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THE FUNCTIONAL EQUATION WITH AN EXPONENTIAL POLYNOMIAL SOLUTION AND ITS STABILITY

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ABSTRACT. In this paper, we prove that the unique continuous solution of the functional equation

$$f(x+y) = a^{-p_n(0)+q_n(x,y)} f(x)f(y)$$

is

$$f(x) = a^{p_n(x)},$$

where $p_n(x)$ is a polynomial with degree n and

$$q_n(x,y) = \sum_{j=2}^{n} \left(\frac{p_n^{(j)}(0)}{j!} \sum_{i=1}^{j-1} \binom{j}{i} x^i y^{j-i} \right).$$

We also obtain the superstability and stability of the functional equation with the following forms, respectively:

$$\left| f(x+y) - a^{-p_n(0)+q_n(x,y)} f(x)f(y) \right| \le \delta$$

and

$$\left| \frac{f(x+y)}{a^{-p_n(0)+q_n(x,y)} f(x)f(y)} - 1 \right| \le \delta.$$

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1. INTRODUCTION

In 1940, S.M. Ulam gave a wide ranging talk in the Mathematical Club of the University of Wisconsin in which he discussed a number of important unsolved problems [8]. One of those was the question concerning the stability of homomorphisms :

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all x, $y \in G_1$, then there exists a homomorphism H : $G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In the next year, D.H. Hyers [5] answered Ulam's question for the case of the additive mapping on Banach spaces G_1, G_2. Thereafter, the result of Hyers has been generalized by Th. M. Rassias [7]. Since then, the stability problems of various functional equations have been investigated by many authors (see [1], [3], [4], [6]).

In particular, J. Baker, J. Lawrence and F. Zorzitto [2] introduced the stability of the exponential functional equation in the following form: if f satisfies the inequality $|f(x+y) - f(x)f(y)| \le \delta$, then either f is bounded or $f(x+y) = f(x)f(y)$. This type is frequently referred as *superstability*.

Throughout this paper, we let $a \ge 1$ and $p_n(x) = t_0 + t_1 x + \cdots + t_n x^n$ be a polynomial with degree n $(t_0, t_1, \cdots, t_n \in R)$, and for all $x, y \in R$, we let

$$q_n(x,y) = \sum_{j=2}^{n} \left(\frac{p_n^{(j)}(0)}{j!} \sum_{i=1}^{j-1} \binom{j}{i} x^i y^{j-i} \right).$$

In section 2, we solve the following generalized exponential functional equation:

$$f(x+y) = a^{-p_n(0)+q_n(x,y)} f(x)f(y). \tag{1}$$

We illustrate this equation further by using two examples.

(a) if $p_2(x) = x^2$, then $-p_2(0) = 0, p_2''(0) = 2$, and $g_2(x,y) = 2xy$. Thus, we obtain the following functional equation:

$$f(x+y) = a^{2xy} f(x)f(y),$$

and $f(x) = a^{x^2}$ is a solution of this equation.

(b) If $p_2(x) = t_0 + t_1 x + t_2 x^2$, then $-p_2(0) = -t_0, p_2''(0) = 2t_2$, and $g_2(x,y) = 2t_2 xy$. Thus, we have a functional equation:

$$f(x+y) = a^{-t_0 + 2t_2 xy} f(x)f(y),$$

and $f(x) = a^{t_0 + t_1 x + t_2 x^2}$ is a solution of this equation.

In section 3, we obtain the superstability of the functional equation (1). It means a generalization of the superstability of the exponential functional equation given by J. Baker et al. [2].

In section 4, we investigate an asymptotic stability of functional equation (1) in the sense of R. Ger [4].

2. SOLUTION OF FUNCTIONAL EQUATION (1)

In this section, we investigate a solution of the functional equation (1).

Theorem 2.1. *The continuous solution on R of the functional equation* (1) *is $a^{p_n(x)}$. In particular, if a condition with values $a^{p_n(1)}$ at 1 and $a^{p_n(-1)}$ at -1 is added, then it is unique.*

Proof. Let $f(x) = a^{p_n(x)}$ for all $x \in R$. Then, for all $x, y \in R$,

$$\begin{aligned} f(x+y) &= a^{p_n(x+y)} \\ &= a^{-p_n(0)+\sum_{j=2}^{n}\left(t_j \sum_{i=1}^{j-1}\binom{j}{i}x^i y^{j-i}\right)} f(x)f(y) \\ &= a^{-p_n(0)+q_n(x,y)} f(x)f(y). \end{aligned}$$

Thus, $f(x)$ is a solution of the functional equation (1).

Suppose that g is an another solution of the functional equation (1) and that $g(1) = a^{p_n(1)}$ and $g(-1) = a^{p_n(-1)}$. Let

$$S = \{x \in R | g(x) = a^{p_n(x)}\}.$$

Then $-1, 1 \in S$ and $x + y \in S$ for all $x, y \in S$, because

$$\begin{aligned} g(x+y) &= a^{-p_n(0)+q_n(x,y)} g(x)g(y) \\ &= a^{t_0+t_1(x+y)+t_2(x^2+2xy+y^2)+\cdots+t_n\sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i}} \\ &= a^{p_n(x+y)}. \end{aligned}$$

Hence, $n \in S$ for every integer n. Since

$$\begin{aligned} g\left(\frac{1}{2}\right)^2 &= g\left(\frac{1}{2}+\frac{1}{2}\right) a^{p_n(0)-q_n(\frac{1}{2},\frac{1}{2})} \\ &= a^{t_0+t_1+\cdots+t_n+t_0-\left(t_2\binom{2}{1}(\frac{1}{2})^2+\cdots+t_n\sum_{i=1}^{n-1}\binom{n}{i}(\frac{1}{2})^n\right)} \\ &= a^{2t_0+t_1+t_2(1-\frac{1}{2})+\cdots+t_n\left(1-\left(\left(\frac{1}{2}+\frac{1}{2}\right)^n-\frac{1}{2^{n-1}}\right)\right)} \\ &= a^{2t_0+t_1+\frac{t_2}{2}+\cdots+\frac{t_n}{2^{n-1}}}, \end{aligned}$$

we have

$$g\left(\frac{1}{2}\right) = a^{t_0+\frac{t_1}{2}+\frac{t_2}{2^2}+\cdots+\frac{t_n}{2^n}},$$

hence $\frac{1}{2} \in S$. If $\frac{1}{2^m} \in S$ for some nonnegative integer m, then $g(\frac{1}{2^m}) = a^{p(\frac{1}{2^m})}$, and

$$\begin{aligned} g\left(\frac{1}{2^{m+1}}\right)^2 &= g\left(\frac{1}{2^m}\right) a^{p_n(0)-q_n(\frac{1}{2^{m+1}},\frac{1}{2^{m+1}})} \\ &= a^{t_0+t_1(\frac{1}{2^m})+\cdots+t_n(\frac{1}{2^m})^n+t_0-\left(t_2\binom{2}{1}(\frac{1}{2^{m+1}})^2+\cdots+t_n\sum_{i=1}^{n-1}\binom{n}{i}(\frac{1}{2^{m+1}})^n\right)} \\ &= a^{2t_0+t_1(\frac{1}{2^m})+t_2((\frac{1}{2^m})^2-2(\frac{1}{2^{m+1}})^2)+\cdots+t_n\left((\frac{1}{2^m})^n-\left(\left(\frac{1}{2^{m+1}}+\frac{1}{2^{m+1}}\right)^n-2(\frac{1}{2^{m+1}})^n\right)\right)} \\ &= a^{2t_0+2t_1(\frac{1}{2^{m+1}})+2t_2(\frac{1}{2^{m+1}})^2+\cdots+2t_n(\frac{1}{2^{m+1}})^n}. \end{aligned}$$

Therefore, $g(\frac{1}{2^{m+1}}) = a^{p(\frac{1}{2^{m+1}})}$. Hence $\frac{1}{2^{m+1}} \in S$. By induction, we have $\frac{1}{2^n} \in S$ for every nonnegative integer n, and similarly, $-\frac{1}{2^n} \in S$. Note that for every nonnegative integer m, $m = a_0 2^0 + a_1 2^1 + \cdots + a_k 2^k$, where $a_i = 0$ or 1 for each $i = 0, 1, \cdots, k$. Then we have

$$\frac{m}{2^n} = \begin{cases} \frac{a_0}{2^n} + \frac{a_1}{2^{n-1}} + \cdots + a_n + a_{n+1}2^0 + \cdots + a_k 2^{k-n} & \text{if } n \le k, \\ \frac{a_0}{2^n} + \frac{a_1}{2^{n-1}} + \cdots + \frac{a_k}{2^{n-k}} & \text{if } n > k. \end{cases}$$

In any case, we have $\frac{m}{2^n} \in S$, and similarly, $-\frac{m}{2^n} \in S$ for every nonnegative integers m and n. Thus, $\bar{S} = R$. For every $r \in R$ and a given integer sequence $\{k\}$, there exists a sequence $\{s_k\}$ in S such that

$$|s_k - r| \leq \frac{1}{k}.$$

Since g is continuous,

$$g(r) = \lim_{k\to\infty} g(s_k) = \lim_{k\to\infty} a^{p_n(s_k)} = a^{p_n(r)} = f(r)$$

for every $r \in R$. Therefore, f is unique. ∎

3. SUPERSTABILITY OF FUNCTIONAL EQUATION (1)

In this section, we let $D = R$ or $(0, \infty)$. From the following theorem, we know that the functional equation (1) has a superstability such as the result of J. Baker et al. [2].

Theorem 3.1. *Let $\delta \geq 0$, and let $q_n(x,y) \geq p_n(0)$ for all $x, y \in D$. If f is an unbounded functional on D (in particular, $|f(m)| \geq max\{2, 2\sqrt{\delta}\}$ for some positive integer m in D) and satisfies the inequality*

$$\left|f(x+y) - a^{-p_n(0)+q_n(x,y)} f(x)f(y)\right| \leq \delta \tag{2}$$

for all $x, y \in D$, then

$$f(x+y) = a^{-p_n(0)+q_n(x,y)} f(x)f(y)$$

for all $x, y \in D$.

Proof. If we replace x and y by m in (2) simultaneously, we obtain

$$\left|f(2m) - a^{-p_n(0)+q_n(m,m)} f(m)^2\right| \leq \delta.$$

An induction argument implies that for all $m \geq 2$,

$$\begin{aligned}
&\left|f(km) - \prod_{i=1}^{k-1} a^{-p_n(0)+q_n(m,im)} f(m)^k\right| \\
&\leq \delta\Big(1 + |f(m)|a^{-p_n(0)+q_n(m,(k-1)m)} \\
&+ |f(m)|^2 a^{-p_n(0)+q_n(m,(k-1)m)} \cdot a^{-p_n(0)+q_n(m,(k-2)m)} \\
&+ \cdots + |f(m)|^{n-2} \prod_{i=1}^{k-2} a^{-p_n(0)+q_n(m,(k-i)m)}\Big) \\
&= \delta\left(1 + \sum_{i=1}^{k-2}\left(|f(m)|^i \prod_{j=1}^{i} a^{-p_n(0)+q_n(m,(k-i)m)}\right)\right). \qquad (3)
\end{aligned}$$

Indeed, if inequality (3) holds, we have

$$\begin{aligned}&\Big|f((k+1)m)-\prod_{i=1}^{k}a^{-p_n(0)+q_n(m,im)}f(m)^{k+1}\Big|\\&\le\Big|f((k+1)m)-a^{-p_n(0)+q_n(m,km)}f(km)f(m)\Big|\\&+\Big|f(km)-\prod_{i=1}^{k-1}a^{-p_n(0)+q_n(m,im)}f(m)^k\Big|\Big|f(m)\Big|a^{-p_n(0)+q_n(m,km)}\\&=\delta\Bigg(1+\sum_{i=1}^{k-1}\Big(|f(m)|^i\prod_{j=1}^{i}a^{-p_n(0)+q_n(m,(k+1-i)m)}\Big)\Bigg).\end{aligned}$$

for all $n\ge 2$. By (3), we obtain

$$\begin{aligned}&\left|\frac{f(km)}{\prod_{i=1}^{k-1}a^{-p_n(0)+q_n(m,im)}f(m)^k}-1\right|\\&\le\Big(\frac{1}{|f(m)|^k}+\frac{1}{|f(m)|^{k-1}}+\cdots+\frac{1}{|f(m)|^2}\Big)\delta\\&\le\frac{1}{|f(m)|^2}\Big(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\Big)\delta=\frac{2\delta}{|f(m)|^2}\le\frac{1}{2}\end{aligned}$$

for all positive integer k. Since $|f(m)|^k\to\infty$ as $k\to\infty$, and

$$\prod_{i=1}^{k-1}a^{-p_n(0)+q_n(m,im)}\ge 1$$

for all $i=1,2,\cdots,k-1$, we obtain

$$f(km)\to\infty\ \text{ as }\ k\to\infty.$$

Note that for all $n\ge 2$ and for all $x,y,z\in D$, we have

$$q_n(y,z)+q_n(x,y+z)-q_n(z,x+y)=q_n(x,y) \tag{4}$$

because

$$q_n(x,y)=\sum_{j=2}^{n}t_j\Big(\sum_{i=1}^{j-i}\binom{j}{i}x^iy^{j-i}\Big)=\sum_{j=2}^{n}t_j\Big((x+y)^j-x^j-y^j\Big).$$

By (2) and (4), we have

$$\begin{aligned}&\Big|f(km)\Big|\Big|f(x+y)-a^{-p_n(0)+q_n(x,y)}f(x)f(y)\Big|\\&\le\Big|a^{-p_n(0)+q_n(km,x+y)}f(km)f(x+y)-f(km+x+y)\Big|\frac{1}{a^{-p_n(0)+q_n(km,x+y)}}\\&+\Big|f(km+x+y)-a^{-p_n(0)+q_n(x,y+km)}f(x)f(y+km)\Big|\frac{1}{a^{-p_n(0)+q_n(km,x+y)}}\\&+\Big|f(y+km)-a^{-p_n(0)+q_n(y,km)}f(y)f(km)\Big|\Big|f(x)\Big|\frac{a^{q_n(x,y)}}{a^{q_n(y,km)}}\end{aligned}$$

for all $x, y \in D$. Therefore,

$$\begin{aligned}&\left|f(x+y) - a^{-p_n(0)+q_n(x,y)} f(x)f(y)\right| \\ &\leq \frac{2\delta}{f(km)a^{-p_n(0)+q_n(y,km)}} + \frac{\delta|f(x)|a^{-p_n(0)+q_n(x,y)}}{|f(km)|a^{-p_n(0)+q_n(y,km)}} \to 0\end{aligned}$$

as $k \to \infty$. Thus, it follows that

$$f(x+y) = a^{-p_n(0)+q_n(x,y)} f(x)f(y)$$

for any $x, y \in D$. ∎

Corollary 3.2. *Let $\delta \geq 0$, $a \geq 1$ and let $p_n(x) = t_0 + t_1 x + \cdots + t_n x^n$ be a polynomial with degree n with $t_i \geq 0$ for each $i = 1, 2, \cdots, n$ and $t_0 \leq 0$. If f is an unbounded functional on $(0, \infty)$ (in particular, $|f(m)| \geq max\{2, 2\sqrt{\delta}\}$ for some positive integer m) and satisfies the inequality*

$$\left|f(x+y) - a^{-p_n(0)+q_n(x,y)} f(x)f(y)\right| \leq \delta$$

for all $x, y \in (0, \infty)$, then

$$f(x+y) = a^{-p_n(0)+q_n(x,y)} f(x)f(y)$$

for all $x, y \in (0, \infty)$.

Proof. Since $p_n(0) = t_0 \leq 0$ and $t_i \geq 0 (i = 1, 2, \cdots, n)$,

$$q_n(x, y) \geq 0 \geq p_n(0)$$

for all $x, y \in (0, \infty)$. From Theorem 3.1 with $D = (0, \infty)$, the required result is established. ∎

Example 3.1. *Let $\delta \geq 0$ and $a \geq 1$. If f is an unbounded functional on $(0, \infty)$ (in particular, $|f(m)| \geq max\{2, 2\sqrt{\delta}\}$ for some positive integer m) and satisfies the inequality*

$$|f(x+y) - a^{xy} f(x)f(y)| \leq \delta$$

for all $x, y \in (0, \infty)$, then

$$f(x+y) = a^{xy} f(x)f(y)$$

for all $x, y \in (0, \infty)$, because $p_2(x) = \frac{x^2}{2}, p_2(0) = 0$, and $q_2(x, y) = xy \geq 0$.

Example 3.2. *Let $\delta \geq 0$ and $a \geq 1$. If f is an unbounded functional on $(0, \infty)$ (in particular, $|f(m)| \geq max\{2, 2\sqrt{\delta}\}$ for some positive integer m) and satisfies the inequality*

$$\left|f(x+y) - a^{1+6xy} f(x)f(y)\right| \leq \delta$$

for all $x, y \in (0, \infty)$, then

$$f(x+y) = a^{1+6xy} f(x)f(y)$$

for all $x, y \in (0, \infty)$, because $p_2(x) = -1+2x+3x^2, p_2(0) = -1$, and $q_2(x, y) = 6xy \geq p_2(0)$.

4. STABILITY OF FUNCTIONAL EQUATION (1)

R. Ger [4] introduced a stability for the exponential equation in the asymptotic type :

$$\left|\frac{f(x+y)}{f(x)f(y)} - 1\right| \leq \delta.$$

Now we prove an asymptotic stability of the functional equation (1) in the sense of Ger.

Theorem 4.1. *Let $0 < \delta < 1$ and $a \geq 1$ be given. If a function $f : R \to (0, \infty)$ satisfies the inequality*

$$\left| \frac{f(x+y)}{a^{-p_n(0)+q_n(x,y)} f(x) f(y)} - 1 \right| \leq \delta \tag{5}$$

for all $x, y \in R$, then there exists an unique function $F : R \to (0, \infty)$ such that

$$F(x+y) = a^{-p_n(0)+q_n(x,y)} F(x) F(y)$$

for all $x, y \in R$ and

$$\left| \frac{F(x)}{f(x)} - 1 \right| \leq \delta$$

for all $x \in R$.

Proof. If we define a function $G : R \to R$ by

$$G(x) = \ln f(x)$$

for all $x \in R$, then the equality (5) may be transformed into

$$|G(x+y) - \ln a^{-p_n(0)+q_n(x,y)} - G(x) - G(y)| \leq \ln(1-\delta) := \theta$$

for all $x, y \in R$. Replacing y by x and dividing by 2, we get

$$\left| \frac{G(2x)}{2} - \ln \left(a^{-p_n(0)+q_n(x,x)} \right)^{\frac{1}{2}} - G(x) \right| \leq \frac{\theta}{2} \tag{6}$$

for all $x \in R$. We apply induction on k to prove that

$$\left| \frac{G(2^k x)}{2^k} - \ln \prod_{i=0}^{k-1} \left(a^{-p_n(0)+q_n(2^i x, 2^i x)} \right)^{\frac{1}{2^{i+1}}} - G(x) \right| \leq \theta \sum_{i=1}^{k} \frac{1}{2^i} \tag{7}$$

for all $x \in R$. Based on (6), the inequality holds for $k = 1$. Suppose that inequality (7) holds true for some $k > 1$. Then, both (6) and (7) imply

$$\begin{aligned}
&\left| \frac{G(2^{k+1} x)}{2^{k+1}} - \ln \prod_{i=0}^{k} \left(a^{-p_n(0)+q_n(2^i x, 2^i x)} \right)^{\frac{1}{2^{i+1}}} - G(x) \right| \\
&\leq \left| \frac{G(2^k 2x)}{2^{k+1}} - \frac{1}{2} \ln \prod_{i=0}^{k-1} \left(a^{-p_n(0)+q_n(2^i 2x, 2^i 2x)} \right)^{\frac{1}{2^{i+1}}} - \frac{G(2x)}{2} \right| \\
&+ \left| \frac{G(2x)}{2} - \ln (a^{-p_n(0)+q_n(x,x)})^{\frac{1}{2}} - G(x) \right| \\
&\leq \frac{1}{2} (\theta \sum_{i=1}^{k} \frac{1}{2^i}) + \frac{\theta}{2} = \theta \sum_{i=1}^{k+1} \frac{1}{2^i},
\end{aligned}$$

which ends the proof of (7). For any $x \in R$ and for every positive integer k, we define

$$T_k(x) = \frac{G(2^k x)}{2^k} - \ln \prod_{i=0}^{k-1} \left(a^{-p_n(0)+q_n(2^i x, 2^i x)} \right)^{\frac{1}{2^{i+1}}}.$$

Let $k, m > 0$ be integers with $k > m$. Then it follows from (7) that

$$
\begin{aligned}
&|T_k(x) - T_m(x)| \\
&= \frac{1}{2^m}\left|\frac{G(2^{k-m}(2^m x))}{2^{k-m}} - \ln\prod_{i=m}^{k-1}\left(a^{-p_n(0)+q_n(2^i x, 2^i x)}\right)^{\frac{1}{2^{i+1-m}}} - G(2^m x)\right| \\
&= \frac{1}{2^m}\left|\frac{G(2^{k-m}(2^m x))}{2^{k-m}} - \ln\prod_{i=0}^{k-m-1}\left(a^{-p_n(0)+q_n(2^i 2^m x, 2^i 2^m x)}\right)^{\frac{1}{2^{i+1}}} - G(2^m x)\right| \\
&\le \frac{\theta}{2^m}\sum_{i=1}^{k-m}\frac{1}{2^i} = \theta\sum_{i=m+1}^{k}\frac{1}{2^i} \to 0
\end{aligned}
$$

as $m \to \infty$. Therefore, the sequence $\{T_k(x)\}$ is a Cauchy sequence, and we may define a function $L : R \to R$ by

$$L(x) := \lim_{n\to\infty} T_k(x)$$

and

$$F(x) := e^{L(x)} = \lim_{k\to\infty}\frac{f(2^k x)^{\frac{1}{2^k}}}{\prod_{i=0}^{k-1}\left(a^{-p_n(0)+q_n(2^i x, 2^i x)}\right)^{\frac{1}{2^{i+1}}}}$$

for all $x \in R$. Note that for all $n \ge 2$ and $x, y \in R$,

$$
\begin{aligned}
&q_n(x, x) + q_n(y, y) - q_n(x+y, x+y) \\
&= \sum_{j=2}^{n} t_j\Big(2^j x^i - 2x^j + 2^j y^i - 2y^j - (2x+2y)^j + 2(x+y)^j\Big) \\
&= \sum_{j=2}^{n} t_j\Big(\big(2(x+y)^j - 2x^j - 2y^j\big) - \big((2x+2y)^j - 2^j x^j - 2^j y^j\big)\Big) \\
&= 2q_n(x, y) - q_n(2x, 2y).
\end{aligned}
$$

Thus, for all $x, y \in R$, we have

$$
\begin{aligned}
&\frac{F(x+y)}{a^{-p_n(0)+q_n(x,y)}F(x)F(y)} \\
&= \lim_{k\to\infty}\frac{f(2^k x + 2^k y)^{\frac{1}{2^k}}\prod_{i=0}^{k-1}\left(a^{-p_n(0)+q_n(2^i x, 2^i x)+q_n(2^i y, 2^i y)}\right)^{\frac{1}{2^{i+1}}}}{a^{-p_n(0)+q_n(x,y)}\big(f(2^k x)f(2^k y)\big)^{\frac{1}{2^k}}\prod_{i=0}^{k-1}\left(a^{q_n(2^i(x+y), 2^i(x+y))}\right)^{\frac{1}{2^{i+1}}}} \\
&= \lim_{k\to\infty}\frac{f(2^k x + 2^k y)^{\frac{1}{2^k}}\prod_{i=0}^{k-1}\left(a^{-p_n(0)+2q_n(2^i x, 2^i y)-q_n(2^{i+1}x, 2^{i+1}y)}\right)^{\frac{1}{2^{i+1}}}}{a^{-p_n(0)+q_n(x,y)}\big(f(2^k x)f(2^k y)\big)^{\frac{1}{2^k}}} \\
&= \lim_{k\to\infty}\left(\frac{f(2^k x + 2^k y)}{f(2^k x)f(2^k y)}\right)^{\frac{1}{2^k}}\cdot\frac{a^{-p_n(0)+\frac{1}{2^k}p_n(0)+q_n(x,y)-\frac{1}{2^k}q_n(2^k x, 2^k y)}}{a^{-p_n(0)+q_n(x,y)}} \\
&= \lim_{k\to\infty}\left(\frac{f(2^k x + 2^k y)}{a^{-p_n(0)+q_n(2^k x, 2^k y)}f(2^k x)f(2^k y)}\right)^{\frac{1}{2^k}},
\end{aligned}
$$

and by (5) we obtain

$$(1-\delta)^{\frac{1}{2^k}} \le \left(\frac{f(2^k x+2^k y)}{a^{-p_n(0)+q_n(2^k x,2^k y)}f(2^k x)f(2^k y)}\right)^{\frac{1}{2^k}} \le (1+\delta)^{\frac{1}{2^k}}$$

for all $x, y \in R$ and for every positive integer k. Therefore, letting $k \to \infty$, we have

$$F(x+y) = a^{-p_n(0)+q_n(x,y)}F(x)F(y)$$

for all $x, y \in R$. It can be easily seen from (5) that

$$(1-\delta)^{\frac{1}{2^k}} \le \frac{f(2^k x)^{\frac{1}{2^k}}}{a^{-\frac{1}{2^k}p_n(0)+\frac{1}{2^k}q_n(2^{k-1}x,2^{k-1}x)}f(2^{k-1}x)^{\frac{1}{2^{k-1}}}} \le (1+\delta)^{\frac{1}{2^k}}$$

for all $x \in R$ and for all positive integer k. Also we have

$$\begin{aligned}
&\frac{f(2^k x)^{\frac{1}{2^k}}}{\prod_{i=0}^{k-1}\left(a^{-p_n(0)+q_n(2^i x,2^i y)}\right)^{\frac{1}{2^{i+1}}}}\\
&= \frac{f(2^k x)^{\frac{1}{2^k}}}{a^{-\frac{1}{2^k}p_n(0)+\frac{1}{2^k}q_n(2^{k-1}x,2^{k-2}x)}f(2^{k-1}x)^{\frac{1}{2^{k-1}}}}\\
&\quad\cdot\frac{f(2^{k-1}x)^{\frac{1}{2^{k-1}}}}{a^{-\frac{1}{2^{k-1}}p_n(0)+\frac{1}{2^{k-1}}q_n(2^{k-2}x,2^{k-2}x)}f(2^{k-1}x)^{\frac{1}{2^{k-1}}}}\\
&\quad\cdot\quad\cdots\quad\cdot\frac{f(2x)^{\frac{1}{2}}}{a^{-\frac{1}{2}p_n(0)+\frac{1}{2}q_n(x,x)}f(x)}.
\end{aligned}$$

for all $x \in R$ and for all positive integer k. Thus, we have

$$\begin{aligned}
(1-\delta)^{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^k}} &\le \frac{f(2^k x)^{\frac{1}{2^k}}}{\prod_{i=0}^{k-1}\left(a^{-p_n(0)+q_n(2^i x,2^i y)}\right)^{\frac{1}{2^{i+1}}}f(x)}\\
&\le (1+\delta)^{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^k}}
\end{aligned}$$

for all $x \in R$ and for all positive integer k, and so

$$1-\delta \le \frac{F(x)}{f(x)} \le 1+\delta$$

for all $x \in R$. To show that F is unique, let $W : R \to (0,\infty)$ be an another such function with

$$W(x+y) = a^{-p_n(0)+q_n(x,y)}W(x)W(y)$$

for all $x, y \in R$ and

$$1-\delta \le \frac{W(x)}{f(x)} \le 1+\delta$$

for all $x \in R$. Note that for all $x \in R$ and for all positive integer k,

$$\frac{F(2x)}{W(2x)} = \frac{F(x)^2}{W(x)^2}, \cdots, \frac{F(2^k x)}{W(2^k x)} = \frac{F(x)^{2^k}}{W(x)^{2^k}}.$$

Thus, we have

$$(1-\delta)^{\frac{1}{2^k}}(\frac{1}{1+\delta})^{\frac{1}{2^k}} \le \frac{F(x)}{W(x)} \le \Big(\frac{F(2^kx)}{f(2^kx)}\Big)^{\frac{1}{2^k}} \cdot \Big(\frac{f(2^kx)}{W(2^kx)}\Big)^{\frac{1}{2^k}}$$
$$\le (1+\delta)^{\frac{1}{2^k}}(\frac{1}{1-\delta})^{\frac{1}{2^k}}.$$

Letting $k \to \infty$, we have $F(x) = W(x)$ for all $x \in R$. ∎

By Theorem 3, we know that the following examples hold:

Example 4.1. *Let* $0 < \delta < 1$ *and* $a \ge 1$ *be given. If a function* $f : R \to (0, \infty)$ *satisfies the inequality*

$$\left|\frac{f(x+y)}{a^{-t_0+2xyt_2+(3x^2y+3xy^2)t_3} f(x)f(y)} - 1\right| \le \delta$$

for all $x, y \in R$*, then there exists an unique function* $F : R \to (0, \infty)$ *such that*

$$F(x+y) = a^{-t_0+2xyt_2+(3x^2y+3xy^2)t_3} F(x)F(y)$$

for all $x, y \in R$ *and*

$$\left|\frac{F(x)}{f(x)} - 1\right| \le \delta$$

for all $x \in R$*.*

Example 4.2. *Let* $0 < \delta < 1$ *and* $a \ge 1$ *be given. If a function* $f : R \to (0, \infty)$ *satisfies the inequality*

$$\left|\frac{f(x+y)}{a^{1+6xy} f(x)f(y)} - 1\right| \le \delta$$

for all $x, y \in R$*, then there exists an unique function* $F : R \to (0, \infty)$ *such that*

$$F(x+y) = a^{1+6xy} F(x)F(y)$$

for all $x, y \in R$ *and*

$$\left|\frac{F(x)}{f(x)} - 1\right| \le \delta$$

for all $x \in R$*.*

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