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POLYNOMIAL DICHOTOMY OF C_0-QUASI SEMIGROUPS IN BANACH SPACES

SUTRIMA[1,2], CHRISTIANA RINI INDRATI[2], LINA ARYATI[2]

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[1] DEPARTMENT OF MATHEMATICS, UNIVERSITAS SEBELAS MARET, PO BOX 57126, SURAKARTA, INDONESIA
sutrima@mipa.uns.ac.id

[2] DEPARTMENT OF MATHEMATICS, UNIVERSITAS GADJAH MADA, PO BOX 55281, YOGYAKARTA, INDONESIA
rinii@ugm.ac.id, lina@ugm.ac.id

ABSTRACT. Stability of solutions of the problems is an important aspect for application purposes. Since its introduction by Datko [7], the concept of exponential stability has been developed in various types of stability by various approaches. The existing conditions use evolution operator, evolution semigroup, and quasi semigroup approach for the non-autonomous problems and a semigroup approach for the autonomous cases. However, the polynomial stability based on C_0-quasi semigroups has not been discussed in the references. In this paper we propose a new stability for C_0-quasi semigroups on Banach spaces i.e the polynomial stability and polynomial dichotomy. As the results, the sufficient and necessary conditions for the polynomial and uniform polynomial stability are established as well as the sufficiency for the polynomial dichotomy. The results are also confirmed by the examples.

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1. Introduction

There are many ways of defining stability of mild and classical solutions of the linear non-autonomous Cauchy problem

$$\begin{cases} \dot{x}(t) & = A(t)x(t), \quad t \geq 0 \\ x(0) & = x_0, \qquad\qquad x_0 \in X, \end{cases} \tag{1.1}$$

on a Banach space X. For example, by the evolution operator approach Datko [7] initiated in characterizing of the stability which is equivalent to the stability of family of the evolution operators $\{U(t,s)\}_{t\geq s\geq 0}$. In two years later Pazy [21] has completed the Datko's theory, and so the result is well-known by Datko-Pazy Theorem.

The Datko-Pazy Theorem can be generalized to the various concepts of stability. The obtained generalizations are found in [4], [5], [16], and [18] for exponential stability, in [15] and [17] for exponential instability, and in [19], [22], and [24] for exponential dichotomy. In particular for autonomous case, the family $\{U(t,s)\}_{t\geq s\geq 0}$ turns into a C_0-semigroup $\{T(t)\}_{t\geq 0}$. Some generalizations of the stability of C_0-semigroups are found in [3], [8], [9], [10], [11], and [23].

The Datko-Pazy Theorem and all its generalizations imply the exponential convergence to 0 of the solutions of (1.1). However, the exponential stability is not enough yet covering all problems even for autonomous cases [2]. Bátkai *et al.* [2] show that if (1.1) is weakly damped systems of linear wave equations, then the classical solutions of (1.1) converges to 0 polynomially, but not exponentially. This phenomenon drives the polynomially stable concept. Related to this concept, Megan *et al.* [14] investigate the polynomial stability of evolution operators in Banach spaces and Paunonen [20] develops the stability for perturbation of Riesz-spectral operators in Hilbert spaces.

A quasi semigroup approach can be implemented to investigate the stability of the solutions of (1.1). In this context, the stability of the solutions depends on the stability of C_0-quasi semigroup $\{R(t,s)\}_{s,t\geq 0}$ which is generated by $\{A(t)\}_{t\geq 0}$ on X. We follow the definition of strongly continuous quasi semigroup in [1] and [12]. The definition implies that every C_0-semigroup is a C_0-quasi semigroup, but it is not conversely. The comprehensively discussion of properties of C_0-quasi semigroups can be found in [1], [12], [13], and [26]. In particular, if $\{R(t,s)\}_{s,t\geq 0}$ is a C_0-quasi semigroup on a Banach space X with its dual $R'(t,s)$ on X', then for every $x \in X$ and $x' \in X'$ the map $t \mapsto \|R(t,s)x\|$ and $s \mapsto \|R'(t,s)x'\|$ are measurable. These are prerequisites to characterize the stability of C_0-quasi semigroups.

The following descriptions discuss the studies of stability relating to C_0-quasi semigroups. Megan and Cuc [13] generalize the Datko's result for the exponential stability of C_0-quasi semigroups in Banach spaces. They have constructed the sufficient and necessary conditions for exponential stability using an admissible increasing function as the upper bound of integral of norm of the quasi semigroup either using an admissible increasing sequence. Next, Cuc [6] has constructed sufficient and necessary conditions for the uniform exponential dichotomy of C_0-quasi semigroup in Banach spaces. The results generalized the similar theorem which are obtained by Datko [7] and Pazy [21] for the exponential stability and Preda and Megan ([22], [23], and [24]) for the exponential dichotomy of C_0-quasi semigroups, respectively. Finally, Sutrima *et al.* [25] construct the sufficient and necessary conditions for the strong stability of C_0-quasi semigroup in Banach spaces. Currently there is no research that addresses the polynomial stability of C_0-quasi semigroups in Banach spaces.

In this paper we propose new concepts of stability, namely the polynomial stability and polynomial dichotomy of C_0-quasi semigroups in Banach spaces. The organization of this paper is as follows. Section 2 deals with the sufficiency and necessity for the polynomial stability of C_0-quasi semigroups. Investigation of the sufficiency is given in Section 3.

2. POLYNOMIAL STABILITY

In this section we concern on the polynomial stability of C_0-quasi semigroups which is an elaboration of the exponential stability. We first give the definition of a strongly continuous quasi semigroup following [1] and [12].

Definition 2.1. Let X be a Banach space and $\mathcal{L}(X)$ be the set of all bounded linear operators on X. A two-parameter commutative family $\{R(t,s)\}_{s,t\geq 0}$ in $\mathcal{L}(X)$ is called a strongly continuous quasi semigroup, in short C_0-quasi semigroup, on X if for each $r,s,t\geq 0$ and $x\in X$:

(a) $R(t,0)=I$, the identity operator on X,
(b) $R(t,s+r)=R(t+r,s)R(t,r)$,
(c) $\lim_{s\to 0^+}\|R(t,s)x-x\|=0$,
(d) there is a continuous increasing function $M:[0,\infty)\to[1,\infty)$ such that

$$\|R(t,s)\|\leq M(s),$$

for all $t,s\geq 0$.

In the sequel we denote the quasi semigroup $\{R(t,s)\}_{s,t\geq 0}$ by $R(t,s)$ and M is a function satisfying condition (d) of Definition 2.1 for some C_0-quasi semigroup $R(t,s)$.

We see that if $R(t,s)$ is exponentially stable then the classical solutions $x(t)=R(0,t)x$ of (1.1) converge to 0 exponentially as $t\to\infty$. The following definitions describe the types of exponential stability following [6], [13], and [25].

Definition 2.2. A C_0-quasi semigroup $R(t,s)$ on a Banach space X is said to be:

(a) exponentially stable on X if there are constant $\alpha>0$ and increasing function $N:\mathbb{R}^+\to[1,\infty)$ such that

$$e^{\alpha s}\|R(t,s)x\|\leq N(t)\|x\|$$

for all $t,s\geq 0$ and $x\in X$;
(b) uniformly exponentially stable on X if there are constant $\alpha>0$ and $N\geq 1$ such that

$$e^{\alpha s}\|R(t,s)x\|\leq N\|x\|$$

for all $t,s\geq 0$ and $x\in X$;
(c) exponentially stable in the Barreira-Valls sense if there are constant $N\geq 1$, $\beta\geq\alpha>0$ such that

$$e^{\alpha s}\|R(t,s)x\|\leq Ne^{(\beta-\alpha)t}\|x\|$$

for all $t,s\geq 0$ and $x\in X$.

In the Definition 2.2 we see the importance of the function or constant $N\geq 1$ to guarantee the solution $x(t)=R(0,t)x$ converges to 0 as $t\to\infty$. The polynomially stable concept is based on this condition.

Definition 2.3. A C_0-quasi semigroup $R(t,s)$ on a Banach space X is said to be polynomially stable if there exist $\alpha>0$, $t_0>0$, and an increasing function $N:\mathbb{R}^+\to[1,\infty)$ such that

$$(t+s)^{\alpha}\|R(t,s)x\|\leq N(t)\|x\|,$$

for all $t,s\geq 0$, $t\geq t_0$, and $x\in X$.

Lemma 2.1. *If C_0-quasi semigroup $R(t,s)$ is exponentially stable on a Banach space X, then $R(t,s)$ is polynomially stable on X.*

Proof. Definition 2.2 gives that there are $\alpha > 0$, $t_0 > 0$, an increasing function $N, g : \mathbb{R}^+ \to [1, \infty)$ with $g(t) := N(t)e^{-\alpha t}$ such that

$$e^{\alpha s}\|R(t,s)x\| \leq g(t)\|x\|,$$

for all $t, s \geq 0$ with $t \geq t_0$ and $x \in X$. Hence

$$(t+s)^\alpha \|R(t,s)x\| \leq e^{\alpha t}e^{\alpha s}\|R(t,s)x\| \leq N(t)\|x\|$$

for all $t, s \geq 0$ with $t \geq t_0$ and $x \in X$. Thus, the C_0-quasi semigroup $R(t,s)$ is polynomially stable on X. ∎

The following example shows that the converse of Lemma 2.1 is not valid.

Example 2.1. *A C_0-quasi semigroup defined by*

$$R(t,s)x := \frac{1+t}{1+t+s}x,$$

for all $t, s \geq 0$ and $x \in \mathbb{R}$ is polynomially stable on $\mathbb{R}$ but it is not exponentially stable.

Theorem 2.2. *Let $R(t,s)$ be a C_0-quasi semigroup on a Banach space X. If there exist $\gamma > 0$, $t_0 \geq 1$, and $H : \mathbb{R}^+ \to [1,\infty)$ such that*

$$\int_t^\infty (t+v)^\gamma \|R(t,v)x\|dv \leq H(t)\|x\|, \tag{2.1}$$

for all $t \geq t_0$ and $x \in X$, then $R(t,s)$ is polynomially stable on X.

Proof. Given any $x \in X$. We use facts that the function h, where $h(t) = e^t/t$, is increasing on interval $(0,\infty)$ and there is a function M satisfying condition (d) of Definition 2.1. For $s \geq 1$ and $t \geq t_0$ we have

$$\begin{aligned}(t+s)^\gamma \|R(t,s)x\| &= \int_{s-1}^s (t+s)^\gamma \|R(t,s)x\|dv \\ &\leq \int_{s-1}^s M(s-v)e^{\gamma(s-v)}(t+v)^\gamma \|R(t,v)x\|dv \\ &\leq M(1)e^\gamma \int_{s-1}^s (t+v)^\gamma \|R(t,v)x\|dv.\end{aligned}$$

For $s \geq t+1$ we obtain

$$\begin{aligned}M(1)e^\gamma \int_{s-1}^s (t+v)^\gamma \|R(t,v)x\|dv &\leq M(1)e^\gamma \int_t^s (t+v)^\gamma \|R(t,v)x\|dv \\ &\leq M(1)e^\gamma \int_t^\infty (t+v)^\gamma \|R(t,v)x\|dv \\ &\leq M(1)e^\gamma H(t)\|x\| = N_1(t)\|x\|,\end{aligned}$$

where $N_1(t) = M(1)e^\gamma H(t)$.

Next, for $t \leq s < t+1$ by the Mean Value Theorem for integral, we have

$$\begin{aligned}M(1)e^\gamma \int_{s-1}^s (t+v)^\gamma \|R(t,v)x\|dv &\leq M(1)e^\gamma \int_{t-1}^s (t+v)^\gamma \|R(t,v)x\|dv \\ &\leq M(1)e^\gamma \int_{t-1}^t (t+v)^\gamma M(v)\|x\|dv + \\ &\quad M(1)e^\gamma \int_t^\infty (t+v)^\gamma \|R(t,v)x\|dv \\ &\leq M(1)e^\gamma \left(2^\gamma t^\gamma M(\zeta) + H(t)\right)\|x\| = N_2(t)\|x\|,\end{aligned}$$

for some $\zeta \in [t-1,t]$ where $N_2(t) = M(1)e^{\gamma}\left(2^{\gamma}t^{\gamma}M(\zeta) + H(t)\right)$. We can choose $N(t) = \max\{N_1(t), N_2(t)\}$ such that

$$(t+s)^{\gamma}\|R(t,s)x\| \leq N(t)\|x\|$$

for all $t, s \geq 0$, $t \geq t_0$, and $x \in X$. Thus, $R(t,s)$ is polynomially stable on X. ∎

The converse of Theorem 2.2 is valid when $\alpha > 1$ in Definition 2.3.

Theorem 2.3. *If $R(t,s)$ is a polynomially stable C_0-quasi semigroup on a Banach space X with $\alpha > 1$, then $R(t,s)$ satisfies the inequality* (2.1).

Proof. By hypothesis there exist $\alpha > 1$, $t_0 > 0$, and an increasing function $N : \mathbb{R}^+ \to [1,\infty)$ such that

$$(t+s)^{\alpha}\|R(t,s)x\| \leq N(t)\|x\|,$$

for all $t, s \geq 0$, $t \geq t_0$, and $x \in X$. Choose $\gamma \in (0, \alpha-1)$. Since γ can be stated as $\gamma = \alpha - 1 - \delta$ for some $\delta > 0$, we have

$$\begin{aligned}
\int_t^{\infty}(t+v)^{\gamma}\|R(t,v)x\|dv &= \int_t^{\infty}(t+v)^{\alpha-(1+\delta)}\|R(t,v)x\|dv \\
&\leq N(t)\|x\|\int_t^{\infty}(t+v)^{-(1+\delta)}dv \\
&= \frac{N(t)}{\delta(2t)^{\delta}}\|x\| := H(t)\|x\|.
\end{aligned}$$

This proves the required assertion. ∎

If $\alpha = 1$ in Definition 2.3, then $R(t,s)$ may be polynomially stable although it does not satisfy (2.1). Example 2.1 confirms this statement. Moreover, if N in Definition 2.3 is a constant function, then the stability is uniform stability.

Definition 2.4. A C_0-quasi semigroup $R(t,s)$ on a Banach space X is said to be uniformly polynomially stable if there exist $\alpha > 0$, $t_0 > 0$, and $N \geq 1$ such that

$$(t+s)^{\alpha}\|R(t,s)x\| \leq Nt^{\alpha}\|x\|,$$

for all $t, s \geq 0$, $t \geq t_0$, and $x \in X$.

Definition 2.4 states that if $R(t,s)$ is uniformly polynomially stable, then $R(t,s)$ is polynomially stable.

Lemma 2.4. *If a C_0-quasi semigroup $R(t,s)$ is uniformly exponentially stable on a Banach space X, then $R(t,s)$ is uniformly polynomially stable on X.*

Proof. By Definition 2.2, there exist $\alpha > 0$ and $N \geq 1$ such that

$$e^{\alpha s}\|R(t,s)x\| \leq N\|x\|,$$

for all $t, s \geq 0$ and $x \in X$. Hence, if $t \geq t_0$ for some $t_0 \geq 1$, then

$$(t+s)^{\alpha}\|R(t,s)x\| \leq t^{\alpha}e^{\alpha s}\|R(t,s)x\| \leq Nt^{\alpha}\|x\|$$

for all $t, s \geq 0$ and $x \in X$. So, $R(t,s)$ is uniformly polynomially stable on X. ∎

The following example shows that there exists a C_0-quasi semigroup which is uniformly polynomially stable but it is not uniformly exponentially stable.

Example 2.2. *A C_0-quasi semigroup $R(t,s)$ is defined by*

$$R(t,s)x = \frac{1+t^2}{1+(t+s)^2}x,$$

for all $t,s \geq 0$ and $x \in \mathbb{R}$, is uniformly polynomially stable. However it is not uniformly exponentially stable.

The $R(t,s)$ is uniformly polynomially stable on $\mathbb{R}$ with $\alpha = 2$. Suppose $R(t,s)$ is uniformly exponentially stable. There exist $N \geq 1$ and $\alpha > 0$ such that

$$e^{\alpha s}(1+t^2) \leq N(1+(t+s)^2),$$

for all $t,s \geq 0$. If we choose $t = 0$, then

$$e^{\alpha s} \leq (1+s^2)N,$$

which is a contradiction for sufficiently large s.

By inspecting the proof of Theorem 2.2 we can construct a sufficient condition for uniformly polynomially stable of C_0-quasi semigroup.

Corollary 2.5. *Let $R(t,s)$ be a C_0-quasi semigroup on a Banach space X. If there exist $\gamma > 0$, $t_0 \geq 1$, and $H \geq 1$ such that*

$$\int_t^\infty (t+v)^\gamma \|R(t,v)x\| dv \leq H\|x\|,$$

for all $t \geq t_0$ and $x \in X$, then $R(t,s)$ is uniformly polynomially stable on X.

Proof. Proof follows the proof of Theorem 2.2 with facts that for $t \geq t_0$ we have

$$M(1)e^\gamma H\|x\| \leq M(1)e^\gamma H t^\gamma \|x\|$$

and

$$M(1)e^\gamma \left(2^\gamma t^\gamma M(\zeta) + H\right)\|x\| \leq M(1)e^\gamma \left(2^\gamma M(\zeta) + H\right)t^\gamma \|x\|.$$

∎

In Definition 2.4, if $\alpha > 1$, then the converse of Corollary 2.5 is still true. The following example confirms Corollary 2.5.

Example 2.3. *A C_0-quasi semigroup $R(t,s)$ defined by*

$$R(t,s)x = e^{-s}\frac{t}{t+s}x,$$

for all $t,s \geq 0$ and $x \in \mathbb{R}$, is uniformly polynomially stable on $\mathbb{R}$.

We verify that there is $t_1 > 1$ such that the function $u(t) = e^{-t}(2t^2+t), t > 0$, attains its maximum at t_1. Hence, we have

$$\begin{aligned}\int_t^\infty (t+v)^2 |R(t,v)x| dv &\leq \int_t^\infty (t+v)te^{-v}|x| dv \\ &= e^{-t}(2t^2+t)|x| \leq u(t_1)|x| := H|x|.\end{aligned}$$

Corollary 2.5 implies that $R(t,s)$ is uniformly polynomially stable on $\mathbb{R}$.

By implementing the dual principle we have an alternative sufficiency for the uniform polynomial stability of $R(t,s)$.

Theorem 2.6. *Let $R(t,s)$ be a C_0-quasi semigroup on a Banach space X. If there exist $\gamma \geq 1$, $t_0 \geq 1$, and $H \geq 1$ such that*

$$\int_t^s \left(\frac{t+s}{v}\right)^\gamma \|R'(t,v)x'\|dv \leq H\|x'\|,$$

for all $s \geq 0$, $t \geq t_0$, and $x' \in X'$, then $R(t,s)$ is uniformly polynomially stable on X.

Proof. Given any $x \in X$ and $x' \in X'$. If $s \geq t+1$ and $t \geq t_0 \geq 1$, then

$$\begin{aligned}\left(\frac{t+s}{t}\right)^\gamma |x'(R(t,s)x)| &= \int_{s-1}^s \left(\frac{t+s}{v}\right)^\gamma \left(\frac{v}{t}\right)^\gamma |R'(t,v)x'(R(t+v,s-v)x)|dv \\ &\leq 2^\gamma M(1)\|x\| \int_t^s \left(\frac{t+s}{v}\right)^\gamma \|R'(t,v)x'\|dv \\ &\leq 2^\gamma M(1)H\|x\|\|x'\| := N\|x\|\|x'\|.\end{aligned}$$

Hence we have

$$\left(\frac{t+s}{t}\right)^\gamma \|R(t,s)x\| \leq N\|x\|.$$

Next, if $t \leq s < t+1$ with $t \geq t_0$, then

$$\left(\frac{t+s}{t}\right)^\gamma \|R(t,s)x\| \leq 3^\gamma M(s)\|x\| \leq 3^\gamma M(2)\|x\| := N\|x\|.$$

Therefore, for all $t,s \geq 0$ with $t \geq t_0$ and $x \in X$ we conclude that

$$(t+s)^\gamma \|R(t,s)x\| \leq Nt^\gamma\|x\|.$$

Hence, $R(t,s)$ is uniformly polynomially stable on X. ∎

As in the exponential stability, we can generalize the polynomially stable concept of C_0-quasi semigroups into the Barreira-Valls sense.

Definition 2.5. A C_0-quasi semigroup $R(t,s)$ is said to be polynomially stable in the Barreira-Valls sense on a Banach space X if there exist $N \geq 1$, $\alpha > 0$, $\beta \geq \alpha$, and $t_0 > 0$ such that

$$(t+s)^\alpha\|R(t,s)x\| \leq Nt^\beta\|x\|,$$

for all $s \geq 0$ and $t \geq t_0$

From Definition 2.5 we conclude that if $R(t,s)$ is polynomially stable in the Barreira-Valls sense, then $R(t,s)$ is polynomially stable. Moreover, if $R(t,s)$ is uniformly polynomially stable, then $R(t,s)$ is polynomially stable in the Berreira-Valls sense. The following two examples show that the converse of both implications is not valid. These examples are modified from [5].

Example 2.4. *Defined a C_0-quasi semigroup $R(t,s)$ on $\mathbb{R}$ by*

$$R(t,s)x = \frac{t^2u(t)}{(t+s)^2u(t+s)}x,$$

for all $t,s \geq 0$ and $x \in \mathbb{R}$, with $u(n) = e^n$ and $u\left(n+\frac{1}{n}\right) = e^2$ for all non-negative integers n.

It is obvious that $R(t,s)$ is polynomially stable on $\mathbb{R}$ with $\alpha = 1$. Suppose $R(t,s)$ is polynomially stable in the Barreira-Valls sense. There exist $N \geq 1$, $\alpha > 0$, $\beta \geq \alpha$, and $t_0 > 0$ such that

$$(t+s)^{\alpha-2}u(t) \leq Nu(t+s)t^{\beta-2},$$

for all $s \geq 0$ and $t \geq t_0$. However, for $t = n$ and $s = \frac{1}{n}$ we have

$$n^{\alpha-\beta} e^n \left(1 + \frac{1}{n^2}\right)^{\alpha-2} \leq Ne^2,$$

which yields a contradiction for sufficiently large n.

Example 2.4 also shows that if $R(t,s)$ is polynomial stable then it does not need uniformly polynomially stable.

Example 2.5. *Defined C_0-quasi semigroup $R(t,s)$ on $\mathbb{R}$ by*

$$R(t,s)x = \frac{(t+1)^2(t+s+1)^{\cos\ln(t+s+1)}}{(t+s+1)^2(t+1)^{\cos\ln(t+1)}} x$$

for all $t, s \geq 0$ and $x \in \mathbb{R}$.

It is easy to show that $R(t,s)$ is polynomially stable in the Barreira-Valls sense on $\mathbb{R}$ with $\alpha = 1$ and $\beta = 3$. Suppose $R(t,s)$ is uniformly polynomially stable on $\mathbb{R}$. There exist $N \geq 1$, $\alpha > 0$, and $t_0 > 0$ such that

$$(t+s)^{\alpha-2}(t+1)^2(t+s+1)^{\cos\ln(t+s+1)} \leq Nt^{\alpha}(t+s+1)^2(t+1)^{\cos\ln(t+1)},$$

for all $t, s \geq 0$ with $t \geq t_0$. For $t = -1 + e^{\frac{4n-1}{2}\pi}$ and $t + s = -1 + e^{2n\pi}$, we have

$$\left(\frac{-1 + e^{2n\pi}}{-1 + e^{\frac{4n-1}{2}\pi}}\right)^{\alpha} e^{3n\pi} \leq N.$$

This yields a contradiction for sufficiently large n.

As a special case of Theorem 2.2, we have a sufficient condition for the polynomial stability in the Barreira-Valls sense.

Corollary 2.7. *Let $R(t,s)$ be a C_0-quasi semigroup on a Banach space X. If there exist $H \geq 1$, $\delta \geq \gamma > 0$, and $t_0 \geq 1$ such that*

$$\int_t^{\infty} (t+v)^{\gamma} \|R(t,v)x\| dv \leq Ht^{\delta}\|x\|,$$

for all $t \geq t_0$ and $x \in X$, then $R(t,s)$ is polynomially stable in the Barreira-Valls sense on X.

Proof. Proof follows the proof of Theorem 2.2 for $H(t) = t^{\delta}$. ∎

As in the polynomially stable case, if Definition 2.5 holds for $\alpha > 1$, then the converse of Corollary 2.7 is still valid.

Example 2.6. *The C_0-quasi semigroup $R(t,s)$ in Example 2.2 is polynomially stable in the Barreira-Valls sense but it is not exponentially stable in the Barreira-Valls sense.*

From Example 2.2 we have

$$(t+s)|R(t,s)x| \leq 2t^2|x|,$$

for all $t, s \geq 0$ with $t \geq t_0 = 1$ and $x \in X$. This states that $R(t,s)$ is polynomially stable in the Barreira-Valls sense with $\alpha = 1$ and $\beta = 2$. Suppose $R(t,s)$ is exponentially stable in the Barreira-Valls sense. There exist $\alpha > 0$ and a function $N : \mathbb{R}^+ \to [1, \infty)$ such that

$$e^{\alpha s}(1 + t^2) \leq \left(1 + (t+s)^2\right) N(t),$$

for all $t, s \geq 0$. However, if t is fixed, then it yields a contradiction for sufficiently large s.

Theorem 2.8. *Let $R(t,s)$ be a C_0-quasi semigroup on a Banach space X. If there exist $\delta \geq \gamma > 0$, $t_0 \geq 1$, and $H \geq 1$ such that*

$$\int_t^s \left(\frac{t+s}{v}\right)^\gamma \|R'(t,v)x'\|dv \leq Ht^\delta\|x'\|,$$

for all $t,s \geq 0$ with $t \geq t_0$ and $x' \in X'$, then $R(t,s)$ is polynomially stable in the Barreira-Valls sense on X.

Proof. Proof follows the proof of Theorem 2.6. Let it be given any $x \in X$ and $x' \in X'$. If $s \geq t+1$ and $t \geq t_0 \geq 1$, then

$$\begin{aligned}\left(\frac{t+s}{t}\right)^\gamma |x'(R(t,s)x)| &= \int_{s-1}^s \left(\frac{t+s}{v}\right)^\gamma \left(\frac{v}{t}\right)^\gamma |R'(t,v)x'(R(t+v,s-v)x)|dv \\ &\leq 2^\gamma M(1)\|x\| \int_t^s \left(\frac{t+s}{v}\right)^\gamma \|R'(t,v)x'\|dv \\ &\leq 2^\gamma M(1)Ht^\delta\|x\|\|x'\| := Nt^\delta\|x\|\|x'\|.\end{aligned}$$

Hence we have

$$(t+s)^\gamma\|R(t,s)x\| \leq Nt^{\delta+\gamma}\|x\|.$$

Next, if $t \leq s < t+1$ with $t \geq t_0$, then

$$\begin{aligned}(t+s)^\gamma\|R(t,s)x\| &\leq \left(\frac{t+s}{t}\right)^\gamma M(2)t^\gamma\|x\| \\ &\leq 3^\gamma M(2)t^\gamma\|x\| \leq 3^\gamma M(2)t^{\delta+\gamma}\|x\| := Nt^{\delta+\gamma}\|x\|.\end{aligned}$$

Therefore we conclude that

$$(t+s)^\gamma \|R(t,s)x\| \leq Nt^{\delta+\gamma}\|x\|,$$

for all $t,s \geq 0$ with $t \geq t_0$ and $x \in X$. So, $R(t,s)$ is polynomially stable in the Barreira-Valls sense on X. ∎

3. POLYNOMIAL DICHOTOMY

In this section we characterize the polynomial dichotomy of C_0-quasi semigroups on a Banach space. The concept is a blend of the polynomial terminology and the exponential dichotomy of semigroups and evolution operators.

Definition 3.1. A C_0-quasi semigroup $R(t,s)$ is said to be polynomially dichotomic on a Banach space X if there exist $\alpha > 0$, $t_0 > 0$, a function $H : \mathbb{R}^+ \to [1,\infty)$, and a strongly continuous projection-valued function $P : \mathbb{R}^+ \to \mathcal{L}(X)$ such that

(a) $R(t,s)P(t) = P(t+s)R(t,s)$;
(b) $(t+s)^\alpha\|R(t,s)x\| \leq H(t)\|x\|$;
(c) $H(t+s)\|R(t,s)y\| \geq (t+s)^\alpha\|y\|$;

for all $t,s \geq 0$ with $t \geq t_0$, $x \in \operatorname{ran} P(t)$, and $y \in \ker P(t)$, where $\operatorname{ran} Q$ and $\ker Q$ denote the range and kernel of Q, respectively.

We see if the function P in Definition 3.1 exists, then $\operatorname{ran} P(t)$ and $\ker P(t)$ are not empty. Moreover, conditions (b) and (c) of Definition 3.1 are equivalent with

(b)′ $(t+r+s)^\alpha\|R(t,r+s)x\| \leq H(t+r)\|R(t,r)x\|$,
(c)′ $(t+r+s)^\alpha\|R(t,r)y\| \leq H(t+r+s)\|R(t,r+s)y\|$,

respectively. The last conditions are useful in applications.

Example 3.1. *Let X be a Banach space of $\mathbb{R}^2$ with norm $\|(x_1,x_2)\| = |x_1| + |x_2|$. We define a C_0-quasi semigroup $R(t,s)$ on X by*

$$R(t,s)x = (e^{a(t+s)-a(t)}x_1, e^{b(t+s)-b(t)}x_2),$$

where $x = (x_1,x_2)$, $a(t) = -2t + t\sin t$ and $b(t) = t - 4t\sin t$. If we define the projection $P_1 : X \to X$ with $P_1(x_1,x_2) = (x_1,0)$ and $P(t) = P_1$ for all $t \geq 0$, then $R(t,s)$ is polynomially dichotomic on X.

In this context we have ran $P(t) = \{(x_1,0) : x_1 \in \mathbb{R}\}$ and ker $P(t) = \{(0,x_2) : x_2 \in \mathbb{R}\}$. We observe that

$$R(t,s)P(t)z = P(t+s)R(t,s)z = (e^{a(t+s)-a(t)}z_1, 0),$$
$$(t+s)\|R(t,s)x\| \leq (t+s)e^{-s}e^{8t}\|x\| \leq te^{8t}\|x\|,$$
$$(t+s)e^{8(t+s)}\|R(t,s)y\| \geq (t+s)\|y\|,$$

for all $z \in X$, $x \in$ ran $P(t)$, $y \in$ ker $P(t)$, and $t,s \geq 0$ with $t \geq t_0$ for any $t_0 > 0$. This shows that $R(t,s)$ is polynomially dichotomic on X with $\alpha = 1$ and $H(t) = te^{8t}$.

Definition 3.2. A C_0-quasi semigroup $R(t,s)$ is said to be uniformly polynomially dichotomic on a Banach space X if there exist $\alpha > 0$, $t_0 > 0$, $H \geq 1$, and a strongly continuous projection-valued function $P : \mathbb{R}^+ \to \mathcal{L}(X)$ such that

(a) $R(t,s)P(t) = P(t+s)R(t,s)$;
(b) $(t+s)^\alpha\|R(t,s)x\| \leq Ht^\alpha\|x\|$;
(c) $Ht^\alpha\|R(t,s)y\| \geq (t+s)^\alpha\|y\|$;

for all $t,s \geq 0$ with $t \geq t_0$, $x \in$ ran $P(t)$, and $y \in$ ker $P(t)$.

It is obvious that if $R(t,s)$ is uniformly polynomially dichotomic, then it is polynomially dichotomic. Example 3.1 shows that the converse is not valid.

As before, conditions (b) and (c) of Definition 3.2 can be replaced by

(b)$''$ $(t+r+s)^\alpha\|R(t,r+s)x\| \leq H(t+r)^\alpha\|R(t,r)x\|$,
(c)$''$ $(t+r+s)^\alpha\|R(t,s)y\| \leq H(t+s)^\alpha\|R(t,r+s)y\|$,

respectively.

Example 3.2. *Let X be a Banach space of $\mathbb{R}^2$ with norm $\|(x_1,x_2)\| = |x_1| + |x_2|$. We define C_0-quasi semigroup $R(t,s)$ on X by*

$$R(t,s)x = \left(e^{-s}\frac{t}{t+s}x_1, e^{s}\frac{t+s}{t}x_2\right),$$

where $x = (x_1,x_2)$. If $P(t) = P_1$ is the projection as in Example 3.1, then $R(t,s)$ is uniformly polynomially dichotomic on X.

We observe that

$$(t+s)^2\|R(t,s)x\| \leq (t+s)te^{-s}\|x\| \leq t^2\|x\|,$$
$$t^2\|R(t,s)y\| \geq (t+s)^2\|y\|,$$

for all $x \in$ ran $P(t)$, $y \in$ ker $P(t)$, and $t,s \geq 0$ with $t \geq t_0$ for any $t_0 > 0$. So $R(t,s)$ is uniformly polynomially dichotomic on X with $\alpha = 2$ and $H = 1$.

Now we give the sufficient condition for the uniform polynomial dichotomy of C_0-quasi semigroups on a Banach space. The sufficiency follows Corollary 2.5 for the uniform polynomial stability.

Theorem 3.1. *Let $R(t,s)$ be a C_0-quasi semigroup on a Banach space X. If there exist $\gamma > 0$, $t_0 \geq 1$, $H \geq 1$, a function $G : \mathbb{R}^+ \to (0,\infty)$, and a strongly continuous projection-valued function $P : \mathbb{R}^+ \to \mathcal{L}(X)$ such that*

$$R(t,s)P(t) = P(t+s)R(t,s), \tag{3.1}$$

$$\int_t^\infty (t+v)^\gamma \|R(t,v)x\| dv \leq H\|x\|, \tag{3.2}$$

$$\int_0^s (t-r+v)^\gamma \|R(t,v)y\| dv \leq HG(r)(t-r+s)^\gamma \|R(t,s)y\|, \tag{3.3}$$

$$(t+r+s)^\gamma \|R(t,s)y\| \leq HG(r)(t+s)^\gamma \|R(t,s+1)y\|, \tag{3.4}$$

for all $x \in \operatorname{ran} P(t)$, $y \in \ker P(t)$, and $t,s \geq 0$ with $t \geq t_0$, then $R(t,s)$ is uniformly polynomially dichotomic on X.

Proof. If condition (3.2) is satisfied, then by Corollary 2.5 $R(t,s)$ is uniformly polynomially dichotomic on $\operatorname{ran} P(t)$.

If $s \geq 1$, $t \geq t_0$, and $r \geq 0$ such that $t+v > r$ with $s-1 \leq v \leq s$ for some $v \geq 0$, then by (3.3) we obtain

$$\begin{aligned}
(t+r+s)^\gamma \|R(t,s)y\| &= \int_{s-1}^s (t+r+s)^\gamma (t-r+v)^{-\gamma} (t-r+v)^\gamma \|R(t,s)y\| dv \\
&\leq \int_{s-1}^s M(s-v) e^{\gamma(2t+s+v)} (t-r+v)^\gamma \|R(t,v)y\| dv \\
&\leq M(1) e^{\gamma(2r+1)} \int_0^{r+s} (t-r+v)^\gamma \|R(t,v)y\| dv \\
&\leq (t+s)^\gamma H \|R(t,r+s)y\|,
\end{aligned}$$

for all $y \in \ker P(t)$ where $G(r) = e^{-\gamma(2r+1)}/M(1)$. Hence, $R(t,s)$ fulfils (b)$''$.

Next, we consider $0 \leq s < 1$. For $0 \leq r \leq 1$ and $y \in \ker P(t)$ by (3.4) we have

$$(t+r+s)^\alpha \|R(t,s)y\| \leq HM(1)e^{\gamma r}(t+s)^\alpha \|R(t,r+s)y\|.$$

While for $r \geq 1$ by (3.4) and Lemma 2.4 of [6] we obtain

$$(t+r+s)^\alpha \|R(t,s)y\| \leq mHe^{\gamma r}(t+s)^\alpha \|R(t,r+s)y\|.$$

For each case we can choose a suitable function G. These show that $R(t,s)$ fulfils (b)$''$. We conclude that $R(t,s)$ is uniformly polynomially dichotomic on X. ∎

Remark 3.1. Corollary 2.5 is a special case of Theorem 3.1 when $\ker P(t) = \{0\}$ for all $t \geq 0$. If $R(t,s)$ is uniformly exponentially dichotomic, then Theorem 3.1 gives a characterization of uniform exponential dichotomy of C_0-quasi semigroups. This characterization is a result obtained by Cuc [6]. Moreover, if $R(t,s)$ is a C_0-semigroup, the characterization had been done by Preda and Megan [23]. There is an open problem of investigation of the polynomial stability and polynomial dichotomy of C_0-quasi semigroups using the infinitesimal generator $A(t)$ similar to C_0-semigroups case.

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