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WAVELET FRAMES IN HIGHER DIMENSIONAL SOBOLEV SPACES

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ABSTRACT. In this paper, we present sufficient condition for the sequence of vectors $\{\varphi_{j,k}^{(j)}, \psi_{j,k,l}^{(j)}\}_{l=1}^{2^d-1}$ to be a frame for $H^s(\mathbb{R}^d)$ are derived. Necessary and sufficient conditions for the sequence of vectors $\{\varphi_{j,k}^{(j)}, \psi_{j,k,l}^{(j)}\}_{l=1}^{2^d-1}$ to be tight wavelet frames in $H^s(\mathbb{R}^d)$ are obtained. Further, as an application an example of tight wavelet frames for $H^s(\mathbb{R}^2)$ as bivariate box spline over 3-direction are given.

Key words and phrases: Wavelets; wavelet frames; multiresolution analysis; Sobolev space.

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1. INTRODUCTION

Wavelet analysis has become extensively instrumental in the functionalities of applied mathematics for recent technologies, like biomedical signal processing, image processing and technologies related to it, which have Impact on our day to day lives. In [1], Hernandez and Weiss are establishing specific new set of crucial and adequate conditions which make integer translates and dilations of L^2 function an orthonormal basis. One of the most prominent example related to applications of wavelets is image compression via employing orthonormal or bi-orthogonal wavelet bases created by the MRA as given in [2, 3]. Another well established and widely prevalent instance of applications of wavelets is noise removal using redundant wavelet systems by [4, 5].

The wide purview of applications of frames can be observed in the initially released literature on applications of Gabor and wavelet frames (see e.g. [6, 7, 8]). Such applications include time frequency analysis for signal processing, coherent state in quantum mechanics, filter bank design in electrical engineering, edge and singularity detection in image processing, and etc. Because of the self duality and the potentiality of redundant representation, tight wavelet frames have a lot of applications. In [9, 10], some new applications of tight wavelet frames are intoduced.

Wavelets and their properties in Sobolev space were instigated by Bastin et al. [11, 12, 13], Dayong and Dengfeng [14], walter [15, 16] and pathak [17]. Han and Shen [18], Ehler [19] founded a new concept to simplify the construction of wavelet systems by constructing a pair of dual wavelet frames for a pair of Sobolev spaces.

But the theory of wavelets in Sobolev space has got immense scope and it needs further research. One most important aspect and purpose of this article is to build a more natural framework for wavelets and wavelet frames in the higher dimensional scenarios.

Organization of the paper. In section 2, Sufficient condition for the sequence of vectors $\{\varphi_{j,k}^{(j)}, \psi_{j,k,l}^{(j)}\}_{l=1}^{2^d-1}$ to be a frame for $H^s(\mathbb{R}^d)$ are derived. Necessary and sufficient conditions for the sequence of vectors $\{\varphi_{j,k}^{(j)}, \psi_{j,k,l}^{(j)}\}_{l=1}^{2^d-1}$ to be tight wavelet frames in $H^s(\mathbb{R}^d)$ are given. Also, we give an example of tight wavelet frames for $H^s(\mathbb{R}^2)$ as bivariate box spline over 3-direction meshes.

2. PRELIMINARIES

2.1. **Sobolev space $H^s(\mathbb{R}^d)$.** For any real number s, the Sobolev space $H^s(\mathbb{R}^d)$ consists of tempered distributions in $S'(\mathbb{R}^d)$ which satisfy:

$$\|f\|_s^2 := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(1+\|\xi\|^2)^s|\hat{f}(\xi)|^2 d\xi,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$ and the corresponding inner product is as follows

$$\langle f, g\rangle_s := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(1+\|\xi\|^2)^s \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi, \qquad f, g \in H^s(\mathbb{R}^d).$$

Note that $H^s(\mathbb{R}^d)$ is a Hilbert space under this inner product.

The Fourier transform $\hat{f}$, for $f \in L^1(\mathbb{R}^d)$ is defined below

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} f(x)dx,$$

where $\langle x, \xi\rangle$ is the inner product of two vectors x and ξ in $\mathbb{R}^d$.
For $f, g \in H^s(\mathbb{R}^d)$, the bracket product of f and g is defined by

$$[f,g]_s := \sum_{k\in\mathbb{Z}^d} \hat{f}(\xi+2\pi k)\overline{\hat{g}(\xi+2\pi k)}(1+\|\xi+2\pi k\|^2)^s, \quad \xi\in\mathbb{R}^d \text{ and } s\in\mathbb{R}$$

.

2.2. **Frame.** For a Hilbert space H a family $\{f_n\}_{n\in\mathbb{Z}}$ of elements is said to be frame for H if there exist positive constants A and B such that

$$A\|f\|^2 \le \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 \le B\|f\|^2$$

for all $f\in H$.

If $A=B$ we call $\{f_n\}_{n\in\mathbb{Z}}$ a tight frame. Thus for the tight frame we have

$$\sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 = B\|f\|^2$$

for all $f\in H$.

3. Wavelet Frames in $H^s(\mathbb{R}^d)$

In this section we give sufficient conditions for wavelet frames in Sobolev space $H^s(\mathbb{R}^d)$. To construct wavelet frame system, firstly we define scaling function and wavelet function. Let $\{a_k^{(j)}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$ be a sequence of real numbers. A distribution $\varphi^{(j)}$ on $\mathbb{R}^d$ is called refinable function associated with $\{a_k^{(j)}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ if it satisfies the refinement equation

$$\varphi^{(j)}(x) = 2^d \sum_{k\in\mathbb{Z}^d} a_k^{(j+1)} \varphi^{(j+1)}(2x-k), \quad x\in\mathbb{R}^d.$$

The Fourier transform of $\varphi^{(j)}$ can be written as

$$\hat{\varphi}^{(j)}(\xi) = m_0^{(j+1)}(\xi/2)\hat{\varphi}^{(j+1)}(\xi/2), \quad \xi\in\mathbb{R}^d,$$

where

$$m_0^{(j)}(\xi) = \sum_{k\in\mathbb{Z}^d} a_k^{(j+1)} e^{-i\langle k,\xi\rangle} \quad \xi\in\mathbb{R}^d.$$

Now, we define the Wavelet functions associated with scaling function $\varphi^{(j)}$ as follows

$$\psi_l^{(j)}(x) = 2^d \sum_{k\in\mathbb{Z}^d} b_{l,k}^{(j+1)} \varphi^{(j+1)}(2x-k), \quad l=1,2,\dots,2^d-1,\ x\in\mathbb{R}^d.$$

The index j means that we work with one scaling function $\varphi^{(j)}$ to each scale. In the d-variate setting, for dilation matrix $2I_{d\times d}$, one expects to find 2^d-1 wavelets (cf.[20]).

In the Fourier domain, the above equation can be written as

$$\hat{\psi}_l^{(j)}(\xi) = m_l^{(j+1)}(\xi/2)\hat{\varphi}^{(j+1)}(\xi/2), \quad l=1,2,\dots,2^d-1,\ x\in\mathbb{R}^d,$$

where $m_0^{(j)}, m_l^{(j)},\ l = 1,2,\dots,2^d-1$ are $2\pi\mathbb{Z}^d$-periodic functions and are in $L^2(\mathbb{T}^d)$; $\mathbb{T}^d = [-\pi,\pi)^d$.

For given $\varphi^{(j)}, \psi_1^{(j)}, \dots, \psi_{2^d-1}^{(j)} \in H^s(\mathbb{R}^d)$, a properly normalized wavelet system in $H^s(\mathbb{R}^d)$ is defined as:

$$\{\varphi_{j,k}^{(j)}, \psi_{j,k,l}^{(j)}\}_{l=1}^{2^d-1} := \{\varphi_{j,k}^{(j)} :\ j\in\mathbb{Z},\ k\in\mathbb{Z}^d\} \bigcup \{\psi_{j,k,l}^{(j)} :\ j\in\mathbb{Z},\ k\in\mathbb{Z}^d,\ l=1,2,\dots,2^d-1\}$$

$$\text{with } \varphi_{j,k}^{(j)} := 2^{jd/2}\varphi^{(j)}(2^j\cdot -k) \text{ and } \psi_{j,k,l}^{(j)} := 2^{jd/2}\psi_l^{(j)}(2^j\cdot -k),\ j\in\mathbb{Z},\ k\in\mathbb{Z}^d.$$

With the help of above defined functions,we define

$$\beta_{\varphi}^{q}(\xi) := \sum_{p'=0}^{\infty} \overline{(1+2^{2p}\|\xi\|^2)^{s/2}}(1+2^{2p}\|\xi+2q\pi\|^2)^{s/2}\hat{\varphi}^{(p-p')}(2^{p'}\xi)\overline{\hat{\varphi}^{(p-p')}(2^{p'}(\xi+2q\pi))},$$

$$\beta_{\psi}^{q}(\xi) := \sum_{l=1}^{2^d-1}\sum_{p'=0}^{\infty} \overline{(1+2^{2p}\|\xi\|^2)^{s/2}}(1+2^{2p}\|\xi+2q\pi\|^2)^{s/2}\hat{\psi}_l^{(p-p')}(2^{p'}\xi)\overline{\hat{\psi}_l^{(p-p')}(2^{p'}(\xi+2q\pi))}.$$

where $p, p' \in \mathbb{Z}$ and $q \in \mathcal{R} := \mathbb{Z}^d/(2\mathbb{Z}^d)$ means that at least one component q_i is odd.

$$\Gamma_{\varphi}(\xi) = \sum_{j}(1+\|\xi\|^2)|^s \left|\hat{\varphi}^{(j)}(2^{-j}\xi)\right|^2, \ \xi \in \mathbb{R}^d,$$

$$\Gamma_{\psi}(\xi) = \sum_{l=1}^{2^d-1}\sum_{j}(1+\|\xi\|^2)|^s \left|\hat{\psi}_l^{(j)}(2^{-j}\xi)\right|^2, \ \xi \in \mathbb{R}^d.$$

Consider

$$\Gamma^- = ess \inf_{\xi\in\mathbb{R}^d}\Gamma_{\varphi}(\xi) + ess \inf_{\xi\in\mathbb{R}^d}\Gamma_{\psi}(\xi)$$
$$\Gamma^+ = ess \sup_{\xi\in\mathbb{R}^d}\Gamma_{\varphi}(\xi) + ess \sup_{\xi\in\mathbb{R}^d}\Gamma_{\psi}(\xi)$$

and

$$\delta_{\varphi}(r) = ess \sup_{\xi\in\mathbb{R}^d}\sum_{k\in\mathbb{Z}}\left|\beta_{\varphi}^{r}(2^k\xi)\right|,$$

$$\delta_{\psi}(r) = ess \sup_{\xi\in\mathbb{R}^d}\sum_{k\in\mathbb{Z}}\left|\beta_{\psi}^{r}(2^k\xi)\right|.$$

The following theorem gives sufficient condition for the sequence of vectors $\{\varphi_{j,k}^{(j)}, \psi_{j,k,l}^{(j)}\}_{l=1}^{2^d-1}$ to be a frame for $H^s(\mathbb{R}^d)$.

Theorem 3.1. *Let $\{\varphi^{(j)}, \psi_l^{(j)}\}_{l=1}^{2^d-1}$ be the scaling function and wavelet function such that*

$$A = \Gamma^- - \sum_{q\in\mathcal{R}}\left[\delta_{\varphi}(q)\delta_{\varphi}(-q)\right]^{1/2} - \sum_{q\in\mathcal{R}}\left[\delta_{\psi}(q)\delta_{\psi}(-q)\right]^{1/2} > 0,$$

and

$$B = \Gamma^+ + \sum_{q\in\mathcal{R}}\left[\delta_{\varphi}(q)\delta_{\varphi}(-q)\right]^{1/2} + \sum_{q\in\mathcal{R}}\left[\delta_{\psi}(q)\delta_{\psi}(-q)\right]^{1/2} < \infty.$$

Then $\{\varphi_{j,k}^{(j)}, \psi_{j,k,l}^{(j)}\}_{l=1}^{2^d-1}$ constitutes wavelet frames for $H^s(\mathbb{R}^d)$ with frame bounds A and B.

Proof. For the all those functions $f \in H^s(\mathbb{R}^d)$ we define the class Λ such that $\hat{f} \in L^{\infty}(\mathbb{R}^d)$ and $\hat{f}$ is compactly supported in $\mathbb{R}^d \setminus \{0\}^d$.

$$\begin{aligned}\langle f, \varphi_{j,k}^{(j)}\rangle_s &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(1+\|\xi\|^2)^s\hat{f}(\xi)\overline{\hat{\varphi}_{j,k}^{(j)}(\xi)}d\xi \\ &= \frac{2^{-jd/2}}{(2\pi)^d}\int_{\mathbb{R}^d}(1+\|\xi\|^2)^s\hat{f}(\xi)\overline{\hat{\varphi}^{(j)}(2^{-j}\xi)}e^{-i2^{-j}\langle k,\xi\rangle}d\xi \\ &= \frac{2^{jd/2}}{(2\pi)^d}\int_{\mathbb{R}^d}(1+\|2^j\xi\|^2)^s\hat{f}(2^j\xi)\overline{\hat{\varphi}^{(j)}(\xi)}e^{i\langle k,\xi\rangle}d\xi.\end{aligned}$$

Therefore

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left|\langle f,\varphi_{j,k}^{(j)}\rangle_s\right|^2=\frac{1}{(2\pi)^d}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}2^{jd}\left|\int_{\mathbb{R}^d}(1+\|2^j\xi\|^2)^s\hat{f}(2^j\xi)\overline{\hat{\varphi}^{(j)}(\xi)}e^{i\langle k,\xi\rangle}d\xi\right|^2 \tag{3.1}$$

Let $F_j(\xi)=(1+\|2^j\xi\|^2)^s\hat{f}(2^j\xi)\overline{\hat{\varphi}^{(j)}(\xi)}$ for $j\in\mathbb{Z}$. Each F_j is compactly supported in $\mathbb{R}^d\setminus\{0\}^d$ and belongs to $H^s(\mathbb{R}^d)$. If F is such a function

$$\sum_{k\in\mathbb{Z}}F(\xi+2k\pi)$$

which is 2π-periodic. Then by Poisson sum formula we have,

$$\sum_{k\in\mathbb{Z}^d}F(\xi+2k\pi)=\frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d}\hat{F}(k)e^{i\langle k,\xi\rangle},$$

where $1/(2\pi)^d\hat{F}(k),\ k\in\mathbb{Z}$ is Fourier coefficients of F.
Thus, we have

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\overline{F(\xi)}\sum_{k\in\mathbb{Z}^d}\hat{F}(k)e^{i\langle k,\xi\rangle}d\xi=\int_{\mathbb{R}^d}\overline{F(\xi)}\sum_{k\in\mathbb{Z}^d}F(\xi+2k\pi)d\xi$$

Now,

$$\begin{aligned}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\overline{F(\xi)}\sum_{k\in\mathbb{Z}^d}\hat{F}(k)e^{i\langle k,\xi\rangle}d\xi&=\frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d}\hat{F}(k)\int_{\mathbb{R}^d}\overline{F(\xi)e^{-i\langle k,\xi\rangle}}d\xi\\&=\frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d}\hat{F}(k)\overline{\hat{F}(k)}\\&=\frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d}|\hat{F}(k)|^2.\end{aligned}$$

It follows that

$$\frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d}|\hat{F}(k)|^2=\int_{\mathbb{R}^d}\overline{\hat{F}(\xi)}\sum_{k\in\mathbb{Z}^d}F(\xi+2\pi k)d\xi. \tag{3.2}$$

By using (3.1) and (3.2), $F_j=F$, we have

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left|\langle f,\varphi_{j,k}^{(j)}\rangle_s\right|^2\\&\quad=\frac{1}{(2\pi)^d}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}2^{jd}\int_{\mathbb{R}^d}\overline{(1+2^{2j}\|\xi\|^2)^s\hat{f}(2^j\xi)}\hat{\varphi}^{(j)}(\xi)\\&\qquad\qquad\times\sum_{k\in\mathbb{Z}^d}(1+2^{2j}\|\xi+2k\pi\|^2)^s\hat{f}(2^j(\xi+2k\pi)\overline{\hat{\varphi}^{(j)}(\xi+2k\pi)}d\xi\\&\quad=\frac{1}{(2\pi)^d}\sum_{j\in\mathbb{Z}}2^{jd}\int_{\mathbb{R}^d}\overline{(1+2^{2j}\|\xi\|^2)^s\hat{f}(2^j\xi)}\hat{\varphi}^{(j)}(\xi)\{(1+2^{2j}\|\xi\|^2)^s\hat{f}(2^j\xi)\overline{\hat{\varphi}^{(j)}(\xi)}\\&\qquad\qquad+\sum_{k\in\mathbb{Z}^d\setminus\{0\}^d}(1+2^{2j}\|\xi+2k\pi\|^2)^s\hat{f}(2^j(\xi+2k\pi)\overline{\hat{\varphi}^{(j)}(\xi+2k\pi)}\}d\xi.\end{aligned}$$

Similarly, we express

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{l=1}^{2^d-1}\left|\langle f,\psi_{j,k,l}^{(j)}\rangle_s\right|^2\\
&\quad=\frac{1}{(2\pi)^d}\sum_{l=1}^{2^d-1}\sum_{j\in\mathbb{Z}}2^{jd}\int_{\mathbb{R}^d}\overline{(1+2^{2j}\|\xi\|^2)^s\hat{f}(2^j\xi)}\hat{\psi}_l^{(j)}(\xi)\{(1+2^{2j}\|\xi\|^2)^s\hat{f}(2^j\xi)\overline{\hat{\psi}_l^{(j)}(\xi)}\\
&\qquad\qquad+\sum_{l=1}^{2^d-1}\sum_{k\in\mathbb{Z}^d\setminus\{0\}^d}(1+2^{2j}\|\xi+2k\pi\|^2)^s\hat{f}(2^j(\xi+2k\pi)\overline{\hat{\psi}_l^{(j)}(\xi+2k\pi)}\}d\xi.
\end{aligned}
$$

Now we define sum

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left|\langle f,\varphi_{j,k}^{(j)}\rangle_s\right|^2+\sum_{l=1}^{2^d-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left|\langle f,\psi_{j,k,l}^{(j)}\rangle_s\right|^2\\
&\quad=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(1+\|\xi\|^2)^{2s}|\hat{f}(\xi)|^2\sum_{j\in\mathbb{Z}}\left(|\hat{\varphi}^{(j)}(2^{-j}\xi)|^2+\sum_{l=1}^{2^d-1}|\hat{\psi}_l^{(j)}(2^{-j}\xi)|^2\right)d\xi+I_1+I_2.
\end{aligned}
$$

where

$$
\begin{aligned}
I_1=&\frac{1}{(2\pi)^d}\sum_{j\in\mathbb{Z}}2^{jd}\int_{\mathbb{R}^d}\overline{(1+2^{2j}\|\xi\|^2)^s\hat{f}(2^j\xi)}\hat{\varphi}^{(j)}(\xi)\\
&\qquad\times\sum_{k\in\mathbb{Z}^d\setminus\{0\}^d}(1+2^{2j}\|\xi+2k\pi\|^2)^s\hat{f}(2^j(\xi+2k\pi)\overline{\hat{\varphi}^{(j)}(\xi+2k\pi)}d\xi.
\end{aligned}
$$

and

$$
\begin{aligned}
I_2=&\frac{1}{(2\pi)^d}\sum_{l=1}^{2^d-1}\sum_{j\in\mathbb{Z}}2^{jd}\int_{\mathbb{R}^d}\overline{(1+2^{2j}\|\xi\|^2)^s\hat{f}(2^j\xi)}\hat{\psi}_l^{(j)}(\xi)\\
&\qquad\times\sum_{k\in\mathbb{Z}^d\setminus\{0\}^d}(1+2^{2j}\|\xi+2k\pi\|^2)^s\hat{f}(2^j(\xi+2k\pi)\overline{\hat{\psi}_l^{(j)}(\xi+2k\pi)}d\xi.
\end{aligned}
$$

In the expression for I, for every k $(k\neq\{0\}^d)$ there is a unique non-negative integer p' and a unique $q\in\mathcal{R}$.

$$
\begin{aligned}
(2\pi)^dI_1=&\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\overline{(1+\|\xi\|^2)^s\hat{f}(\xi)}\hat{\varphi}^{(j)}(2^{-j}\xi)\sum_{p'=0}^{\infty}\sum_{q\in\mathcal{R}}(1+\|\xi+2^{j+p'}2q\pi\|^2)^s\\
&\qquad\times\hat{f}(\xi+2^{j+p'}2q\pi)\overline{\hat{\varphi}^{(j+l)}(2^{p'}(2^{-j-p'}\xi+2q\pi))}d\xi\\
=&\int_{\mathbb{R}^d}\overline{(1+\|\xi\|^2)^{s/2}\hat{f}(\xi)}\sum_{p'=0}^{\infty}\sum_{q\in\mathcal{R}}\sum_{p\in\mathbb{Z}}(1+\|\xi+2^p2q\pi\|^2)^{s/2}\hat{f}(\xi+2^p2q\pi)\\
&\times\overline{(1+\|\xi\|^2)^{s/2}}(1+\|\xi+2^p2q\pi\|^2)^{s/2}\hat{\varphi}^{(p-p')}(2^{p'}2^{-p}\xi)\overline{\hat{\varphi}^{(p-p')}(2^{p'}(2^{-p}\xi+2q\pi))}d\xi\\
=&\int_{\mathbb{R}^d}\overline{(1+\|\xi\|^2)^{s/2}\hat{f}(\xi)}\sum_{q\in\mathcal{R}}\sum_{p\in\mathbb{Z}}(1+\|\xi+2^p2q\pi\|^2)^{s/2}\hat{f}(\xi+2^p2q\pi)\beta_{\varphi}^{q}(2^{-p}\xi)d\xi.
\end{aligned}
$$

Similarly,

$$(2\pi)^d I_2 = \int_{\mathbb{R}^d} \overline{(1+\|\xi\|^2)^{s/2}\hat{f}(\xi)} \sum_{q\in\mathcal{R}} \sum_{p\in\mathbb{Z}} (1+\|\xi+2^p 2q\pi\|^2)^{s/2} \hat{f}(\xi+2^p 2q\pi)\beta_\psi^q(2^{-p}\xi)d\xi$$

where

$$\beta_\varphi^q(2^{-p}\xi) := \sum_{p'=0}^{\infty} \overline{(1+\|\xi\|^2)^{s/2}}(1+\|\xi+2^p 2q\pi\|^2)^{s/2}\hat{\varphi}^{(p-p')}(2^{p'}2^{-p}\xi)\overline{\hat{\varphi}^{(p-p')}(2^{p'}(2^{-p}\xi+2q\pi))},$$

$$\beta_\psi^q(2^{-p}\xi) := \sum_{l=1}^{2^d-1} \sum_{p'=0}^{\infty} \overline{(1+\|\xi\|^2)^{s/2}}(1+\|\xi+2^p 2q\pi\|^2)^{s/2}\hat{\psi}_l^{(p-p')}(2^{p'}2^{-p}\xi)\overline{\hat{\psi}_l^{(p-p')}(2^{p'}(2^{-p}\xi+2q\pi))}.$$

Thus,

$$\begin{aligned}
(3.3)\qquad & \sum_{q\in\mathcal{R}} [\delta_\varphi(q)\delta_\varphi(-q)]^{1/2}\|f\|_s^2 \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |(1+\|\xi\|^2)|^{2s}|\hat{f}(\xi)|^2 \sum_{j\in\mathbb{Z}} |\hat{\varphi}^{(j)}(2^{-j}\xi)|^2 d\xi \\
&\quad + \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \overline{(1+\|\xi\|^2)^s\hat{f}(\xi)} \sum_{q\in\mathcal{R}}\sum_{p\in\mathbb{Z}} (1+\|\xi+2^p 2q\pi\|^2)^s \hat{f}(\xi+2^p 2q\pi)\beta_\varphi^q(2^{-p}\xi)d\xi \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |(1+\|\xi\|^2)|^{2s}|\hat{f}(\xi)|^2 \sum_{j\in\mathbb{Z}} |\hat{\varphi}^{(j)}(2^{-j}\xi)|^2 d\xi + \frac{1}{(2\pi)^d}P_1(f)
\end{aligned}$$

where $P_1(f) = (2\pi)^d I_1$.
By using Schwartz's inequality, for all $f \in \Lambda$, we have

$$\begin{aligned}
|P_1(f)| \le \sum_{q\in\mathcal{R}}\sum_{p\in\mathbb{Z}} &\left(\int_{\mathbb{R}^d} (1+\|\eta\|^2)^s |\hat{f}(\eta)|^2 |\beta_\varphi^q(2^{-p}\eta)|d\eta\right)^{1/2} \\
&\times \left(\int_{\mathbb{R}^d} (1+\|\eta+2^p 2q\pi\|^2)^s |\hat{f}(\eta+2^p 2q\pi)|^2 |\beta_\varphi^q(2^{-p}\eta)|d\eta\right)^{1/2}.
\end{aligned}$$

We know that $\beta_\varphi^q(\xi-2q\pi) = \overline{\beta_\varphi^{-q}(\xi)}$ and by changing variables in second integral and applying Schwarz's inequality for series we have

$$\begin{aligned}
|P_1(f)| &\le \sum_{q\in\mathcal{R}} \left(\sum_{p\in\mathbb{Z}} \int_{\mathbb{R}^d} (1+\|\eta\|^2)^s |\hat{f}(\eta)|^2 |\beta_\varphi^q(2^{-p}\eta)|d\eta\right)^{1/2} \\
&\qquad \times \left(\sum_{p\in\mathbb{Z}} \int_{\mathbb{R}^d} (1+\|\eta\|^2)^s |\hat{f}(\eta)|^2 |\beta_\varphi^q(2^{-p}\eta)|d\eta\right)^{1/2} \\
&\le \sum_{q\in\mathcal{R}} [\delta_\varphi(q)\delta_\varphi(-q)]^{1/2}\|f\|_s^2.
\end{aligned}$$

Hence,

$$-\sum_{q\in\mathcal{R}} [\delta_\varphi(q)\delta_\varphi(-q)]^{1/2}\|f\|_s^2 \le P_1(f) \le \sum_{q\in\mathcal{R}} [\delta_\varphi(q)\delta_\varphi(-q)]^{1/2}\|f\|_s^2$$

likewise for $P_2(f) = (2\pi)^d I_2$,

$$-\sum_{q\in\mathcal{R}} [\delta_\psi(q)\delta_\psi(-q)]^{1/2} \, \|f\|_s^2 \le P_2(f) \le \sum_{q\in\mathcal{R}} [\delta_\psi(q)\delta_\psi(-q)]^{1/2} \, \|f\|_s^2$$

These inequality together with (3.3)

$$A\|f\|_s^2 \le \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left| \langle f, \varphi_{j,k}^{(j)} \rangle_s \right|^2 + \sum_{l=1}^{2^d-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left| \langle f, \psi_{j,k,l}^{(j)} \rangle_s \right|^2 \le B\|f\|_s^2. \tag{3.4}$$

Since Λ is dense in $H^s(\mathbb{R}^d)$, the above inequality holds for all $f \in H^s(\mathbb{R}^d)$. ∎

We already know that $(1+\|\xi\|^2)^{s/2}\hat{f}(\xi), (1+\|\xi\|^2)^{s/2}\hat{\psi}_l^{(j)}(2^{-j}\xi) \in L^2(\mathbb{R}^d),\ l = 0, 1 \ldots, 2^{d-1}$, by the definition of $H^s(\mathbb{R}^d)$.

Assume D_j is the set of Lebesgue points for $(1+|2^j\xi|^2)^s|\hat{\psi}_l^{(j)}(\xi)|^2,\ j \in \mathbb{Z}$, then $|D_j^c| = 0$. Thus $|\bigcup_j D_j^c| = 0$. That is, almost every point of $\mathbb{R}^d$ is a Lebesgue point for all $(1+|2^j\xi|^2)^s|\hat{\psi}_l^{(j)}(\xi)|^2$, $j \in \mathbb{Z}$.

Let ξ_0 be an arbitrary Lebesgue point for all $(1+|2^j\xi|^2)^s|\hat{\psi}_l^{(j)}(\xi)|^2,\ j \in \mathbb{Z}$. For every fixed positive integer M, set

$$\hat{f}(\xi) = \frac{(2\pi)^{d/2}\chi_{Q(\xi_0,\delta)}}{(\Delta)^{1/2}(1+\|\xi\|^2)^{s/2}}, \tag{3.5}$$

where $Q(\xi_0, \delta)$ is an unit ball centered at δ_0 with radius δ and it has measure Δ. Choose $\delta < \frac{\pi}{2^M}$, obviously, $\|f\|_s = 1$.

For $|j| \le M$, from Theorem 3.1, we have

$$\begin{aligned}
&\sum_{|j|\le M} \sum_{l=0}^{2^{d-1}} \sum_{k\in\mathbb{Z}^d} \left| \langle f, \psi_{j,k,l}^{(j)} \rangle_s \right|^2 \\
&\quad = \frac{1}{(2\pi)^d} \sum_{l=0}^{2^d-1} \sum_{|j|\le M} \int_{\mathbb{R}^d} (1+\|\xi\|^2)^{2s} |\hat{f}(\xi)|^2 |\hat{\psi}_l^{(j)}(2^{-j}\xi)|^2 d\xi \\
&\qquad + \frac{1}{(2\pi)^d} \sum_{l=0}^{2^d-1} \sum_{|j|\le M} \int_{\mathbb{R}^d} \overline{(1+\|\xi\|^2)^s \hat{f}(\xi)|^2} \hat{\psi}_l^{(j)}(2^{-j}\xi) \\
&\qquad \times \sum_{k\in\mathbb{Z}^d\setminus\{0\}^d} (1+2^{2j}\|\xi+2k\pi\|^2)^s \hat{f}(2^j(\xi+2k\pi) \overline{\hat{\psi}_l^{(j)}(\xi+2k\pi)}\} d\xi
\end{aligned}$$

by using (3.4) and (3.5), we get

$$\begin{aligned}
\sum_{|j|\le M} \sum_{l=0}^{2^{d-1}} \sum_{k\in\mathbb{Z}^d} \left| \langle f, \psi_{j,k,l}^{(j)} \rangle_s \right|^2 &= \frac{1}{(2\pi)^d} \sum_{|j|\le M} \sum_{l=0}^{2^{d-1}} \int_{Q(\xi_0,\delta)} \frac{(2\pi)^d}{\Delta} (1+\|\xi\|^2)^s |\hat{\psi}_l^{(j)}(2^{-j}\xi)|^2 d\xi \\
&= \frac{1}{\Delta} \sum_{|j|\le M} \sum_{l=0}^{2^{d-1}} \int_{Q(\xi_0,\delta)} (1+\|\xi\|^2)^s |\hat{\psi}_l^{(j)}(2^{-j}\xi)|^2 d\xi \\
&\le B.
\end{aligned}$$

Let $\delta \to 0$ and $M \to \infty$ consecutively, by Lebesgue differential theorem, we have

$$\sum_{j\in\mathbb{Z}} \sum_{l=0}^{2^{d-1}} (1+\|\xi\|^2)^s |\hat{\psi}_l^{(j)}(2^{-j}\xi)|^2 \le B.$$

For $A = B = 1$ in (3.4), we have

$$\sum_{j\in\mathbb{Z}} (1+\|\xi\|^2)^s |\hat{\varphi}^{(j)}(2^{-j}\xi)|^2 + \sum_{l=1}^{2^{d-1}} \sum_{j\in\mathbb{Z}} (1+\|\xi\|^2)^s |\hat{\psi}_l^{(j)}(2^{-j}\xi)|^2 = 1. \tag{3.6}$$

For $A = B = 1$, we can rewrite (3.4) as

$$\begin{aligned}
\|f\|_s^2 &= \sum_{l=0}^{2^{d-1}} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left| \langle f, \psi_{j,k,l}^{(j)} \rangle_s \right|^2 \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |(1+\|\xi\|^2)|^{2s} |\hat{f}(\xi)|^2 \sum_{l=0}^{2^{d-1}} \sum_{j\in\mathbb{Z}} |\hat{\psi}^{(j)}(2^{-j}\xi)|^2 d\xi \\
&\quad + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \overline{(1+\|\xi\|^2)^s \hat{f}(\xi)} \sum_{q\in\mathcal{R}} \sum_{p\in\mathbb{Z}} (1+\|\xi+2^p 2q\pi\|^2)^s \hat{f}(\xi+2^p 2q\pi) \beta_\psi^q(2^{-p}\xi) d\xi
\end{aligned}$$

by using (3.6), we obtain

$$\sum_{q\in\mathcal{R}} \sum_{p\in\mathbb{Z}} (1+\|\xi+2^p 2q\pi\|^2)^s \hat{f}(\xi+2^p 2q\pi) \beta_\psi^q(2^{-p}\xi) = 0$$

Define $\hat{f}(\xi)$ as (3.5), we obtain, for a.e. $\xi \in \mathbb{R}^d$,

$$\begin{aligned}
0 &= \beta_\psi^q(2^{-p}\xi) \\
&= \sum_{l=0}^{2^{d-1}} \sum_{l=0}^{\infty} \overline{(1+2^{2p}\|\xi\|^2)^{s/2}} (1+2^{2p}\|\xi+2q\pi\|^2)^{s/2} \hat{\psi}^{(p-l)}(2^l\xi) \overline{\hat{\psi}^{(p-l)}(2^l(\xi+2q\pi))}.
\end{aligned} \tag{3.7}$$

By using (3.6) and (3.7) we give following results:

Theorem 3.2. *The set $\{\varphi_{j,k}^{(j)}, \psi_{j,k,l}^{(j)}\}_{l=1}^{2^d-1}$ is a tight frame of $H^s(\mathbb{R}^d)$ if and only if*

$$\sum_{j\in\mathbb{Z}} (1+\|\xi\|^2)^s |\hat{\varphi}^{(j)}(2^{-j}\xi)|^2 + \sum_{l=1}^{2^{d-1}} \sum_{j\in\mathbb{Z}} (1+\|\xi\|^2)^s |\hat{\psi}_l^{(j)}(2^{-j}\xi)|^2 = 1, \ a.e.$$

and

$$\sum_{l=0}^{2^{d-1}} \sum_{l=0}^{\infty} \overline{(1+2^{2p}\|\xi\|^2)^{s/2}} (1+2^{2p}\|\xi+2q\pi\|^2)^{s/2} \hat{\psi}^{(p-l)}(2^l\xi) \overline{\hat{\psi}^{(p-l)}(2^l(\xi+2q\pi))} = 0, \ a.e.$$

If, in addition, $\|\psi_l^{(j)}\|_s = 1$, the frame $\{\varphi_{j,k}^{(j)}, \psi_{j,k,l}^{(j)}\}_{l=1}^{2^d-1}$ is an orthonormal basis of $H^s(\mathbb{R}^d)$.

Now, we give an example as an application of Theorem 3.2 for $H^s(\mathbb{R}^2)$ as bivariate box spline over 3-direction meshes. For fix $s \ge 0$ and natural numbers m_1, m_2, m_3 such that

$$min\{m_1+m_2-2, m_1+m_3-2, m_2+m_3-2\} > s.$$

By using standard unit vectors $e_1 = (1,0)^T, e_2 = (0,1)^T$ in $\mathbb{R}^2$, we define a 3-direction mesh bivariate box spline in terms of Fourier transform as follows :

$$\hat{M}_{m_1,m_2,m_3}(\xi_1,\xi_2) := \left(\frac{1-e^{-i\xi_1}}{i\xi_1}\right)^{m_1}\left(\frac{1-e^{-i\xi_2}}{i\xi_2}\right)^{m_2}\left(\frac{1-e^{-i(\xi_1+\xi_2)}}{i(\xi_1+\xi_2)}\right)^{m_3}$$
$$= e^{i\xi_1(m_1+m_3)/2}e^{i\xi_2(m_2+m_3)/2}\left(\frac{\sin\xi_1/2}{\xi_1/2}\right)^{m_1}\left(\frac{\sin\xi_2/2}{\xi_2/2}\right)^{m_2}\left(\frac{\sin(\xi_1+\xi_2)/2}{(\xi_1+\xi_2)/2}\right)^{m_3}$$

Then, $M_{m_1,m_2,m_3} \in H^s(\mathbb{R}^2)$(cf. [21] and [22]). Let

$$\omega^{(j)}_{m_1,m_2,m_3}(\xi_1,\xi_2) = \sum_{(k_1,k_2)\in\mathbb{Z}^d}(1+2^{2j}((\xi_1+2\pi k_1)+(\xi_2+2\pi k_2)))^s|\hat{M}_{m_1,m_2,m_3}(\xi_1+2\pi k_1,\xi_2+2\pi k_2)|^2.$$

There exist two constants c and C such that

$$0 < c \le \omega^{(j)}_{m_1,m_2,m_3}(\xi_1,\xi_2) \le C < \infty.$$

Now, we define for every $j \in \mathbb{Z}$,

$$\hat{\varphi}^{(j)}(\xi_1,\xi_2) = \frac{\hat{M}_{m_1,m_2,m_3}(\xi_1,\xi_2)}{\sqrt{\omega^{(j)}_{m_1,m_2,m_3}(\xi_1,\xi_2)}}$$

where $\varphi^{(j)} \in H^s(\mathbb{R}^2)$ (cf. [23]).

Let $\psi^{(j)}_l,\ l = 1,2,3$ be a associated wavelets with the scaling functions $\varphi^{(j)},\ j \in \mathbb{Z}$, and their Fourier transform is given by

$$\hat{\psi}^{(j)}_l(\xi_1,\xi_2) = m^{(j+1)}_l(\xi_1/2,\xi_2/2)\hat{\varphi}^{(j+1)}(\xi_1/2,\xi_2/2),\ l = 1,2,3.$$

where $m^{(j)}_l,\ l = 1,2,3$ are $2\pi\mathbb{Z}^2$ periodic functions belongs to $\ell^2(\mathbb{Z}^2)$, such that

$$\begin{bmatrix} m^{(j)}_0(\xi_1,\xi_2) & m^{(j)}_0(\xi_1+\pi,\xi_2) & m^{(j)}_0(\xi_1,\xi_2+\pi) & m^{(j)}_0(\xi_1+\pi,\xi_2+\pi) \\ m^{(j)}_1(\xi_1,\xi_2) & m^{(j)}_1(\xi_1+\pi,\xi_2) & m^{(j)}_1(\xi_1,\xi_2+\pi) & m^{(j)}_1(\xi_1+\pi,\xi_2+\pi) \\ m^{(j)}_2(\xi_1,\xi_2) & m^{(j)}_2(\xi_1+\pi,\xi_2) & m^{(j)}_2(\xi_1,\xi_2+\pi) & m^{(j)}_2(\xi_1+\pi,\xi_2+\pi) \\ m^{(j)}_3(\xi_1,\xi_2) & m^{(j)}_3\xi_1+\pi,\xi_2) & m^{(j)}_3(\xi_1,\xi_2+\pi) & m^{(j)}_3(\xi_1+\pi,\xi_2+\pi) \end{bmatrix} \tag{3.8}$$

are unitary for $j \in \mathbb{Z}$.

Together with (3.8) and orthonormality of $\varphi^{(j)},\ \psi^{(j)},\ l = 1,2,3$, we have

$$\begin{aligned} &\sum_{j\in\mathbb{Z}}(1+(\xi_1)^2+(\xi_2)^2)^s|\hat{\varphi}^{(j)}(2^{-j}(\xi_1,\xi_2))|^2 + \sum_{l=1}^{3}\sum_{j\in\mathbb{Z}}(1+(\xi_1)^2+(\xi_2)^2)^s|\hat{\psi}^{(j)}_l(2^{-j}(\xi_1,\xi_2))|^2 \\ &= \sum_{j\in\mathbb{Z}}(1+(\xi_1)^2+(\xi_2)^2)^s|m^{(j+1)}_0(\xi_1/2,\xi_2/2)|^2|\hat{\varphi}^{(j+1)}(2^{-j-1}(\xi_1,\xi_2)|^2 \\ &\qquad + \sum_{l=1}^{3}\sum_{j\in\mathbb{Z}}(1+(\xi_1)^2+(\xi_2)^2)^s|m^{(j+1)}_l(\xi_1/2,\xi_2/2)|^2|\hat{\varphi}^{(j+1)}(2^{-j-1}(\xi_1,\xi_2)|^2 \\ &= \sum_{j\in\mathbb{Z}}(1+(\xi_1)^2+(\xi_2)^2)^s|\hat{\varphi}^{(j+1)}(2^{-j-1}(\xi_1,\xi_2))|^2 \\ &= 1 \end{aligned}$$

and

$$\sum_{l=0}^{3}\sum_{p'=0}^{\infty}\overline{(1+2^{2p}(\xi_1)^2+2^{2p}(\xi_2)^2)^{s/2}}(1+2^{2p}(\xi_1+2q\pi)^2+2^{2p}(\xi_2+2q\pi)^2)^{s/2}$$
$$\times\,\hat{\psi}^{(p-p')}(2^{p'}\xi_1,2^{p'}\xi_2)\overline{\hat{\psi}^{(p-p')}(2^{p'}(\xi_1+2q\pi,\xi_2+2q\pi))}$$
$$=\sum_{l=0}^{3}\sum_{p'=0}^{\infty}\overline{(1+2^{2p}(\xi_1)^2+2^{2p}(\xi_2)^2)^{s/2}}(1+2^{2p}(\xi_1+2q\pi)^2+2^{2p}(\xi_2+2q\pi)^2)^{s/2}$$
$$\times\sum_{l=0}^{3}m_l^{(p-p'+1)}(2^{p'-1}\xi_1,2^{p'-1}\xi_2)\overline{m_l^{(p-p'+1)}(2^{p'-1}(\xi_1+2q\pi,\xi_2+2q\pi))}$$
$$\times\,\hat{\varphi}^{(p-p'+1)}(2^{p'-1}\xi_1,2^{p'-1}\xi_2)\overline{\hat{\varphi}^{(p-p'+1)}(2^{p'-1}(\xi_1+2q\pi,\xi_2+2q\pi))}$$
$$=0$$

hence, by Theorem 3.2 it is a tight wavelet frame for $H^s(\mathbb{R}^2)$.

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