The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 16, Issue 1, Article 14, pp. 1-14, 2019

CUBIC ALTERNATING HARMONIC NUMBER SUMS

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Received 12 September, 2018; accepted 23 January, 2019; published 27 May, 2019.

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ABSTRACT. We develop new closed form representations of sums of cubic alternating harmonic numbers and reciprocal binomial coefficients. We also identify a new integral representation for the $\zeta(4)$ constant.

Key words and phrases: Polylogarithm function, Alternating cubic harmonic numbers, Combinatorial series identities, Summation formulas, Partial fraction approach, Binomial coefficients.

2000 *Mathematics Subject Classification.* Primary 05A19, 33C20. Secondary 11M06, 11B83.

ISSN (electronic): 1449-5910

1. INTRODUCTION AND PRELIMINARIES

In this paper we will develop identities, closed form representations of alternating cubic harmonic numbers and reciprocal binomial coefficients, including integral representations, of the form:

$$\Omega(k,p)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\ H_n^3}{n^p\begin{pmatrix} n+k\\ k\end{pmatrix}}, \tag{1.1}$$

for $p=0$, 1 and $k\in\mathbb{N}_0$. Here, the n^{th} harmonic number

$$H_n=\sum_{r=1}^{n}\frac{1}{r}=\gamma+\psi(n+1)=\int_0^1\frac{1-t^n}{1-t}\,dt, \qquad H_0:=0 \tag{1.2}$$

and as usual, γ denotes the Euler-Mascheroni constant and $\psi(z)$ is the Psi (or Digamma) function defined by

$$\psi(z):=\frac{d}{dz}\{\log\Gamma(z)\}=\frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log\Gamma(z)=\int_1^z\psi(t)\,dt.$$

For sums of harmonic numbers with positive terms [10], [27], [28] and [29] have given many results, including sums of the form

$$\sum_{n=1}^{\infty}\frac{H_n^3}{n^p\begin{pmatrix} n+k\\ k\end{pmatrix}}.$$

Other results are given by [5] and [17]. Let $\mathbb{R}$ and $\mathbb{C}$ denote, respectively the sets of real and complex numbers and let $\mathbb{N}:=\{1,2,3,\cdots\}$ be the set of positive integers, and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. A generalized binomial coefficient $\binom{\lambda}{\mu}$ $(\lambda,\mu\in\mathbb{C})$ is defined, in terms of the familiar gamma function, by

$$\binom{\lambda}{\mu}:=\frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\,\Gamma(\lambda-\mu+1)}, \qquad \lambda,\mu\in\mathbb{C}.$$

The Pochhammer symbol $(\lambda)_\nu$ $(\lambda,\nu\in\mathbb{C})$ is also defined in terms of the gamma function, by

$$(\lambda)_\nu:=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}=\begin{cases}1 & (\nu=0;\ \lambda\in\mathbb{C}\setminus\{0\})\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}),\end{cases}.$$

A generalized harmonic number $H_n^{(m)}$ of order m is defined, for positive integers n and m, as follows:

$$H_n^{(m)}:=\sum_{r=1}^{n}\frac{1}{r^m}\ ,\ m,n\in\mathbb{N} \qquad \text{and} \qquad H_0^{(m)}:=0, \qquad m\in\mathbb{N}.$$

In the case of *non-integer* values of n such as (for example) a value $\rho\in\mathbb{R}$, the generalized harmonic numbers $H_\rho^{(m+1)}$ may be defined, in terms of the polygamma functions

$$\psi^{(n)}(z):=\frac{d^n}{dz^n}\{\psi(z)\}=\frac{d^{n+1}}{dz^{n+1}}\{\log\Gamma(z)\}, \qquad n\in\mathbb{N}_0,$$

by

$$H_\rho^{(m+1)}=\zeta(m+1)+\frac{(-1)^m}{m!}\,\psi^{(m)}(\rho+1)\,,\ H_0^{(m+1)}=0 \tag{1.3}$$

$$(\rho\in\mathbb{R}\setminus\{-1,-2,-3,\cdots\}\,;\ m\in\mathbb{N})\,,$$

where $\zeta(z)$ is the Riemann zeta function. The evaluation of the polygamma function $\psi^{(\alpha)}\left(\frac{r}{a}\right)$ at rational values of the argument can be explicitly done via a formula as given by Kölbig [7], or Choi and Cvijovic [2] in terms of the polylogarithmic or other special functions. Some specific values are listed in the books[15], [21] and [22]. The polylogarithm or de-Jonquière function $Li_p(z)$, is defined as,

$$Li_p(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^p},\ p \in \mathbb{C} \text{ when } |z| < 1;\ \Re(p) > 1 \text{ when } |z| = 1.$$

Some results for sums of alternating and non-alternating harmonic numbers may be seen in the works of [3], [4], [12], [13], [14], [16], [17], [18], [19], [20], [23], [24], [26] and references therein. Some explicit, and closely related results may also be seen in the well presented papers [9] and [25].

The following lemma will be useful in the development of the main theorems.

Lemma 1.1. *Let r be a positive integer and $p \in \mathbb{N}$. Then:*

$$F(p,r) = \sum_{j=1}^{r} \frac{(-1)^j}{j^p} = \frac{1}{2^p}\left(H^{(p)}_{\left[\frac{r}{2}\right]} + H^{(p)}_{\left[\frac{r-1}{2}\right]}\right) - H^{(p)}_{2\left[\frac{r+1}{2}\right]-1} \tag{1.4}$$

where $[x]$ is the integer part of x, and when $p = 1$,

$$F(1,r) = \sum_{j=1}^{r} \frac{(-1)^j}{j} = H_{\left[\frac{r}{2}\right]} - H_r. \tag{1.5}$$

For $p = 2$,

$$F(2,r) = \sum_{j=1}^{r} \frac{(-1)^j}{j^2} = \frac{1}{4}\left(H^{(2)}_{\left[\frac{r}{2}\right]} - H^{(2)}_{\left[\frac{r+1}{2}\right]-\frac{1}{2}}\right) - \frac{1}{2}\zeta(2). \tag{1.6}$$

Proof. The proof is given in the paper [11]. ∎

Lemma 1.2. *The following identity holds,*

$$X(0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^3}{n} = \frac{5}{8}\zeta(4) + \frac{3}{4}\zeta(2)\ln^2 2 - \frac{1}{4}\ln^4 2 - \frac{9}{8}\zeta(3)\ln 2, \tag{1.7}$$

$$= \int_0^1 \frac{\log(1-x)\left(\log^2(1+x) + Li_2(-x)\right)}{x(1+x)} dx, \tag{1.8}$$

where $Li_p(\cdot)$ is the polylogarithm function.

Proof. Consider

$$\begin{aligned} V(j,t) &= \sum_{n=1}^{\infty} \frac{t^n}{n\binom{n+j}{j}} \\ &= \sum_{n=1}^{\infty} \frac{t^n\,\Gamma(n)\,\Gamma(j+1)}{\Gamma(n+j+1)} = \sum_{n=1}^{\infty} t^n\, B(n, j+1), \end{aligned} \tag{1.9}$$

where $\Gamma(\cdot)$ is the gamma function and $B(\cdot,\cdot)$ is the beta function. Now

$$V(j,t)=\int_0^1 \frac{(1-x)^j}{x}\sum_{n=1}^{\infty}(tx)^n\ dx = t\int_0^1 \frac{(1-x)^j}{1-tx}dx,$$

and differentiating p times with respect to j and then letting $j\to 0$ with $t=-1$, results in

$$V(0,-1)^{(p)} = \int_0^1 \frac{\log^p(1-x)}{1+x}dx = (-1)^p\, p!Li_{p+1}\left(\frac{1}{2}\right).$$

For $p=3$

$$V(0,-1)^{(3)} = -6Li_4\left(\frac{1}{2}\right) \tag{1.10}$$

From (1.9) we also have

$$V(0,-1)^{(p)} = \sum_{n=1}^{\infty}\frac{(-1)^n\sigma_n}{n}$$

where

$$\sigma_n = \lim_{j\to 0}\left[\frac{d^p}{dj^p}\left(\binom{n+j}{j}^{-1}\right)\right],$$

when $p=3$,

$$V(0,-1)^{(3)} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\phi_n}{n} = 6Li_4\left(\frac{1}{2}\right) \tag{1.11}$$

and $\phi_n = H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)}$. From the paper [8], we have the results

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n^{(3)}}{n} = \frac{19}{16}\zeta(4) - \frac{3}{4}\zeta(3)\ln 2,$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_nH_n^{(2)}}{n} &= 2Li_4\left(\frac{1}{2}\right) - \zeta(4) - 4\zeta(2)\ln^2 2\\ &\quad +\frac{1}{4}\ln^4 2 + \frac{7}{8}\zeta(3)\ln 2,\end{aligned}$$

and substituting into (1.11) we have the result (1.7). The representation of the integral (1.8) is obtained in the following way. From [11], we can express, for $p\in\mathbb{N}$

$$(-1)^p\frac{H_n^p}{n^p} = \int_0^1\cdots\int_0^1 \left(\prod_{j=1}^p x_j\right)^{n-1}\prod_{j=1}^p \ln(1-x_j)\ dx_j$$

where $\int_0^1\cdots\int_0^1$ is a p- fold integration procedure, for $p=3$,

$$-\frac{H_n^3}{n^3} = \int_0^1\int_0^1\int_0^1 \frac{(x_1x_2x_3)^n\ln(1-x_1)\ln(1-x_2)\ln(1-x_3)}{x_1x_2x_3}dx_1dx_2dx_3. \tag{1.12}$$

Now,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\ H_n^3}{n} &= -\int_0^1\int_0^1\int_0^1 \ln(1-x)\ln(1-y)\ln(1-z) \\ &\quad\times\sum_{n=1}^{\infty} n^2(-xyz)^{n-1}\ dxdydz\end{aligned}$$

$$\begin{aligned}&= -\int_0^1\int_0^1\int_0^1 \frac{\ln(1-x)\ln(1-y)\ln(1-z)(1-xyz)}{(1+xyz)^2}dxdydz \\ &= \int_0^1\int_0^1 \frac{\ln(1-x)\ln(1-y)(1-\ln(1+xy))}{(1+xy)^2}dxdy \\ &= \int_0^1 \frac{\ln(1-x)}{x(1+x)}\left(\frac{\ln^2(1+x)}{2}-Li_2\left(\frac{x}{1+x}\right)\right)dx\end{aligned}$$

and applying Landen's identity

$$Li_2\left(\frac{x}{1+x}\right) = -\frac{\ln^2(1+x)}{2} - Li_2(-x)$$

we obtain (1.8). By association we conclude

$$\begin{aligned}\int_0^1 \frac{\log(1-x)\left(\log^2(1+x)+Li_2(-x)\right)}{x(1+x)}dx &= \frac{5}{8}\zeta(4)+\frac{3}{4}\zeta(2)\ln^2 2 \\ &\quad -\frac{1}{4}\ln^4 2-\frac{9}{8}\zeta(3)\ln 2 \\ &= X(0) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n^3}{n}.\end{aligned}$$

∎

An alternate manipulation of (1.7) leads to the following result and adds to the results on cubic sums some of which are published in [10].

Lemma 1.3.

$$\begin{aligned}\sum_{n\geq 1}\frac{H_{2n}^3}{2n(2n-1)} &= \frac{5}{8}\zeta(4)+\frac{67}{16}\zeta(3)+2\zeta(2)+\frac{3}{4}\zeta(2)\ln^2 2-\frac{3}{2}\zeta(2)\ln 2 \\ &\quad -\frac{9}{8}\zeta(3)\ln 2+\ln 2-\frac{1}{4}\ln^4 2+\ln^3 2-\frac{3}{2}\ln^2 2.\end{aligned}$$

(1.13)

Proof. By re-arrangement

$$\begin{aligned}X(0) &= \sum_{n=1}^{\infty}\left(\frac{\left(H_{2n}-\frac{1}{2n}\right)^3}{2n-1}-\frac{H_{2n}^3}{2n}\right) \\ &= \sum_{n\geq 1}\frac{H_{2n}^3-3H_{2n}^2}{2n(2n-1)}+\sum_{n\geq 1}\frac{3H_{2n}}{4n^2(2n-1)}-\sum_{n\geq 1}\frac{1}{8n^3(2n-1)}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{n\geq1}\frac{H_{2n}^3-3H_{2n}^2}{2n\left(2n-1\right)} &= \frac{5}{8}\zeta\left(4\right)+\frac{3}{4}\zeta\left(2\right)\ln^2 2-\frac{1}{4}\ln^4 2-\frac{9}{8}\zeta\left(3\right)\ln 2\\ &\quad +\ln 2+\frac{33}{16}\zeta\left(3\right)-\frac{3}{2}\zeta\left(2\right)+\frac{3}{2}\ln^2 2\\ &\quad -\frac{1}{8}\zeta\left(3\right)-3\ln 2-\frac{1}{4}\zeta\left(2\right).\end{aligned}$$

Now we can evaluate

$$\sum_{n\geq1}\frac{H_{2n}^2}{2n\left(2n-1\right)}=\frac{5}{4}\zeta\left(2\right)+\ln 2-\ln^2 2-\frac{1}{2}\zeta\left(2\right)\ln 2+\frac{1}{3}\ln^3 2+\frac{3}{4}\zeta\left(3\right),$$

therefore (1.13) follows. ∎

Lemma 1.4. *The following identity holds,*

$$X\left(1\right)=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}H_n^3}{n+1}$$

$$\begin{aligned} &= -\frac{5}{16}\zeta\left(4\right)-\frac{3}{4}\zeta\left(2\right)\ln^2 2+\frac{1}{4}\ln^4 2+\frac{9}{8}\zeta\left(3\right)\ln 2\\ &= -X\left(0\right)+\frac{5}{16}\zeta\left(4\right),\end{aligned} \tag{1.14}$$

$$= \frac{5}{16}\zeta\left(4\right)-\int_0^1\frac{\log\left(1-x\right)\left(\log^2\left(1+x\right)+Li_2\left(-x\right)\right)}{x\left(1+x\right)}dx. \tag{1.15}$$

Proof. Consider $X\left(1\right)$ and by a change of summation index

$$\begin{aligned}X\left(1\right) &= \sum_{n=1}^{\infty}\frac{\left(-1\right)^{n}H_{n-1}^3}{n}=-\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}}{n}\left(H_n-\frac{1}{n}\right)^3\\ &= -\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}H_n^3}{n}-3\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}H_nH_{n-1}}{n^2}+\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}}{n^4}.\end{aligned}$$

Using lemma 1.2

$$X\left(1\right)=-X\left(0\right)-\frac{9}{16}\zeta\left(4\right)+\frac{7}{8}\zeta\left(4\right)$$

and (1.14) follows. The integral identity (1.15) follows directly from (1.8). We can also note that a similar calculation as that of lemma 1.3, yields the identity

$$\begin{aligned}\sum_{n\geq1}\left(\frac{H_{2n}^3}{2n\left(2n+1\right)}+\frac{3H_{2n}H_{2n-1}}{4n^2\left(2n-1\right)}\right) &= -X\left(0\right)+\frac{3}{8}\zeta\left(4\right)\\ &\quad +\frac{9}{4}\zeta\left(2\right)-\frac{3}{2}\ln^2 2-\frac{3}{2}\zeta\left(2\right)\ln 2+\ln^3 2+\frac{69}{16}\zeta\left(3\right).\end{aligned}$$

∎

Remark 1.1. It may be noticed from lemma 1.2 and lemma 1.4 that

$$X(0) + X(1) = \frac{5}{16}\zeta(4)$$

in which case manipulating the integral identities we obtain

$$\zeta(4) = -\frac{8}{3}\int_0^1 \frac{\log(1-x)\log^2(1+x)}{x}dx. \tag{1.16}$$

Further extraction from $X(0)$ and $X(1)$ gives us the results

$$\begin{aligned}\int_0^1 \frac{\log(1-x)\log^2(1+x)}{1+x}dx &= 2\zeta(3)\ln 2 + 2Li_4\left(\frac{1}{2}\right) + \frac{1}{3}\ln^4 2 \\ &\quad -2\zeta(4) - \zeta(2)\ln^2 2,\end{aligned} \tag{1.17}$$

$$\begin{aligned}\int_0^1 \frac{\log(1-x)Li_2(-x)}{x(1+x)}dx &= \frac{7}{8}\zeta(3)\ln 2 + 2Li_4\left(\frac{1}{2}\right) + \frac{1}{12}\ln^4 2 \\ &\quad -\zeta(4) - \frac{1}{4}\zeta(2)\ln^2 2\end{aligned} \tag{1.18}$$

and

$$\begin{aligned}\int_0^1 \frac{\log(1-x)Li_2(-x)}{x}dx &= -\frac{7}{4}\zeta(3)\ln 2 - 2Li_4\left(\frac{1}{2}\right) - \frac{1}{12}\ln^4 2 \\ &\quad +\frac{11}{4}\zeta(4) + \frac{1}{2}\zeta(2)\ln^2 2 \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n^3}.\end{aligned}$$

The integrals (1.16), (1.17) and (1.18) cannot be analytically evaluated by "Mathematica". There are many integral representations of powers of Pi, for example

$$\pi = 4\int_0^1 \sqrt{1-x^2}dx, \; (-1)^p p!\zeta(p+1) = \int_0^1 \frac{\log^p(1-x)}{x}dx.$$

Guillera and Sondow [6] gave

$$\pi^2 = 8\int_0^1\int_0^1 \frac{dydx}{1-x^2y^2}, \; \pi^3 = -16\int_0^1\int_0^1 \frac{\ln xy\, dydx}{1+x^2y^2},$$

and amongst many other results [1] obtained the intriguing representation (after some manipulation)

$$\ln\left(\frac{\pi}{2}\right) = \sum_{n=1}^{\infty}(-1)^{n+1}\ln\left(1+\frac{1}{n}\right),$$

however the author believes (1.16) is a new representation.

Lemma 1.5. *Let $r \geq 2$ be a positive integer, defining*

$$S(r) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n+r} \tag{1.19}$$

then $S(r)$ has the recurrence relation

$$S(r) + S(r-1) = \frac{1}{r-1}\left(1 + (-1)^r\right) \ln 2 - \frac{(-1)^{r+1}}{r-1} F(1, r-1)$$

with representation

$$\begin{aligned} S(r) &= (-1)^{r+1} S(1) + (-1)^r \left(2H_{r-1} - H_{\left[\frac{r-1}{2}\right]}\right) \ln 2 \\ &\quad + (-1)^r \sum_{j=1}^{r-1} \frac{1}{j} \left(H_{\left[\frac{j}{2}\right]} - H_j\right), \end{aligned} \tag{1.20}$$

where $S(0) = \frac{1}{2}\left(\zeta(2) - \ln^2 2\right)$, $S(1) = \frac{1}{2} \ln^2 2$ and $F(1, r-1)$ is given by (1.5).

Proof. The proof is detailed in [11]. ∎

Lemma 1.6. *For a positive integer $r \geq 2$, we define*

$$T(r) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^2}{n+r}$$

then $T(r)$ has the recurrence relation

$$\begin{aligned} T(r) + T(r-1) &= -\frac{2S_{r-1}}{r-1} + \frac{\zeta(2)}{2(r-1)} - \frac{\ln^2 2}{r-1} + \frac{\ln 2}{(r-1)^2} \\ &\quad - \frac{1}{2(r-1)^2}\left(H_{\frac{r-1}{2}} - H_{\frac{r-2}{2}}\right), \end{aligned}$$

with representation

$$\begin{aligned} T(r) &= (-1)^{r+1} T(1) + (-1)^{r+1} \frac{1}{2} F(1, r-1) \zeta(2) \\ &\quad + (-1)^r F(1, r-1) \ln^2 2 + (-1)^{r+1} F(2, r-1) \ln 2 \\ &\quad + (-1)^r \sum_{j=1}^{r-1} \frac{(-1)^j}{j} \left(2S(j) + \frac{H_{\frac{j}{2}} - H_{\frac{j-1}{2}}}{2j}\right), \end{aligned} \tag{1.21}$$

with $T(0) = \frac{3}{4}\zeta(3) + \frac{1}{3}\ln^3 2 - \frac{1}{2}\zeta(2)\ln 2$, $T(1) = \frac{1}{2}\zeta(2)\ln 2 - \frac{1}{3}\ln^3 2 - \frac{1}{4}\zeta(3)$, $S(j)$ is given by (1.19) and $F(\cdot, r-1)$ is given by (1.4).

Proof. The proof is detailed in [11]. ∎

Lemma 1.7. *Let* $r \in \mathbb{N} \backslash \{1\}$, *then*

$$Y(r-1) = 3\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n H_{n-1}}{n(n+r-1)}$$

$$\begin{aligned} &= \frac{3}{(r-1)}\left(\frac{1}{8}\zeta(3) + \frac{1}{3}\ln^3 2 - \frac{1}{2}\zeta(2)\ln 2 - T(r-1)\right) \\ &+ \frac{3}{2(r-1)^2}\left(\zeta(2) - \ln^2 2 - 2S(r-1)\right), \end{aligned} \tag{1.22}$$

where $S(r-1)$ *is given by (1.19) and* $T(r-1)$ *is given by (1.21). For* $r = 1$ *we have*

$$Y(0) = 3\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n H_{n-1}}{n^2} = -\frac{9}{16}\zeta(4). \tag{1.23}$$

Proof. Consider

$$\begin{aligned} Y(r-1) &= 3\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n H_{n-1}}{n(n+r-1)} \\ &= \frac{3}{r-1}\sum_{n=1}^{\infty} (-1)^{n+1} H_n \left(H_n - \frac{1}{n}\right)\left(\frac{1}{n} - \frac{1}{n+r-1}\right) \\ &= \frac{3}{r-1}\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{H_n^2}{n} - \frac{H_n}{n^2} - \frac{H_n^2}{n+r-1} + \frac{H_n}{n(n+r-1)}\right) \end{aligned}$$

$$\begin{aligned} &= \frac{3}{r-1}\left(\frac{3}{4}\zeta(3) + \frac{1}{3}\ln^3 2 - \frac{1}{2}\zeta(2)\ln 2 - \frac{5}{8}\zeta(3) - T(r-1)\right) \\ &+ \frac{3}{r-1}\sum_{n=1}^{\infty} (-1)^{n+1} \frac{H_n}{r-1}\left(\frac{1}{n} - \frac{1}{n+r-1}\right) \\ &= \frac{3}{r-1}\left(\frac{3}{4}\zeta(3) + \frac{1}{3}\ln^3 2 - \frac{1}{2}\zeta(2)\ln 2 - \frac{5}{8}\zeta(3) - T(r-1)\right) \\ &+ \frac{3}{(r-1)^2}\left(\frac{1}{2}\zeta(2) - \frac{1}{2}\ln^2 2 - S(r-1)\right) \end{aligned}$$

and (1.22) follows. For the case $r = 1$, we have

$$\begin{aligned} Y(0) &= 3\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n H_{n-1}}{n^2} = 3\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n \left(H_n - \frac{1}{n}\right)}{n^2} \\ &= 3\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{H_n^2}{n^2} - \frac{H_n}{n^3}\right) \end{aligned}$$

and (1.23) follows. ∎

Lemma 1.8. *Let $r \in \mathbb{N}$, then*

$$X(r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^3}{n+r}$$

then

$$\begin{aligned} X(r) &= -(-1)^r X(1) + 3(-1)^{1+r} \left(\frac{3\zeta(3)}{8} + \frac{\ln^3 2}{3} - \frac{\zeta(2)\ln 2}{2} \right) F(1, r-1) \\ &+ \frac{(-1)^{r+1}}{2} \left(2\zeta(2) - 3\ln^2 2 \right) F(2, r-1) + (-1)^{1+r} \ln 2 F(3, r-1) \\ &+ (-1)^{1+r} \sum_{j=1}^{r-1} (-1)^j \left(\frac{1}{2j^3} \left(H_{\frac{j-1}{2}} - H_{\frac{j}{2}} \right) - \frac{3}{j} \left(T(j) + \frac{S(j)}{j} \right) \right) \end{aligned}$$

(1.24)

where $X(0)$ is given by (1.8), $X(1)$ is given by (1.14), $S(j)$ is given by (1.20), $T(j)$ is given by (1.21) and $F(\cdot, r-1)$ is given by (1.4).

Proof. Consider

$$X(r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^3}{n+r}$$

and by a change of summation index

$$\begin{aligned} X(r) &= \sum_{n=2}^{\infty} \frac{(-1)^n H_{n-1}^3}{n+r-1} = -X(r-1) + 3\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n H_{n-1}}{n(n+r-1)} \\ &+ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3(n+r-1)}. \end{aligned}$$

Now using lemma 1.5, 1.6 and 1.7

$$\begin{aligned} X(r) &= -X(r-1) + 3Y(r-1) + \frac{3\zeta(3)}{4(r-1)} - \frac{\zeta(2)}{2(r-1)^2} + \frac{\ln 2}{(r-1)^3} \\ &+ \frac{1}{2(r-1)^3} \left(H_{\frac{r-2}{2}} - H_{\frac{r-1}{2}} \right), \end{aligned}$$

hence we have the recurrence relation

$$\begin{aligned} X(r) + X(r-1) &= \frac{3}{r-1} \left(\frac{3}{8}\zeta(3) + \frac{1}{3}\ln^3 2 - \frac{1}{2}\zeta(2)\ln 2 \right) - \frac{3T(r-1)}{r-1} \\ &+ \frac{1}{2(r-1)^2} \left(2\zeta(2) - 3\ln^2 2 \right) + \frac{\ln 2}{(r-1)^3} \\ &- \frac{3S(r-1)}{(r-1)^2} + \frac{1}{2(r-1)^3} \left(H_{\frac{r-2}{2}} - H_{\frac{r-1}{2}} \right) \end{aligned}$$

for $r \geq 2$. The recurrence relation is solved by the subsequent reduction of the $X(r)$, $X(r-1)$, ..., $X(1)$ terms and using lemma 1.1, finally arriving at the relation (1.24). ∎

The next few theorems relate the main results of this investigation, namely the closed form and integral representation of (1.1).

2. CLOSED FORM AND INTEGRAL IDENTITIES

We now prove the following theorems.

Theorem 2.1. *Let $k \geq 1$ be real positive integer, then from (1.1) with $p = 0$ we have*

$$\Omega(k,0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^3}{\binom{n+k}{k}} = \sum_{r=1}^{k} (-1)^{r+1} r \binom{k}{r} X(r) \tag{2.1}$$

$$= \int_0^1 \int_0^1 \int_0^1 \frac{\ln(1-x)\ln(1-y)\ln(1-z)}{1+k} \, {}_4F_3 \left[\begin{matrix} 2,2,2,2 \\ 1,1,2+k \end{matrix} \middle| -xyz \right] dxdydz$$

where $X(r)$ is given by (1.24).

Proof. Consider the expansion

$$\Omega(k,0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^3}{\binom{n+k}{k}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} k! \, H_n^3}{(n+1)_{1+k}}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} k! \, H_n^3 \sum_{r=1}^{k} \frac{\Phi_r}{n+r}$$

where

$$\Phi_r = \lim_{n \to -r} \left\{ \frac{n+r}{\prod_{r=1}^{k} n+r} \right\} = \frac{(-1)^{r+1} r}{k!} \binom{k}{r},$$

hence

$$\begin{aligned} \Omega(k,0) &= \sum_{r=1}^{k} (-1)^{r+1} r \binom{k}{r} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^3}{n+r} \\ &= \sum_{r=1}^{k} (-1)^{r+1} r \binom{k}{r} X(r), \end{aligned}$$

and using lemma 1.8, gives

$$\Omega(k,0) = \sum_{r=1}^{k} (-1)^{r+1} r \binom{k}{r} X(r), \tag{2.2}$$

hence (2.1) follows. For the integral representation we employ the same technique as lemma 1.2. ∎

The other case of $\Omega(k,1)$, can be evaluated in a similar fashion. We list the result in the next theorem.

Theorem 2.2. *Under the assumptions of Theorem 2.1, we have,*

$$\Omega(k,1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ H_n^3}{n \begin{pmatrix} n+k \\ k \end{pmatrix}}$$

$$= \int_0^1 \int_0^1 \int_0^1 \frac{\ln(1-x)\ln(1-y)\ln(1-z)}{1+k}\ {}_3F_2 \left[\begin{array}{c} 2,2,2 \\ 1,2+k \end{array} \middle| -xyz \right] dxdydz,$$

$$(2.3) \qquad = X(0) - \sum_{r=1}^{k} (-1)^{r+1} \begin{pmatrix} k \\ r \end{pmatrix} X(r).$$

Proof. The proof follows directly from theorem 2.1 and using the same technique. ∎

Example 2.1. *Some illustrative examples follow.*

$$\Omega(3,0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ H_n^3}{\begin{pmatrix} n+3 \\ 3 \end{pmatrix}} = \frac{255}{16} - \frac{21}{2}\zeta(2) - \frac{261}{16}\zeta(3) - \frac{15}{4}\zeta(4)$$

$$+27\zeta(2)\ln 2 + 36\ln^2 2 - 18\ln^3(2) - \frac{63}{2}\ln 2$$

$$-9\zeta(2)\ln^2 2 + 3\ln^4 2 + \frac{27}{2}\zeta(3)\ln 2,$$

$$\Omega(4,1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ H_n^3}{n \begin{pmatrix} n+4 \\ 4 \end{pmatrix}} = -\frac{32575}{1296} + \frac{499}{36}\zeta(2) + \frac{325}{16}\zeta(3) + \frac{85}{16}\zeta(4)$$

$$-34\zeta(2)\ln 2 - \frac{151}{3}\ln^2 2 + \frac{68}{3}\ln^3(2) + \frac{10636}{216}\ln 2$$

$$+12\zeta(2)\ln^2 2 - 4\ln^4 2 - 18\zeta(3)\ln 2.$$

Remark 2.1. Following a similar argument as that of Lemma 1.3, we note first from (2.3), with $k=2$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ H_n^3}{n(n+1)(n+2)} = \frac{25}{32}\zeta(4) - \frac{1}{2} + \frac{1}{2}\zeta(2) + \frac{15}{16}\zeta(3) + \ln 2$$

$$-\frac{3}{2}\zeta(2)\ln 2 - \frac{3}{2}\ln^2 2 - \frac{1}{2}\ln^4 2 + \ln^3 2 + \frac{3}{2}\zeta(2)\ln^2 2 - \frac{9}{4}\zeta(3)\ln 2.$$

Also

$$\sum_{n=1}^{\infty} \frac{H_n^3}{n(n+1)(n+2)} = 5\zeta(4) - 2\zeta(3) - \zeta(2) - \frac{1}{2},$$

and hence,

$$\sum_{n=1}^{\infty} \frac{H_{2n-1}^3}{n(2n-1)(2n+1)} = \frac{185}{32}\zeta(4) - 1 - \frac{1}{2}\zeta(2) - \frac{17}{16}\zeta(3) + \ln 2$$

$$-\frac{3}{2}\zeta(2)\ln 2-\frac{3}{2}\ln^2 2-\frac{1}{2}\ln^4 2+\ln^3 2+\frac{3}{2}\zeta(2)\ln^2 2-\frac{9}{4}\zeta(3)\ln 2.$$

In the following remark we list, without proof, an extension of the results related to Lemma 1.2.

Remark 2.2. Let $\phi_n = H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}$, then

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\phi_n}{n(n+1)} &= \frac{3}{2}\zeta(2)\ln^2 2-\frac{21}{4}\zeta(3)\ln 2+6Li_4\left(\frac{1}{2}\right)\\ &\quad -\frac{1}{4}\ln^4 2+\frac{15}{8}\zeta(4).\end{aligned}$$

$$\begin{aligned}\int_0^1\frac{\log^3(1-x)\log(1+x)}{x^2}dx &= \frac{3}{2}\zeta(2)\ln^2 2-\frac{21}{4}\zeta(3)\ln 2+6Li_4\left(\frac{1}{2}\right)\\ &\quad -\frac{1}{4}\ln^4 2-\frac{33}{8}\zeta(4).\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\phi_n}{n+1} &= \int_0^1\frac{\log^3(1-x)}{1+x}\left(\frac{\log(1+x)}{x}-\frac{1}{1+x}\right)dx\\ &= \frac{21}{4}\zeta(3)\ln 2-\frac{3}{2}\zeta(2)\ln^2 2+\frac{1}{4}\ln^4 2-\frac{15}{8}\zeta(4).\end{aligned}$$

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