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ITERATIVE ALGORITHM FOR SPLIT GENERALIZED MIXED EQUILIBRIUM PROBLEM INVOLVING RELAXED MONOTONE MAPPINGS IN REAL HILBERT SPACES

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ABSTRACT. The main purpose of this paper is to introduce a certain class of split generalized mixed equilibrium problem involving relaxed monotone mappings. To solve our proposed problem, we introduce an iterative algorithm and obtain its strong convergence to a solution of the split generalized mixed equilibrium problems in Hilbert spaces. As special cases of the proposed problem, we studied the proximal split feasibility problem and variational inclusion problem.

Key words and phrases: Relaxed monotone mappings, split generalized mixed equilibrium problems, variational inclusion problems, variational inequality problems, maximal monotone mapping.

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1. INTRODUCTION

Let C be a nonempty closed and convex subset of a real Hilbert space H and $S : C \to C$ be any nonlinear mapping. Then, S is called *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\| \; \forall \; x, y \in C,$$

if $L = 1$, then S is called *nonexpansive*. A point $x \in C$ is called a *fixed point* of S if $Sx = x$. Throughout this paper, we shall denote the set of fixed points of S by $\mathcal{F}(S)$. A mapping $S : C \to C$ is said to be

(i) *monotone*, if

$$\langle Sx - Sy, x - y\rangle \geq 0, \; \forall x, y \in C,$$

(ii) μ*-strongly monotone*, if there exists a constant $\mu > 0$ such that

$$\langle Sx - Sy, x - y\rangle \geq \mu\|x - y\|^2, \; \forall x, y \in C,$$

(iii) μ*-inverse strongly monotone*, if there exists a constant $\mu > 0$ such that

$$\langle Sx - Sy, x - y\rangle \geq \mu\|Sx - Sy\|^2, \; \forall x, y \in C,$$

(iv) *firmly nonexpansive*, if

$$\langle Sx - Sy, x - y\rangle \geq \|Sx - Sy\|^2, \; \forall x, y \in C.$$

A mapping $T : C \to H$ is said to be relaxed $\eta - \alpha$ monotone (see [8]), if there exists a mapping $\eta : C \times C \to H$ and a function $\alpha : H \to \mathbb{R}$ positively homogeneous of degree p (i.e., $\alpha(tz) = t^p\alpha(z)$ for all $t > 0$ and $z \in H$, where $p > 1$) such that

$$\langle Tx - Ty, \eta(x, y)\rangle \geq \alpha(x - y) \; \forall x, y \in C.$$

In particular, if $\eta(x, y) = x - y, \; \forall x, y \in C$, T is called relaxed α-monotone. Furthermore, if $\eta(x, y) = x - y, \; \forall x, y \in C$ and $\alpha(z) = \mu||z||^p$, where $p > 1$ and $\mu > 0$ are constants, then T is called p-monotone [12, 23]. In fact, if $p = 2$, then T is called μ-strictly monotone (see [24]). Clearly, every monotone mapping is relaxed η-α monotone with $\eta(x, y) = x - y \; \forall x, y \in C$ and $\alpha = 0$. Thus, inverse strongly monotone mappings are relaxed η-α monotone. The following is an example of a relaxed η-α monotone mapping.

Example 1.1. [7] *Let* $H = \mathbb{R}^2$ *and* $C = [0, 1] \times [0, 1]$. *Define a mapping* $T : C \to H$ *by* $T(x_1, x_2) = (x_1, x_2) \; \forall(x_1, x_2) \in C$, $\alpha : H \to \mathbb{R}$ *by* $\alpha(x_1, x_2) = 3x_1^2 + 3x_2^2$ *and* $\eta : C \times C \to H$ *by* $\eta((x_1, x_2), (y_1, y_2)) = (4(x_1 - y_1), 4(x_2 - y_2)) \; \forall(x_1, x_2) \times (y_1, y_2) \in C \times C$. *Then,* T *is relaxed* η-α *monotone.*

Recall that a mapping $F : C \to C$ is said to be averaged nonexpansive if $\forall x, y \in C, \; F = (1 - \beta)I + \beta S$ holds for a nonexpansive operator $S : C \to C$ and $\beta \in (0, 1)$. In this case, we say that F is β-averaged. The term "averaged mapping" was coined by Biallon *et al.* [4]. Moreover, F is firmly nonexpansive if and only if F can be expressed as $F = \frac{1}{2}(I + S)$, where S is nonexpansive (see [20]). Thus, we make the following remark which can be easily verified (see, also [13, 14]).

Remark 1.1. In a Hilbert space, F is firmly nonexpansive if and only if it is averaged with $\beta = \frac{1}{2}$.

The metric projection P_C is a map defined on H onto C which assigns to each $x \in H$, the unique point in C, denoted by P_Cx such that

$$||x - P_Cx|| = \inf\{||x - y|| : \; y \in C\}.$$

It is well known that P_Cx is characterized by the inequality $\langle x - P_Cx, z - P_Cx\rangle \leq 0,\ \forall z \in C$ and P_C is a firmly nonexpansive mapping. Thus, P_C is nonexpansive. For more information on metric projections, see [10, 6].
The Equilibrium Problem (EP) (in the sense of Blum and Oettli [1]) is to find $x \in C$ such that

$$\phi(x, y) \geq 0\ \forall y \in C, \tag{1.1}$$

where $\phi : C \times C \to \mathbb{R}$ is a bifunction. We denote the solution set of EP (1.1) by $G(\phi)$. To solve the EP, the bifuncton ϕ is assumed to satisfy the following conditions:

(A1) $\phi(x, x) = 0$ for all $x \in C$;
(A2) ϕ is monotone; that is $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y \in C$, $\lim\limits_{t\to 0} \phi(tz + (1 - t)x, y) \leq \phi(x, y)$;
(A4) for all $x \in C$, $\phi(x, .)$ is convex and lower semicontinuous.

The Mixed Equilibrium Problem (MEP) is to find $x \in C$ such that

$$\phi(x, y) + \langle Tx, y - x\rangle + f(y) - f(x) \geq 0\ \forall y \in C, \tag{1.2}$$

where $\phi : C \times C \to \mathbb{R}$ is a bifunction, T is some nonlinear mapping and $f : C \to (-\infty, +\infty]$ is a proper convex and lower semi continuous function. The solution set of (1.2) is denoted by $G(\phi, T, f)$.
Equilibrium problems and mixed equilibrium problems are known to be one of the most successful tools in many fields such as physics, economics, engineering, computer science, among others for solving problems like linear and nonlinear programming, variational inequality problems, fixed point problems, optimization problems and others (for example, see [3, 9, 17, 18]). The MEP have been studied widely by many authors in the case where T is an inverse strongly monotone mapping (for example, see [3, 9] and the references therein). Since the introduction of the relaxed monotone mapping by Fang and Huang [8], authors are now beginning to study MEP for the case where T is a relaxed monotone mapping. For instance, Wang *et al.* [24] introduced the following iterative algorithm for solving MEP (in the case where $f = 0$) and fixed point problem for a nonexpansive mapping in Hilbert space:

$$\begin{cases} x_1 \in C \ chosen\ arbitrarily, \\ \phi(u_n, y) + \langle Tu_n, \eta(y, u_n)\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0\ \forall y \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)\beta_n Sx_n + (1 - \alpha_n)(1 - \beta_n)u_n, \\ C_n = \{z \in C : ||y_n - z|| \leq ||x_n - z||\}, \\ Q_n = \cap_{j=1}^n C_j, \\ x_{n+1} = P_{Q_n} x_1,\ n \geq 1, \end{cases} \tag{1.3}$$

where ϕ is a bifunction satisfying $(A1)$-$(A4)$, T is a relaxed η-α monotone mapping and $S : C \to C$ is nonexpansive. Under some conditions on the control sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$, they obtained strong convergence of Algorithm (1.3) to a solution of the mixed equilibrium problem (in which $f = 0$), which is also a fixed point of S.
Recently, Chen *et al.* [7] studied the MEP with the relaxed monotone mapping in uniformly convex and uniformly smooth Banach space. They proposed the following algorithm to approximate a common solution of the MEP and fixed point problem for quasi-ϕ nonexpansive mapping:

(1.4)
$$\begin{cases} x_1 = x \in C \textit{ is chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \\ u_n \in C \textit{ such that} \\ \phi(u_n, y) + \langle Au_n, \eta(y, u_n)\rangle + f(y) - f(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \ \forall y \in C, \\ C_n = \{z \in C : ||y_n - z|| \leq ||x_n - z||\}, \\ Q_n = \cap_{j=1}^{n} C_j, \\ x_{n+1} = P_{Q_n} x_1, n \geq 1, \end{cases}$$

where ϕ is a bifunction satisfying $(A1)$-$(A4)$, T is a relaxed η-α monotone mapping, $f : C \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semi continuous function and S is a quasi-ϕ nonexpansive mapping from C to C. Under some certain assumptions on the parameter sequences $\{\alpha_n\}$ and $\{r_n\}$, they obtained strong convergence of (1.4) to common solution of MEP and fixed point problem for S.
Motivated by the works of Wang *et al.* [24] and Chen *et al.* [7], we introduce and study the following Split Generalized Mixed Equilibrium Problem (SGMEP) which involves relaxed monotone mappings:

$$\text{Find } x \in C_1 \text{ such that } x \in G(\phi_1, T_1, f_1, F), \tag{1.5}$$

$$\text{and } Ax = y \in C_2 \text{ such that } y \in \left(G(\phi_2, T_2, f_2) \cap \mathcal{F}(S)\right), \tag{1.6}$$

where C_1 and C_2 are nonempty closed and convex subsets of H_1 and H_2 respectively, $A : C_1 \to C_2$ is a bounded linear mapping, $\phi_1 : C_1 \times C_1 \to \mathbb{R}$ and $\phi_2 : C_2 \times C_2 \to \mathbb{R}$ are bifunctions, $T_1 : C_1 \to C_1$ and $T_2 : C_2 \to C_2$ are relaxed η-α monotone mappings, $f_1 : C_1 \to (-\infty, +\infty]$ and $f_2 : C_2 \to (-\infty, +\infty]$ are proper convex and lower semicontinuous functions, $S : C_2 \to C_2$ is a nonlinear mapping and $F : C_1 \to C_1$ is a μ-inverse strongly monotone mapping. Throughout this paper, we denote by Γ, the solution set of SGMEP (1.5)-(1.6). If we consider SGMEP (1.5)-(1.6) separately, then we denote by $G(\phi_1, T_1, f_1, F)$ the solution set of the problem: Find $x \in C$ such that

$$\phi(x, z) + \langle Tx - \eta(z, x)\rangle + f(z) - f(x) + \langle Fx, z - x\rangle \geq 0 \ \forall z \in C,$$

and by $G(\phi_1, T_1, f_1)$ the solution set of the problem: Find $y \in C$ such that

$$\phi(y, z) + \langle Ty - \eta(z, y)\rangle + f(z) - f(y) + \langle Fy, z - y\rangle \geq 0 \ \forall z \in C.$$

Remark 1.2. We observe that, to prove strong convergence results for MEP and other related optimization problems, the CQ (modified Haugazeau) algorithms are often used. In some other cases (where algorithms other than the CQ algorithm are used), some compactness conditions are assumed on the operators under consideration, or the proof maybe divided into two cases which may result to a very long proof (see, for example [7, 13, 14, 15, 16, 21, 24, 25, 26] and the references therein). On this note, Shehu and Iyiola [22] in 2017, proposed the following modified proximal split feasibility iterative algorithm:

Algorithm 1.1. (1) *Given the initial points* $x_1, u \in H_1$
(2) *Set* $n = 1$ *and compute:*
(3) $y_n = \alpha_n u + (1-\alpha_n)x_n$
(4) $\Theta(y_n) = ||A^*(I - \textit{prox}_{\lambda g})Ay_n + (I - \textit{prox}_{\lambda f})y_n||$
(5) $z_n = y_n - \rho_n \frac{h(y_n)+l(y_n)}{\Theta^2(y_n)}\left(A^*(I - \textit{prox}_{\lambda g})Ay_n + (I - \textit{prox}_{\lambda f})y_n\right)$
(6) $x_{n+1} = (1-\beta_n)y_n + \beta_n z_n.$

(7) *If* $A^*(I - prox_{\lambda g})Ay_n = 0 = (I - prox_{\lambda f})y_n$ *and* $x_{n+1} = x_n$*, then stop, otherwise*
(8) *set* $n = n + 1$ *and repeat step (3)-(6),*

where $h(y_n) := \frac{1}{2}||(I - prox_{\lambda g})Ay_n||^2$*,* $l(y_n) := \frac{1}{2}||(I - prox_{\lambda f})y_n||^2$*, and the sequences* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\rho_n\}$ *satisfy the following conditions:*

(i) $\{\alpha_n\} \subset (0, 1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*, for all* $n \in \mathbb{N}$.
(iii) $\liminf_{n\to\infty} \rho_n(4 - \rho_n) > 0$.

Furthermore, Shehu and Iyiola [22] obtained strong convergence of Algorithm 1.1 to a solution of the following Proximal Split Feasibility Problem (PSFP): Find $x \in H_1$ such that

$$\min_{x\in H_1} \{f(x) + g(Ax)\}, \tag{1.7}$$

where $A : H_1 \to H_2$ is a bounded linear mapping, $f : H_1 \to (-\infty, +\infty]$ and $g : H_2 \to (-\infty, +\infty]$ are proper convex and lower semi-continuous functions.

Remark 1.3. As observed by Shehu and Iyiola [22], the termination test in the above algorithm (Algorithm 1.1) is justified by the fact that, if $A^*(I - \text{prox}_{\lambda g})Ay_n = 0 = (I - \text{prox}_{\lambda f})y_n$ and $x_{n+1} = x_n$, then x_n solves (1.7). This is because $A^*(I - \text{prox}_{\lambda g})Ay_n = 0 = (I - \text{prox}_{\lambda f})y_n$ implies that y_n is a solution of (1.7). Also, from Algorithm 1.1, $A^*(I - \text{prox}_{\lambda g})Ay_n = 0 = (I - \text{prox}_{\lambda f})y_n$ implies that $z_n = y_n$ and $x_{n+1} = y_n$. So that, if $x_{n+1} = x_n$, then we get that $x_n = y_n$ and hence, x_n is a solution of (1.7). Therefore, Algorithm 1.1 is well-defined.

Inspired by the above work of Shehu and Iyiola [22], we obtain strong convergence results for solving our proposed SGMEP (1.5)-(1.6) without using any of the methods mentioned in Remark 1.2, and the method of proof which we adopted appears to be more shorter and easier to read. Our results extends and improves the results of Wang *et al.* [24], Chen *et al.* [7], Shehu and Iyiola [22], and many other results in literature.

2. PRELIMINARIES

We state some useful results which will be needed in proving our main results.

Lemma 2.1. [5][11] *Let* H *be a real Hilbert space, then for all* $x, y \in H$ *and* $\alpha \in (0, 1)$*, the following hold:*

(i) $2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x - y||^2 = ||x + y||^2 - ||x||^2 - ||y||^2$,
(ii) $||x + y||^2 \leq ||y||^2 + 2\langle x, x + y\rangle$,
(iii) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$.

Lemma 2.2. [27] *Let* H *be a real Hilbert space and* $T : H \to H$ *be a nonlinear mapping, then* T *is nonexpansive if and only if* $I - T$ *is* $\frac{1}{2}$*-inverse strongly monotone.*

Lemma 2.3. [7] *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H *and* $T : C \to H$ *be a relaxed* $\eta - \alpha$ *monotone mapping. Let* $\phi : C \times C \to \mathbb{R}$ *be a bifunction satisfying* $(A1) - (A4)$ *and* $f : C \to \mathbb{R} \cup \{+\infty\}$ *be a proper convex function. For* $r > 0$*, define the resolvent mapping* $T_r : H \to C$ *associated with* ϕ*,* T *and* f *by*

$$T_r x = \{z \in C : \phi(z, y) + \langle Tz, \eta(y, z)\rangle + f(y) - f(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ y \in C\}, \tag{2.1}$$

for all $x \in H$*, and assume that*

(i) $\eta(x, y) + \eta(y, x) = 0\ \forall\ x, y \in C$,
(ii) *for any* $x, y \in C$, $\alpha(x - y) + \alpha(y - x) \geq 0$.

Then the following hold:

(1) T_r *is single-valued,*
(2) $F(T_r) = G(\phi, T, f)$.

Lemma 2.4. [28] *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $S : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0 (i.e., if $\{x_n\}$ converges weakly to $x \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0, then $x = Tx$).*

Lemma 2.5. [19] *Let $\{a_n\}$ be a sequence of non-negative numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \gamma_n,$$

where $\{\gamma_n\}$ is a sequence of real numbers bounded from above and $\{\alpha_n\} \subset [0, 1]$ satisfies $\sum \alpha_n = \infty$. Then,

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} \gamma_n.$$

3. MAIN RESULTS

Lemma 3.1. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $T : C \to C$ be a relaxed η-α-monotone mapping and $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A2)$. Let $f : C \to (-\infty, +\infty]$ be a proper convex function and $F : C \to C$ be a μ-inverse strongly monotone mapping. Assume that the following conditions are satisfied:*

(i) $\eta(x, y) + \eta(y, x) = 0 \; \forall x, y \in C$,
(ii) *for any $x, y \in C$, $\alpha(x - y) + \alpha(y - x) \geq 0$.*

Then, for each $r > 0$,

(i) T_r *is nonexpansive,*
(ii) $||T_rx - y||^2 + ||T_rx - x||^2 \leq ||x - y||^2 \; \forall x \in H$ *and* $y \in \mathcal{F}(T_r)$,
(iii) *for $0 < r \leq s$, we have that $||T_rx - T_sx|| \leq ||x - T_sx|| \; \forall x \in H$,*
(iv) $z \in G(\phi, T, f, F)$ *if and only if* $z = T_r(I - rF)z$,
(v) *for $r \in (0, 2\mu)$, $T_r(I - rF)$ is averaged.*

Proof. (i) Let $x, y \in H$, then we obtain from (2.1) that

$$\phi(T_rx, w) + \langle T(T_rx), \eta(w, T_rx)\rangle + f(w) - f(T_rx) + \frac{1}{r}\langle w - T_rx, T_rx - x\rangle \geq 0 \; \forall \; w \in C.$$

In particular, we have

$$\phi(T_rx, T_ry) + \langle T(T_rx), \eta(T_ry, T_rx)\rangle + f(T_ry) - f(T_rx) + \frac{1}{r}\langle T_ry - T_rx, T_rx - x\rangle \geq 0.$$

Similarly, we have that

$$\phi(T_ry, T_rx) + \langle T(T_ry), \eta(T_rx, T_ry)\rangle + f(T_rx) - f(T_ry) + \frac{1}{r}\langle T_rx - T_ry, T_ry - y\rangle \geq 0.$$

Adding both inequalities, and using assumption (i) and (A2), we obtain

$$\langle T(T_rx) - T(T_ry), \eta(T_ry, T_rx)\rangle + \frac{1}{r}\langle T_ry - T_rx, T_rx - x - T_ry + y\rangle \geq 0.$$

Since T is relaxed η-α monotone, we obtain that

$$\begin{aligned} \langle T_ry - T_rx, (T_rx - x) - (T_ry - y)\rangle &\geq r\langle T(T_ry) - T(T_rx), \eta(T_ry, T_rx)\rangle \\ &\geq r\alpha(T_ry - T_rx). \end{aligned} \tag{3.1}$$

By exchanging x and y in (3.1), we obtain

$$\langle T_rx - T_ry, (T_ry - y) - (T_rx - x)\rangle \geq r\alpha(T_rx - T_ry). \tag{3.2}$$

Adding (3.1) and (3.2), and using assumption (ii), we obtain

$$2\langle T_rx - T_ry, (T_ry - y) - (T_rx - x)\rangle \geq 0.$$

That is,

$$\langle T_rx - T_ry, T_rx - T_ry\rangle \leq \langle T_rx - T_ry, x - y\rangle, \tag{3.3}$$

which implies

$$||T_rx - T_ry||^2 \leq ||T_rx - T_ry||||x - y||,$$

and this gives that

$$||T_rx - T_ry|| \leq ||x - y||.$$

(ii) From (3.3), we obtain that

$$||T_rx - T_ry||^2 \leq \langle T_rx - T_ry, x - y\rangle.$$

That is, T_r is firmly nonexpansive. Thus, for each $x \in H,\ y \in F(T_r)$, we obtain from Lemma 2.1(i) that

$$\begin{aligned} ||T_rx - y||^2 &\leq \langle T_rx - y, x - y\rangle \\ &= \frac{1}{2}\left(||T_rx - y||^2 + ||y - x||^2 - ||T_rx - x||^2\right). \end{aligned} \tag{3.4}$$

That is,

$$||T_rx - y||^2 + ||T_rx - x||^2 \leq ||y - x||^2.$$

(iii) Let $z = T_rx$ and $w = T_sx,$ from (2.1), we have

$$\phi(z, w) + \langle Az, \eta(w, z)\rangle + f(w) - f(z) + \frac{1}{r}\langle w - z, z - x\rangle \geq 0. \tag{3.5}$$

Similarly we obtain that

$$\phi(w, z) + \langle Aw, \eta(z, w)\rangle + f(z) - f(w) + \frac{1}{s}\langle z - w, w - x\rangle \geq 0. \tag{3.6}$$

Adding equation (3.5) and (3.6), we obtain from assumption (i) that

$$\phi(z, w) + \phi(w, z) + \langle Az - Aw, \eta(w, z)\rangle + \frac{1}{r}\langle w - z, z - x\rangle + \frac{1}{s}\langle z - w, w - x\rangle \geq 0. \tag{3.7}$$

Using condition $(A2)$, we have

$$\frac{1}{r}\langle w - z, z - x\rangle + \frac{1}{s}\langle z - w, w - x\rangle \geq \langle Aw - Az, \mu(w, z)\rangle \geq \alpha(w - z), \tag{3.8}$$

Observe that adding (3.5) and (3.6), and using assumption (i) and (A2), one can also get that

$$\langle Aw - Az, \eta(z, w)\rangle + \frac{1}{r}\langle w - z, z - x\rangle + \frac{1}{s}\langle z - w, w - x\rangle \geq 0,$$

which by the definition of T implies

$$\frac{1}{s}\langle z - w, w - x\rangle + \frac{1}{r}\langle w - z, z - x\rangle \geq \alpha(z - w). \tag{3.9}$$

Adding (3.8) and (3.9), and using condition (ii), we have

$$2\left(\frac{1}{r}\langle w - z, z - x\rangle + \frac{1}{s}\langle z - w, w - x\rangle\right) \geq 0, \tag{3.10}$$

which implies that

$$\langle x - z, z - w\rangle \geq \frac{r}{s}\langle x - w, z - w\rangle.$$

Thus, from Lemma 2.1 (i), we have that

$$||x-w||^2 - ||x-z||^2 - ||z-w||^2 \geq \frac{r}{s}\left(||x-w||^2 + ||w-z||^2 - ||x-z||^2\right). \tag{3.11}$$

Since $\frac{r}{s} \leq 1$, we obtain that

$$\left(1+\frac{r}{s}\right)||z-w||^2 \leq \left(1-\frac{r}{s}\right)||x-w||^2.$$

So that

$$||z-w||^2 \leq \left(\frac{s-r}{s+r}\right)||x-w||^2 \leq ||x-w||^2. \tag{3.12}$$

Hence, $||T_r x - T_s x|| \leq ||x - T_s x||\ \forall x \in H$.

(iv)

$$\begin{aligned} z \in G(\phi, T, f, F) &\iff \phi(z,y) + \langle Tz - \eta(y,z)\rangle + f(y) - f(z) + \langle Fz, y-z\rangle \geq 0\ \forall y \in C \\ &\iff \phi(z,y) + \langle Tz, \eta(y,z)\rangle + f(y) - f(z) + \frac{1}{r}\langle z - (z - rFz), y - z\rangle \geq 0 \\ &\iff \phi(z,y) + \langle Tz, \eta(y,z)\rangle + f(y) - f(z) + \frac{1}{r}\langle z - (I - rF)z, y - z\rangle \geq 0 \\ &\iff z = T_r(I - rF)z. \end{aligned}$$

(v) We first observe that for $r \in (0, 2\mu)$, $(I - rF)$ is $\frac{r}{2\mu}$-averaged. Also, since T_r is firmly nonexpansive, we have that T_r is averaged. Hence, the composition $T_r(I - rF)$ is averaged for $r \in (0, 2\mu)$. ∎

Under the assumptions of Lemma 3.1, we make the following remark.

Remark 3.1. (i)Since every averaged mapping is nonexpansive, we have from Lemma 3.1 (v) that $T_r(I - rF)$ is nonexpansive for $r \in (0, 2\mu)$.
(ii) For $r \in (0, 2\mu)$, we obtain from Remark 1.1 and Lemma 3.1 (v) that $T_r(I - rF)$ is firmly nonexpansive. Thus, for any $x \in H$ and $y \in F(T_r(I - rF))$ with $r \in (0, 2\mu)$, we have from Lemma 2.1 (i) that

$$\begin{aligned} ||T_r(I-rF)x - y||^2 &\leq \langle T_r(I-rF)x - y, x - y\rangle \\ &= \frac{1}{2}\left[||T_r(I-rF)x - y||^2 + ||x-y||^2 - ||T_r(I-rF)x - x||^2\right], \end{aligned}$$

which implies

$$||y - T_r(I-rF)x||^2 + ||x - T_r(I-rF)x||^2 \leq ||y-x||^2.$$

Lemma 3.2. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $T: C \to H$ be a relaxed η-α-monotone mapping and $\phi: C \times C \to \mathbb{R}$ be a bifunction satisfying $(A2)$. Let $f: C \to (-\infty, +\infty]$ be a proper convex function and $F: C \to H$ be any nonlinear mapping. Assume that the following conditions are satisfied:*

(i) $\eta(x,y) + \eta(y,x) = 0\ \forall x, y \in C$,
(ii) *for any* $x, y \in C$, $\alpha(x-y) + \alpha(y-x) \geq 0$.

Then, for $0 < r \leq s$, *we have that* $||T_r(I-rF)x - T_s(I-sF)x|| \leq ||x - T_s(I-sF)x||\ \forall x \in H$.

Proof. Let $z = T_r(I-rF)x$ and $w = T_s(I-sF)x$, from (2.1), we have

$$\phi(z,w) + \langle Az, \eta(w,z)\rangle + f(w) - f(z) + \frac{1}{r}\langle w - z, z - (I-rF)x\rangle \geq 0. \tag{3.13}$$

Similarly, we obtain that

$$\phi(w,z)+\langle Aw,\eta(z,w)\rangle+f(z)-f(w)+\frac{1}{s}\langle z-w,w-(I-sF)x\rangle\geq 0. \tag{3.14}$$

Thus, following the same line of arguments as in (3.7)-(3.10), we obtain that

$$2\left(\frac{1}{r}\langle w-z,z-(I-rF)x\rangle+\frac{1}{s}\langle z-w,w-(I-sF)x\rangle\right)\geq 0.$$

That is,

$$\langle x-z-rFx,z-w\rangle-\frac{r}{s}\langle x-w-sFx,z-w\rangle\geq 0.$$

Hence,

$$\langle (x-rFx-z)-(\frac{r}{s}x-rFx-\frac{r}{s}w),z-w\rangle\geq 0,$$

which implies that

$$\langle x-z,z-w\rangle\geq\frac{r}{s}\langle x-w,z-w\rangle.$$

By the same line of arguments as in (3.11)-(3.12), we obtain the desired result ∎

Throughout this paper, we shall write $T_r^{(1)}$ for the resolvent mapping associated with ϕ_1, T_1 and f_1, and $T_r^{(2)}$ for the resolvent mapping associated with ϕ_2, T_2 and f_2. We also make the following assumptions

Assumption 3.1. *Assume that $\{\alpha_n\},\ \{\beta_n\}$ and $\{t_n\}$ are sequences of real numbers satisfying the following:*

(i) $\{\alpha_n\}\subset(0,1)$ *such that* $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$.

(ii) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$, *for all* $n\in\mathbb{N}$.

(iii) $\liminf_{n\to\infty}t_n(2-t_n)>0$.

Let $h(x):=\frac{1}{2}||(I-ST_r^{(2)})Ax||^2$ *and* $l(x):=\frac{1}{2}||(I-T_r^{(1)}(I-rF))x||^2$*. Then, we consider the following algorithm to study problem* (1.5)-(1.6).

Algorithm 3.1. (1) *Let $\{\alpha_n\},\ \{\beta_n\}$ and $\{t_n\}$ be such that Assumption 3.1 is satisfied*

(2) *Given the initial point* $x_1\in C_1$

(3) *Set* $n=1$ *and compute:*

(4) $y_n=\alpha_n g(x_n)+(1-\alpha_n)x_n$

(5) $\Theta(y_n)=||A^*(I-ST_{r_n}^{(2)})Ay_n+(I-T_{r_n}^{(1)}(I-r_nF))y_n||$

(6) $z_n=y_n-t_n\frac{h(y_n)+l(y_n)}{\Theta^2(y_n)}\left(A^*(I-ST_{r_n}^{(2)})Ay_n+(I-T_{r_n}^{(1)}(I-r_nF))y_n\right)$

(7) $x_{n+1}=(1-\beta_n)y_n+\beta_n z_n$.

(8) *If* $A^*(I-ST_{r_n}^{(2)})Ay_n=0=(I-T_{r_n}^{(1)}(I-r_nF))y_n$ *and* $x_{n+1}=x_n$*, then stop, otherwise*

(9) *set* $n=n+1$ *and repeat step (4)-(7).*

We observe here that, by similar argument as in Remark 1.3, one can easily see that Algorithm 3.1 is well defined. Therefore, using Algorithm 3.1, we present in what follows, our strong convergence theorem for solving problem (1.5)-(1.6).

Theorem 3.3. *Let C_1 and C_2 be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, and $A:C_1\to C_2$ be a bounded linear mapping. Let $\phi_1:C_1\times C_1\to\mathbb{R}$, $\phi_2:C_2\times C_2\to\mathbb{R}$ be bifunctions satisfying* $(A1)$-$(A4)$ *and $T_1:C_1\to C_1$, $T_2:C_2\to C_2$ be η-hemicontinuous and relaxed η-α monotone mappings. Let $f_1:C_1\to(-\infty,+\infty]$, $f_2:C_2\to(-\infty,+\infty]$ be proper convex and lower semicontinuous functions and $F:C_1\to C_1$ be a μ-inverse strongly monotone mapping. Let $S:C_2\to C_2$ be a nonexpansive mapping and*

$g : C_1 \to C_1$ be a contraction with constant k. Suppose that $\Gamma \neq \emptyset$ and $\{r_n\}$ is a real sequence such that $0 < r \leq r_n \leq b < 2\mu$. Then, the sequence generated by Algorithm 3.1 converges strongly to $z \in \Gamma$, where $z = P_\Gamma g(z)$.

Proof. Let $z \in P_\Gamma g(z)$ and $J_{r_n} = T_{r_n}^{(1)}(I - r_nF))$, then $z = J_{r_n}z$ and $Az = ST_{r_n}^{(2)}(Az)$. Also, since $0 < r \leq r_n \leq b < 2\mu$, we have from Remark 3.1(i) that J_{r_n} is nonexpansive. Again, from Lemma 3.1 (i), we obtain that $S \circ T_{r_n}^{(2)}$ is nonexpansive. Thus, by Lemma 2.2, we obtain that

$$\begin{aligned}
\langle (I - ST_{r_n}^{(2)})Ay_n, Ay_n - Az\rangle &= \langle (I - ST_{r_n}^{(2)})Ay_n - (I - ST_{r_n}^{(2)})Az, Ay_n - Az\rangle \\
&\geq \frac{1}{2}||(I - ST_{r_n}^{(2)})Ay_n - (I - ST_{r_n}^{(2)})Az||^2 \\
&= h(y_n). \qquad (3.15)
\end{aligned}$$

Similarly, we obtain that

$$\langle (I - J_{r_n})y_n, y_n - z\rangle \geq l(y_n). \qquad (3.16)$$

From Lemma 2.1 (i), (3.15), (3.16) and Algorithm 3.1, we obtain

$$\begin{aligned}
||z_n - z||^2 &= ||y_n - z||^2 - 2t_n\frac{h(y_n) + l(y_n)}{\Theta^2(y_n)}\langle A^*(I - ST_{r_n}^{(2)})Ay_n + (I - J_{r_n})y_n, y_n - z\rangle \\
&\quad + \frac{t_n^2(h(y_n) + l(y_n))^2}{\Theta^4(y_n)}||A^*(I - ST_{r_n}^{(2)})Ay_n + (I - J_{r_n})y_n||^2 \\
&= ||y_n - z||^2 - 2t_n\frac{h(y_n) + l(y_n)}{\Theta^2(y_n)}\left[\langle (I - ST_{r_n}^{(2)})Ay_n, Ay_n - Az\rangle + \langle (I - J_{r_n})y_n, y_n - z\rangle\right] \\
&\quad + \frac{t_n^2(h(y_n) + l(y_n))^2}{\Theta^2(y_n)} \\
&\leq ||y_n - z||^2 - 2t_n\frac{h(y_n) + l(y_n)}{\Theta^2(y_n)}(h(y) + l(y_n)) + \frac{t_n^2(h(y_n) + l(y_n))^2}{\Theta^2(y_n)} \\
&= ||y_n - z||^2 - t_n(2 - t_n)\left[\frac{(h(y_n) + l(y_n))^2}{\Theta^2(y_n)}\right]. \qquad (3.17)
\end{aligned}$$

Now, observe from Algorithm 3.1 that

$$x_{n+1} - y_n = \beta_n(z_n - y_n). \qquad (3.18)$$

Thus, we obtain from Algorithm 3.1 that

$$\begin{aligned}
||x_{n+1} - z||^2 &= ||(y_n - z) - \beta_n(y_n - z_n)||^2 \\
&= ||y_n - z||^2 - 2\beta_n\langle y_n - z, y_n - z_n\rangle + \beta_n^2||y_n - z_n||^2 \\
&\leq ||y_n - z||^2 - \beta_n(1 - \beta_n)||y_n - z_n||^2 \\
&= ||y_n - z||^2 - \frac{1}{\beta_n}(1 - \beta_n)||x_{n+1} - y_n||^2. \qquad (3.19)
\end{aligned}$$

From Algorithm 3.1, we obtain

$$\begin{aligned}
||x_{n+1} - z|| &= ||\alpha_n(g(x_n) - g(z)) + \alpha_n(g(z) - z) + (1 - \alpha_n)(x_n - z)|| \\
&\leq \alpha_n k||x_n - z|| + \alpha_n||g(z) - z|| + (1 - \alpha_n)||x_n - z|| \\
&= (1 - \alpha_n(1 - k))||x_n - z|| + \alpha_n||g(z) - z|| \\
&\leq \max\left\{||x_n - z||,\ \frac{||g(z) - z||}{1 - k}\right\} \\
&\ \vdots \\
&\leq \max\left\{||x_1 - z||,\ \frac{||g(z) - z||}{1 - k}\right\}.
\end{aligned}$$

Hence, $\{x_n\}$ is bounded. So are $\{y_n\}$ and $\{z_n\}$. Now, from (3.18), we obtain

$$||z_n - y_n||^2 = \frac{1}{\beta_n^2}||x_{n+1} - y_n||^2 = \frac{\alpha_n}{\beta_n}\left(\frac{||x_{n+1} - y_n||^2}{\alpha_n\beta_n}\right). \tag{3.20}$$

Also, from Algorithm 3.1 and Lemma 2.1 (ii), we obtain

$$\begin{aligned}
||y_n - z||^2 &= ||\alpha_n(g(x_n) - g(z)) + \alpha_n(g(z) - z) + (1 - \alpha_n)(x_n - z)||^2 \\
&\leq ||\alpha_n(g(x_n) - g(z)) + (1 - \alpha_n)(x_n - z)||^2 + 2\alpha_n\langle g(z) - z, y_n - z\rangle \\
&\leq \alpha_n^2 k^2||x_n - z||^2 + (1 - \alpha_n)^2||x_n - z||^2 \\
&\quad +2\alpha_n(1 - \alpha_n)\langle g(x_n) - g(z), x_n - z\rangle + 2\alpha_n\langle g(z) - z, y_n - z\rangle \\
&\leq (1 - \alpha_n)^2||x_n - z||^2 + k^2\alpha_n^2||x_n - z||^2 + 2\alpha_n\langle g(z) - z, y_n - z\rangle \\
&\quad +2\alpha_n(1 - \alpha_n)||g(x_n) - g(z)||||x_n - z|| \\
&\leq (1 - \alpha_n)^2||x_n - z||^2 + \alpha_n^2 k^2||x_n - z||^2 + 2\alpha_n\langle g(z) - z, y_n - z\rangle \\
&\quad +2\alpha_n(1 - \alpha_n)k||x_n - z||^2 \\
&= (1 - 2\alpha_n(1 - k(1 - \alpha_n)))||x_n - z||^2 \\
&\quad +\alpha_n^2(1 + k^2)||x_n - z||^2 + 2\alpha_n\langle g(z) - z, y_n - z\rangle \\
&\leq (1 - 2\alpha_n(1 - k))||x_n - z||^2 - 2\alpha_n\left[\langle g(z) - z, z - y_n\rangle - \frac{\alpha_n(1 + k^2)}{2}||x_n - z||^2\right] \qquad (3.21)
\end{aligned}$$

From (3.19) and (3.21), we obtain that

$$\begin{aligned}
||x_{n+1} - z||^2 &\leq (1 - 2\alpha_n(1 - k))||x_n - z||^2 - 2\alpha_n\left[\langle g(z) - z, z - y_n\rangle - \frac{\alpha_n(1 + k^2)}{2}||x_n - z||^2\right] \\
&\quad -\frac{1}{\beta_n}(1 - \beta_n)||x_{n+1} - y_n||^2 \\
&= (1 - 2\alpha_n(1 - k))||x_n - z||^2 \\
&\quad -2\alpha_n\left[\langle g(z) - z, z - y_n\rangle + \frac{1}{2\alpha_n\beta_n}(1 - \beta_n)||x_{n+1} - y_n||^2 - \frac{\alpha_n(1 + k^2)}{2}||x_n - z||^2\right] \qquad (3.22)
\end{aligned}$$

Let $\gamma_n = \langle g(z) - z, z - y_n\rangle + \frac{1}{2\alpha_n\beta_n}(1 - \beta_n)||x_{n+1} - y_n||^2 - \frac{\alpha_n(1+k^2)}{2}||x_n - z||^2$. Then, (3.22) becomes

$$\begin{aligned}
||x_{n+1} - z||^2 &\leq (1 - 2\alpha_n(1 - k))||x_n - z||^2 - 2\alpha_n\gamma_n \\
&\leq (1 - 2\alpha_n(1 - k))||x_n - z||^2 + 2\alpha_n(1 - k)(-\gamma_n). \qquad (3.23)
\end{aligned}$$

Let $\delta_n = 2\alpha_n(1 - k)$. Then, it follows from Assumption 3.1 (i) that $\sum_{n=1}^{\infty}\delta_n = \infty$. Also, we know that $\{x_n\}$ is bounded below (so is $\{y_n\}$), thus $(-\gamma_n)$ is bounded above. Hence, applying

Lemma 2.5 in (3.23), we obtain that

$$\begin{aligned}\limsup_{n\to\infty}||x_n - z||^2 &\leq \limsup(-\gamma_n)\\ &= -\liminf_{n\to\infty}\gamma_n. \end{aligned} \tag{3.24}$$

That is,

$$\liminf_{n\to\infty}\gamma_n \leq -\limsup_{n\to\infty}||x_n - z||^2.$$

Thus, $\liminf\limits_{n\to\infty}\gamma_n$ exists. Also, by Assumption 3.1 (i), we obtain that

$$\liminf_{n\to\infty}\gamma_n = \liminf_{n\to\infty}\left(\langle g(z) - z, z - y_n\rangle + \frac{1}{2\alpha_n\beta_n}(1-\beta_n)||x_{n+1} - y_n||^2\right).$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to a point $x^* \in C_1$, and

$$\liminf_{n\to\infty}\gamma_n = \lim_{k\to\infty}\left(\langle g(z) - z, z - y_{n_k}\rangle + \frac{1}{2\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})||x_{n_k+1} - y_{n_k}||^2\right). \tag{3.25}$$

Hence, $\left\{\frac{1}{2\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})||x_{n_k+1} - y_{n_k}||^2\right\}$ is bounded. Furthermore, Assumption 3.1 implies that there exists $b \in (0,1)$ such that $\beta_n \leq b \leq 1$. Thus,

$$\frac{1}{2\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k}) \geq \frac{1}{2\alpha_{n_k}\beta_{n_k}}(1-b) > 0,$$

which implies that $\left\{\frac{1}{2\alpha_{n_k}\beta_{n_k}}||x_{n_k+1} - y_{n_k}||^2\right\}$ is bounded. Also, Assumption 3.1, implies that there exists $a \in (0,1)$ such that $0 < a \leq \beta_n$. Thus, $0 < \frac{\alpha_{n_k}}{\beta_{n_k}} \leq \frac{\alpha_{n_k}}{a} \to 0,\ k\to\infty$. Hence, we obtain from (3.20) and the fact that $\left\{\frac{1}{\alpha_{n_k}\beta_{n_k}}||x_{n_k+1} - y_{n_k}||^2\right\}$ is bounded that

$$\lim_{k\to\infty}||z_{n_k} - y_{n_k}|| = 0. \tag{3.26}$$

From Algorithm 3.1 and (3.26), we obtain that

$$||x_{n_k+1} - y_{n_k}|| = \beta_{n_k}||z_{n_k} - y_{n_k}|| \to 0,\ k\to\infty. \tag{3.27}$$

Again, we obtain from Algorithm 3.1 that

$$||y_{n_k} - x_{n_k}|| = \alpha_{n_k}||g(x_{n_k}) - x_{n_k}|| \to 0,\ k\to\infty. \tag{3.28}$$

From (3.27) and (3.28), we obtain that

$$\lim_{k\to\infty}||x_{n_k+1} - x_{n_k}|| = 0. \tag{3.29}$$

Also, from (3.17) and (3.26), we obtain that

$$\begin{aligned} t_{n_k}(2-t_{n_k})\left(\frac{(h(y_{n_k}) + l(y_{n_k}))^2}{\Theta^2(y_{n_k})}\right) &\leq ||y_{n_k} - z||^2 - ||z_{n_k} - z||^2\\ &\leq ||y_{n_k} - z_{n_k}||^2 + 2||y_{n_k} - z_{n_k}||||z_{n_k} - z|| \to 0,\ k\to\infty. \end{aligned}$$

By Assumption 3.1, we obtain that $\lim\limits_{k\to\infty}\frac{(h(y_{n_k})+l(y_{n_k}))^2}{\Theta^2(y_{n_k})} = 0$. Consequently, we obtain that

$$\lim_{k\to\infty}(h(y_{n_k}) + l(y_{n_k})) = 0 \Longleftrightarrow \lim_{k\to\infty}h(y_{n_k}) = 0 \text{ and } \lim_{k\to\infty}l(y_{n_k}) = 0.$$

That is,

$$\lim_{k\to\infty}||Ay_{n_k} - ST^{(2)}_{r_{n_k}}Ay_{n_k}|| = 0, \text{ and} \tag{3.30}$$

$$\lim_{k\to\infty} ||y_{n_k} - J_{r_{n_k}} y_{n_k}|| = 0. \tag{3.31}$$

Now, set $v_n = T^{(2)}_{r_n} Ay_n$, then (3.30) becomes $\lim_{k\to\infty} ||Ay_{n_k} - Sv_{n_k}|| = 0$. Thus, from Lemma 3.1 (ii), we obtain that

$$\begin{aligned} ||Ay_{n_k} - v_{n_k}||^2 &\le ||Ay_{n_k} - Az||^2 - ||v_{n_k} - Az||^2 \\ &\le ||Ay_{n_k} - Az||^2 - ||Sv_{n_k} - SAz||^2 \\ &\le ||Ay_{n_k} - Sv_{n_k}||^2 + 2||Ay_{n_k} - Sv_{n_k}||||Sv_{n_k} - SAz|| \to 0,\ k \to \infty. \end{aligned}$$

That is,

$$\lim_{k\to\infty} ||Ay_{n_k} - T^{(2)}_{r_{n_k}} Ay_{n_k}|| = 0. \tag{3.32}$$

Also,

$$\begin{aligned} ||Ay_{n_k} - SAy_{n_k}|| &\le ||Ay_{n_k} - Sv_{n_k}|| + ||Sv_{n_k} - SAy_{n_k}|| \\ &\le ||Ay_{n_k} - Sv_{n_k}|| + ||v_{n_k} - Ay_{n_k}|| \to 0,\ k \to \infty. \end{aligned} \tag{3.33}$$

From (3.32) and Lemma 3.1(iii), we obtain that

$$\begin{aligned} ||Ay_{n_k} - T^{(2)}_{r} Ay_{n_k}|| &\le ||Ay_{n_k} - T^{(2)}_{r_{n_k}} Ay_{n_k}|| + ||T^{(2)}_{r_{n_k}} Ay_{n_k} - T^{(2)}_{r} Ay_{n_k}|| \\ &\le 2||Ay_{n_k} - T^{(2)}_{r_{n_k}} Ay_{n_k}|| \to 0,\ k \to \infty. \end{aligned} \tag{3.34}$$

Again, by Lemma 3.2, we obtain that

$$\begin{aligned} ||y_{n_k} - J_r y_{n_k}|| &\le ||y_{n_k} - J_{r_{n_k}} y_{n_k}|| + ||J_{r_{n_k}} y_{n_k} - J_r y_{n_k}|| \\ &\le 2||y_{n_k} - J_{r_{n_k}} y_{n_k}|| \to 0,\ k \to \infty. \end{aligned} \tag{3.35}$$

Since $\{x_n\}$ converges weakly to $x^* \in C_1$, we have from (3.28) that there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\}$ converges weakly to $x^* \in C_1$. Also, since A is a bounded linear mapping, we have that there exists a subsequence $\{Ay_{n_k}\}$ of $\{Ay_n\}$ that converges weakly to $Ax^* \in C_2$. It then follows from Lemma 2.4, (3.33), (3.34) and (3.35) that $Ax^* \in \left(F(S) \cap F(T^{(2)}_r)\right)$ and $x^* \in F(J_r)$. Hence, $x^* \in \Gamma$.

We now show that $\{x_n\}$ converges strongly to z. Now, from (3.25), (3.27) and by the property of the metric projection P_C, we obtain

$$\begin{aligned} \liminf_{n\to\infty} \gamma_n &= \lim_{k\to\infty} \langle g(z) - z, z - y_{n_k} \rangle \\ &= \langle g(z) - z, z - x^* \rangle \\ &\ge 0. \end{aligned}$$

Thus, from (3.24), we obtain that $\limsup_{n\to\infty} ||x_n - z||^2 \le 0$. Hence, $\limsup_{n\to\infty} ||x_n - z||^2 = 0$. Therefore, we conclude that $\{x_n\}$ converges strongly to z. ∎

Consider the following Split Mixed Equilibrium Problem:

$$\text{Find } x \in C_1 \text{ such that } x \in G(\phi_1, T_1, f_1), \tag{3.36}$$

$$\text{and } Ax = y \in C_2 \text{ such that } y \in G(\phi_2, T_2, f_2), \tag{3.37}$$

where $\phi_1, T_1, f_1, \phi_2, T_2, f_2$ are as defined in Theorem 3.3.

As corollary of our main results, we can solve Problem (3.36)-(3.37) by setting $S = I$ and $F = 0$ in Algorithm 3.1. Also, by setting $\phi_1 = \phi_2 = T_1 = T_2 = F = 0$ and $S = I$ in Algorithm 3.1, we can apply Theorem 3.3 to solve the Proximal Split Feasibility Problem studied in [22].

4. SPLIT GENERALIZED MIXED EQUILIBRIUM PROBLEM OVER THE SOLUTION SET OF VARIATIONAL INCLUSIONS

Recall that a multivalued mapping $M : H \to 2^H$ is called monotone, if

$$\langle x-y, u-v\rangle \geq 0 \, \forall x, y \in H, \; u \in M(x), \; v \in M(y),$$

and maximal monotone if the graph $G(M)$ of M defined by

$$G(M) =: \{(x,y) \in H \times H : y \in M(x)\}$$

is not properly contained in the graph of any other monotone mapping. The resolvent operator J_λ^M associated with a mapping M and λ is the mapping $J_\lambda^M : H \to 2^H$ defined by

$$J_\lambda^M(x) = (I + \lambda M)^{-1}x, \; x \in H, \lambda > 0. \tag{4.1}$$

It is known that if the mapping M is monotone, then J_λ^M is single valued and firmly nonexpansive (see [2]).
Now, consider the following Monotone Variational Inclusion Problem (MVIP): Find

$$x \in H \text{ such that } 0 \in M_1(x) + F_2(x), \tag{4.2}$$

where $M_1 : H \to 2^H$ is a multivalued mapping and $F_2 : H \to 2^H$ is a single valued mapping. We shall denote the solution set of problem (4.2) by $(M_1 + F_2)^{-1}(0)$. In [20], Moudafi proved that $x \in (M_1+F_2)^{-1}(0)$ if and only if $x = J_\lambda^{M_1}(I-\lambda F_2)(x), \; \forall \lambda > 0$. It was also shown in [20] that, if F_2 is a μ-inverse strongly monotone mapping and M_1 is a maximal monotone mapping, then $J_\lambda^{M_1}(I - \lambda F_2)$ is averaged with $0 < \lambda < 2\mu$. Hence, $J_\lambda^{M_1}(I - \lambda F_2)$ is a nonexpansive mapping with $0 < \lambda < 2\mu$.
Thus, by setting $S = J_\lambda^{M_1}(I - \lambda F_2)$ in Algorithm 3.1, we can apply Theorem 3.3 to solve the following SGMEP over the solution set of MVIP:

$$\text{Find } x \in C_1 \text{ such that } x \in G(\phi_1, T_1, f_1, F_1), \tag{4.3}$$

$$\text{and } Ax = y \in C_2 \text{ such that } y \in \left(G(\phi_2, T_2, f_2) \cap (M_1 + F_2)^{-1}(0)\right), \tag{4.4}$$

where $\phi_1, T_1, f_1, F_1, \phi_2, T_2, f_2$ are as defined in Theorem 3.3.

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