The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 15, Issue 1, Article 9, pp. 1-15, 2018

FRACTIONAL CLASS OF ANALYTIC FUNCTIONS DEFINED USING q-DIFFERENTIAL OPERATOR

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Received 27 May, 2017; accepted 10 November, 2017; published 24 May, 2018.

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ABSTRACT. We define a q-differential fractional operator, which generalizes Sălăgean and Ruscheweyh differential operators. We introduce and study a new class of analytic functions involving q-differential fractional operator. We also determine the necessary and sufficient conditions for functions to be in the class. Further, we obtain the coefficient estimates, extreme points, growth and distortion bounds.

Key words and phrases: q-calculus, Factional calculus, Sălăgean differential operator, Subordination, Starlike function, Convex function.

2000 *Mathematics Subject Classification.* Primary 30C45, Secondary 30C80.

ISSN (electronic): 1449-5910

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Recently, one of the substantive issues in many applications of geometric function theory is how to employ the fractional operators to analytic univalent functions and what are the advantages of this operators. So far, many mathematicians in different stages considered this issues and gave numerous applications based on certain fractional operators of analytic function in physics, engineering and mathematical applications (see [9]). Also, the area of the q-analysis has attracted serious attention of researchers. The great interest is due to its applications in various branches of mathematics and physics, as for example, in the areas of ordinary fractional calculus, optimal control problems, q-difference and $q-$integral equations and in q-transform analysis. The generalized $q-$Taylor formula in the fractional $q-$calculus was introduced by Purohit and Raina [23]. The application of $q-$calculus was initiated by Jackson [13, 14]. He was the first to develop the $q-$integral and $q-$derivative in a systematic way. Later, geometrical interpretation of the $q-$analysis has been recognized through studies on quantum groups. Simply, the quantum calculus is ordinary classical calculus without the notion of limits. It defines $q-$calculus and $h-$calculus. Here h ostensibly stands for Planck's constant, while q stands for quantum. Mohammed and Darus [20] studied approximation and geometric properties of these $q-$operators in some subclasses of analytic functions in compact disk. Recently, Purohit and Raina [23, 24] have used the fractional $q-$calculus operators in investigating certain classes of functions which are analytic in the open disk. Also Purohit [22] also studied these $q-$operators, defined by using the convolution of normalized analytic functions and $q-$hypergeometric functions. A comprehensive study on the applications of $q-$calculus in the operator theory may be found in [1].

The q-difference operator denoted as $D_q f(z)$ is defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}\,, (f \in \mathcal{A}\,,\ z \in \mathcal{U} - \{0\}),$$

and $D_q f(0) = f'(0)$, where $q \in (0,1)$. It can be easily seen that $D_q f(z) \to f'(z)$ as $q \to 1^-$.

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathcal{U}$. We denote by $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{C}^*$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$. Our favorite references of the field are [6, 7, 8] which covers most of the topics in a lucid and economical style.

The concept of starlike functions and convex functions were further extended as follows.

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : Re \frac{zf'(z)}{f(z)} > \alpha, z \in \mathcal{U} \right\},$$

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{A} : Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, z \in \mathcal{U} \right\}.$$

We note that

$$f \in \mathcal{C}(\alpha) \Leftrightarrow zf' \in \mathcal{S}^*(\alpha),$$

where $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ are respectively, the classes of starlike of order α and convex of order α in $\mathcal{U}$ (see Robertson [25]).

Similarly, close-to convex functions and quasi-convex functions were further extended as follows.

$$\mathcal{K}(\alpha, \beta) = \left\{ f \in \mathcal{A} : Re \frac{zf'(z)}{g(z)} > \alpha,\ g \in \mathcal{S}^*(\beta),\ z \in \mathcal{U} \right\},$$

$$\mathcal{C}^*(\alpha, \beta) = \left\{ f \in \mathcal{A} : Re \left(\frac{\left(zf'(z)\right)'}{g'(z)} \right) > \alpha,\ g \in \mathcal{C}(\beta),\ z \in \mathcal{U} \right\},$$

where $\mathcal{K}(\alpha,\beta)$ and $\mathcal{C}^*(\alpha,\beta)$ are respectively, the classes of close-to-convex of order α type β and quasi-convex of order α and type β in $\mathcal{U}$ (see K. I. Noor and D. K. Thomas [21]).

The Koebe function which plays a pivotal role in the study of univalent function theory is given by

$$f(z)=\frac{z}{(1-z)^2}=\sum_{n=2}^{\infty} n\,z^n.$$

The rotated Koebe function is

$$f_\alpha(z)=\frac{z}{(1-\alpha z)^2}=\sum_{n=2}^{\infty} n\alpha^{n-1}\,z^n,$$

with α being a complex number of absolute value 1. The Koebe function and its rotations are univalent and achieving the normalization $f(0)=0$ and $f'(0)=1$.

Srivastava et al (see [27]), introduced a fractional analytic function as follows:

$$f(z)=\frac{z^{\alpha+1}}{(1-z)^2},\ \ \alpha\in\mathcal{R}.$$

In this effort, we let $\mathcal{A}_\mu$ to denote the class of functional fractional analytic functions $F_\mu(z)$ in unit disk $\mathcal{U}=\{z\in\mathcal{C};|z|<1\}$ as follows:

$$F_\mu(z)=\frac{z^\mu}{1-z^\mu}, \tag{1.1}$$

where $\mu:=\frac{n+m-1}{m}$, $n,\,m\in N$. Hence $\mu=1$, when $n=1$ and has the formal power series:

$$F_\mu\,(z)=z+\sum_{n=2}^{\infty}a_n\,z^{\mu n},\qquad (\mu\geq 1;n\in\mathcal{N},z\in\mathcal{U}). \tag{1.2}$$

which is normalized by $F_\mu\,(z)=1$ and $F'_\mu\,(z)=1$ at $z=0$ for all $z\in\mathcal{U}$. Recall that a function $F_\mu\in\mathcal{A}_\mu$ is called bounded turning if it satisfies the following inequality:

$$Re(F'_\mu\,(z))>\psi\qquad (0\ \leq\ \psi\ <\ 1) \tag{1.3}$$

and a function $F_\mu\in\mathcal{A}_\mu$ is starlike function in $\mathcal{U}$ if it satisfies

$$Re\left(\frac{z\,F'_\mu\,(z)}{F_\mu\,(z)}\right)>\psi\qquad (0\ \leq\ \psi\ <\ 1). \tag{1.4}$$

Furthermore, a function $F_\mu\in\mathcal{A}_\mu$ is convex in $\mathcal{U}$ if it satisfies

$$Re\left(1+\frac{z\,F''_\mu\,(z)}{F'_\mu\,(z)}\right)>\psi\qquad (0\ \leq\ \psi\ <\ 1). \tag{1.5}$$

For more details about the subclasses of univalent functions defined using functions in $\mathcal{A}_\mu$, we refer to [5] and [10].

Next, if the function $F_\mu\,(z)$ of form (1.2) and

$$G_\mu\,(z)=z+\sum_{n=2}^{\infty}b_n\,z^{\mu n}$$

are two functions in the class $\mathcal{A}_\mu$, then the convolution (or Hadamard product) of two analytic functions is denoted by $F_\mu(z) * G_\mu(z)$ and is given by

$$F_\mu(z) * G_\mu(z) = z + \sum_{n=2}^{\infty} a_n b_n\, z^{\mu n}.$$

and satisfy $[F_\mu(z) * G_\mu(z)] = 0$ and $(F_\mu(z) * G_\mu(z))' = 1$ at $z = 0$.

Now let a functional function $\Theta_\mu(z)$ defined as follows:

$$\Theta_\mu(z) = \frac{\mu z^\mu}{(1-z^\mu)^2} + \frac{\mu z^\mu}{1-z^\mu} = z + \sum_{n=2}^{\infty} (\mu n)\, z^{\mu n}\ ,\ (z \in \mathcal{U}).$$

By employing Hadamard product of analytic function $\Theta_\mu(z)$, we obtain

$$\begin{aligned}\Theta_{\mu,k}(z) &= F_\mu(z) * \ldots * F_\mu(z)\ ,\ k \text{ times}\\ &= z + \sum_{n=2}^{\infty} (\mu n)^k\, z^{\mu n}\ ,\ (z \in \mathcal{U}).\end{aligned}$$

The q-analogue of Sălăgean differential operator (see [31]) $R_q^m f(z) : \mathcal{A} \to \mathcal{A}$ for $m \in N$, is formed as follows.

$$\begin{aligned} R_q^0 f(z) &= f(z)\\ R_q^1 f(z) &= z(D_q f(z))\\ &\ \ \vdots\\ R_q^m f(z) &= R_q^1\left(R_q^{m-1} f(z)\right).\end{aligned}$$

From the definition of $R_q^m f(z)$, we get

$$R_q^m f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1-q^n}{1-q}\right)^m a_n z^n, \qquad (z \in \mathcal{U}). \tag{1.6}$$

It can be seen that if we let $q \to 1^-$, then $R_q^m f(z)$ reduces to the well-known Sălăgean differential operator [31].

For $F_\mu \in \mathcal{A}_\mu$, analogous to $R_q^m f(z)$ we define a differential operator $R_q^m F_\mu(z) : \mathcal{A}_\mu \to \mathcal{A}_\mu$ for $m \in N$, as follows.

$$R_q^m F_\mu(z) = z + \sum_{n=2}^{\infty} \left(\frac{1-q^{\mu n}}{1-q}\right)^m a_n z^{\mu n}, \tag{1.7}$$

$$(\mu \geq 1; n \in \mathbb{N}\setminus\{1\};\ m \in \mathbb{N}_0; z \in \mathcal{U}).$$

Remark 1.1. If we let $q \to 1^-$ in (1.7), then it can be easily seen that $R_q^m F_\mu(z)$ reduces to the operator defined by Abdulnaby et. al. (see [33]).

Let f and g be analytic in the open unit disk $\mathcal{U}$. The function f is subordinate to g written as $f \prec g$ in $\mathcal{U}$, if there exist a function w analytic in $\mathcal{U}$ with $w(0) = 0$ and $\mid w(z) \mid < 1$; $(z \in \mathcal{U})$ such that $f(z) = g(w(z))$, $(z \in \mathcal{U})$.

Now we define a new class of analytic functions and investigate several interesting results.

Definition 1.1. Let the functions

$$\Phi_\mu(z) = z + \sum_{n=2}^{\infty} \vartheta_n\, z^{\mu n} \text{ and } \Psi_\mu(z) = z + \sum_{n=2}^{\infty} \lambda_n\, z^{\mu n},$$

be analytic in the open unit disk $\mathcal{U}$ where

$$\vartheta_n \geq 0, \lambda_n \geq 0 \text{ and } \vartheta_n \geq \lambda_n \ (n \in \mathbb{N}\backslash\{0, 1\}).$$

Then a function $F_\mu(z) \in A_\mu(z)$ is said to be in the class $\varepsilon_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$ if and only if

$$\frac{R_q^k (F_\mu * \Phi_\mu)(z)}{R_q^m (F_\mu * \Psi_\mu)(z)} \prec (1-\gamma)\frac{1+Az}{1+Bz} + \gamma, \quad (z \in \mathcal{U}),$$

where $F_\mu(z) * \Psi_\mu(z) \neq 0$, A and B are arbitrarily fixed numbers such that $-1 \leq B < A \leq 1$ and $-1 \leq B < 0$ with $-1 \leq \gamma \leq 1$ and $k \geq m \ (k, m \in \mathbb{N}_0)$.

Let χ_μ be the class of analytic functions $F_\mu(z)$ in unit disk $\mathcal{U}$ of the following form

(1.8)
$$F_\mu(z) = z - \sum_{n=2}^{\infty} a_n z^{\mu n},$$
$$(a_n \geq 1; \mu \geq 1; n \in \mathbb{N} \setminus \{1\})$$

which satisfies $F_\mu(z)\,|_{z=0} = 1$ and $F'_\mu(z)\,|_{z=0} = 1$ for all $z \in \mathcal{U}$. We let

$$\tilde{\varepsilon}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma) = \varepsilon_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma) \cap \chi_\mu.$$

For suitable choices of Φ and Ψ , we obtain various subclasses of $\mathcal{A}_\mu$. For example, we have the following:

(1.9)
$$\tilde{\varepsilon}_{0,0}\left(\frac{\mu z^\mu}{(1-z^\mu)^2}, \frac{z^\mu}{1-z^\mu}, 1, -1, \mu, \gamma\right) = S^*_\mu(\gamma),$$

and

(1.10)
$$\tilde{\varepsilon}_{0,0}\left(\frac{\mu^2 z^\mu + \mu^2 z^2 \mu}{(1-z^\mu)^3}, \frac{\mu z^\mu}{(1-z^\mu)^2}, 1, -1, \mu, \gamma\right) = \kappa^*_\mu(\gamma).$$

If $\mu = 1$ in (1.9) and (1.10) respectively, we obtain the well known subclasses S^*_μ and κ^*_μ which were investigated by Silverman in [18].

2. CHARACTERIZATION PROPERTIES

In this section, we consider several properties for the function $F_\mu(z) \in \tilde{\varepsilon}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$. We will divide this section into five subsections.

Theorem 2.1. *If $F_\mu(z) \in A_\mu$ satisfies the following inequality*

(2.1)
$$\sum_{n=2}^{\infty}\left[(1-B)\left[\left(\frac{1-q^{\mu n}}{1-q}\right)^k \vartheta_n - \left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n\right] + (A-B)(1-\gamma)\left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n\right]|a_n| \leq (A-B)(1-\lambda).$$

$$(\vartheta_n \geq 0, \lambda_n \geq 0; \vartheta_n \geq \lambda_n (n \in \mathbb{N}\backslash\{1\}); \mu \geq 1; 0 \leq \gamma < 1; k \geq m, k, m \in \mathbb{N}_0)$$

Then

$$F_\mu(z) \in \varepsilon_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma).$$

Proof. Let the condition (2.1) holds, then we obtain

$$\left|R_q^k\left(F_\mu * \Phi_\mu\right)(z) - R_q^m\left(F_\mu * \Psi_\mu\right)(z)\right| - \left|(A-B)(1-\gamma)R_q^m\left(F_\mu * \Psi_\mu\right)(z) - B[R_q^k\left(F_\mu * \Phi_\mu\right)(z)\right.$$
$$\left.-R_q^m\left(F_\mu * \Psi_\mu\right)(z)]\right| = \left|\sum_{n=2}^{\infty}\left[\left(\frac{1-q^{\mu n}}{1-q}\right)^k \vartheta_n - \left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n\right] a_n z^{\mu n}\right| - |(A-B)(1-\gamma)z$$
$$+(A-B)(1-\gamma)\sum_{n=2}^{\infty}\left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n a_n z^{\mu n} - B\sum_{n=2}^{\infty}\left[\left(\frac{1-q^{\mu n}}{1-q}\right)^k \vartheta_n - \left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n\right] a_n z^{\mu n}\Bigg|.$$

$$\tag{2.2} \leq \sum_{n=2}^{\infty}\left[\left(\frac{1-q^{\mu n}}{1-q}\right)^k \vartheta_n - \left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n\right]|a_n|\, r^{\mu n} + (A-B)(1-\gamma)r$$
$$+(A-B)(1-\gamma)\sum_{n=2}^{\infty}\left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n |a_n|\, r^{\mu n} + |B|\sum_{n=2}^{\infty}\left[\left(\frac{1-q^{\mu n}}{1-q}\right)^k \vartheta_n - \left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n\right]|a_n|\, r^{\mu n}$$

$$\tag{2.3} \begin{aligned} \leq \sum_{n=2}^{\infty}\Bigg[(1-B)&\left(\left(\frac{1-q^{\mu n}}{1-q}\right)^k \vartheta_n - \left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n\right) \\ &+(A-B)(1-\gamma)\left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n\Bigg]|a_n| - (A-B)(1-\gamma) \leq 0. \end{aligned}$$

For all $r(0 \leq r \leq 1)$, the inequality in (2.2) holds true. Thus, letting $r \to 1-$ in (2.2), we obtain (2.3). This completes the proof of the Theorem 2.1.

Remark 2.1. If we let $q \to 1^-$ in Theorem 2.1, then we get the result obtained by Zainab E. Abdulnaby (see [33]).

Theorem 2.2. *If $F_\mu(z) \in \chi_\mu$ satisfies the following inequality*

$$\tag{2.4} \begin{aligned} \sum_{n=2}^{\infty}\Bigg[(1-B)&\left[\left(\frac{1-q^{\mu n}}{1-q}\right)^k \vartheta_n - \left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n\right] \\ &+(A-B)(1-\lambda)\left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n\Bigg] a_n \leq (A-B)(1-\lambda). \end{aligned}$$

$$(\vartheta_n \geq 0,\, \lambda_n \geq 0;\, \vartheta_n \geq \lambda_n (n \in \mathbb{N}\backslash\{1\}); \mu \geq 1; 0 \leq \gamma < 1;\, k \geq m,\, k,\, m \in \mathbb{N}_0)$$

Then

$$F_\mu(z) \in \tilde{\varepsilon}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma).$$

Proof. Since

$$\tilde{\varepsilon}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma) \subset \varepsilon_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma),$$

we only need to prove the only if part of Theorem 2.2, for function $F_\mu(z) \in \chi_\mu$ we can write

$$\left|\frac{\frac{R_q^k(F_\mu*\Phi_\mu)(z)}{R_q^m(F_\mu*\Psi_\mu)(z)}-1}{(A-B)(1-\gamma)-B\left(\frac{R_q^k(F_\mu*\Phi_\mu)(z)}{R_q^m(F_\mu*\Psi_\mu)(z)}-1\right)}\right|$$
$$=\left|\frac{R_q^k\left(F_\mu*\Phi_\mu\right)(z)-R_q^m\left(F_\mu*\Psi_\mu\right)(z)}{(A-B)(1-\gamma)(R_q^m\left(F_\mu*\Psi_\mu\right)(z))-B\left(R_q^k\left(F_\mu*\Phi_\mu\right)(z)-R_q^m\left(F_\mu*\Psi_\mu\right)(z)\right)}\right|$$
$$=\left|\frac{\sum_{n=2}^{\infty}\left[\left(\frac{1-q^{\mu n}}{1-q}\right)^k\vartheta_n-\left(\frac{1-q^{\mu n}}{1-q}\right)^m\lambda_n\right]a_nz^{\mu n-1}}{(A-B)(1-\gamma)-(A-B)(1-\gamma)\sum_{n=2}^{\infty}\left(\frac{1-q^{\mu n}}{1-q}\right)^m\lambda_na_nz^{\mu n-1}}\right.$$
$$\left.+B\sum_{n=2}^{\infty}\left(\left(\frac{1-q^{\mu n}}{1-q}\right)^k\vartheta_n-\left(\frac{1-q^{\mu n}}{1-q}\right)^m\lambda_n\right)a_nz^{\mu n-1}\right|.$$

Since $Re(z)\leq|z|$ for all $z\in\mathcal{U},$ we have

$$Re\left[\frac{\sum_{n=2}^{\infty}\left[\left(\frac{1-q^{\mu n}}{1-q}\right)^k\vartheta_n-\left(\frac{1-q^{\mu n}}{1-q}\right)^m\lambda_n\right]a_nz^{\mu n-1}}{(A-B)(1-\gamma)-(A-B)(1-\gamma)\sum_{n=2}^{\infty}\left(\frac{1-q^{\mu n}}{1-q}\right)^m\lambda_na_nz^{\mu n-1}}\right.$$
$$\left.+B\sum_{n=2}^{\infty}\left(\left(\frac{1-q^{\mu n}}{1-q}\right)^k\vartheta_n-\left(\frac{1-q^{\mu n}}{1-q}\right)^m\lambda_n\right)a_nz^{\mu n-1}\right]\leq 1.$$

If we choose z to be real and let $z\to 1-,$ we obtain

$$\sum_{n=2}^{\infty}\left[(1-B)\left[\left(\frac{1-q^{\mu n}}{1-q}\right)^k\vartheta_n-\left(\frac{1-q^{\mu n}}{1-q}\right)^m\lambda_n\right]+(A-B)(1-\lambda)\left(\frac{1-q^{\mu n}}{1-q}\right)^m\lambda_n\right]a_n$$
$$\leq(A-B)(1-\gamma),$$

which is equivalent to (2.4). The result is sharp for functions $F_\mu(z)$ given by

(2.5)
$$F_\mu(z)=z-\frac{(A-B)(1-\gamma)}{(1-B)\left(\left(\frac{1-q^{\mu n}}{1-q}\right)^k\vartheta_n-\left(\frac{1-q^{\mu n}}{1-q}\right)^m\lambda_n\right)+(A-B)(1-\gamma)\left(\frac{1-q^{\mu n}}{1-q}\right)^m\lambda_n}z^{\mu n}\quad(n\geq 2).$$

This completes the proof of Theorem 2.2.

Corollary 2.3. *Let a function $F_\mu(z)$ defined by (1.8) belongs to $\tilde{\varepsilon}_{k,m}(\Phi_\mu,\Psi_\mu,A,B,\mu,\gamma)$. Then*

(2.6)
$$a_n\leq\frac{(A-B)(1-\gamma)}{(1-B)\left(\left(\frac{1-q^{\mu n}}{1-q}\right)^k\vartheta_n-\left(\frac{1-q^{\mu n}}{1-q}\right)^m\lambda_n\right)+(A-B)(1-\gamma)\left(\frac{1-q^{\mu n}}{1-q}\right)^m\lambda_n}\quad(n\geq 2).$$

Remark 2.2. By taking different choices for the functions $\Phi_\mu(z)$ and $\Psi_\mu(z)$ same as stated in (1.9) and (1.10), Theorem 2.2 leads us to the necessary and sufficient conditions for a function $F_\mu(z)$ to be in the following classes $S_\mu^*(\gamma)$ and $\kappa_\mu^*(\gamma)$.

3. DISTORTION THEOREMS

Theorem 3.1. *Let the function* $F_\mu(z)$ *defined by (1.8), be in* $\tilde{\varepsilon}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$. *Then*

(3.1)
$$F_\mu(z) \geq |z| - \frac{(A-B)(1-\gamma)}{(1-B)\left(\left(\frac{1-q^{2\mu}}{1-q}\right)^k \vartheta_n - \left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_n\right) + (A-B)(1-\gamma)\left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_n} |z|^{2\mu}$$
$$\leq |z| + \frac{(A-B)(1-\gamma)}{(1-B)\left(\left(\frac{1-q^{2\mu}}{1-q}\right)^k \vartheta_n - \left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_n\right) + (A-B)(1-\gamma)\left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_n} |z|^{2\mu}.$$

The result is sharp.

Proof. By considering Theorem 2.1, since

$$\Xi(n) = (1-B)\left[\left(\frac{1-q^{\mu n}}{1-q}\right)^k \vartheta_n - \left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n\right] + (A-B)(1-\gamma)\left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n$$

is an increasing function of $n(n \geq 2)$, we get

$$\Xi(2)\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}\Xi(n)\,|a_n| \leq (A-B)(1-\gamma),$$

that is

$$\sum_{n=2}^{\infty}|a_n| \leq \frac{(A-B)(1-\gamma)}{\Xi(2)}.$$

Therefore, we have

$$|F_\mu(z)| \leq |z| + |z|^{2\mu}\sum_{n=2}^{\infty}|a_n|,$$

$$|F_\mu(z)| \leq |z| + \frac{(A-B)(1-\gamma)}{(1-B)\left(\left(\frac{1-q^{2\mu}}{1-q}\right)^k \vartheta_2 - \left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_2\right) + (A-B)(1-\gamma)\left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_2} |z|^{2\mu}.$$

Similarly, we have

$$|F_\mu(z)| \geq |z| - |z|^{2\mu}\sum_{n=2}^{\infty}|a_n|,$$

$$|F_\mu(z)| \geq |z| - \frac{(A-B)(1-\gamma)}{(1-B)\left(\left(\frac{1-q^{2\mu}}{1-q}\right)^k \vartheta_2 - \left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_2\right) + (A-B)(1-\gamma)\left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_2} |z|^{2\mu}.$$

The result is sharp for the function

(3.2)
$$F_\mu(z) = |z| - \frac{(A-B)(1-\gamma)}{(1-B)\left(\left(\frac{1-q^{2\mu}}{1-q}\right)^k \vartheta_2 - \left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_2\right) + (A-B)(1-\gamma)\left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_2} |z|^{2\mu}.$$

This completes the proof of Theorem 3.1.

Theorem 3.2. *Let $F_\mu(z)$ defined by (1.8) be in the class $\tilde{\varepsilon}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$, then*

$$\left|F'_\mu(z)\right| \geq 1 - \frac{\left(\frac{1-q^{2\mu}}{1-q}\right)(A-B)(1-\gamma)}{(1-B)\left(\left(\frac{1-q^{2\mu}}{1-q}\right)^k \lambda_2 - \left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_2\right) + (A-B)(1-\gamma)\left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_2} |z|^{2\mu-1}$$
$$\leq 1 + \frac{\left(\frac{1-q^{2\mu}}{1-q}\right)(A-B)(1-\gamma)}{(1-B)\left(\left(\frac{1-q^{2\mu}}{1-q}\right)^k \lambda_2 - \left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_2\right) + (A-B)(1-\gamma)\left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_2} |z|^{2\mu-1}.$$

The result is sharp.

Proof. Similarly $\frac{\Xi(n)}{n}$ is an increasing function of $n (n \geq 1)$,

(3.3)
$$\frac{\Xi(2)}{\left(\frac{1-q^{2\mu}}{1-q}\right)} \sum_{n=2}^{\infty} \left(\frac{1-q^{\mu n}}{1-q}\right) |a_n| \leq \sum_{n=2}^{\infty} \frac{\Xi(n)}{\left(\frac{1-q^{\mu n}}{1-q}\right)} \left(\frac{1-q^{\mu n}}{1-q}\right) |a_n| = \sum_{n=2}^{\infty} \Xi(n) |a_n| \leq (A-B)(1-\gamma),$$

that is

$$\sum_{n=2}^{\infty} \left(\frac{1-q^{\mu n}}{1-q}\right) |a_n| \leq \frac{\left(\frac{1-q^{2\mu}}{1-q}\right)(A-B)(1-\gamma)}{\Xi(2)}. \tag{3.4}$$

Then, we get

$$\left|F'_\mu(z)\right| \leq 1 + |z|^{2\mu-1} \sum_{n=2}^{\infty} \left(\frac{1-q^{\mu n}}{1-q}\right) |a_n|,$$

(3.5)
$$\left|F'_\mu(z)\right| \leq 1 + \frac{\left(\frac{1-q^{\mu n}}{1-q}\right)(A-B)(1-\gamma)}{(1-B)\left(\left(\frac{1-q^{2\mu}}{1-q}\right)^k \vartheta_2 - \left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_2\right) + (A-B)(1-\gamma)\left(\frac{1-q^{2\mu}}{1-q}\right)^m \vartheta_2} |z|^{2\mu-1}.$$

Similarly, we have

$$\left|F'_\mu(z)\right| \geq 1 - |z|^{2\mu-1} \sum_{n=2}^{\infty} \left(\frac{1-q^{\mu n}}{1-q}\right) |a_n|,$$

(3.6)
$$\left|F'_\mu(z)\right| \geq 1 - \frac{\left(\frac{1-q^{\mu n}}{1-q}\right)(A-B)(1-\gamma)}{(1-B)\left(\left(\frac{1-q^{2\mu}}{1-q}\right)^k \vartheta_2 - \left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_2\right) + (A-B)(1-\gamma)\left(\frac{1-q^{2\mu}}{1-q}\right)^m \lambda_2} |z|^{2\mu-1}.$$

It is clear that the assertions of Theorem 3.2 are sharp for the function $F_\mu(z)$ given by (3.2). This complete the proof of Theorem 3.2. ∎

4. THE RADII SUBCLASSES OF CLASS $\tilde{\varepsilon}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$

In this section radii of bounded turning, convexity and starlikeness for functions $F_\mu(z) \in \tilde{\varepsilon}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$ are studied.

The real number

$$r_\rho^*(f) = \sup\{r > 0\,|Re\,(k(z)) > \rho \text{ for all } z \in \mathcal{U}_r\}$$

is called the radius of starlikeness of order ρ of the function f when $k(z) = \frac{zf'(z)}{f(z)}$. Note that $r_\rho^*(f) = r_0^*(f)$ is in fact the largest radius such that the image region $f\left(\mathcal{U}_{r^*(f)}\right)$ is a starlike domain with respect to the origin. Similar definition is used to define radius of convexity and close to convexity by equivalently replacing $k(z)$ with $1 + \frac{zf'(z)}{f(z)}$ and $\frac{f'(z)}{g'(z)}$ respectively. For the study of various radius problems, we refer to [2, 15, 16, 26, 32].

Hereafter, we let

$$\Omega_n^q\,(A, B;\,\gamma) = (1-B)\left(\left(\frac{1-q^{\mu n}}{1-q}\right)^k \vartheta_n - \left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n\right) + (A-B)(1-\gamma)\left(\frac{1-q^{\mu n}}{1-q}\right)^m \lambda_n.$$

Theorem 4.1. *Let $F_\mu(z)$ given by (1.8) be in the class $\tilde{\varepsilon}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$, then*

(i) *$F_\mu(z)$ is starlike of order $\psi (0 \le \psi < 1)$ in $|z| < r_1$, where*

$$r_1 = \inf_{n\ge 2}\left[\frac{\Omega_n^q\,(A, B;\,\gamma)}{(A-B)(1-\gamma)} \times \left(\frac{1-\psi}{\mu n - \psi}\right)\right]^{1/(\mu n - 1)}. \tag{4.1}$$

(ii) *$F_\mu(z)$ is convex of order $\psi (0 \le \psi < 1)$ in $|z| < r_2$, where*

$$r_2 = \inf_{n\ge 2}\left[\frac{\Omega_n^q\,(A, B;\,\gamma)}{(A-B)(1-\gamma)} \times \left(\frac{1-\psi}{\mu n(\mu n - \psi)}\right)\right]^{1/(\mu n - 1)}. \tag{4.2}$$

(iii) *$F_\mu(z)$ is close to convex of order $\psi (0 \le \psi < 1)$ in $|z| < r_3$, where*

$$r_3 = \inf_{n\ge 2}\left[\frac{\Omega_n^q\,(A, B;\,\gamma)}{(A-B)(1-\gamma)} \times \left(\frac{1-\psi}{\mu n}\right)\right]^{1/(\mu n - 1)}. \tag{4.3}$$

Each of the results are sharp for the function $F_\mu(z)$ given by (3.2).

Proof. It is sufficient show that

$$\left|\frac{zF_\mu'(z)}{F_\mu'(z)}\right| \le 1 - \psi, \quad \text{for } |z| < r_1, \tag{4.4}$$

where r_1 is defined by (4.1). Further, we find from (1.8) that

$$\left|\frac{zF_\mu'(z)}{F_\mu'(z)} - 1\right| \le \frac{\sum_{n=2}^{\infty}(\mu n - 1)a_n\,|z|^\mu\,n - 1}{\sum_{n=2}^{\infty} a_n\,|z|^\mu\,n - 1}.$$

Therefore, we satisfy (4.4) if and only if

$$\sum_{n=2}^{\infty} \frac{(\mu-\psi)a_n \left|z\right|^{\mu} n - 1}{1-\psi} \leq 1. \tag{4.5}$$

Nevertheless, from Theorem (2.1), inequality (4.5) it will be true that if

$$\frac{(\mu-\psi) \left|z\right|^{\mu} n - 1}{1-\psi} \leq \frac{\Omega_n^q (A, B; \gamma)}{(A-B)(1-\gamma)}$$

this is, if

$$|z| \leq \left[\frac{\Omega_n^q (A, B; \gamma)}{(A-B)(1-\gamma)} \left(\frac{1-\psi}{\mu n - \psi} \right) \right]^{1/(\mu n - 1)} \tag{4.6}$$

Or equivalent to

$$r_1 = \inf_{n \geq 2} \left[\frac{\Omega_n^q (A, B; \gamma)}{(A-B)(1-\gamma)} \times \left(\frac{1-\psi}{\mu n - \psi} \right) \right]^{1/(\mu n - 1)}$$

This completes the proof of (4.1). To prove (4.2) and (4.3) respectively, it is sufficient to show that

$$\left| 1 + \frac{zF_{\mu}''(z)}{F_{\mu}'(z)} - 1 \right| \leq 1 - \psi \quad (|z| < r_2; 0 \leq \psi < 1),$$

and

$$\left| F_{\mu}'(z) - 1 \right| \leq 1 - \psi \quad (|z| < r_3; 0 \leq \psi < 1).$$

But we choose to omit the details of the proof as it is analogous to the proof of (4.1). ∎

5. EXTREME POINT

The study of the convex hulls and extreme points of various families of univalent functions was initiated by L. Brickman, T. H. MacGregor, and D. R. Wilken in [3]. The importance of determining the extreme points of a compact family F lies in the fact that the maximum or minimum value of any continuous linear functional defined over the set of analytic functions occurs at one of the extreme points of the closed convex hull of F. There have been numerous papers recently dealing with the extreme points for the closed convex hull of several compact families of univalent functions, but for the classical analysis of the significance of extreme points can be found in [3, 4, 30] . But we employ the technique adopted by Silverman in [30] to find the extreme points for our class.

Theorem 5.1. *Let $F_1(z) = z$, and*

$$F_{\mu n}(z) = z - \frac{(A-B)(1-\gamma)}{\Omega_n^q (A, B; \gamma)} z^{\mu n} \quad (n \geq 2). \tag{5.1}$$

Then $F_{\mu}(z) \in \tilde{\varepsilon}_{k,m}(\Phi, \Psi, A, B, \mu, \gamma)$ if and only if it can be expressed in the following form:

$$F(z) = \sum_{n=1}^{\infty} \eta_n F_{\mu n}(z)$$

where $\eta_n \geq 0$ and $\sum_{n=1}^{\infty} \eta_n = 1$.

Proof. Assuming that,

$$F_\mu(z) = \sum_{n=1}^{\infty} \eta_n F_{\mu n}(z),$$

where

$$F_{\mu n}(z) = z - \sum_{n=1}^{\infty} \frac{(A-B)(1-\gamma)}{\Omega_n^q\left(A, B;\, \gamma\right)} z^{\mu n}.$$

Then, from Theorem 2.1, we obtain

$$\sum_{n=2}^{\infty} \left(\left[\Omega_n^q\left(A, B;\, \gamma\right)\right] \times \frac{(A-B)(1-\gamma)}{\Omega_n^q\left(A, B;\, \gamma\right)} z^{\mu n} \right)$$
$$= (A-B)(1-\gamma) \sum_{n=1}^{\infty} \eta_n = (A-B)(1-\gamma)(1-\eta_1) \leq (A-B)(1-\gamma).$$

Therefore in view of Theorem 2.1, we find that $F(z) \in \tilde{\varepsilon}_{k,m}(\Phi, \Psi, A, B, \mu, \gamma)$. Conversely, let us suppose that $F(z) \in \tilde{\varepsilon}_{k,m}(\Phi, \Psi, A, B, \mu, \gamma)$, then

$$a_n \leq \sum_{n=1}^{\infty} \frac{(A-B)(1-\gamma)}{\Omega_n^q\left(A, B;\, \gamma\right)}.$$

By setting $\eta_1 = 1 - \sum_{n=2}^{\infty} \eta_n$, where

$$\eta_n = \frac{\Omega_n^q\left(A, B;\, \gamma\right)}{(A-B)(1-\gamma)} a_n \quad (n = 2, 3, ...)$$

Therefore, we have

$$F(z) = \sum_{n=1}^{\infty} \eta_n F_{\mu n}(z).$$

By this, we complete the proof of Theorem 5.1.

∎

Corollary 5.2. *The extreme point of the class* $\tilde{\varepsilon}_{k,m}(\Phi, \Psi, A, B, \mu, \gamma)$ *are given by*

$$F_{\mu n}(z) = z - \frac{(A-B)(1-\gamma)}{\Omega_n^q\left(A, B;\, \gamma\right)} z^{\mu n}.$$

6. INTEGRAL MEANS INEQUALITY

In this section, we consider some result due to Littlewood subordination (see [19]).

Lemma 6.1. *If the functions* f *and* g *are analytic in open unit disk* $\mathcal{U}$ *with*

$$f(z) \prec g(w(z)), \quad (z \in \mathcal{U}) \tag{6.1}$$

then, for $q > 0$ *and* $z = e^{i\theta} \quad (0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^q \, d\theta \leq \int_0^{2\pi} |g(z)|^q \, d\theta. \tag{6.2}$$

Now, we use Lemma 6.1 to prove the following Theorem.

Theorem 6.2. *Assume that* $F_\mu(z) \in \tilde{\varepsilon}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma),\ \ q > 0\ , -1 \leq B \leq A \leq 1, k; m \in \mathbb{N}_0, and\, F_{2\mu}(z)$ *is defined by*

$$F_{2\mu}(z) = z - \frac{(A-B)(1-\gamma)}{\Omega_2^q\,(A,B;\,\gamma)} z^{2\mu} \tag{6.3}$$

then $z = re^{i\theta}(0 < r < 1)$, *we obtain*

$$\int_0^{2\pi} |f(z)|^q \, d\theta \leq \int_0^{2\pi} |g(z)|^q \, d\theta.$$

Proof. For $F_\mu(z) = z - \sum_{n=2}^{\infty} a_n z^{\mu n}\ \ (a_n \geq 0)$ and by (6.2) it is equivalent to prove,

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n z^{\mu n - 1} \right|^q d\theta \leq \int_0^{2\pi} \left| 1 - \frac{(A-B)(1-\gamma)}{\Omega_2^q\,(A,B;\,\gamma)} z^{2\mu-1} \right|^q d\theta. \tag{6.4}$$

By applying Lemma 6.1, it would suffice to show that

$$1 - \sum_{n=2}^{\infty} a_n z^{\mu n - 1} \prec 1 - \frac{(A-B)(1-\gamma)}{\Omega_2^q\,(A,B;\,\gamma)} z^{2\mu-1}.$$

By putting

$$1 - \sum_{n=2}^{\infty} a_n z^{\mu n - 1} = 1 - \frac{(A-B)(1-\gamma)}{\Omega_2^q\,(A,B;\,\gamma)} \Theta(z)$$

and by using Theorem 2.1, we have

$$\left| \sum_{n=0}^{\infty} \frac{\Omega_2^q\,(A,B;\,\gamma)}{(A-B)(1-\gamma)} a_n z^{2\mu-1} \right|^q \leq \left| z^{2\mu-1} \right| \sum_{n=0}^{\infty} \frac{\Omega_2^q\,(A,B;\,\gamma)}{(A-B)(1-\gamma)} a_n \leq |z| \leq 1.$$

This completes the proof of the Theorem. ∎

7. CONCLUSION

In unit disk, we derived a new class of fractional power $\mathcal{A}_\mu$ and consider this class to define a generalized differential operator, also we employed this operator to define a new subclasses in open unit disk. Further, we studied some characteristic properties including that of certain fractional calculus operators.

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