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APPLICATIONS OF THE STRUCTURE THEOREM OF (w_1, w_2)-TEMPERED ULTRADISTRIBUTIONS

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ABSTRACT. Using a previously obtained structure theorem for (w_1, w_2)-tempered ultradistributions, we prove that these ultradistributions can be represented as initial values of solutions of the heat equation.

Key words and phrases: Short-time Fourier transform; Tempered Ultradistributions; Structure Theorem.

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1. INTRODUCTION

In mathematical analysis, distributions (generalized functions) are objects which generalize functions. They extend the concept of derivative to all integrable functions and beyond, and are used to formulate generalized solutions of partial differential equations. They play a crucial role in physics and engineering where many non-continuous problems naturally lead to differential equations whose solutions are distributions, such as the Dirac delta distribution. In late forties, L. Schwartz gave his formulation of distribution theory. This formulation leads to extensive applications in mathematical analysis, mathematical physics, and engineering. Recently, the theory of distributions devised by L. Schwartz is used in microlocal analysis, signal processing, image processing and wavelets.

The Schwartz space $\mathfrak{S}$, as defined by L. Schwartz (see [10]), consists of all $C^{\infty}(\mathbb{R}^n)$ functions φ such that $\left\|x^{\alpha}\partial^{\beta}\varphi\right\|_{\infty} < \infty$ for all $\alpha, \beta \in \mathbb{N}^n$. The topological dual space of $\mathfrak{S}$, is a space of generalized functions, called tempered distributions. In 1963, A. Beurling presented his generalization of tempered distributions. The aim of this generalization was to find an appropriate context for his work on pseudo-analytic extensions (see [2]).

In 1967 (see [3]), G. Björck studied and expanded the theory of Beurling on ultradistributions to extend the work of Hörmander on existence, nonexistence, and regularity of solutions of differential equations with constant coefficient and also he studied the convolution . The Beurling-Björck space $\mathfrak{S}_w$, as defined by G. Björck, consists all $C^{\infty}(\mathbb{R}^n)$ functions φ such that $\left\|e^{kw(x)}\partial^{\beta}\varphi\right\|_{\infty} < \infty$ and $\left\|e^{kw(x)}\partial^{\beta}\widehat{\varphi}\right\|_{\infty} < \infty$ for all $\alpha, \beta \in \mathbb{N}^n$, where w is a subadditive weight function satisfying the classical Beurling conditions. The topological dual $\mathfrak{S}'_w$ of $\mathfrak{S}_w$ is a space of generalized functions, called $w-$tempered ultradistributions. When $w(x) = \log(1 + |x|)$, the Beurling- Björck space $\mathfrak{S}_w$ becomes the Schwartz space $\mathfrak{S}$ (see [1] and [4]).

In [7], the authors introduced the space $\mathfrak{S}_{w_1,w_2}$ of all $C^{\infty}(\mathbb{R}^n)$ functions φ such that $\left\|e^{kw_1(x)}\partial^{\beta}\varphi\right\|_{\infty} < \infty$ and $\left\|e^{kw_2(x)}\partial^{\beta}\widehat{\varphi}\right\|_{\infty} < \infty$ for all $k \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$, where w_1 and w_2 are two weights satisfying the classical Beurling conditions. The topological dual $\mathfrak{S}'_{w_1,w_2}$ of $\mathfrak{S}_{w_1,w_2}$ is a space of generalized functions, called (w_1, w_2)-tempered ultradistributions. Moreover, they proved a structure theorem for functionals $T \in \mathfrak{S}'_{w_1,w_2}$ using the classical Riesz representation theorem.

In this paper, we will use the structure theorem obtained in [7] to prove that the convolution of an ultradistribution $T \in \mathfrak{S}'_{w_1,w_2}$ and a test function $\varphi \in \mathfrak{S}_{w_1,w_2}$, defined by $T*\varphi = \langle T, \varphi(x-\cdot)\rangle$, coincides with the classical definition of the convolution of two integrable distributions. As a result, we prove that (w_1, w_2)-tempered ultradistributions can be represented as boundary values of solutions of the heat equation.

The symbols C^{∞}, C_0^{∞}, L^p, etc., denote the usual spaces of functions defined on $\mathbb{R}^n$, with complex values. We denote $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$, while $\|\cdot\|_p$ indicates the p-norm in the space L^p, where $1 \leq p \leq \infty$. In general, we work on the Euclidean space R^n unless we indicate other than that as appropriate. Partial derivatives will be denoted by ∂^{α}, where α is a multi-index $(\alpha_1, ..., \alpha_n)$ in $\mathbb{N}_0^n$. We will use the standard abbreviations $|\alpha| = \alpha_1 + ... + \alpha_n$, $x^{\alpha} = x_1^{\alpha_1}...x_n^{\alpha_n}$. The Fourier transform of a function f will be denoted by $\mathcal{F}(f)$ or $\widehat{f}$ and it will be defined as $\int_{\mathbb{R}^n} e^{-2\pi i x\xi} f(x)\,dx$. With $\mathcal{C}_0$ we denote the Banach space of continuous functions vanishing at infinity with supremum norm.

2. PRELIMINARY DEFINITIONS AND RESULTS

In this section, we start with the definition of the space of admissible functions $\mathcal{M}_c$ as they were introduced by Björck.

Definition 2.1. ([3]) With $\mathcal{M}_c$ we indicate the space of functions $w : \mathbb{R}^n \to \mathbb{R}$ of the form $w(x) = \Omega(|x|)$, where

(1) $\Omega : [0,\infty) \to [0,\infty)$ is increasing, continuous and concave,
(2) $\Omega(0) = 0$,
(3) $\int_{\mathbb{R}} \frac{\Omega(t)}{(1+t^2)} dt < \infty$,
(4) $\Omega(t) \geq a + b\ln(1+t)$ for some $a \in \mathbb{R}$ and some $b > 0$.

Standard classes of functions w in $\mathcal{M}_c$ are given by

$$w(x) = |x|^d \text{ for } 0 < d < 1, \text{ and } w(x) = p\ln(1+|x|) \text{ for } p > 0.$$

Remark 2.1. Let us observe for future use that if we take an integer $N > \frac{n}{b}$, then

$$C_N = \int_{\mathbb{R}^n} e^{-Nw(x)} dx < \infty, \text{ for all } w \in \mathcal{M}_c,$$

where b is the constant in Condition 4 of Definition 2.1.

In the following theorem, we state the characterization of the space $\mathfrak{S}_{w_1,w_2}$.

Theorem 2.1. *([7]) Given $w_1, w_2 \in \mathcal{M}_c$, the space $\mathfrak{S}_{w_1,w_2}$ can be described as a set as well as topologically by*

$$\mathfrak{S}_{w_1,w_2} = \left\{ \begin{array}{c} \varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \textit{ is continuous and for all} \\ k = 0,1,2,..., p_k(\varphi) < \infty, q_k(\varphi) < \infty \end{array} \right\},$$

where $p_k(\varphi) = \left\| e^{kw_1}\varphi \right\|_\infty$, $q_k(\varphi) = \left\| e^{kw_2}\widehat{\varphi} \right\|_\infty$.

The space $\mathfrak{S}_{w_1,w_2}$, equipped with the family of semi-norms

$$\mathcal{N} = \{p_k, q_k : k \in \mathbb{N}_0\},$$

is a Fréchet space.

In [7], the authors employ the above theorem to prove the following structure theorem for functionals $T \in \mathfrak{S}'_{w_1,w_2}$.

Theorem 2.2. *([7]) If $T \in \mathfrak{S}'_{w_1,w_2}$, then there exist two regular complex Borel measures μ_1 and μ_2 of finite total variation and $k \in \mathbb{N}_0$ such that*

$$T = e^{kw_1}\mu_1 + \mathcal{F}(e^{kw_1}\mu_2) \tag{2.1}$$

in the sense of $\mathfrak{S}'_{w_1,w_2}$.

Lemma 2.3. *Let $\varphi \in \mathfrak{S}_{w_1,w_2}$. Then $\varphi(x+y) \in \mathfrak{S}_{w_1,w_2}$ for each $y \in \mathbb{R}^n$.*

Proof. Fix $y \in \mathbb{R}^n$ and let $\varphi \in \mathfrak{S}_{w_1,w_2}$. First, let us prove that $\left\| e^{kw_1(x)}\varphi(x+y) \right\|_\infty < \infty$. To do so, we use concavity property of w_1 as follows:

$$\begin{aligned} e^{kw_1(x)} |\varphi(x+y)| &= e^{kw_1(x)} e^{-2kw_1(x+y)} e^{2kw_1(x+y)} |\varphi(x+y)| \\ &\leq e^{kw_1(x)} e^{-kw_1(x+y)} \left\| e^{2kw_1(x+y)}\varphi(x+y) \right\|_\infty \\ &\leq Ce^{2k(\frac{w_1(x)}{2} - 2w_1(x+y))} \leq e^{2k(\frac{w_1(x)}{2} - w_1(\frac{x+y}{2}))} \\ &\leq Ce^{2k(-\frac{w_1(y)}{2})} \leq Ce^{-kw_1(y)} < \infty. \end{aligned}$$

This proves that $\left\| e^{kw_1(x)}\varphi(x+y) \right\|_\infty < \infty$. Similarly, $\left\| e^{kw_2(x)}\widehat{\varphi}(x+y) \right\|_\infty < \infty$. This completes the proof of Lemma 2.3. □

Given two functionals T and S that are integrable functions, the classical definition of convolution of T and S is given by

$$\langle T * S, \phi\rangle = \langle T_x, \langle S_y, \phi(y+x)\rangle.$$

In Definition 1.7.1 of [3], G. Björck defines the convolution $T * \varphi$ for a functional defined on the space of test functions with compact support $\mathfrak{D}_w \subset \mathfrak{S}_w$ and $\varphi \in \mathfrak{D}_w$ as a function by $\langle T, \varphi(x - \cdot)\rangle$. Using this definition, Definition 1.6.11, and results from Section 1.7 of [3], he proves that if $T \in \mathfrak{S}'_w$ and $\varphi \in \mathfrak{S}_w$, then the functional $T * \varphi \in \mathfrak{S}'_w$.

Theorem 2.4. *If $T \in \mathfrak{S}'_{w_1,w_2}$ and $\varphi \in \mathfrak{S}_{w_1,w_2}$, then the functional $T * \varphi \in \mathfrak{S}'_{w_1,w_2}$ and given by $\langle T, \varphi(x - \cdot)\rangle$.*

Proof. Since $\mathfrak{S}_{w_1,w_2}$ is a natural generalization of the Beurling-Björck space $\mathfrak{S}_w$, the proof of Theorem 2.4 mimics the proof of Theorem 1.8.12 of [3] and so we will omit it. $\square$

We end this section with the definition of operator semigroup on a Banach space that we will use in applications in the next section.

Definition 2.2. [8] Let $\mathfrak{B}$ be a Banach space. An operator semigroup on $\mathfrak{B}$ is a family $(T_t : t \in \mathbb{R}^+)$ of bounded linear operators on $\mathfrak{B}$ such that

i): $T_0 = I$,
ii): $T_sT_t = T_{s+t}$ for all $t, s \in \mathbb{R}^+$.

3. APPLICATIONS

In this section, we study some applications of the structure theorem of (w_1, w_2)-tempered ultradistributions stated in Theorem 2.2 by proving some results on a semi-group acting on the Fréchet space $\mathfrak{S}_{w_1,w_2}$ and extend it to its dual $\mathfrak{S}'_{w_1,w_2}$. We start this section by proving that the convolution in Theorem 2.4 coincides with classical definition of convolution of two integrable functionals.

Theorem 3.1. *If $T \in \mathfrak{S}'_{w_1,w_2}$ and $\varphi \in \mathfrak{S}_{w_1,w_2}$, then the functional $T * \varphi$ defined by*

$$\langle T * \varphi, \phi\rangle = \langle T_y, (\varphi_z, \phi(x+y)\rangle$$

coincides with the functional given by integration against the function $\psi(x) = \langle T_y, \varphi(x-y)\rangle$.

Proof. Using (2.1) in Theorem 2.2, we can write for each x

$$\psi(x) = \langle T_y, \varphi(x-y)\rangle = \int_{\mathbb{R}^n} e^{kw_1(y)}\varphi(x+y)d\mu_1(y) + \int_{\mathbb{R}^n} e^{kw_2(\xi)}e^{-2\pi iy.\xi}\mathcal{F}^{-1}(\varphi)(\xi)d\mu_2(\xi).$$

So,

$$\begin{aligned}
\langle T * \varphi, \phi\rangle &= \langle T_y, (\varphi_z, \phi(x+y)\rangle \\
&= \int_{\mathbb{R}^n} e^{kw_1(y)}(\int_{\mathbb{R}^n} \varphi(x-y)\phi(y)d\mu_1(y)) + \int_{\mathbb{R}^n} e^{kw_2(\xi)}\mathcal{F}^{-1}(\varphi)(\xi)\widehat{\phi}(\xi)d\mu_2(\xi) \\
&= \int_{\mathbb{R}^n} e^{kw_1(y)}(\int_{\mathbb{R}^n} \varphi(x-y)\phi(y)d\mu_1(y)) + \int_{\mathbb{R}^n} e^{kw_2(\xi)}\mathcal{F}(\check{\varphi} * \phi)(\xi)d\mu_2(\xi) \\
&= \langle e^{kw_1}\mu_1(y), \langle\varphi(x-y), \phi(x)\rangle\rangle + \langle\mathcal{F}(e^{kw_2}\mu_2)(y), \langle\varphi(x-y), \phi(x)\rangle\rangle \\
&= \langle e^{kw_1}\mu_1(y) + \mathcal{F}(e^{kw_2}\mu_2)(y), \langle\varphi(x-y), \phi(x)\rangle\rangle \\
&= \langle T_y, \langle\varphi(x-y), \phi(x)\rangle\rangle
\end{aligned}$$

for all $\phi \in \mathfrak{S}_{w_1,w_2}$. This completes the proof of Theorem 3.1.

$\square$

Now we employ the above theorem to describe the action of the semi-group defined by the convolution kernel $t^{-n}T(\frac{x-y}{t})$, where $t>0$ on $\mathfrak{S}'_{w_1,w_2}$.

Theorem 3.2. *Let $T\in\mathfrak{S}_{w_1,w_2}$ and $\{P_t\}_{t\geq 0}$ be a semi-group defined by the convolution kernel $t^{-n}T(\frac{x-y}{t})$, where $t>0$. Then, the action of P_t on $\mathfrak{S}'_{w_1,w_2}$ is given by the integration against the function*

$$\rho(x)=\langle S_y, t^{-n}T(\frac{x-y}{t})\rangle, \tag{3.1}$$

where $S_y\in\mathfrak{S}'_{w_1,w_2}$ and y indicates on which variable the functional S acts.

Proof. Using Lemma 2.3 and Theorem 3.1, it is enough to show that $T(\frac{\cdot}{t})\in\mathfrak{S}_{w_1,w_2}$ for each $t>0$. Note that

$$\begin{aligned}\left|e^{kw_1(x)}T(\frac{x}{t})\right| &\leq \left|e^{ktw_1(\frac{x}{t})}T(\frac{x}{t})\right|\\ &\leq \left|e^{([kt]+1)w_1(\frac{x}{t})}T(\frac{x}{t})\right|\\ &= \left|e^{mw_1(\frac{x}{t})}T(\frac{x}{t})\right|\\ &\leq ||e^{mw_1}T||_\infty\end{aligned}$$

and

$$\left|e^{kw_2(\xi)}\widehat{T}(\frac{x}{t})(\xi)\right|=\left|e^{kw_2(\xi)}t\widehat{T}(t\xi)\right|=C_t\left|e^{kw_2(\xi)}\widehat{T}(t\xi)\right|.$$

Now if $t\geq 1$, then $w_2(\xi)\leq w_2(t\xi)$ and therefore

$$\begin{aligned}\left|e^{kw_2(\xi)}\widehat{T}(t\xi)\right| &\leq \left|e^{kw_2(t\xi)}\widehat{T}(t\xi)\right|\\ &\leq \left|\left|e^{kw_2}\widehat{T}\right|\right|_\infty.\end{aligned}$$

For $0<t<1$, we have

$$\begin{aligned}\left|e^{kw_2(\xi)}\widehat{T}(t\xi)\right| &\leq \left|e^{kNw_2(t\xi)}\widehat{T}(t\xi)\right|\\ &\leq \left|\left|e^{kNw_2}\widehat{T}\right|\right|_\infty,\end{aligned}$$

where N is an integer such that $N\geq\frac{1}{t}$. This completes the proof of Theorem 3.2. □

Theorem 3.3. *Let B be a bounded subset of $\mathfrak{S}_{w_1,w_2}$. Then $\varphi_t(x)=\langle t^{-n}T(\frac{x-y}{t}),\varphi(x)\rangle=\int_{\mathbb{R}^n}t^{-n}T(\frac{x-y}{t})\varphi(y)dy\to\varphi$ in $\mathfrak{S}_{w_1,w_2}$as $t\to 0^+$ uniformly on B.*

Proof. We note that $\varphi_t\in\mathfrak{S}_{w_1,w_2}\subset\mathfrak{S}'_{w_1,w_2}$ for each $t>0$. If $0<t<1$ and $z=\frac{x-y}{t}$, then for any $\delta>0$, we can write

$$\begin{aligned}e^{kw_1(y)}\left|\varphi_t(x)-\varphi(y)\right| &= \int_{\mathbb{R}^n}e^{kw_1(y)}T(z)\left|\varphi(y+tz)-\varphi(y)\right|dz\\ &\leq I_1+I_2+I_3,\end{aligned}$$

where

$$\begin{aligned}I_1 &= \int_{|y|\leq\delta}e^{kw_1(y)}T(z)\left|\varphi(y+tz)-\varphi(y)\right|dz,\\ I_2 &= \int_{|y|\geq\delta}e^{kw_1(y)}T(z)\left|\varphi(y+tz)\right|dz,\\ I_3 &= \int_{|y|\geq\delta}e^{kw_1(y)}T(z)\left|\varphi(y)\right|dz.\end{aligned}$$

We begin estimating I_1. For each $0 < t < 1$ and $z \in \mathbb{R}^n$, there exists $C > 0$ such that

$$e^{kw_1(y)} \left|\varphi(y+tz) - \varphi(y)\right| \leq Ct\left|z\right|.$$

We note that Condition 4 in Definition 2.1 implies that there exist $N \in \mathbb{N}$ and $C > 0$ such that $|z| \leq Ce^{Nw_1(z)}$. Substituting this into I_1, we obtain the estimate

$$\begin{aligned} I_1 &\leq \int_{|y|\leq\delta} Ct\left|z\right| T(z)dz \\ &\leq C\int_{|y|\leq\delta} te^{Nw_1(z)}T(z)dz \\ &\leq C\delta t\left\|e^{Nw_1}T\right\|_\infty. \end{aligned} \tag{3.2}$$

Next, we estimate I_2. Using the subaddivity of w_1 and $0 < t < 1$, we obtain

$$\begin{aligned} I_2 &\leq \int_{|y|\geq\delta} e^{kw_1(y)}T(z)\left|\varphi(y+tz)\right| dz \\ &\leq \int_{|y|\geq\delta} e^{kw_1(y+tz-tz)}T(z)\left|\varphi(y+tz)\right| dz \\ &\leq \int_{|y|\geq\delta} e^{kw_1(tz)}T(z)\left|e^{kw_1(y+tz)}\varphi(y+tz)\right| dz \\ &\leq \left\|e^{Nw_1}\varphi\right\|_\infty \int_{|z|\geq\delta} e^{kw_1(z)}T(z)dz \\ &\leq C\int_{|z|\geq\delta} e^{kw_1(z)}T(z)dz. \end{aligned} \tag{3.3}$$

Finally, let us estimate I_3. We have

$$\begin{aligned} I_3 &= \int_{|z|\geq\delta} e^{kw_1(y)}T(z)\left|\varphi(y)\right| dz \\ &\leq \left\|e^{kw_1}\varphi\right\|_\infty \int_{|z|\geq\delta} e^{kw_1(z)}T(z)dz. \end{aligned} \tag{3.4}$$

Therefore if we choose δ to be sufficiently large and t sufficiently small then the estimates in (3.2), (3.3) and (3.4) imply that $\left\|e^{kw_1(y)}\left(\varphi_t(x) - \varphi(y)\right)\right\|_\infty$ converges to 0 as $t \to 0^+$. Now to prove that $\left\|e^{kw_2(\xi)}\mathcal{F}\left(\varphi_t(x) - \varphi(y)\right)(\xi)\right\|_\infty$ converges to 0 as $t \to 0^+$, we consider

$$\begin{aligned} e^{kw_2(\xi)}\left|\mathcal{F}\left(\varphi_t(x) - \varphi(y)\right)(\xi)\right| &= e^{kw_2(\xi)}\left|\mathcal{F}\left(\int_{\mathbb{R}^n} t^{-n}T\left(\frac{x-y}{t}\right)\varphi(y)dy\right)(\xi) - \mathcal{F}(\varphi(y))(\xi)\right| \\ &= e^{kw_2(\xi)}\left|\mathcal{F}\left(\int_{\mathbb{R}^n} t^{-n}T\left(\frac{x-y}{t}\right)\varphi(y)dy\right)(\xi) - \mathcal{F}(\varphi(y))(\xi)\right| \\ &= e^{kw_2(\xi)}\widehat{\varphi}(\xi)\left|\mathcal{F}(T)(t\xi) - 1\right| \\ &\leq \left\|e^{kw_2}\widehat{\varphi}\right\|_\infty \left|\mathcal{F}(T)(t\xi) - 1\right| \leq C\left|\mathcal{F}(T)(t\xi) - 1\right|. \end{aligned}$$

Now by uniform continuity of $\mathcal{F}(T)(t\xi)$, we observe that $\mathcal{F}(T)(t\xi) \to \mathcal{F}(T)(0) = \left\|t^{-n}T(\frac{\cdot}{t})\right\|_1 = 1$, which implies that $C\left|\mathcal{F}(T)(t\xi) - 1\right| \to 0$ as $t \to 0^+$ uniformly on compact subsets of $\mathbb{R}^n$. Thus $\left\|e^{kw_2}\mathcal{F}(\varphi_t(x) - \varphi(y))\right\|_\infty$ converges to 0 uniformly on B. This completes the proof of Theorem 3.3. $\square$

Example 3.1. *Consider the heat kernel*

$$E(x,t)=\begin{cases}(4\pi t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}},\ \textit{for } t>0,\\ 0,\ \textit{for } t<0.\end{cases}$$

It is known that $\|E(\cdot,t)\|_1=1$ *for* $t>0$. *Also, consider the Gauss-Weierstrass semigroups* $\{GW_t\}_{t\geq 0}$ *defined by the integration with respect to the heat kernel*

$$GW_{\sqrt{t}}(\varphi)(x)=\langle E(x-y,t),\varphi(y)\rangle=\langle t^{-n/2}T(\frac{x-y}{\sqrt{t}}),\varphi(y)\rangle.$$

This semigroup generated by the Laplacian on $\mathbb{R}^n$ *and the function* $u(x,t)=GW_{\sqrt{t}}(\varphi)(x)$ *is a solution of the equation* $u_t-\triangle u=0$ *with* $u(x,0)=\varphi(x)$ *for an appropriate* φ. *That is, the convolution*

$$u(x,t)=E*\varphi$$

is the solution to the heat equation and

$$u(x,0)=\varphi(x)=\lim_{t\to 0^+}GW_{\sqrt{t}}(\varphi)(x)$$

and the convergence is uniform on bounded subsets of $\mathbb{R}^n$. *Now it is clear that* $E(x,t)\in\mathfrak{S}_{w_1,w_2}$ *for all* $w_1,w_2\in\mathcal{M}_c$ *since* $E(x,t)$ *is exponentially decreasing and using Theorem 3.2. Moreover, Theorem 3.2 implies that the action of* $GW_{\sqrt{t}}$ *on* $L\in\mathfrak{S}'_{w_1,w_2}$ *for all* $w_1,w_2\in\mathcal{M}_c$ *can be defined by the integral against the function* $\rho(x)$ *given in (3.1) and by using Theorem 3.1, we conclude that this is equivalent to*

$$GW_{\sqrt{t}}(T)=\langle L_y,\langle t^{-n/2}T(\frac{x-y}{\sqrt{t}}),\phi(x)\rangle\rangle$$

which implies that $\lim_{t\to 0^+}GW_{\sqrt{t}}(T)=T$ *in the sense of* $\mathfrak{S}'_{w_1,w_2}$ *and this is equivalent to*

$$\langle t^{-n/2}T(\frac{x-y}{\sqrt{t}}),\phi(x)\rangle\rangle\to\varphi\ \textit{in}\ \mathfrak{S}'_{w_1,w_2}\textit{as}\ t\to 0^+.$$

As a result, the (w_1,w_2)*-tempered ultradistributions can be considered as boundary values of the equation* $u_t-Au=0$.

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