The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 4, Issue 2, Article 4, pp. 1-14, 2017

ITERATIVE APPROXIMATION OF ZEROS OF ACCRETIVE TYPE MAPS, WITH APPLICATIONS

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Received 1 November, 2016; accepted 20 April, 2017; published 8 August, 2017.

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ABSTRACT. Let E be a reflexive real Banach space with uniformly Gâteaux differentiable norm. Let $J : E \to E^*$ be the normalized duality map on E and let $A : E^* \to E$ be a map such that AJ is an accretive and uniformly continuous map. Suppose that $(AJ)^{-1}(0)$ in nonempty. Then, an iterative sequence is constructed and proved to converge strongly to some u^* in $(AJ)^{-1}(0)$. Application of our theorem in the case that E is a real Hilbert space yields a sequence which converges strongly to a zero of A. Finally, non-trivial examples of maps A for which AJ is accretive are presented.

Key words and phrases: Accretive map, Modulus of continuity, uniformly continuous map, Uniformly Gâteaux differentiable norm.

2010 *Mathematics Subject Classification.* 47H04, 47H05, 46N10, 47H06, 47J25.

ISSN (electronic): 1449-5910

Research supported from ACBF Research Grant Funds to AUST..

1. INTRODUCTION

Let $(X, \|.\|_X)$, $(Y, \|.\|_Y)$ be real normed spaces. A map $T : X \to Y$ is said to be *Lipschitz* if there exists $L \geq 0$ such that for each $x, y \in X$, $\|Tx - Ty\|_Y \leq L\|x - y\|_X$. If $L = 1$, the map T is called *nonexpansive*.

Let E be a real normed space with dual space E^*. A map $J : E \to 2^{E^*}$ defined by

$$J(x) := \Big\{x^* \in E^* : \big\langle x, x^* \big\rangle = \|x\|^2,\ \|x^*\| = \|x\| \ \forall\, x \in E\Big\},$$

is called the *normalized duality map* on E, where $\big\langle \cdot, \cdot \big\rangle$ denotes the duality pairing between elements of E and those of E^*. The following are some properties of the duality map which will be needed in the sequel (see e.g., Ibaraki and Takahashi, [12]).

- If E is strictly convex, then J is one-to-one.
- If E is reflexive, then J is onto.
- If E is smooth, then J is single-valued.
- In a Hilbert space, H, the duality map J is the identity map on H.

A map $A : E \to E$ is called *accretive* if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that $\big\langle Ax - Ay, j(x - y)\big\rangle \geq 0$. A map $T : E \to E$ is called *pseudocontractive* if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that $\big\langle Tx - Ty, j(x - y)\big\rangle \leq \|x - y\|^2$. It is easy to see that A is accretive if and only if $T := (I - A)$ is pseudocontractive, where I is the identity map on E.

The *modulus of convexity* of a space E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| = \|y\| = 1;\ \epsilon = \|x - y\|\right\}.$$

The space E is called *uniformly convex* if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. Let $S := \{z \in E : \|z\| = 1\}$. The space E is said to have a *Gâteaux differentiable norm* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists for all $x, y \in S$ and is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in S$, the limit (1.1) exists and is attained uniformly, for $x \in S$. Let E be a real normed space of dimension ≥ 2. The *modulus of smoothness* of E, $\rho_E : [0, \infty) \to [0, \infty)$, is defined by:

$$\rho_E(\tau) := \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau,\ \tau > 0\right\}.$$

The space E is called *smooth* if $\rho_E(\tau) > 0\ \forall\ \tau > 0$ and is called *uniformly smooth* if $\lim\limits_{t \to 0^+} \dfrac{\rho_E(t)}{t} = 0$. If there exist a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be q-*uniformly smooth*. Typical examples of such spaces are the Lebesgue spaces L_p, ℓ_p and Sobolev spaces, W_p^m for $1 < p < \infty$ where,

$$L_p\ (or\ l_p)\ or\ W_p^m \text{ is } \begin{cases} 2 - \text{uniformly smooth} & \text{if } \ 2 \leq p < \infty; \\ p - \text{uniformly smooth} & \text{if } \ 1 < p < 2. \end{cases}$$

A map $A : E \to 2^{E^*}$ is said to be *monotone* if for each $x, y \in E$, the following inequality holds: $\big\langle x - y, x^* - y^*\big\rangle \geq 0\ \forall\ x^* \in Ax,\ y^* \in Ay$. It is called *maximal monotone* if, in addition, the graph of A is not properly contained in the graph of any other monotone operator.

Let $f : E \to \mathbb{R}$ be a convex function. The *subdifferential* of f at $x \in E$ denoted by $\partial f : E \to 2^{E^*}$ is defined by $\partial f(x) = \big\{x^* \in E : f(y) - f(x) \geq \big\langle y - x, x^* \big\rangle \; \forall \; y \in E\big\}$. It is easy to check that ∂f is a *monotone operator* on E, and that $0 \in \partial f(x)$ if and only if x is a minimizer of f. Setting $\partial f \equiv A$, it follows that solving the inclusion $0 \in Au$, in this case, is solving for a minimizer of f.

In general, the following problem is of interest and has been studied extensively by numerous authors. Find $u \in E$ such that

$$0 \in Au, \tag{1.2}$$

where $A : E \to 2^{E^*}$ is a monotone-type map.

If H is a real Hilbert space, a well known method for approximating a solution of $0 \in Au$, where $A : H \to 2^H$ is a maximal monotone map is the *proximal point algorithm (PPA)* introduced by Martinet [17] and studied extensively by Rockafellar [22] and a host of other authors (see for e.g., Reich and Sabach, [21]; Xu [24]; Brézis and Lions, [4]; Bruck [5]; Kamimura and Takahashi, [14]; Kamimura and Takahashi, [13]; Reich [20]; Takahashi and Ueda, [26]; Chidume [7]; Chidume and Djitte, [8]; Chidume *et al.* [9]). The *PPA* is defined by $x_1 \in H$,

$$x_{n+1} = J_{r_n} x_n, \; n \geq 1 \tag{1.3}$$

where $\{r_n\}_{n=1}^{\infty} \subset (0, \infty)$ and $J_{r_n} := (I + r_n A)^{-1}$. Martinet [17] considered a special case $A : H \to 2^H$ in which A is defined as follows:

$$A(v) = \begin{cases} A_0(v) + N_D(v) & \text{if} \quad v \in \; D, \\ \emptyset & \text{if} \quad v \notin \; D, \end{cases} \tag{1.4}$$

where D is a nonempty closed convex subset of H, $A_0 : D \to H$ is a single-valued, monotone map and $N_D(v) := \{w^* \in H : \langle v - u, w^* \rangle \geq 0 \; \forall \; u \in D\}$. In this case, the relation $0 \in A(v)$ reduces to $-A_0(v) \in N_D(v)$ which is the *variational inequality problem*:

$$\text{find } v \in D \text{ such that } \langle v - u, A_0(v) \rangle \leq 0 \; \forall \; u \in D.$$

Martinet proved that the sequence $\{x_n\}_{n=1}^{\infty}$ converges *weakly* to an element of $A^{-1}(0)$ if D is bounded.

Rockafellar [22] improved this result of Martinet. He considered the case of a general monotone operator, $A : H \to 2^H$ and proved that if $\liminf\limits_{n\to\infty} r_n > 0$ and $A^{-1}(0) \neq \emptyset$, then, the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.3) converges *weakly* to an element of $A^{-1}(0)$.

He then posed the following problem.
Problem: *Does the proximal point algorithm (1.3) always converges strongly*?

This problem was resolved in the negative by Güler [11] who produced a proper closed convex function g in the infinite dimensional Hilbert space l_2 for which the proximal point algorithm converges *weakly* but *not strongly*, (see also Bauschke *et al.*, [3]). Several authors modified the proximal point algorithm to obtain *strong* convergence (see e.g., Bruck [5]; Kamimura and Takahashi [14]; Lehdili and Moudafi [16]; Reich [20]; Solodov and Svaiter [23]; Xu [24]). We remark that in every one of these modifications, the recursion formula developed involved either the computation of $(I + r_n A)^{-1}(x_n)$ at each point of the iteration process or the construction, *at each iteration*, of two subsets of the space, intersecting them and projecting the initial vector onto the intersection. Clearly, none of these processes is convenient in any possible application.

Typical of such results obtained is the following theorem.

Theorem 1.1 (Kamimura and Takahashi, [13]). *Let H be a real Hilbert space, let $A \subset H \times H$ be a maximal monotone map and let $J_r := (I + rA)^{-1}$, for $r > 0$. For $x_1, u \in H$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined by*

$$\begin{cases} y_n = J_{r_n} x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \ n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$ and $\{r_n\}_{n=1}^{\infty} \subset (0,\infty)$ satisfy $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\lim\limits_{n\to\infty} r_n = \infty$. Suppose the criterion for the approximate computation of y_n is $\|y_n - J_{r_n} x_n\| < \delta_n$ where $\sum\limits_{n=1}^{\infty} \delta_n < \infty$ and $A^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to Pu, where P is the metric projection of H onto $A^{-1}(0)$.

Kohsaka and Takahashi, [15] extended Theorem 1.1 to the framework of Banach spaces that are both smooth and uniformly convex. In particular, they proved the following theorem.

Theorem 1.2 (Kohsaka and Takahashi, [15]). *Let E be a smooth and uniformly convex Banach space and let $A \subset E \times E^*$ be a maximal monotone mapping. Let $J_r := (J + rA)^{-1} J$ for all $r > 0$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined by*

$$x_1 = u \in E, \ x_{n+1} = J^{-1}(\alpha_n Ju + (1 - \alpha_n) J J_{r_n} x_n), \ n \geq 1, \tag{1.6}$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$ and $\{r_n\}_{n=1}^{\infty} \subset (0,\infty)$ satisfy $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\lim\limits_{n\to\infty} r_n = \infty$. If $A^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to $P_{A^{-1}(0)} x$, where $P_{A^{-1}(0)}$ is the generalized projection from E onto $A^{-1}(0)$.

We remark that the iterative sequence (1.6) involves the resolvent map, $J_{r_n} := (J + r_n A)^{-1} J$, and this, as has been remarked, is generally not convenient in any possible applications because *at each step of the iteration process,* one has to compute $\Big(J + r_n A\Big)^{-1} J(x_n)$.

Zegeye observed the inclusion of this resolvent map in the recursion formula (1.6). In an attempt to propose an iteration process which will not involve the resolvent map, he developed the following theorem for approximating a zero of a monotone map A.

Theorem 1.3 (Zegeye, [27]). *Let E be a uniformly convex and 2-uniformly smooth real Banach space with dual E^*. Let $A : E^* \to E$ be a Lipschitz continuous monotone map with constant $L \geq 0$ and $A^{-1}(0) \neq \emptyset$. For given $u, x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be generated by the algorithm*

$$x_{n+1} = \beta_n u + (1 - \beta_n)(x_n - \alpha_n A J x_n), \ n \geq 1, \tag{1.7}$$

where J is the normalized duality map from E to E^*. Suppose that $B_{min} \cap (AJ)^{-1}(0) \neq \emptyset$. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $Ru := x^* \in (AJ)^{-1}(0)$ and that $Jx^* \in A^{-1}(0)$, where R is the sunny generalized nonexpansive retraction of E onto $(AJ)^{-1}(0)$, where $\{\alpha_n\}_{n=1}^{\infty}$ and

$\{\beta_n\}_{n=1}^{\infty}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*

$$(i) \lim_{n\to\infty} \beta_n = 0, \quad (ii) \sum_{n=1}^{\infty} \beta_n = \infty, \quad (iii) \lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0.$$

It turns out that there is a gap in the proof of Theorem 1.3. Araka [2] gave an *example* to illustrate the existence of this gap.

Before we state this example, we need the following lemma.

Lemma 1.4. *Let* $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, *and* $\{c_n\}_{n=1}^{\infty}$ *be sequences of non-negative real numbers satisfying the following relation*

$$a_{n+1} \leq (1-\sigma_n)\, a_n + b_n + c_n,\ n \geq 1, \tag{1.8}$$

where $\{\sigma_n\}_{n=1}^{\infty} \subset [0,1]$. *If* (i) $\sum\limits_{n=1}^{\infty} \sigma_n = \infty$, (ii) $\frac{b_n}{\sigma_n} \to 0$, *as* $n \to \infty$, (iii) $\sum\limits_{n=1}^{\infty} c_n < \infty$. *Then,* $\lim\limits_{n\to\infty} a_n = 0$.

The Example

Take $A = J^{-1}$, then $AJ = I$, where I is the identity map on E and choose the iterative parameters $\alpha_n = n^{-2}$ and $\beta_n = n^{-1}$ which both satisfy the conditions of the parameters in Theorem 1.3. Let $u \neq 0$ be an arbitrary element of E. Then, using the recursion formula (1.7), one obtains the following.

$$\begin{aligned} x_{n+1} &= \frac{1}{n}u + \Big(1 - \frac{1}{n}\Big)\Big(x_n - \frac{1}{n^2}x_n\Big) \\ &= \Big(1 - \frac{1}{n^2} - \frac{1}{n} + \frac{1}{n^3}\Big)x_n + \frac{1}{n}u \\ &= \Big(1 - \frac{1}{n^2} - \frac{1}{n} + \frac{1}{n^3}\Big)(x_n - u) + \Big(1 - \frac{1}{n^2} - \frac{1}{n} + \frac{1}{n^3}\Big)u + \frac{1}{n}u. \end{aligned}$$

Hence,

$$\|x_{n+1} - u\| \leq \Big(1 - \frac{1}{n}\Big)\|x_n - u\| + \frac{2}{n^2}\|u\|. \tag{1.9}$$

Now, setting $\lambda_n = \frac{1}{n}$ and $\sigma_n = \frac{2}{n^2}\|u\|$, it then follows from inequality (1.9) that

$$\|x_{n+1} - u\| \leq (1 - \lambda_n)\|x_n - u\| + \sigma_n, \tag{1.10}$$

which yields, by Lemma 1.4, that

$$x_n \to u \in I^{-1}(0), \text{ as } n \to \infty, \text{ but } u \neq 0.$$

Araka proved the following analogue of Theorem 1.3.

Theorem 1.5 (Araka, [2])**.** *Let* E *be a reflexive real Banach space with Uniformly Gâteaux differentiable norm. Let* $A : E^* \to E$. *For any* $x_1 \in E$, *let* $\{x_n\}_{n=1}^{\infty}$ *be the sequence iteratively generated by*

$$x_{n+1} = x_n - \beta_n AJx_n - \alpha_n\beta_n x_n,\ n \geq 1, \tag{1.11}$$

where J *is the normalized duality map from* E *to* E^*. *Suppose that* $(AJ)^{-1}(0) \neq \emptyset$*; and suppose that* AJ *is an accretive Lipschitz map with Lipschitz constant* $L \geq 0$, *then the sequence* $\{x_n\}_{n=1}^{\infty}$

converges strongly to $u^* \in (AJ)^{-1}(0)$ *with* $Ju^* \in (A)^{-1}(0)$, *where* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*

$$(i) \lim_{n\to\infty} \beta_n = 0, \quad (ii)\ \beta_n(1+\alpha_n) < 1, \quad \sum_{n=1}^{\infty} \alpha_n\beta_n = \infty,$$

$$(iii) \lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0 \quad \textit{iff}\ \beta_n = \circ(\alpha_n), \quad (iv) \lim_{n\to\infty} \beta_n^{-1}\alpha_n^{-2}(\alpha_{n-1} - \alpha_n) = 0.$$

Remark 1.1. An example of an operator A, for which AJ is accretive, given in Araka [2] is $A := J^{-1}$. This is a *trivial example*.

It is our purpose in this paper to extend Theorem 1.5 from the class of accretive *Lipschitz* maps to the more general class of accretive *uniformly continuous* maps. Furthermore, the sequence in our theorem will, as in Theorem 1.5, converge strongly to an element of $(AJ)^{-1}(0)$. Finally, we present *non-trivial examples* of maps $A : E^* \to E$ for which AJ is accretive.

2. PRELIMINARIES

Lemma 2.1 (Moore and Nnoli, [18]). *Let* $\{\rho_n\}_{n=1}^{\infty}$ *be a sequence of non-negative real numbers satisfying the following relation:*

$$\rho_{n+1} \le \rho_n - \alpha_n\psi(\rho_{n+1}) + \sigma_n,\ n \ge 1, \tag{2.1}$$

where $(i)\ 0 < \alpha_n < 1$ $(ii)\ \sum_{n=1}^{\infty}\alpha_n = \infty$ $(iii)\ \psi : [0,\infty) \to [0,\infty)$ *is a strictly increasing function with* $\psi(0) = 0$. *Suppose that* $\sigma_n = o(\alpha_n)$. *Then,* $\rho_n \to 0$ *as* $n \to \infty$.

Lemma 2.2 (Chidume, [6]). *Let E be a real normed space, and* $J : E \to 2^{E^*}$ *be the normalized duality map. Then, for any* $x,\ y \in E$, *the following inequality holds:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \tag{2.2}$$

for all $j(x+y) \in J(x+y)$.

Lemma 2.3 (Morales and Jung, [19]). *Let K be a closed convex subset of a reflexive Banach space E with Uniformly Gâteaux differentiable norm. Let* $T : K \to K$ *be continuous pseudo-contractive map with* $F(T) \neq \emptyset$. *Suppose that every nonempty closed convex and bounded subset of K has the fixed point property for nonexpansive self-map. Then, for* $u \in K$, *the path* $t \to y_t \in K$, $t \in (0,1]$, *satisfying*

$$y_t = (1-t)Ty_t + tu, \tag{2.3}$$

converges strongly to a fixed point Qu of T as $t \to 0^+$, *where Q is the unique sunny nonexpansive retract from K onto* $F(T)$.

We obtain the following corollary from Lemma 2.3 if $0 \in K$ and $u = 0$.

Corollary 2.4. *Let K be a closed subset of a reflexive Banach space E with a uniformly Gâteaux differentiable norm such that* $0 \in K$. *Let* $T : K \to K$ *be a continuous pseudocontractive map with* $F(T) \neq \emptyset$. *Suppose that every nonempty closed convex and bounded subset of K has the fixed point property of nonexpansive self-map. Then, the path* $t \to y_t$, $t \in (0,1]$, *satisfying,*

$$y_t = (1-t)Ty_t, \tag{2.4}$$

converges strongly to a fixed point of $Q(0)$ of T as $t \to 0^+$, *where Q is the unique sunny nonexpansive retract from K onto* $F(T)$.

Remark 2.1. In corollary 2.4, if we define, $y_n := y_{t_n}$ and $t_n = \frac{1}{1+\alpha_n}\ \forall\ n \geq 1$, where $\alpha_n \to 0$, as $n \to \infty$, we obtain,

$$y_n = \frac{1}{1+\alpha_n} T y_n. \tag{2.5}$$

It follows from equation (2.5), that

$$\alpha_n y_n + (I-T)y_n = 0 \tag{2.6}$$

and

$$-\alpha_{n-1}y_{n-1} = -\alpha_n y_n + (I-T)y_{n-1} - (I-T)y_n. \tag{2.7}$$

We observe that since T is a pseudocontractive map, then $(I-T)$ is an accretive map, where I is the identity map.

3. MAIN RESULTS

For the rest of this paper, the sequences $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$, $\{\beta_n\}_{n=1}^{\infty} \subset (0,1)$ are assumed to satisfy the following conditions:

$(i)\ \sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$,

$(ii)\ \lim_{n\to\infty} \Big[\frac{|\alpha_{n-1}-\alpha_n|}{\alpha_n^2 \beta_n} + \beta_n + \frac{\xi_n}{\alpha_n}\Big] = 0$, where $\xi_n := \|AJx_{n+1} - AJx_n\|$.

We now prove the following theorem.

Theorem 3.1. *Let E be a real Banach space with dual space E^*, such that the normalized duality map J from E to E^* is single-valued. Let $A : E^* \to E$ be a map such that AJ is an accretive and uniformly continuous map. For arbitrary $x_1 \in E$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively defined by*

$$x_{n+1} = x_n - \beta_n AJx_n - \alpha_n\beta_n x_n, \ \ n \geq 1. \tag{3.1}$$

Assume that $(AJ)^{-1}(0) \neq \emptyset$. Then, $\{x_n\}_{n=1}^{\infty}$ is bounded.

Proof. Let $x^* \in (AJ)^{-1}(0)$. Then, there exists $r > 0$ such that for $B := \overline{B_r(x^*)}$, the following conditions hold:

(1) $x_1 \in \overline{B_{\frac{r}{2}}(x^*)}$,
(2) $\|x^*\| \leq \frac{r}{4}$,
(3) $\xi_n < \frac{r}{8}\alpha_n$,
(4) $\beta_n M_0 \leq \frac{r}{8}$, where $M_0 := \sup\{\|AJx + \alpha x\| : x \in \overline{B_r(x^*)}, \alpha \in (0,1)\}$.

We prove that $x_n \in B\ \forall\ n \geq 1$. The proof is by induction. Clearly, $x_1 \in B$ by construction. Assume that $x_n \in B$, for some $n \geq 1$. We prove that $x_{n+1} \in B$. Suppose that $x_{n+1} \notin B$ i.e., suppose $\|x_{n+1} - x^*\| > r$. Then, by lemma 2.2 and the recursion formula (3.1), we have that

$$\begin{aligned} r^2 &< \|x_{n+1} - x^*\|^2 \\ &= \|x_n - x^* - \beta_n(AJx_n + \alpha_n x_n)\|^2 \\ &\leq \|x_n - x^*\|^2 - 2\beta_n\langle AJx_n + \alpha_n x_n, J(x_{n+1} - x^*)\rangle \\ &= \|x_n - x^*\|^2 - 2\beta_n\langle \alpha_n(x_{n+1} - x^*) - \alpha_n(x_{n+1} - x_n) + AJx_n + \alpha_n x^* \\ &\quad +(AJx_{n+1} - AJx_{n+1}), J(x_{n+1} - x^*)\rangle \\ &= \|x_n - x^*\|^2 - 2\beta_n\alpha_n\|x_{n+1} - x^*\|^2 + 2\beta_n\langle \alpha_n(x_{n+1} - x_n) - \alpha_n x^* \\ &\quad +(AJx_{n+1} - AJx_n) - AJx_{n+1}, J(x_{n+1} - x^*)\rangle. \end{aligned}$$

Thus, using the recursion formula (3.1) again and the fact that AJ is an accretive and uniformly continuous map, we obtain that

$$\begin{aligned} r^2 &< \|x_{n+1} - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - 2\beta_n\alpha_n\|x_{n+1} - x^*\|^2 + 2\beta_n\langle \alpha_n(x_{n+1} - x_n) - \alpha_n x^* \\ &\quad +(AJx_{n+1} - AJx_n), J(x_{n+1} - x^*)\rangle \\ &\leq \|x_n - x^*\|^2 - 2\beta_n\alpha_n\|x_{n+1} - x^*\|^2 + 2\beta_n\Big[\alpha_n\|x_{n+1} - x_n\| + \alpha_n\|x^*\| \\ &\quad +\|AJx_{n+1} - AJx_n\|\Big]\|x_{n+1} - x^*\| \\ &\leq \|x_n - x^*\|^2 - 2\beta_n\alpha_n\|x_{n+1} - x^*\|^2 + 2\beta_n\Big[\alpha_n\beta_n M_0 + \alpha_n\|x^*\| + \xi_n\Big]\|x_{n+1} - x^*\|. \end{aligned}$$

But, by induction hypothesis, $x_n \in B$, and using the conditions on the sequences $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$, we obtain that

$$\begin{aligned} r^2 &< \|x_{n+1} - x^*\|^2 \\ &\leq r^2 - 2\beta_n\alpha_n\|x_{n+1} - x^*\|^2 + 2\beta_n\Big(\frac{r}{8}\alpha_n + \frac{r}{4}\alpha_n + \frac{r}{8}\alpha_n\Big)\|x_{n+1} - x^*\|. \end{aligned}$$

This yields that, $\quad 2\beta_n\alpha_n\|x_{n+1} - x^*\|^2 \leq 2\beta_n\Big(\frac{r}{8}\alpha_n + \frac{r}{4}\alpha_n + \frac{r}{8}\alpha_n\Big)\|x_{n+1} - x^*\|$.
Hence, $\|x_{n+1} - x^*\| \leq \frac{r}{4} + \frac{r}{4} = \frac{r}{2}$, a contradiction. Therefore, $x_n \in B \ \forall\, n \geq 1$, and so $\{x_n\}_{n=1}^\infty$ is bounded. ∎

We now prove the following strong convergence theorem.

Theorem 3.2. *Let E be a reflexive real Banach space with uniformly Gâteaux differentiable norm. Let $A : E^* \to E$ be a map such that AJ is an accretive and uniformly continuous map. For any $x_1 \in E$, let the sequence $\{x_n\}_{n=1}^\infty$ be iteratively defined by*

$$x_{n+1} = x_n - \beta_n AJx_n - \alpha_n\beta_n x_n, \ n \geq 1, \tag{3.2}$$

where $J : E \to E^*$ is the single-valued normalized duality map. Suppose that $(AJ)^{-1}(0) \neq \emptyset$. Then, the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to some $u^* \in (AJ)^{-1}(0)$.

Proof. From the recursion formula (3.2), Lemma 2.2 and Corollary 2.4 (with y_n as in Corollary 2.4), we have that

$$\begin{aligned}
\|x_{n+1}-y_n\|^2 &= \|x_n-y_n-\beta_n(AJx_n+\alpha_nx_n)\|^2\\
&\leq \|x_n-y_n\|^2-2\beta_n\langle AJx_n+\alpha_nx_n, J(x_{n+1}-y_n)\rangle\\
&= \|x_n-y_n\|^2-2\beta_n\langle \alpha_n(x_{n+1}-y_n)-\alpha_n(x_{n+1}-y_n)\\
&\quad +AJx_n+\alpha_nx_n, J(x_{n+1}-y_n)\rangle\\
&= \|x_n-y_n\|^2-2\beta_n\alpha_n\langle x_{n+1}-y_n, J(x_{n+1}-y_n)\rangle\\
&\quad +2\beta_n\langle \alpha_n(x_{n+1}-y_n)-AJx_n-\alpha_nx_n, J(x_{n+1}-y_n)\rangle\\
&= \|x_n-y_n\|^2-2\beta_n\alpha_n\|x_{n+1}-y_n\|^2+2\beta_n\langle\alpha_n(x_{n+1}-x_n)\\
&\quad +(AJx_{n+1}-AJx_n)-(\alpha_ny_n+AJy_n)-(AJx_{n+1}-AJy_n), J(x_{n+1}-y_n)\rangle.
\end{aligned}$$

Thus, using equation (2.6) and the fact that AJ is an accretive and uniformly continuous map, we have that

$$\begin{aligned}
\|x_{n+1}-y_n\|^2 &\leq \|x_n-y_n\|^2-2\beta_n\alpha_n\|x_{n+1}-y_n\|^2\\
&\quad +2\beta_n\langle\alpha_n(x_{n+1}-x_n)+(AJx_{n+1}-AJx_n), J(x_{n+1}-y_n)\rangle\\
&\leq \|x_n-y_n\|^2-2\beta_n\alpha_n\|x_{n+1}-y_n\|^2\\
&\quad +2\beta_n\Big[\alpha_n\beta_nM_0+\xi_n\Big]\|x_{n+1}-y_n\|.
\end{aligned}\tag{3.3}$$

Since AJ is accretive, using equation (2.7), we observe that

$$\begin{aligned}
\|y_{n-1}-y_n\| &\leq \|y_{n-1}-y_n+\alpha_n^{-1}(AJy_{n-1}-AJy_n)\|\\
&= |\alpha_n^{-1}(\alpha_{n-1}-\alpha_n)|\|y_{n-1}\|.
\end{aligned}\tag{3.4}$$

By the boundedness of the sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$, one can easily see that

$$\|x_n-y_n\|^2\leq\|x_n-y_{n-1}\|^2+M_1\|y_{n-1}-y_n\|,\tag{3.5}$$

for some positive constant M_1. Using inequalities (3.3), (3.4), (3.5) and the fact that the sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are bounded, it follows that for some positive constants M_0, M_1, M_2, M_3 and M, we have that

$$\begin{aligned}
\|x_{n+1}-y_n\|^2 &\leq \|x_n-y_{n-1}\|^2-2\beta_n\alpha_n\|x_{n+1}-y_n\|^2\\
&\quad +M_1\|y_{n-1}-y_n\|+2\beta_n\Big[\alpha_n\beta_nM_0+\xi_n\Big]M_2\\
&\leq \|x_n-y_{n-1}\|^2-2\beta_n\alpha_n\|x_{n+1}-y_n\|^2+|\alpha_n^{-1}(\alpha_{n-1}-\alpha_n)|M_3\\
&\quad +2\beta_n\Big[\alpha_n\beta_nM_0+\xi_n\Big]M_2\\
&\leq \|x_n-y_{n-1}\|^2-2\beta_n\alpha_n\|x_{n+1}-y_n\|^2\\
&\quad + M\Big[|\alpha_n^{-1}(\alpha_{n-1}-\alpha_n)|+\alpha_n\beta_n^2+\beta_n\xi_n\Big].
\end{aligned}$$

Thus, by Lemma 2.1 and the conditions on $\{\alpha_n\}$ and $\{\beta_n\}$, we get that $\|x_{n+1}-y_n\|\to 0$ as $n\to\infty$. Consequently, $\|x_n-y_{n-1}\|\to 0$ as $n\to\infty$. Since $\{y_n\}_{n=1}^\infty$ converges strongly to some $u^*\in(AJ)^{-1}(0)$, we get that $\{x_n\}_{n=1}^\infty$ converges strongly to some $u^*\in(AJ)^{-1}(0)$, completing the proof. ∎

Corollary 3.3 below which is a result of Araka [2] is an immediate consequence of Theorem 3.1.

Corollary 3.3. *Let E be a reflexive real Banach space with uniformly Gâteaux differentiable norm with dual space E^*. Let $A : E^* \to E$ be any map. For any $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence iteratively defined by*

$$x_{n+1} = x_n - \beta_n AJx_n - \alpha_n\beta_n x_n, \; n \geq 1, \tag{3.6}$$

where $J : E \to E^*$ is the normalized duality map. Suppose that $(AJ)^{-1}(0) \neq \emptyset$; and suppose that AJ is an accretive Lipschitz map with Lipschitz constant $L \geq 0$. Then, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to some $u^* \in (AJ)^{-1}(0)$.

Remark 3.1. Since AJ is Lipschitz, let the Lipschitz constant of AJ be L. Then, we have

$$\frac{\xi_n}{\alpha_n} = \frac{\|AJx_{n+1} - AJx_n\|}{\alpha_n} \leq \frac{L\beta_n M_o}{\alpha_n}. \tag{3.7}$$

Xu and Roach [25] proved that in a 2-uniformly smooth space, the normalized duality map $J : E \to E^*$ is Lipschitz. Therefore, the assumption that AJ is a Lipschitz map can be dispensed with when A is Lischitz. Hence, we obtain the following corollary.

Corollary 3.4. *Let E be a uniformly convex and 2-uniformly smooth real Banach space with dual E^*. Let $A : E^* \to E$ be a Lipschitz maximal monotone map with $A^{-1}(0) \neq \emptyset$. For any $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence iteratively defined by*

$$x_{n+1} = x_n - \beta_n AJx_n - \alpha_n\beta_n x_n, \; n \geq 1. \tag{3.8}$$

Suppose that AJ is an accretive map and $(AJ)^{-1}0 \neq \emptyset$. Then, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to some $u^* \in (AJ)^{-1}(0)$.

Remark 3.2. Corollary 3.4 is a significant improvement of Theorem 1.3, in the sense that this corollary fills the gap in the theorem as identified by Araka.

AN APPLICATION

Corollary 3.5. *Let H be a real Hilbert space and $A : H \to H$ be a uniformly continuous and monotone map such that $A^{-1}(0) \neq \emptyset$. For any $x_1 \in H$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence iteratively defined by*

$$x_{n+1} = x_n - \beta_n Ax_n - \alpha_n\beta_n x_n, \; n \geq 1. \tag{3.9}$$

Then, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to some $u^* \in (A)^{-1}(0)$.

Remark 3.3. In corollary 3.5, the map A is uniformly continuous, monotone and has H as its domain, so it is maximal monotone (see e.g., Cioranescu [10]).

Remark 3.4. Corollary 3.5 complements Theorem 1.1 in the case that A is uniformly continuous by providing a sequence which converges strongly to an element of $A^{-1}(0)$, where the recursion formula of the sequence does not involve the resolvent map, $(I + rA)^{-1}$. Furthermore, the condition "*... suppose the criterion for the approximate computation of y_n is $\|y_n - J_{r_n}x_n\| < \delta_n$ where $\sum_{n=1}^{\infty}\delta_n < \infty$ ...*" imposed in Theorem 1.1 is dispensed with.

Remark 3.5. In the case where $A : E \to E^*$ is any map such that $J^{-1}A : E \to E$ is an accretive and uniformly continuous map, following the technique of the proof of Theorem 3.2 and replacing AJ by $J^{-1}A$, the following theorem is easily proved.

Theorem 3.6. *Let E be a reflexive real Banach space with uniformly Gâteaux differentiable norm. Let $A : E \to E^*$ be a map such that $J^{-1}A : E \to E$ is an accretive and uniformly continuous map. For any $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence defined iteratively by*

$$x_{n+1} = x_n - \beta_n J^{-1}Ax_n - \alpha_n\beta_n x_n,\ n \geq 1, \tag{3.10}$$

where $J : E \to E^*$ is the normalized duality map such that $J^{-1} : E^* \to E$ exists. Suppose that $(J^{-1}A)^{-1}(0) \neq \emptyset$. Then, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to some $u^* \in (J^{-1}A)^{-1}(0)$.

Remark 3.6. Prototypes of sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ satisfying conditions of Corollaries 3.3 and 3.4 are:

$$\alpha_n = (n+1)^{-a} \text{ and } \beta_n = (n+1)^{-b},\ n \geq 1,$$

where $0 < a < b$ and $a + b < 1$.

4. EXAMPLES OF MAPS A FOR WHICH AJ IS ACCRETIVE

Remark 4.1. (see e.g., Alber and Ryazantseva, [1]; p. 36) The analytical representations of duality maps are known in a number of Banach spaces. For instance, in the spaces l_p, $L_p(G)$ and $W_m^p(G)$, $p \in (1, \infty)$, $p^{-1} + q^{-1} = 1$, respectively,
$Jx = \|x\|_{l_p}^{2-p} y \in l_q,\ y = \{|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, ...\},\ x = \{x_1, x_2, ...\},$
$J^{-1}x = \|x\|_{l_q}^{2-q} y \in l_p,\ y = \{|x_1|^{q-2}x_1, |x_2|^{q-2}x_2, ...\},\ x = \{x_1, x_2, ...\},$
$Jx = \|x\|_{L_p}^{2-p}|x(s)|^{p-2}x(s) \in L_q(G),\ s \in G,$
$J^{-1}x = \|x\|_{L_q}^{2-q}|x(s)|^{q-2}x(s) \in L_p(G),\ s \in G,$ and
$Jx = \|x\|_{W_m^p}^{2-p}\sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^{\alpha}(|D^{\alpha}x(s)|^{p-2}D^{\alpha}x(s)) \in W_{-m}^q(G), m > 0, s \in G.$

Remark 4.2. We now give non-trivial examples of maps, $A : E^* \to E$, such that AJ is an accretive map.

Let $A : \ell_p \to \ell_q,\ 1 < q < p,\ \frac{1}{p} + \frac{1}{q} = 1.$
Recall (Remark 4.1) that $J : \ell_p \to (\ell_p)^*$ is defined by:

$$J(x) = \|x\|^{2-p}\Big(x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, x_3|x_3|^{p-2}, \cdots\Big). \tag{4.1}$$

Example 4.1. *Now, define A by*

$$A(y) := \|y\|^{2-q}\Big(0, y_2|y_2|^{q-2}, y_2|y_2|^{q-2}, y_3|y_3|^{q-2}, \cdots\Big). \tag{4.2}$$

We compute AJ.

$$\begin{aligned}(AJ)(x) &= A\Big[\|x\|^{2-p}\Big(x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, x_3|x_3|^{p-2}, \cdots\Big)\Big]\\ &= \|x\|^{2-p}\|x\|^{2-q}\Big(0, x_2|x_2|^{p-2}\Big|\Big(x_2|x_2|^{p-2}\|x\|^{2-p}\Big)\Big|^{q-2},\\ &\qquad x_3|x_3|^{p-2}\Big|\Big(x_3|x_3|^{p-2}\|x\|^{2-p}\Big)\Big|^{q-2}, \cdots\Big)\\ &= (0, x_2, x_3, \cdots).\end{aligned}$$

Thus, $(AJ)(x) = (0, x_2, x_3, \cdots)$.
We now show that AJ *is accretive. Since,* AJ *is linear we compute as follows.*

$$\begin{aligned}\langle AJx, Jx\rangle &= \Big\langle (0, x_2, x_3, \cdots), \|x\|^{2-p}\Big(x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, x_3|x_3|^{p-2}, \cdots\Big)\Big\rangle \\ &= \|x\|^{2-p}\Big(\sum_{n=2}^{\infty}|x_n|^p\Big) \geq 0.\end{aligned}$$

Therefore, $\langle AJx, Jx\rangle \geq 0$. *Hence,* AJ *is accretive.*

Example 4.2. *Define* A *by*

$$A(y) := \|y\|^{2-q}\Big(y_1|y_1|^{q-2}, \frac{y_2|y_2|^{q-2}}{2}, \frac{y_3|y_3|^{q-2}}{3}, \cdots\Big). \tag{4.3}$$

We compute AJ.

$$\begin{aligned}(AJ)(x) &= A\Big[\|x\|^{2-p}\big(x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, x_3|x_3|^{p-2}, \cdots\big)\Big] \\ &= \|x\|^{2-p}\|x\|^{2-q}\Big(x_1|x_1|^{p-2}\Big|\big(x_1|x_1|^{p-2}\|x\|^{2-p}\big)\Big|^{q-2}, \frac{x_2|x_2|^{p-2}\Big|\big(x_2|x_1|^{p-2}\|x\|^{2-p}\big)\Big|^{q-2}}{2}, \\ &\quad \frac{x_3|x_3|^{p-2}\Big|\big(x_3|x_3|^{p-2}\|x\|^{2-p}\big)\Big|^{q-2}}{3}, \cdots\Big) \\ &= \Big(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots\Big).\end{aligned}$$

Thus, $(AJ)(x) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots)$.
We now show that AJ *is accretive. Clearly,* AJ *is linear. We compute as follows:*

$$\begin{aligned}\langle AJx, Jx\rangle &= \Big\langle \Big(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots\Big), \|x\|^{2-p}\Big(x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, x_3|x_3|^{p-2}, \cdots\Big)\Big\rangle \\ &= \|x\|^{2-p}\Big(\sum_{n=1}^{\infty}\frac{|x_n|^p}{n}\Big) \geq 0.\end{aligned}$$

Therefore, $\langle AJx, Jx\rangle \geq 0$. *Hence,* AJ *is accretive.*

Example 4.3. *Define* A *by*

$$A(y) := \|y\|^{2-q}\Big(y_1|y_1|^{q-2}, y_2|y_2|^{q-2}, \cdots, y_m|y_m|^{q-2}, 0, 0, 0, \cdots\Big). \tag{4.4}$$

We compute AJ.

$$\begin{aligned}(AJ)(x) &= A\Big[\|x\|^{2-p}\big(x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \cdots, x_m|x_m|^{p-2}, \cdots\big)\Big] \\ &= \|x\|^{2-p}\|x\|^{2-q}\Big(x_1|x_1|^{p-2}\Big|\big(x_1|x_1|^{p-2}\|x\|^{2-p}\big)\Big|^{q-2}, x_2|x_2|^{p-2}\Big|\big(x_2|x_1|^{p-2}\|x\|^{2-p}\big)\Big|^{q-2}, \\ &\quad \cdots, x_m|x_m|^{p-2}\Big|\big(x_m|x_m|^{p-2}\|x\|^{2-p}\big)\Big|^{q-2}, 0, 0, 0, \cdots\Big) \\ &= \Big(x_1, x_2, \cdots, x_m, 0, 0, 0, \cdots\Big).\end{aligned}$$

Thus, $(AJ)(x) = \Big(x_1, x_2, \cdots, x_m, 0, 0, 0, \cdots\Big)$.
We now show that AJ *is accretive. Clearly,* AJ *is linear. We compute as follows:*

$$\begin{aligned} \langle AJx, Jx\rangle &= \Big\langle \Big(x_1, x_2, \cdots, x_m, 0, 0, 0, \cdots\Big), \|x\|^{2-p}\Big(x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, x_3|x_3|^{p-2}, \cdots\Big)\Big\rangle \\ &= \|x\|^{2-p}\Big(\sum_{n=1}^{m}|x_n|^p\Big) \geq 0. \end{aligned}$$

Therefore, $\langle AJx, Jx\rangle \geq 0$. *Hence,* AJ *is accretive.*

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