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ON CLOSED RANGE C^*-MODULAR OPERATORS

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ABSTRACT. In this paper, for the class of the modular operators on Hilbert C^*-modules, we give the conditions to closedness of their ranges. Also, the equivalence conditions for the closedness of the range of the modular projections on Hilbert C^*-modules are discussed. Moreover, the mixed reverse order law for the Moore-Penrose invertible modular operators are given.

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1. INTRODUCTION AND PRELIMINARIES

As we know, Hilbert C^*-modules are extension of Hilbert spaces with the same properties. However, there exist some basic differences. Some well known properties of Hilbert spaces like Pythagoras' equality, self-duality, and even decomposition into orthogonal complements do not hold in the framework Hilbert modules. As an important difference, we observe that a bounded linear operators between Hilbert C^*-modules not necessary adjointable in general. In other words, for any bounded linear operator $T : \mathcal{X} \longrightarrow \mathcal{Y}$ there is not necessary a bounded linear operator $T^* : \mathcal{Y} \longrightarrow \mathcal{X}$, for which $\langle T^*y, x\rangle = \langle y, T\rangle$. This subject has matchwood in dual structure of Hilbert C^*-modules which is not simplicity as Hilbert spaces. We denote the set of all adjointable bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert C^*-modules. It is well known that any element T of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa) = (Tx)a$ for $x \in \mathcal{X}$ and $a \in \mathcal{A}$ [8, Page 8].

It is of fundamental importance to note that, Hilbert C^*-modules form a category in between Banach spaces and Hilbert spaces. The basic idea is to consider module over C^*-algebra instead of linear space and to allow the inner product to take values in a more general C^*-algebra than $\mathbb{C}$. This structure was first used by Kaplansky [7] in 1952 and more carefully investigated by Rieffel [15] and Paschke [14] later in 1972-73. We give only a brief introduction to the theory of Hilbert C^*-modules to make our explanations self-contained.

We use the notations $\mathcal{L}(\mathcal{X})$ instead of $\mathcal{L}(\mathcal{X}, \mathcal{X})$, and $\ker(\cdot)$ and $\operatorname{ran}(\cdot)$ for the kernel and the range of operators, respectively. The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$ or 1 if there is no ambiguity. The readers are referred to [1, 2], [10, 11, 12] and the references cited therein for more details in Hilbert C^*-modules. Throughout the paper $\mathcal{A}$ is a C*-algebra (not necessarily unital) and $\mathcal{X}, \mathcal{Y}$ are Hilbert $\mathcal{A}$-modules.

The paper is organized as follows. Preliminary is presented in the sequal of this section. In Section 2, the main theorem of our work is appeared, which is provide the equivalence conditions for the closedness of the range of the modular projections. Thereafter, in Section 3 by a simple technique, matrix forms of operators, we present the conditions which is the mixed reverses order law hold.

A (right) pre-Hilbert module over a C^*-algebra $\mathcal{A}$ is a complex linear space $\mathcal{X}$, which is an algebraic right $\mathcal{A}$-module and $\lambda(xa) = (\lambda x)a = x(\lambda a)$ equipped with an $\mathcal{A}$-valued inner product $\langle ., .\rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying,

(1) $\langle x, x\rangle \geq 0, and \langle x, x\rangle = 0$ iff $x = 0$,
(2) $\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle$,
(3) $\langle x, ya\rangle = \langle x, y\rangle a$,
(4) $\langle y, x\rangle = \langle x, y\rangle^*$,

for each $x, y, z \in \mathcal{X}$, $\lambda \in \mathbb{C}$, $a \in \mathcal{A}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called a Hilbert $\mathcal{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. Left Hilbert $\mathcal{A}$-modules are defined in a similar way. For example every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with respect to inner product $\langle x, y\rangle = x^*y$, and every Hilbert space is a left Hilbert $\mathbb{C}$-module. Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $\mathcal{Y}$ is a closed submodule of $\mathcal{X}$. We say that $\mathcal{Y}$ is orthogonally complemented if $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}^\perp$, where $\mathcal{Y}^\perp := \{y \in \mathcal{X} : \langle x, y\rangle = 0 \text{ for all } x \in \mathcal{Y}\}$ denotes the orthogonal complement of $\mathcal{Y}$ in $\mathcal{X}$. Recall that a closed submodule in a Hilbert $\mathcal{A}$-module is not necessarily orthogonally complemented, i.e. if $\mathcal{F}$ is a (possibly non-closed) $\mathcal{A}$-submodule of $\mathcal{X}$, then $\mathcal{F}^{\perp}$ is a closed $\mathcal{A}$-submodule of $\mathcal{X}$ and $\overline{\mathcal{F}} \subseteq \mathcal{F}^{\perp\perp}$. However, Lance in [8] proved that certain submodules are orthogonally complemented as follows.

Theorem 1.1. ([8, Theorem 3.2]) Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ has closed range, and

(i) $\ker(T)$ is orthogonally complemented in $\mathcal{X}$, with $(\ker(T))^\perp = \mathrm{ran(T^*)}$.
(ii) $\mathrm{ran(T)}$ is orthogonally complemented in $\mathcal{Y}$, with $(\mathrm{ran(T)})^\perp = \ker(\mathrm{T}^*)$.

A generalized inverse of $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is an operator $T^\times \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that

$$TT^\times T = T \text{ and } T^\times T T^\times = T^\times. \tag{1.1}$$

Definition 1.1. Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The Moore-Penrose inverse $T^\dagger$ of T is an element in $L(\mathcal{Y}, \mathcal{X})$ which satisfies

(1) $TT^\dagger T = T$,
(2) $T^\dagger T T^\dagger = T^\dagger$,
(3) $(TT^\dagger)^* = TT^\dagger$,
(4) $(T^\dagger T)^* = T^\dagger T$.

If $\theta \subseteq \{1, 2, 3, 4\}$, and X satisfies the equations (i) for all $i \in \theta$, then X is an θ-inverse of T. The set of all θ-inverses of T is denoted by $T\{\theta\}$. In particular, $T\{1, 2, 3, 4\} = \{T^\dagger\}$. The reader should be aware of the fact that a bounded adjointable operator may admit an unbounded operator as its Moore-Penrose, see [12] for more detailed information. Motivated by these conditions, $T^\dagger$ is unique and $T^\dagger T$ and $TT^\dagger$ are orthogonal projections, in the sense that those are selfadjoint idempotent operators. In fact, if $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range, then $TT^\dagger = P_{\mathrm{ran(T)}}$ and $T^\dagger T = P_{\mathrm{ran(T^*)}}$.

Clearly, T is Moore-Penrose invertible if and only if T^* is Moore-Penrose invertible, and in this case $(T^*)^\dagger = (T^\dagger)^*$. Let us now turn the attentions to waud the following theorem.

Theorem 1.2. ([16, Theorem 2.2]) Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then the Moore-Penrose inverse $T^\dagger$ of T exists if and only if T has closed range.

By Definition 1.1, we have

$$\begin{aligned} \mathrm{ran(T)} &= \mathrm{ran(T\,T^\dagger)}, & \mathrm{ran(T^\dagger)} &= \mathrm{ran(T^\dagger T)} = \mathrm{ran(T^*)}, \\ \ker(T) &= \ker(T^\dagger T), & \ker(T^\dagger) &= \ker(TT^\dagger) = \ker(T^*), \end{aligned}$$

and by Theorem 1.1, we can conclude

$$\begin{aligned} \mathcal{X} &= \ker(T) \oplus \mathrm{ran(T^\dagger)} = \ker(\mathrm{T^\dagger T}) \oplus \mathrm{ran(T^\dagger T)}, \\ \mathcal{Y} &= \ker(T^\dagger) \oplus \mathrm{ran(T)} = \ker(\mathrm{T\,T^\dagger}) \oplus \mathrm{ran(T\,T^\dagger)}. \end{aligned}$$

2. THE CLOSED RANGE PROJECTIONS

In this section, by using the Lemma 2.2 and Corollaries 2.3 and 2.4 some results about closedness of the range of modular projections are presented. We state the following Lemma given in [9].

Lemma 2.1. *Let T be a non-zero operator in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, then T has closed range if and only if $\ker(T)$ is orthogonally complemented in $\mathcal{X}$ and*

$$\gamma(T) = \inf\{\|Tx\| : x \in Ker(T)^\perp \text{ and } \|x\| = 1\} > 0.$$

In this case, $\gamma(T) = \|T^\dagger\|^{-1}$ and $\gamma(T) = \gamma(T^*)$.

Definition 2.1. The Dixmier (or minimal) angle between submodules $\mathcal{M}$ and $\mathcal{N}$ of a Hilbert C*-module $\mathcal{X}$ is the angle $\alpha_0(\mathcal{M}, \mathcal{N})$ in $[0, \pi/2]$ whose cosine is defined by

$$c_0(\mathcal{M}, \mathcal{N}) = \sup\{\|\langle x, y\rangle\| : x \in \mathcal{M},\ \|x\| \leq 1\ ,\ y \in \mathcal{N},\ \|y\| \leq 1\}.$$

Lemma 2.2. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and T, S be operators in $\mathcal{L}(\mathcal{X})$, then $\gamma(TS) \leq\| T \| \gamma(S)$.*

Proof. By Lemma 2.1 and this fact that $\ker(TS) \subseteq \ker(T)$ and $\ker(S) \subseteq \ker(TS)$ we have $(\ker(TS))^{\perp} \subseteq (\ker(S))^{\perp}$. So,

$$\begin{aligned}\gamma(TS) &= \inf\{\|TSx\| : x \in (\ker(TS))^{\perp} \text{ and } \|x\| = 1\} \\ &\leq\| T \| \inf\{\|Sx\| : x \in (\ker(S))^{\perp} \text{ and } \|x\| = 1\}\end{aligned}$$

In this case, $\gamma(TS) \leq\| T \| \gamma(S)$. ∎

Corollary 2.3. *Let $T \in \mathcal{L}(\mathcal{X})$ be invertible and S has closed range, then TS has closed range.*

Proof. Suppose T is invertible, by using Lemma 2.2 for (T^{-1}/TS) we have,

$$\| T^{-1} \|^{-1} \gamma(S) \leq \gamma(TS) \leq\| T \| \gamma(S).$$

By Lemma 2.1 and the closedness of the range of S, we conclude that TS has closed range. ∎

Corollary 2.4. *Let T, S and TS have closed ranges. Then $\| S^{\dagger} \|\leq\| T \| \|(TS)^{\dagger}\|$. Moreover, if T is invertible, then $\| (TS)^{\dagger} \|\leq\| T^{-1} \| \|S^{\dagger}\|$.*

Proof. The first part is clear. For the second, by assumption that T is invertible we have $\| T^{-1} \|^{-1} \gamma(S) \leq \gamma(TS) \leq\| T \| \gamma(S)$. Hence,

$$\| T^{-1} \|^{-1} \|S^{\dagger}\|^{-1} \leq \gamma(TS) \leq\| T \| \|S^{\dagger}\|^{-1},$$

therefore,

$$(\| T^{-1} \| \|S^{\dagger}\|)^{-1} \leq\| (TS)^{\dagger} \|^{-1}\leq\| T \| \|S^{\dagger}\|^{-1},$$

finally, the first inequality implies that, $\| (TS)^{\dagger} \|\leq\| T^{-1} \| \|S^{\dagger}\|$. ∎

For the proof of the following Lemma see [9].

Lemma 2.5. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and P, Q be orthogonal projections in $\mathcal{L}(\mathcal{X})$. Then the following conditions are equivalent:*

(i) PQ has closed range,
(ii) $1 - P - Q$ has closed range,
(iii) $1 - P + Q$ has closed range,
(iv) $1 - Q + P$ has closed range.

Corollary 2.6. *Let $T \in L(\mathcal{X})$ be partial isometry and $\mathcal{M}$ be a closed submodule of the $\mathrm{ran(T^*)}$ and $P_{\mathcal{M}}$ be the orthogonal projection onto $\mathcal{M}$, then $\| (TP_{\mathcal{M}})^{\dagger} \|= 1$.*

Proof. Since $\mathcal{M}$ is a closed submodule of $\mathrm{ran(T^*)}$ and $P_{\mathcal{M}}$ the orthogonal projection onto $\mathcal{M}$ and T is partial isometry, thus Lemma 2.5 implies that $TP_{\mathcal{M}}$ has closed range and so $(TP_{\mathcal{M}})^{\dagger}$ exist. On the other hand we have $P_{(\ker(T))^{\perp}}P_{\mathcal{M}} = 0$, so by Lemma 2.2 we have $\gamma(TP_{\mathcal{M}})^2 = 1$ and Lemma 2.5 implies that $\| (TP_{\mathcal{M}})^{\dagger} \|= 1$. ∎

Corollary 2.7. *Let P and Q be projections on a Hilbert $\mathcal{A}$-module $\mathcal{X}$. Then the following are equivalent:*

(i) $\| Px \|\leq\| Qx \|$ for all $x \in \mathcal{X}$.
(ii) $\mathrm{ran(P)} \subseteq \mathrm{ran(Q)}$.

Corollary 2.8. *Let $\mathcal{M} \subseteq \mathcal{N}$ be closed orthogonal summand submodules of a Hilbert $\mathcal{A}$-module. Then $P_{\mathcal{M}}P_{\mathcal{N}}$ and $1 - P_{\mathcal{N}} - P_{\mathcal{M}}$ have closed ranges.*

Proof. Since $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ are orthogonal projections such that $\mathcal{M} = \mathrm{ran(P_{\mathcal{M}})} \subseteq \mathrm{ran(P_{\mathcal{N}})} = \mathcal{N}$, implies that $P_{\mathcal{N}}P_{\mathcal{M}} = P_{\mathcal{M}}$, which conclude that $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}$ by taking $*$-operation. Thus, $P_{\mathcal{M}}P_{\mathcal{N}}$ is actually a projection and so it has closed range. Also, by [9, Lemma 3.2], $1-P_{\mathcal{N}}-P_{\mathcal{M}}$ has closed range. ∎

Theorem 2.9. *Let $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{R}$ be closed orthogonal summand submodules of a Hilbert $\mathcal{A}$-module $\mathcal{X}$. Then*

(i) *$P_{\mathcal{N}}P_{\mathcal{R}}$, $P_{\mathcal{M}}P_{\mathcal{N}}$ and $P_{\mathcal{M}}P_{\mathcal{R}}$ have closed ranges,*
(ii) *$1 - P_{\mathcal{R}} - P_{\mathcal{N}}$ has closed range,*
(iii) *$1 - P_{\mathcal{N}} - P_{\mathcal{M}}$ has closed range,*
(iv) *$1 - P_{\mathcal{R}} - P_{\mathcal{N}}$ has closed range,*
(v) *$1 - P_{\mathcal{R}} + P_{\mathcal{N}}$ has closed range,*
(vi) *$1 - P_{\mathcal{R}} + P_{\mathcal{N}}$ has closed range.*

Where $P_{\mathcal{M}}$, $P_{\mathcal{N}}$ and $P_{\mathcal{R}}$ are orthogonal projections onto $\mathcal{M}, \mathcal{N}$ and $\mathcal{R}$ respectively.

Proof. Since $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{R}$ are closed orthogonal summand submodules, then there exists orthogonal projections $P_{\mathcal{M}}$, $P_{\mathcal{N}}$ and $P_{\mathcal{R}}$ onto $\mathcal{M}, \mathcal{N}$ and $\mathcal{R}$ respectively. Hence, $P_{\mathcal{R}}P_{\mathcal{N}} = P_{\mathcal{N}}$, $P_{\mathcal{N}}P_{\mathcal{M}} = P_{\mathcal{M}}$ and $P_{\mathcal{R}}P_{\mathcal{M}} = P_{\mathcal{M}}$ have closed ranges, which means that $P_{\mathcal{N}}P_{\mathcal{R}} = P_{\mathcal{N}}$ $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}$ and $P_{\mathcal{M}}P_{\mathcal{R}} = P_{\mathcal{M}}$ by taking $*$-operation. This is implies that (i) holds. By Lemma 2.5 and Corollary 2.8 we conclude the assertions (ii)-(vi). ∎

Theorem 2.10. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and P, Q be orthogonal projections in $\mathcal{L}(\mathcal{X})$ and $\mathrm{ran(P)} \subseteq \mathrm{ran(Q)}$ and $Q - P$ has closed range, then PQ and $1 - P - Q$ have closed ranges.*

Proof. Since $Q - P$ has closed range, $\ker(Q - P)$ is an orthogonal summand. Hence, Lemma 3.4 of [9] and the fact that, $\ker(P - Q) = \mathrm{ran(P)} + \mathrm{ker(Q)}$, implies that PQ has a closed range. Moreover, Lemma 2.5 implies that $1 - P - Q$ has closed range and so some results as the previous Lemma are appeared. ∎

Suppose $\mathcal{M}$ and $\mathcal{N}$ are closed submodules of a Hilbert $\mathcal{A}$-module $\mathcal{X}$ and $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ are orthogonal projections onto $\mathcal{M}$ and $\mathcal{N}$, respectively. Then $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{M}}$ if and only if $P_{\mathcal{N}}P_{\mathcal{M}} = P_{\mathcal{M}}$, if and only if $\mathcal{M} \subset \mathcal{N}$, see [9] for more detailed information.

Proposition 2.11. *Let $\mathcal{M}$ and $\mathcal{N}$ be closed submodules of a Hilbert $\mathcal{A}$-module $\mathcal{X}$ and $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ are orthogonal projection onto $\mathcal{M}$ and $\mathcal{N}$ respectively. If $\mathcal{M} \subset \mathcal{N}$, then the following assertions are equivalent:*

(i) *$P_{\mathcal{N}}P_{\mathcal{M}}$ has closed range,*
(ii) *$P_{\mathcal{M}}$ has closed range.*

Proof. (i)$\to$(ii) Suppose $\mathcal{M} \subset \mathcal{N}$. Then $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{M}}$ and $P_{\mathcal{N}}P_{\mathcal{M}} = P_{\mathcal{M}}$ and so, $\mathrm{ran(P_{\mathcal{M}}P_{\mathcal{N}})} = \mathrm{ran(P_{\mathcal{M}})}$ and $\ker(P_{\mathcal{M}}P_{\mathcal{N}}) = \ker(P_{\mathcal{M}})$. Therefore,

$$\begin{aligned} \mathcal{X} &= \mathrm{ran(P_{\mathcal{M}}P_{\mathcal{N}})} \oplus (\mathrm{ran(P_{\mathcal{M}}P_{\mathcal{N}})})^{\perp} \\ &= \mathrm{ran(P_{\mathcal{M}}P_{\mathcal{N}})} \oplus \mathrm{ker((P_{\mathcal{M}}P_{\mathcal{N}})^*)} \\ &= \mathrm{ran(P_{\mathcal{M}}P_{\mathcal{N}})} \oplus \mathrm{ker(P_{\mathcal{N}}P_{\mathcal{M}})} \\ &= \mathrm{ran(P_M)} \oplus \mathrm{ker(P_M)}. \end{aligned}$$

On the other hand, since $P_{\mathcal{N}}P_{\mathcal{M}}$ has a closed range, we have by Lemma 2.1

$$\begin{aligned} \gamma(P_{\mathcal{N}}P_{\mathcal{M}}) &= \inf\{\|P_{\mathcal{N}}P_{\mathcal{M}}x\| : x \in \ker(P_{\mathcal{N}}P_{\mathcal{M}})^{\perp} \text{ and } \|x\| = 1\} \\ &= \inf\{\|P_{\mathcal{M}}x\| : x \in \ker(P_{\mathcal{M}})^{\perp} \text{ and } \|x\| = 1\} > 0. \end{aligned}$$

Then, $P_{\mathcal{M}}$ has closed range. By the above relation, the implication (ii)$\to$ (i) is obviously . ∎

The following theorem state the angle between $\ker(T)$ and $\mathrm{ran(S^*)}$ when T, S have closed ranges.

Theorem 2.12. *Let T, S have closed ranges, then* $\alpha_0(\ker(T), \mathrm{ran(S^*)}) = \frac{\pi}{2} - \alpha_0(\mathrm{ker(T)}, \mathrm{ker(S)})$

Proof. We know that, the $\ker(T)$ and $\ker(S)$ are submodules of $\mathcal{X}$. Since $\mathcal{X} = \ker T \oplus (\ker(T))^\perp = \ker(T) \oplus \mathrm{ran(T^*)}$ and $\mathcal{X} = \ker(S) \oplus (\ker(S))^\perp = \ker(S) \oplus \mathrm{ran(S^*)}$, we have

$$\alpha_0(\ker(T), \ker(S)) + \alpha_0(\ker(T), \mathrm{ran(S^*)}) = \frac{\pi}{2}.$$

Thus,

$$(P_{\ker(T)} P_{\mathrm{ran(S^*)}})|_{\ker(T)} = (P_{\ker(T)})|_{\ker(T)} - (P_{\ker(T)} P_{\ker(S)})|_{\ker(T)} = 1_{\ker(T)} - (P_{\ker(T)} P_{\ker(S)})|_{\ker(T)}.$$

Now, by spectral mapping theorem for $f(t) = 1 - t$ we have

$$sp((P_{\ker(T)} P_{\mathrm{ran(S^*)}})|_{\ker(T)}) = 1 - sp((P_{\ker(T)} P_{\ker(S)})|_{\ker(T)}),$$

where the $sp(.)$ stand for the spectrum of the operators as in [13]. Then, the element λ belongs to the spectrum of $(P_{\ker(T)} P_{\ker(S)})|_{\ker(T)}$ if and only if the element $1 - \lambda$ belongs to the spectrum of $(P_{\ker(T)} P_{\mathrm{ran(S^*)}})|_{\ker(T)}$. So, $\alpha_0(\ker(T), \mathrm{ran(S^*)}) = \frac{\pi}{2} - \alpha_0(\mathrm{ker(T)}, \mathrm{ker(S)})$. ∎

Lemma 2.13. *Let $T \in L(\mathcal{X}, \mathcal{Y})$ and $\mathcal{M}$ be a closed submodule of $\mathcal{X}$ and $P_{\mathcal{M}}$ be the orthogonal projection onto $\mathcal{M}$. Then* $\ker(P_{(\ker(T))^\perp} P_{\mathcal{M}}) = \mathcal{M}^\perp \oplus (\mathcal{M} \cap \ker(T))$.

Proof. Let $z \in \mathcal{M}^\perp \oplus (\mathcal{M} \cap \ker(T))$, $z = x \oplus y$ such that $x \in \mathcal{M}^\perp$ and $y \in \mathcal{M} \cap \ker(T)$, so $\langle x, y \rangle = 0$ and $T(y) = 0$ if and only if $P_{(\ker(T))^\perp} P_{\mathcal{M}}(z) = P_{(\ker(T))^\perp} P_{\mathcal{M}}(x) + P_{(\ker(T))^\perp} P_{\mathcal{M}}(y) = P_{(\ker(T))^\perp}(y) = 0$. The latter equation holds since $y \in \ker(T)$ and $P_{(\ker(T))^\perp}(y) = 0$. ∎

Corollary 2.14. *Let $T, S \in L(\mathcal{X}, \mathcal{Y})$ have closed ranges, then* $(\ker(P_{(\ker(T))^\perp} P_{\mathrm{ran(S)}}))^\perp = 0$ *or* $\ker(P_{\mathrm{ran(T^*)}} P_{\mathrm{ran(S)}}) = \mathcal{X}$.

Proof. By Lemma 2.13 and this fact that $(\ker(T))^\perp = \mathrm{ran(T^*)}$, we have $\ker(P_{(\ker(T))^\perp} P_{\mathrm{ran(S)}}) = (\mathrm{ran(S)})^\perp \oplus (\mathrm{ran(S)} \cap \mathrm{ker(T)}) = \mathrm{ran(S^*)} \oplus (\mathrm{ran(S)} \cap \mathrm{ker(T)})$. It follows that,

$$(\ker(P_{(\ker(T))^\perp} P_{\mathrm{ran(S)}}))^\perp = \mathrm{ran(S)} \cap (\mathrm{ran(S)} \cap \mathrm{ker(T)})^\perp$$

which implies that, $(\ker(P_{(\ker(T))^\perp} P_{\mathrm{ran(S)}}))^\perp = 0$. ∎

3. THE MIXED REVERSE ORDER LAW

If $(TS)^\dagger = S^\dagger T^\dagger$, we say reverse order law hold. Reverse order law can be defined for three or more than operators. Moreover, if it combinate by another operators, it called mixed reverse order law. In general, there is no relation between $(TS)^\dagger$ with $T^\dagger$ and $S^\dagger$ except in some especial cases. This problem was first studied by Bouldin and Izumino for bounded operators between Hilbert spaces, see [6]. Recently Sharifi [9] and Mohammadzadeh Karizaki [10, 11, 12] studied Moore -Penrose inverse of product of the operators with closed range in Hilbert C^*-modules. The reader is referred to [1] - [5] and the references cited therein for more details of this discuses on Hilbert C^*-modules. In this section, we present the conditions which is state that, the mixed reverse order law hold.

Definition 3.1. An operator $T \in \mathcal{L}(\mathcal{X})$ is called the EP operator, if $\mathrm{ran(T)} = \mathrm{ran(T^*)}$.

Proposition 3.1. *Let $T, S \in L(\mathcal{X}, \mathcal{Y})$ with $\ker(T) = \ker(S)$ and let S has closed range. If the S is EP, then T has closed range.*

Proof. From the fact that, $\ker(T) \subseteq \ker(S)$, then S and T are injective on $(\ker(T))^{\perp}$. But $(\ker(T))^{\perp} = \mathrm{ran}(S^{\dagger}) = \mathrm{ran}(S^{*}) = \mathrm{ran}(S)$. Since, S has closed range, then $T((\ker(S))^{\perp})$ is closed in $\mathcal{Y}$. Therefore, $\mathrm{ran(T)} = \mathrm{T}(\mathcal{X}) = \mathrm{T}(\ker(S) \oplus \mathrm{ran}(S^*)) = \mathrm{T}(\ker(T) \oplus \mathrm{ran}(S)) = \mathrm{T}(\mathrm{ran}(S))$ is closed in $\mathcal{Y}$. ∎

In the following theorem, we show that the Moore-Penrose inverse of the convergent of a sequence of closed range operators, is converging to the Moore-Penrose inverse of the convergence limit of them.

Proposition 3.2. *Let $T_n \in L(\mathcal{X}, \mathcal{Y})$ be a sequence of closed range operators which is converges to a closed range operator T, then $T_n^{\dagger} \longrightarrow T^{\dagger}$.*

Proof. Let us note that, if $T_n \longrightarrow T$, then $T_n^* \longrightarrow T^*$. By this, we obtain

$$\begin{aligned} T_n^{\dagger} - T^{\dagger} &= -T_n^{\dagger}T_nT^{\dagger} + T_n^{\dagger}TT^{\dagger} + T_n^*(T^{\dagger})^*T^{\dagger} - T^*(T^{\dagger})^*T^{\dagger} - T_n^{\dagger}T_nT_n^*(T^{\dagger})^*T^{\dagger} - T^*(T^{\dagger})^*T^{\dagger} \\ &\quad + T_n^{\dagger}(T_n^{\dagger})^*T_n^* - T_n^{\dagger}(T_n^{\dagger})^*T^* - T_n^{\dagger}(T_n^{\dagger})^*T_n^* + T_n^{\dagger}(T_n^{\dagger})^*T^*TT^{\dagger} \\ &= -T_n^{\dagger}(T_n - T)T^{\dagger} + (1 - T_n^{\dagger}T_n)(T_n^* - T^*)(T^{\dagger})^*T^{\dagger} + T_n^{\dagger}(T_n^{\dagger})^*(T_n^* - T^*)(1 - TT^{\dagger}). \end{aligned}$$

This complete the proof. ∎

Proposition 3.3. *Let $T, S \in L(\mathcal{X}, \mathcal{Y})$ and let S has closed range. If the following property hold and $T(\ker(S)) \subseteq T((\ker(S))^{\perp})$, then T has closed range.*

(3.1) *There exists a constant $c > 0$ such that* $\|Sx\| \leq c\|Tx\|$ *for all* $x \in \mathcal{X}$.

Proof. Since $S \in L(\mathcal{X})$, then there exists $k > 0$ such that $\|Sx\| \geq k\|x\|$ for all $x \in (\ker(S))^{\perp}$. From the property 3.1 $\|Tx\| \geq \frac{k}{c}\|x\|$ for all $x \in (\ker(S))^{\perp}$. Thus $T((\ker(S))^{\perp})$ is closed and so T has closed range. ∎

Theorem 3.4. *Let $\mathcal{X}$ be Hilbert $\mathcal{A}$-module and T, S have closed ranges. Then, $S^{\dagger}T^{\dagger}TSS^{\dagger}T^{\dagger} = S^{\dagger}T^{\dagger}$ if and only if $(SS^{\dagger}T^{\dagger}T)^2 = SS^{\dagger}T^{\dagger}T$.*

Proof. ($\Leftarrow$) By multiplying the $SS^{\dagger}T^{\dagger}TSS^{\dagger}T^{\dagger}T = SS^{\dagger}T^{\dagger}T$ by $S^{\dagger}$ from the left hand and by $T^{\dagger}$ from the right hand, we have

$$S^{\dagger}SS^{\dagger}T^{\dagger}TSS^{\dagger}T^{\dagger}TT^{\dagger} = S^{\dagger}SS^{\dagger}T^{\dagger}TT^{\dagger},$$

so, from the Moore-Penrose conditions we obtain

$$S^{\dagger}T^{\dagger}TSS^{\dagger}T^{\dagger} = S^{\dagger}T^{\dagger}.$$

($\Rightarrow$) Multiplying the $S^{\dagger}T^{\dagger}$ by S from the left side and by T from the right side, we have $S(S^{\dagger}T^{\dagger})T = S(S^{\dagger}T^{\dagger}TSS^{\dagger}T^{\dagger})T = (SS^{\dagger}T^{\dagger}T)(SS^{\dagger}T^{\dagger}T)$. ∎

Theorem 3.5. *Let $\mathcal{X}$ be Hilbert $\mathcal{A}$-module and let T, S have closed ranges. Then, $T^{\dagger}TSS^{\dagger}$ is an idempotent if and only if its Moore-Penrose inverse satisfying the reverse order law. i.e. $(T^{\dagger}TSS^{\dagger})^{\dagger} = SS^{\dagger}T^{\dagger}T$.*

Proof. ($\Rightarrow$) Suppose that $(T^{\dagger}TSS^{\dagger})(T^{\dagger}TSS^{\dagger}) = T^{\dagger}TSS^{\dagger}$, then $T^{\dagger}TSS^{\dagger} = SS^{\dagger}T^{\dagger}T$.
Hence, $(T^{\dagger}TSS^{\dagger})^{\dagger} = SS^{\dagger}T^{\dagger}T$.
($\Leftarrow$) From the Moore-Penrose condition, we have

$$\begin{aligned} T^{\dagger}TSS^{\dagger}T^{\dagger}TSS^{\dagger} &= T^{\dagger}TSS^{\dagger}(SS^{\dagger}T^{\dagger}T)T^{\dagger}TSS^{\dagger} \\ &= T^{\dagger}TSS^{\dagger}(T^{\dagger}TSS^{\dagger})^{\dagger}T^{\dagger}TSS^{\dagger} \\ &= T^{\dagger}TSS^{\dagger}. \end{aligned}$$

∎

Theorem 3.6. *Let $\mathcal{X}$ be Hilbert $\mathcal{A}$-module and let T, S have closed ranges. Then, $TSS^\dagger T^\dagger TS = TS \Leftrightarrow T^\dagger T$ commute with $SS^\dagger$.*

Proof. $(\Rightarrow)$ Multiplying the $TSS^\dagger T^\dagger TS = TS$ by $T^\dagger$ from the left hand, we have $T^\dagger TSS^\dagger T^\dagger TS = T^\dagger TSS^\dagger S$. So, $T^\dagger TSS^\dagger(1 - T^\dagger T)S = 0$.

Now, since $T^\dagger T$ is projection we have,

$$\begin{aligned} T^\dagger T(S^\dagger)^*((1 - T^\dagger T)S)^*(1 - T^\dagger T)S &= T^\dagger T(S^\dagger)^* S^*(1 - T^\dagger T)(1 - T^\dagger T)S \\ &= T^\dagger TSS^\dagger(1 - T^\dagger T)S \\ &= 0. \end{aligned}$$

Therefore, $T^\dagger T(S^\dagger)^*((1 - T^\dagger T)S)^* = 0$, which is implies that,

$$\begin{aligned} T^\dagger TSS^\dagger &= T^\dagger TSS^\dagger T^\dagger T \\ &= (T^\dagger TSS^\dagger TT^\dagger)^* \\ &= (T^\dagger TSS^\dagger)^* \\ &= SS^\dagger T^\dagger T. \end{aligned}$$

$(\Leftarrow)$ By multiplying the $T^\dagger TSS^\dagger = SS^\dagger T^\dagger T$ by the $T^\dagger T$ from the left side and by the $SS^\dagger$ from the right side, we obtain

$$\begin{aligned} T^\dagger T(SS^\dagger T^\dagger)SS^\dagger &= T^\dagger T(T^\dagger TSS^\dagger)SS^\dagger \\ &= T^\dagger TSS^\dagger. \end{aligned}$$

Again, multiplying $T^\dagger TSS^\dagger = T^\dagger TSS^\dagger T^\dagger SS^\dagger$ by the T from the left hand and S from the right hand, we have

$$\begin{aligned} T(T^\dagger TSS^\dagger)S &= T(T^\dagger TSS^\dagger T^\dagger SS^\dagger)S \\ &= TSS^\dagger T^\dagger TS. \end{aligned}$$

Which is complete the proof. ∎

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