(\mathbf{y})}{\partial y^{(i-1)}} - \frac{\partial F_i(\mathbf{y})}{\partial t}}{\xi_t F_i(\mathbf{y}) - \xi_{y^{(i-1)}}F_0(\mathbf{y})},\ i = 1, 2, \cdots, n$$

and

$$\frac{\frac{\partial F_j(\mathbf{y})}{\partial y^{(i-1)}} - \frac{\partial F_i(\mathbf{y})}{\partial y^{(j-1)}}}{\xi_{y^{(j-1)}}F_i(\mathbf{y}) - \xi_{y^{(i-1)}}F_j(\mathbf{y})},\ i = 2, \cdots, n,\ j = 1, 2, \cdots, i-1$$

are all equal and they are functions in ξ. Thus, we have the following theorem:

Theorem 2.2. *Let $\xi = \alpha(t)\prod_{k=1}^{n}\alpha_k(y^{(k-1)})$ where $\alpha(t)$ and $\alpha_k\left(y^{(k-1)}\right), k = 1, 2, \cdots, n$ are differentiable functions. Assume that $F_0(\mathbf{y}), F_1(\mathbf{y}), F_2(\mathbf{y}), \ldots, F_n(\mathbf{y})$ are continuous functions with their first partial derivatives on some simply connected domain $\Omega \subset \mathbb{R}^{n+1}$. Moreover,* *assume that Equation* (1.1) *is a non exact differential equation. Then it admits a non constant integrating factor* $\mu(\xi)$ *if*

$$\frac{\frac{\partial F_0(\mathbf{y})}{\partial y^{(i-1)}} - \frac{\partial F_i(\mathbf{y})}{\partial t}}{\xi_t F_i(\mathbf{y}) - \xi_{y^{(i-1)}} F_0(\mathbf{y})},\ i = 1, 2, \cdots, n$$

and

$$\frac{\frac{\partial F_j(\mathbf{y})}{\partial y^{(i-1)}} - \frac{\partial F_i(\mathbf{y})}{\partial y^{(j-1)}}}{\xi_{y^{(j-1)}} F_i(\mathbf{y}) - \xi_{y^{(i-1)}} F_j(\mathbf{y})},\ i = 2, \cdots, n,\ j = 1, 2, \cdots, i-1$$

are all equal and they are functions in ξ*. In this case, the integrating factor is explicitly given by*

$$\mu(\xi) = \exp\left\{ \int \frac{\frac{\partial F_0(\mathbf{y})}{\partial y} - \frac{\partial F_1(\mathbf{y})}{\partial t}}{\xi_t F_1(\mathbf{y}) - \xi_y F_0(\mathbf{y})} d\xi \right\}. \ \square$$

Corollary 2.3. *Assume that* $F_0(\mathbf{y}), F_1(\mathbf{y}), F_2(\mathbf{y}), \cdots, F_n(\mathbf{y})$ *are continuous functions with their first partial derivatives on some simply connected domain* $\Omega \subset \mathbb{R}^{n+1}$*. Moreover, assume that Equation* (1.1) *is a non exact differential equation. Then it admits a non constant integrating factor* $\mu(t)$ *if*

I) *for* $i = 2, \cdots, n$ *and for* $j = 1, 2, \cdots, i-1$*, we have*

$$\frac{\partial F_j(\mathbf{y})}{\partial y^{(i-1)}} = \frac{\partial F_i(\mathbf{y})}{\partial y^{(j-1)}},$$

and

II) *for* $i = 1, 2, \cdots, n$*, the functions*

$$\left[\frac{\partial F_0(\mathbf{y})}{\partial y^{(i-1)}} - \frac{\partial F_i(\mathbf{y})}{\partial t}\right] / \left[F_i(\mathbf{y})\right]$$

are equal and they are functions in t*.*

In addition, the integrating factor is explicitly given by

$$\mu(\xi) = \exp\left\{ \int \left[\frac{\partial F_0(\mathbf{y})}{\partial y} - \frac{\partial F_1(\mathbf{y})}{\partial t}\right] / \left[F_1(\mathbf{y})\right] d\xi \right\}. \ \square$$

Example 2.1. *Consider the following* n*-th order linear differential equation:*

$$P_n(t)y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_2(t)y'' + P_1(t)y' + P_0(t)y = h(t) \tag{2.4}$$

where $P_i(t),\ i = 0, 1, 2, \cdots, n$ *are non-zero differentiable functions on some open interval* $(a, b) \subset \mathbb{R}$*, and* $h(t)$ *is continuous function on* (a, b)*. Then* $F_n = P_n(t)$*,* $F_{(n-1)} = P_{(n-1)}(t), \cdots, F_1 = P_1(t),\ F_0 = P_0(t)y - h(t)$*. Clearly,*

$$\frac{\partial F_j(\mathbf{y})}{\partial y^{(i-1)}} = \frac{\partial F_i(\mathbf{y})}{\partial y^{(j-1)}} = 0,\ i = 2, \cdots, n \text{ and } j = 1, 2, \cdots, i-1.$$

Moreover, $\frac{\partial F_0(\mathbf{y})}{\partial y^{(i-1)}} = 0,\ \forall i = 2, \cdots, n$*,* $\frac{\partial F_i(\mathbf{y})}{\partial y} = P_i'(t),\ \forall i = 1, \cdots, n$*, and* $\frac{\partial F_0(\mathbf{y})}{\partial y} = P_0(t)$*. Hence, to have an integrating factor in* t*, we must have*

$$\frac{P_n'(t)}{P_n(t)} = \cdots = \frac{P_2'(t)}{P_2(t)} = \frac{P_1'(t) - P_0(t)}{P_1(t)}.$$

Therefore, $P_n(t), \cdots, P_2(t)$ *must be linearly dependent functions. Moreover,* P_0 *and* P_1 *must satisfy* $W(P_1, P_2)(t) = P_0(t)P_2(t)$ *where* $W(P_1, P_2)$ *is the Wronskian's of* P_1 *and* P_2*. In this* *case, the integrating factor* $\mu(t) = \frac{1}{P_n(t)}$. *Hence, for non-zero and differentiable functions* $P(t)$, $P_1(t)$ *and* $P_0(t)$ *the differential equation*

$$a_n P(t)y^{(n)} + a_{n-1}P(t)y^{(n-1)} + \cdots + a_2 P(t)y'' + P_1(t)y' + P_0(t)y = h(t),$$

has an integrating factor $\mu(t) = \frac{1}{P(t)}$ *provided that* $W(P_1, P)(t) = P_0(t)P(t)$. *Hence, we get* $\frac{P_0}{P} = \left(\frac{P1}{P}\right)'$, *and so, the above differential equation becomes*

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2 y'' + \frac{P_1(t)}{P(t)}y' + \left(\frac{P_1(t)}{P(t)}\right)' y = \frac{P(t)}{h(t)}.$$

Thus, the first integral of (2.4) *is given by*

$$a_n y^{(n-1)} + a_{n-1}y^{(n-2)} + \cdots + a_3 y'' + a_2 y' + \frac{P_1(t)}{P(t)}y = \int^t \frac{h(s)}{P(s)}ds + c$$

where c *is the integrating constant.*

Example 2.2. *Consider the Initial value problem*

$$e^{-t}y'' + (\cos y)(1 + 2e^{-t})y' + \sin(y) = 0,\ y\left(\frac{\pi}{2}\right) = y'\left(\frac{\pi}{2}\right) = 0.$$

Then $F_2((t, y, y') = e^{-t},\ F_1((t, y, y')(\cos y)(1 + 2e^{-t})$, *and* $F_0((t, y, y') = \sin(y)$. *From the above corollary, an integrating factor in* t *for this equation if* $\frac{\partial F_1}{\partial y'} = \frac{\partial F_2}{\partial y}$, *and* $\left(\frac{\partial F_0}{\partial y} - \frac{\partial F_1}{\partial t}\right)/F_1 = \left(\frac{\partial F_0}{\partial y'} - \frac{\partial F_2}{\partial t}\right)/F_2$ *and they are functions in* t. *Clearly, these conditions hold and* $\left(\frac{\partial F_0}{\partial y} - \frac{\partial F_1}{\partial t}\right)/F_1 = \left(\frac{\partial F_0}{\partial y'} - \frac{\partial F_2}{\partial t}\right)/F_2 = 1$. *Thus, and integrating factor for this equation exists and it equals to* e^t. *This integrating factor transforms the above equation into*

$$y'' + (\cos y)(e^t + 2)y' + \sin(y)e^t = 0.$$

Due to Theorem 2.1 this equation is exact and its first integral is given by

$$y' + (\sin y)(e^t + 2) = 0$$

which can be solved by separating the variables.

Corollary 2.4. *Assume that* $F_0(\mathbf{y}), F_1(\mathbf{y}), F_2(\mathbf{y}), \ldots, F_n(\mathbf{y})$ *are continuous functions with their first partial derivatives on some simply connected domain* $\Omega \subset \mathbb{R}^{n+1}$. *Moreover, assume that Equation* (1.1) *is a non exact differential equation. Then it admits a non constant integrating factor* $\mu(y^{(k-1)})$, $k = 1, 2, \cdots, n$, *if the following two conditions hold:*

I) $\dfrac{\partial F_i}{\partial t} = \dfrac{\partial F_0}{\partial y^{(i-1)}}$ *for* $i = 1, 2, \cdots, n$, *and* $\dfrac{\partial F_j(\mathbf{y})}{\partial y^{(i-1)}} = \dfrac{\partial F_i(\mathbf{y})}{\partial y^{(j-1)}}$ *for* $i = 2, 3, \cdots, n,\ j = 1, 2, \cdots, i-1,$ *and* $i, j \neq k$; *and*

II) *for* $i = 1, 2, \cdots, k-1, k+1, \cdots, n$ *the functions*

$$\left[\frac{\partial F_k(\mathbf{y})}{\partial y^{(i-1)}} - \frac{\partial F_i(\mathbf{y})}{\partial y^{(k-1)}}\right] / \left[F_i(\mathbf{y})\right]$$

and the function

$$\left[\frac{\partial F_k(\mathbf{y})}{\partial t} - \frac{\partial F_0(\mathbf{y})}{\partial y^{(k-1)}}\right] / \left[F_0(\mathbf{y})\right]$$

are all equal and they are functions in $y^{(k-1)}$.

In addition, the integrating factor is explicitly given by

$$\mu(\xi) = \exp\left\{\int \left[\frac{\partial F_k(\mathbf{y})}{\partial t} - \frac{\partial F_0(\mathbf{y})}{\partial y^{(k-1)}}\right] / \left[F_0(\mathbf{y})\right] d\xi\right\}. \square$$

Example 2.3. *Consider the following third order differential equation:*

$$y^3y''' + y^3y'' - 2ty' + y = 0. \tag{2.5}$$

Then $F_3 = y^3$, $F_2 = y^3$, $F_1 = -2t$, *and* $F_0 = y$. *Hence,* $F_{2y''} = F_{3y'} = 0$, $F_{0y''} = F_{3t} = 0$, $F_{0y'} = F_{2t} = 0$, $0 = F_{1y'} \neq F_{2y} = 3y^2$, $0 = F_{1y''} \neq F_{3y} = 3y^2$, *and* $-2 = F_{1t} \neq F_{0y} = 1$. *By applying the above corollary, an integrating factor in terms of* y *exists for this equation. Particularly,* $\mu(y) = y^{-3}$. *By Multiplying* (2.5) *by* $\mu(y) = y^{-3}$, *we get*

$$y''' + y'' - 2ty^{-3}y' + y^{-2} = 0.$$

Clearly, this differential equation is exact. Moreover, its first integral is

$$y'' + y' + ty^{-2} = c$$

where c *is an integrating constant.*

3. CONCLUDING REMARKS

In this paper, we investigated the existence of integrating factors of the following class of third order differential equations:

$$\begin{aligned} F_n\left(t, y, y', y'', \dots, y^{(n-1)}\right) y^{(n)} + F_{n-1}\left(t, y, y', y'', \dots, y^{(n-1)}\right) y^{(n-1)} + \cdots + \\ + F_1\left(t, y, y', y'', \dots, y^{(n-1)}\right) y' + F_0\left(t, y, y', y'' \dots, y^{(n-1)}\right) \\ = 0 \end{aligned} \tag{3.1}$$

where $F_0, F_1, F_2, \cdots, F_n$ are continuous functions with their first partial derivatives on some simply connected domain $\Omega \subset \mathbb{R}^{n+1}$. Particularly, we proved some results related to the existence of integrating factors of (3.1). We also presented some illustrative examples. We remark that these results not only useful for finding integrating factors for (3.1) analytically but also computationally. In fact, we can check the validity of the conditions in our results by using the symbolic toolboxes in different mathematical softwares, e.g., MAPLE and MATLAB softwares. Also, by using these symbolic toolboxes, we can find the integrating factors of (3.1) by using the explicit forms given in our results. Following the same procedure described in the paper, we can also find different forms of integrating factor for (3.1). For example, $\mu(\xi)$ where $\xi = \alpha(t) + \sum_{k=1}^{n} \alpha_k\left(y^{(k-1)}\right)$.

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