The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 15, Issue 2, Article 4, pp. 1-9, 2018

INVARIANT SUBSPACES CLOSE TO ALMOST INVARIANT SUBSPACES FOR BOUNDED LINEAR OPERATORS

M. A. FARZANEH, A. ASSADI, AND H. M. MOHAMMADINEJAD

Received 2 January, 2018; accepted 28 May, 2018; published 16 July, 2018.

DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF BIRJAND, PO BOX 97175/615, BIRJAND, IRAN.
farzaneh@birjand.ac.ir

DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF BIRJAND, PO BOX 97175/615, BIRJAND, IRAN.
assadi-aman@birjand.ac.ir

DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF BIRJAND, PO BOX 97175/615, BIRJAND, IRAN.
hmohammadin@birjand.ac.ir

ABSTRACT. In this paper, we consider some features of almost invariant subspace notion. At first, we restate the notion of almost invariant subspace and obtain some results. Then we try to achieve an invariant subspace completely close to an almost invariant subspace. Also, we introduce the notion of “almost equivalent subspaces ” to simply the subject related to almost invariant subspaces and apply it.

Key words and phrases: Subspace, Almost invariant subspace, Half-space, Almost equivalent subspaces.

2010 *Mathematics Subject Classification.* Primary 47A15. Secondary 46A22.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Invariant subspace problem is a longstanding problem in functional analysis which has been solved for some classical Banach spaces but still remained open for reflexive Banach spaces, as well as, for Hilbert spaces. A moderate approach in this context which has been introduced by Androulakis, Popov, Tcaciuc and Troitsky [1], and followed by Marcoux and Radjavi [2], is called almost invariant subspace.

For a Banach space X and a bounded linear operator T on X, a closed subspace Y of X is called almost invariant under T, if there exists a finite-dimensional subspace M of X such that $TY \subseteq Y + M$. Furthermore, M can be chosen with minimum dimension. In this situation, M and $d_{Y,T} = \dim M$ are respectively called error and *defect* of Y under T. It is simply seen that every finite-dimensional or finite-codimensional subspace of X is always an almost invariant under every operator on X. So, it is reasonable that the study of almost invariant subspaces restricted only to *half-spaces*, which are subspaces with both infinite dimension and infinite codimension in X. In contrast to the invariant subspace problem, the existence of almost invariant half-space has been proven for all bounded operators on reflexive Banach spaces [4], and also for most important classes of operators on Banach spaces, such as compact operators, conjugated operators, and quasinilpotent operators [5]; however, it is open in general setting. As a last effort, it has been shown that the set of all operators which have at least one almost invariant half-space with $defect \leq 1$ is norm-dense in the algebra of all of bounded operators [6].

In section 2, we propose an equivalent statement for almost invariant subspace. In section 3, we search some conditions for finding an invariant subspace different from an almost invariant subspace by a finite-dimensional subspace.

Throughout the paper, X is a complex Banach space and by $\mathcal{B}(X)$, we denote the set of all bounded linear operators on X. By a "subspace" of Banach space, we always mean a closed subspace. Also by an "operator" on a Banach space, we always mean a bounded linear operator.

2. AN EQUIVALENT RESTATEMENT FOR ALMOST INVARIANT SUBSPACE AND SOME RESULTS

The following result of Androulakis, Popov, et al [1] gives an equivalent statement for almost invariant subspace which relates this notion to invariant subspace of a finite rank perturbation of underlying operator.

Proposition 2.1. *Let $T \in \mathcal{B}(X)$ and Y be a subspace of X. Then Y is an almost invariant subspace under T if and only if Y is invariant under $T + F$ for some finite rank operator F.*

The finite rank operator F is not unique, and can be chosen as $rankF = d_{Y,T}$. Even, there exists a finite rank projection P such that $F = PT$. In fact, where $TY \subseteq Y + M$ and $\dim M = d_{Y,T}$, P is a projection with range M and kernel including Y.

Let Y be an almost invariant subspace under T. There exists a finite dimensional subspace M of X such that $TY \subseteq Y + M$. Thus $T^*(Y + M)^\perp \subseteq Y^\perp$. Since $(Y + M)^\perp = Y^\perp \cap M^\perp$, we conclude that $T^*(Y^\perp \cap M^\perp) \subseteq Y^\perp$ where $M^\perp$ is a finite-codimensional subspace of X^*. This inspires us to look at almost invariant subspace from a new point of view.

Proposition 2.2. *Let $T \in \mathcal{B}(X)$ and Y be a subspace of X. Then Y is an almost invariant subspace under T if and only if there exists a finite-codimentional subspace N of X such that $T(Y \cap N) \subseteq Y$. Moreover, if N is a subspace of the minimum codimension then $codimN = d_{Y,T}$.*

Proof. Given Y an almost invariant subspace under T. Let M be a subspace of the minimum dimension such that $TY \subseteq Y + M$. For the quotient map $q : X \longrightarrow X/Y$, $q(M)$ is a finite-dimensional subspace of X/Y. So there exists a subspace $L' \subseteq X/Y$ such that $L' \oplus q(M) = X/Y$. Since $Y \cap M = \{0\}$, by setting $L = q^{-1}(L')$, we have $M \oplus L = X$ and $L \supseteq Y$.

Now, consider the operator $\tilde{T} : X/T^{-1}L \longrightarrow X/L$ by $\tilde{T}(x + T^{-1}L) = Tx + L$. Since $\tilde{T}$ is one-to-one, $codimT^{-1}L = dimX/T^{-1}L \leq dimX/L < \infty$. Put $N = T^{-1}L$. So

$$T(Y \cap N) = T(Y \cap T^{-1}L) \subseteq TY \cap L \subseteq (Y + M) \cap L = Y.$$

Conversely, given N a finite-codimensional subspace of X such that $T(Y \cap N) \subseteq Y$. Let N be a subspace of the smallest codimension.

$Y + N = X$; because, if $Y + N \neq X$, then there exists $x \in X \setminus (Y + N)$. By setting $N' = N + span\{x\}$, we have $Y \cap N' = Y \cap N$ and so $T(Y \cap N') \subseteq Y$. However $codimN' < codimN$.

Since $Y + N = X$, we can find a finite-dimensional subspace M_1 of X that $M_1 \subseteq Y$ and $M_1 \oplus N = X$. Now

$$TY \subseteq T((Y \cap N) + M_1) \subseteq T(Y \cap N) + TM_1 \subseteq Y + TM_1.$$

For the second part of the proposition, we choose the subspace N of minimum codimension such that $N \supseteq T^{-1}L$, then

$$\begin{aligned} d_{Y,T} &\leq dimTM_1 \leq dimM_1 = codimN \leq codimT^{-1}L \leq codimL \\ &= dimM = d_{Y,T}. \end{aligned}$$

Hence, $codimN = d_{Y,T}$.
∎

The following proposition which is similar to lemma 2.1 in [3] is expressed according to new restatement of almost invariant subspace.

Proposition 2.3. *Let Y be a subspace of X, $\mathcal{C}$ a collection of bounded operators on X, and N a finite-codimensional subspace of X of minimum codimension such that $T(Y \cap N) \subseteq Y$, for all $T \in \mathcal{C}$. Then*

(i) $Y + N = X$;
(ii) *There exists a projection P with kernel N such that $(T - TP)Y \subseteq Y$ for all $T \in \mathcal{C}$;*
(iii) *For the projection P in (ii), $\cap_{T\in\mathcal{C}}(TP)^{-1}Y = N$;*
(iv) *If $\mathcal{C}$ consists of a single operator T then N can be chosen such that $N \supseteq T^{-1}Y$; and*
(v) *If $\mathcal{C}$ is an algebra of operators, and Y a half-space, then $\mathcal{C}$ admits an invariant half-space included in Y.*

Proof. (i) It can be shown as in proposition 2.2, for a single operator.

(ii) Since $Y + N = X$, there exists a finite-dimensional subspace M of X with $M \subseteq Y$ and $M \oplus N = X$. Consider the projection P on X with kernel N and range M. For any $y \in Y$, we have $y = y_M + y_N$ for some $y_M \in M$ and $y_N \in N$. $y_N \in Y$, because $y_M \in M \subseteq Y$ and so

$$(T - TP)y = T(I - P)y = Ty_N \in T(Y \cap N) \subseteq Y$$

for all $T \in \mathcal{C}$.

(iii) Obviously, $N \subseteq (TP)^{-1}(0) \subseteq (TP)^{-1}(Y)$ for all $T \in \mathcal{C}$. Conversely, let $x \notin N$. $x = x_M + x_N$, for some $x_M \in M$ and $x_N \in N$. Set $N' = N + span\{x_M\}$. Since $N' \supsetneq N$, there exists $T \in \mathcal{C}$ such that $T(Y \cap N') \nsubseteq Y$. Choose $y \in Y \cap N'$ with $Ty \notin Y$. $y = z + \alpha x_M$ for some $z \in N$ and scalar α. Also $z \in Y$ because $x_M \in M \subseteq Y$. Now $Ty = Tz + \alpha Tx_M$. But $Ty \notin Y$, and $Tz \in T(Y \cap N) \subseteq Y$. Hence, $TPx = Tx_M \notin Y$ which means $x \notin (TP)^{-1}Y$.

(iv) By subspace M introduced in (ii), $Y = (Y \cap N) + M$, and

$$TY \subseteq T(Y \cap N) + TM \subseteq Y + TM.$$

$Y \cap TM = \{0\}$, because $dimTM \leq dimM = d_{Y,T}$. As we saw in proof of proposition 2.2, there exists a finite-codimensional subspace N_1 of X with $N_1 \oplus TM = X$, and $N_1 \supseteq Y$. Now, $T(Y \cap T^{-1}N_1) \subseteq Y$ and $T^{-1}N_1 \supseteq T^{-1}Y$.

(v) Here, we show that $Y \cap N$ is invariant under $\mathcal{C}$.

Denote $Z = Y \cap (\cap_{T\in\mathcal{C}} T^{-1}Y)$. Since $\mathcal{C}$ is an algebra of operators, the subspace Z is invariant under $\mathcal{C}$. If P is the projection introduced in (ii), then by (iii), $P^{-1}(\cap_{T\in\mathcal{C}} T^{-1}Y) = N$. Hence,

$$Y \cap N = Y \cap P^{-1}(\cap_{T\in\mathcal{C}} T^{-1}Y) = Y \cap (\cap_{T\in\mathcal{C}} T^{-1}Y) = Z.$$

Since Y is a half-space and N of finite codimension, $Y \cap N$ is an invariant half-space under $\mathcal{C}$. ∎

If the subspace Y is almost invariant under T, then $Y^\perp$ is also almost invariant under T^*. In fact, there is a finite-rank operator F with $rankF = d_{Y,T}$ such that $(T + F)Y \subseteq Y$. So, $(T^* + F^*)Y^\perp \subseteq Y^\perp$ and $T^*Y^\perp \subseteq Y^\perp + F^*Y^\perp$. This follows that $Y^\perp$ is almost invariant under T^* and $d_{Y^\perp,T^*} \leq rankF^* = rankF = d_{Y,T}$. Now, we show that $d_{Y^\perp,T^*} = d_{Y,T}$.

Proposition 2.4. *Let $T \in \mathcal{B}(X)$, and Y be a subspace of X. If Y is almost invariant under T, then $Y^\perp$ is almost invariant under T^*. Moreover, $d_{Y^\perp,T^*} = d_{Y,T}$.*

Proof. According to above argument, it is sufficient to show that $d_{Y^\perp,T^*} \geq d_{Y,T}$. Let M be a finite-dimensionsl subspace such that $TY \subseteq Y + M$, and $dimM = d_{Y,T}$. At first, we show that $Y^\perp + M^\perp = X^*$.

Since $Y \cap M = \{0\}$, as we saw in the proof of proposition 2.2, there is a finite-codimension N such that $N \supseteq Y$ and $M \oplus N = X$. Let P be the projection with range M and kernel N. P^* is also a projection,

$$M^\perp = (PX)^\perp = \ker P^*,$$

and

$$N^\perp = (\ker P)^\perp = \overline{P^*X} = P^*X.$$

This means $M^\perp \oplus N^\perp = X^*$. Therefore:

$$Y^\perp + M^\perp \supseteq N^\perp + M^\perp = X^*.$$

Since $TY \subseteq Y + M$, we have $T^*(Y^\perp \cap M^\perp) \subseteq Y^\perp$. Let $N \supsetneq M^\perp$ be a subspace of X^* such that $T^*(Y^\perp \cap N) \subseteq Y^\perp$. Choose $x^* \in N \setminus M^\perp$ and $m \in M$ with $x^*(m) \neq 0$. Since $Y^\perp + M^\perp = X^*$, $x^* = y^* + m^*$ for some $y^* \in Y^\perp$ and $m^* \in M^\perp$, and also from $d_{Y,T} = \dim M$, there exist $y, y_1 \in Y$ such that $Ty = y_1 + m$. We have

$$(T^*y^*)y = y^*(Ty) = y^*(y_1 + m) = y^*(m) = x^*(m) \neq 0.$$

Therefore, $Ty^* \notin Y^\perp$. But $y^* = x^* - m^* \in Y^\perp \cap N$, that is a contradiction. It results that $M^\perp$ is the subspace of minimum codimension such that $T^*(Y^\perp \cap M^\perp) \subseteq Y^\perp$. By proposition 2.2,

$$d_{Y^\perp,T^*} = codimM^\perp = \dim M = d_{Y,T}.$$

∎

3. INVARIANT SUBSPACES ALMOST EQUIVALENT TO ALMOST INVARIANT SUBSPACES

For a subspace Y of X, and an operator T on X, the subspace

$$V := cl(\cup_{n=1}^{\infty}(Y + TY + ... + T^n Y))$$

is invariant under T. In the case Y is invariant under T, V is equal to Y. But in the case Y is almost invariant under T, V enlarges, and will be the smallest invariant subspace including Y. In fact, if Z is an invariant subspace including Y, then $T^n Y \subseteq T^n Z \subseteq Z$, for all n, and so $V \subseteq Z$. In other words, T has a nontrivial invariant subspace including Y if and only if

$$cl(\cup_{n=1}^{\infty}(Y + TY + ... + T^n Y) \neq .X$$

Now, let M be a finite-dimensional subspace with minimum dimension such that $TY \subseteq Y + M$. By lemma 2.1 (iii) in [3], M can be chosen such that $M \subseteq TY$ and so

$$cl(\cup_{n=1}^{\infty}(Y + TY + ... + T^n Y)) = cl(\cup_{n=1}^{\infty}(Y + M + ... + T^{n-1} M)).$$

Also

$$Y \subseteq Y + M \subseteq Y + M + TM \subseteq ...$$

is an ascending chain of almost invariant subspaces. If this chain eventually stops, i.e. $T^{n+1} M \subseteq Y + M + ... + T^n M$, for some integer n, then

$$cl(\cup_{n=1}^{\infty}(Y + TY + ... + T^n Y)) = Y + M + TM + ... + T^n M$$

is the smallest invariant subspace under T including Y which differs from Y by a finite-dimensional subspace.

Similarly, by the new restatement of almost invariant subspace, we can construct the largest invariant subspace included in an almost invariant subspace.

For the subspace Y, and an operator T, the subspace $W := Y \cap (\cap_{n=1}^{\infty} T^{-n} Y)$ is invariant under T. If Y is invariant under T, then $W = Y$. But in the case Y is almost invariant under T, W will be the largest invariant subspace included in Y. Indeed, if Z is invariant subspace included in Y, then $Z \subseteq T^{-n} Z \subseteq T^{-n} Y$, for all n, and so $W \supseteq Z$. In other words, T has a nontrivial invariant subspace included in Y if and only if

$$Y \cap (\cap_{n=1}^{\infty} T^{-n} Y) \neq 0.$$

By proposition 2.2, there exists a finite-codimensional subspace N with minimum codimension such that $T(Y \cap N) \subseteq Y$. Also, by proposition 2.3 (iv), N can be chosen such that $N \supseteq T^{-1} Y$. Therefore

$$Y \cap (\cap_{n=1}^{\infty} T^{-n} Y) = Y \cap (\cap_{n=0}^{\infty} T^{-n} N).$$

Since

$$T(Y \cap N \cap ... \cap T^{-n} N) \subseteq Y \cap N \cap ... \cap T^{-(n-1)} N$$

and $T^{-n} N$ is of finite codimension,

$$Y \supseteq Y \cap N \supseteq Y \cap N \cap T^{-1} N \supseteq ...$$

is a descending chain of almost invariant subspaces included in Y. Also, if this chain eventually stops, i.e. $T^{-(n+1)} N \supseteq Y \cap N \cap ... \cap T^{-n} N$, for some n, then

$$Y \cap (\cap_{n=1}^{\infty} T^{-n} Y) = Y \cap N \cap ... \cap T^{-n} N$$

is the largest invariant subspace under T included in Y which differs from Y by a finite-dimensional subspace.

The above argument leads us to characterize the subspace of X with respect to finite dimensional subspaces.

For the subspaces Y_1 and Y_2, we say Y_1 is almost equivalent to Y_2 if there exist finite-dimensional subspaces M_1 and M_2, such that $Y_1 + M_1 = Y_2 + M_2$. This relationship is clearly an equivalent relation.

Let Y be an almost invariant subspace under T and Y_1 be almost equivalent to Y. If M and M_1 are two finite-dimensional subspaces such that $Y + M = Y_1 + M_1$, then

$$T^n Y_1 \subseteq T^n(Y + M) \subseteq T^n Y + T^n M \subseteq Y + N + T^n M \subseteq Y_1 + M_1 + N + T^n M$$

where $d_{Y,T^n} = \dim N$. So

$$d_{Y_1,T^n} \leq d_{Y,T^n} + \dim M_1 + \dim M.$$

This shows that Y_1 is also almost invariant under T. Similarly,

$$d_{Y,T^n} \leq d_{Y_1,T^n} + \dim M_1 + \dim M$$

and this motivates the following lemma.

Lemma 3.1. *Let Y_1 and Y_2 be almost invariant subspaces under operator T. If Y_1 and Y_2 are almost equivalent, then*

$$\sup_n |d_{Y_1,T^n} - d_{Y_2,T^n}| < \infty$$

For $T \in \mathcal{B}(X)$ and Y as a subspace of X, we denote:

$$D_T(Y) = Y \cap T^{-1}Y$$

$$U_T(Y) = Y + TY$$

By lemma 3.4 in [3], if Y is almost invariant under T then $D_T(Y)$ and $U_T(Y)$ are also almost invariant under T, and $d_{Y,T} = \dim Y/D_T(Y) = \dim U_T(Y)/Y$.

Popov Show that if Y is a half-space of X, almost invariant under T such that $d_{D_T^k(Y),T} \geq d_{Y,T}$, and $d_{U_T^k(Y),T} \geq d_{Y,T}$ for all k, then $d_{Y,T^m} \geq m$ for all m [3].

We modify Popov's proof to obtain the following proposition which is an extension of this lemma having sufficient condition. In fact, If Y is an almost invariant subspace under operator T, then $d_{Y,T^n} \leq n d_{Y,T}$, for all n. The next proposition states what happens if $d_{Y,T^n} = n d_{Y,T}$, for all n.

Proposition 3.2. *Let Y be an almost invariant subspace under operator $T \in \mathcal{B}(X)$. There exists a subspace Z almost equivalent to Y with $d_{Z,T} < d_{Y,T}$ if and only if $d_{Y,T^n} < n d_{Y,T}$, for some n.*

Proof. Suppose that Z is a subspace almost equivalent to Y with $d_{Z,T} < d_{Y,T}$. By lemma 3.1, there is an integer $k > 0$ such that $|d_{Y,T^n} - d_{Z,T^n}| < k$, for all n. So

$$d_{Y,T^n} < d_{Z,T^n} + k \leq n d_{Z,T} + k.$$

Since $d_{Z,T} < d_{Y,T}$, by setting n large enough, we have $d_{Y,T^n} < n d_{Y,T}$.

Conversely, suppose that for every subspace Z almost equivalent to Y, we have $d_{Z,T} \geq d_{Y,T}$. Let $d_{Y,T} = m$ and $\{e_i\}_{i=1}^m$ be linearly independent vectors such that $TY \subseteq Y \oplus span\{e_i\}_{i=1}^m$. There exist linearly independent vectors $\{e_i^1\}_{i=1}^m \subseteq Y$, and vectors $\{y_i^0\}_{i=1}^m \subseteq Y$, that $Te_i^1 = y_i^0 + e_i$. It is clear that $D_T(Y) \cap span\{e_i^1\}_{i=1}^m = \{0\}$ and $\dim Y/D_T(Y) = d_{Y,T} = m$. So

$$T(D_T(Y)) \subseteq Y = D_T(Y) \oplus span\{e_i^1\}_{i=1}^m.$$

Since $D_T^k(Y)$ is almost equivalent to Y for all k, $d_{D_T^k(Y),T} \geq d_{Y,T} = m$ and for each k, we can obtain linearly independent vectors $\{e_i^k\}_{i=1}^m \subseteq D_T^{k-1}(Y)$, and vectors $\{y_i^{k-1}\}_{i=1}^m \subseteq D_T^{k-1}(Y)$, that $Te_i^k = y_i^{k-1} + e_i^{k-1}$. Thus

$$T^k e_i^k = T^{k-1} y_i^{k-1} + ... + T y_i^1 + y_i^0 + e_i.$$

So, for each $1 \leq i \leq m$ and $k \geq 1$ there exists $y_{i,k} \in Y$ such that $T^k e_i^k = y_{i,k} + e_i$.

Now, we prove by induction

$$U_T^{k+1}(Y) = Y \oplus span\{e_i\}_{i=1}^m \oplus ... \oplus span\{T^k e_i\}_{i=1}^m.$$

It's obvious where $k = 0$. Suppose this is true for $k \geq 1$. Denote $M = span\{e_i\}_{i=1}^m$. We have

$$\begin{aligned} U_T^{k+1}(Y) &= U_T(U_T^k(Y)) = U_T^k(Y) + T(U_T^k(Y)) \\ &= Y \oplus M \oplus ... \oplus T^{k-1}M + (TY + TM + ... + T^k M) \\ &= Y \oplus M \oplus ... \oplus T^{k-1}M + T^k M = U_T^k(Y) + T^k M. \end{aligned}$$

Since $U_T^k(Y)$ is almost equivalent to Y, $d_{U_T^k(Y),T} \geq d_{Y,T} = m$. On the other hand,

$$T(U_T^k(Y)) \subseteq U_T^{k+1}(Y) = U_T^k(Y) + T^k M.$$

So, $\dim T^k M = m$, and $U_T^{k+1}(Y) = U_T^k(Y) \oplus T^k M$.

It means that the vectors $\{T^k e_i\}_{i=1}^m \subseteq U_T^{k+1}(Y)$ are linearly independent and $span\{T^k e_i\}_{i=1}^m \cap U_T^k(Y) = \{0\}$, for all k.

Given integers $n \geq 1$ and $1 \leq k \leq n$, we have

$$T^n e_i^k = T^{n-k} T^k e_i^k = T^{n-k}(y_{i,k} + e_i) = T^{n-k} y_{i,k} + T^{n-k} e_i.$$

Since $T^{n-k} y_{i,k} \in U_T^{n-k}(Y)$, the vectors $\{T^{n-k} e_i\}_{i=1}^m \subseteq U_T^{n-k+1}(Y)$ are linearly independent, and $span\{T^{n-k} e_i\}_{i=1}^m \cap U_T^{n-k}(Y) = \{0\}$, so the vectors $\{T^n e_i^k\}_{i=1}^m \subseteq U_T^{n-k+1}(Y)$ are linearly independent, and $span\{T^n e_i^k\}_{i=1}^m \cap U_T^{n-k}(Y) = \{0\}$, for $1 \leq k \leq n$.

It follows that $T^n Y$ contains $n d_{Y,T}$ vectors $\{T^n e_1^k, ..., T^n e_m^k\}_{k=1}^n$ which are linearly independent and $span\{T^n e_1^k, ..., T^n e_m^k\}_{k=1}^n \cap Y = \{0\}$. Therefore, $d_{Y,T^n} \geq n d_{Y,T}$. ∎

The following proposition gives a necessary and sufficient condition for existence of invariant subspaces almost equivalent to almost invariant subspaces.

Proposition 3.3. *Let $T \in \mathcal{B}(X)$ and Y be a subspace of X. T has an invariant subspace almost equivalent to Y if and only if*

$$\sup_m d_{Y,T^m} < \infty.$$

Proof. Given Z as an invariant subspace under T, almost equivalent to Y. Let M_1 and M_2 be two finite-dimensional subspaces such that $Y + M_1 = Z + M_2$. We have

$$T^m Y \subseteq T^m(Z + M_2) = T^m Z + T^m M_2 \subseteq Z + T^m M_2 \subseteq Y + M_1 + T^m M_2$$

So, $d_{Y,T^m} \leq \dim M_1 + \dim M_2$, for all m.

Conversely, suppose that $\sup_m d_{Y,T^m} < \infty$. Then $d_{Y,T^n} < n d_{Y,T}$ for some n. By proposition 3.2, there exists a subspace Y_1 almost equivalent to Y such that $d_{Y_1,T} < d_{Y,T} < \infty$, and by lemma 3.1, $\sup_m d_{Y_1,T^m} < \infty$. By replacing Y_1 with Y, we get a subspace Y_2 almost equivalent to Y_1 such that $d_{Y_2,T} < d_{Y_1,T} < d_{Y,T} < \infty$, and $\sup_m d_{Y_2,T^m} < \infty$. Finally, after the finite steps, we obtain the subspace Z almost equivalent to Y such that $d_{Z,T} = 0$, and Z is invariant under T. ∎

Proposition 3.4. *For $T \in \mathcal{B}(X)$ and subspace Y of X, we have*

$$d_{Y,T^2} \geq d_{D_T(Y),T} + d_{U_T(Y),T}$$

Proof. Let $d_{Y,T} = n$, and $\{e_i\}_{i=1}^n$ be linearly independent vectors such that $TY \subseteq Y + span\{e_i\}_{i=1}^n$. As in the proof of proposition 3.2, there are linearly independent vectors $\{z_i\}_{i=1}^n \subseteq Y$, and vectors $\{y_i\}_{i=1}^n \subseteq Y$ such that $Tz_i = y_i + e_i$, for $1 \leq i \leq n$, and

$$Y = D_T(Y) \oplus span\{z_i\}_{i=1}^n.$$

Suppose that $d_{D_T(Y),T} = k$. We can find linearly independent vectors $w_1, ..., w_k \in span\{z_i\}_{i=1}^n$ such that

$$T(D_T(Y)) \subseteq D_T(Y) \oplus span\{w_i\}_{i=1}^k.$$

We can also choose linearly independent vectors $\{w_i^1\}_{i=1}^k \subseteq D_T(Y)$, and vectors $\{y_i^1\}_{i=1}^k \subseteq D_T(Y)$ such that $Tw_i^1 = y_i^1 + w_i$, for $1 \leq i \leq k$. So

$$T^2 w_i^1 = Ty_i^1 + Tw_i \in Y + span\{e_i\}_{i=1}^n$$

for $1 \leq i \leq k$. It is easily seen that the vectors $\{T^2 w_i^1\}_{i=1}^k \subseteq U_T(Y)$ are linearly independent, and $span\{T^2 w_i^1\}_{i=1}^k \cap Y = \{0\}$.

On the other hand,

$$T(U_T(Y)) \subseteq U_T(Y) + span\{Te_i\}_{i=1}^n.$$

Suppose that $d_{U_T(Y),T} = l$, and $\{u_i\}_{i=1}^l \subseteq span\{Te_i\}_{i=1}^n$ are linearly independent vectors such that

$$T(U_T(Y)) \subseteq U_T(Y) \oplus span\{u_i\}_{i=1}^l.$$

Here, we can choose linearly independent vectors $\{v_i\}_{i=1}^l \subseteq U_T(Y)$, and vectors $\{y_i'\}_{i=1}^l \subseteq U_T(Y)$ such that $Tv_i = y_i' + u_i$, for $1 \leq i \leq l$. Since $v_i \in U_T(Y)$, there exist $v_i^1, y_i'' \in Y$ such that $Tv_i^1 = y_i'' + v_i$, for $1 \leq i \leq l$. Also

$$T^2 v_i^1 = Ty_i'' + Tv_i = Ty_i'' + y_i' + u_i \in U_T(Y) + span\{Te_i\}_{i=1}^n$$

for $1 \leq i \leq l$. It is clear that the vectors $\{T^2 v_i^1\}_{i=1}^l$ are linearly independent and $span\{T^2 v_i^1\}_{i=1}^l \cap U_T(Y) = \{0\}$.

This follows that $T^2 Y$ contains $k + l$ linearly independent vectors $\{T^2 w_i^1\}_{i=1}^k$, and $\{T^2 v_i^1\}_{i=1}^l$ such that no non-zero linear combination of these vectors belongs to Y. Hence, $d_{Y,T^2} \geq k + l = d_{D_T(Y),T} + d_{U_T(Y),T}$. ∎

Continuing the inequality of the previous proposition, the following inequalities are obtained.

$$d_{Y,T^4} \geq d_{D_T D_T^2(Y),T} + d_{U_T D_T^2(Y),T} + d_{D_T U_T^2(Y),T} + d_{U_T U_T^2(Y),T}$$

and in general

$$d_{Y,T^{2^n}} \geq \sum_{S^{(0)},...,S^{(n-1)} \in \{D,U\}} d_{S_T^{(0)} S_{T^2}^{(1)} S_{T^4}^{(2)} \dots S_{T^{2^{n-1}}}^{(n-1)}(Y),T} \tag{3.1}$$

This inequality motivates the next corollary.

Corollary 3.5. *Let $T \in \mathcal{B}(X)$ and Y be a half-space of X. If $d_{Y,T^k} < k$, for some k, then T has an invariant subspace almost equivalent to Y. In this case,* $\sup_m d_{Y,T^m} < \infty$.

Proof. Let $d_{Y,T} = m$ and $d_{Y,T^k} \leq k - 1$, for an integer $k > 0$. We choose an integer $n > 0$ such that $2^n > k^2 m$. Denote $r = [\frac{2^n}{k}]$ and $q = 2^n - rk$. It is clear that $r \geq km$ and $0 \leq q < k$. Since $d_{Y,TS} \leq d_{Y,T} + d_{Y,S}$, we have

$$\begin{aligned} d_{Y,T^{2^n}} &= d_{Y,T^q (T^k)^r} \leq d_{Y,T^q} + d_{Y,(T^k)^r} \leq qm + r(k-1) = qm + 2^n - q - r \\ &\leq qm + 2^n - r < km + 2^n - r \leq 2^n \end{aligned}$$

Therefore, $d_{Y,T^{2^n}} < 2^n$ and by the inequality (3.1), T admits an invariant subspace almost equivalent to Y. Also, by proposition 3.3, $\sup_m d_{Y,T^m} < \infty$. ∎

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