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ON INTERPOLATION OF L^2 FUNCTIONS

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Abstract. In this paper we are interested in polynomial interpolation of irregular functions namely those elements of $L^2(\mathbb{R}, \mu)$ for μ a given probability measure. This is of course doesn't make any sense unless for L^2 functions that, at least, admit a continuous version. To characterize those functions we have, first, constructed, in an abstract fashion, a chain of Sobolev like subspaces of a given Hilbert space $\mathbb{H}_0$. Then we have proved that the chain of Sobolev like subspaces controls the existence of a continuous version for L^2 functions and gives a pointwise polynomial approximation with a quite accurate error estimation.

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1. Introduction

Polynomial interpolation of functions is one of most important problem in numerical analysis, in fact, it has numerous applications either in Mathematical problems or even in computer sciences and in engineering sciences in general. In this paper we have considered polynomial interpolation of functions elements of $L^2(\mathbb{R}, \mu)$ for μ a probability measure. Since it is meaningless to speak about values of an L^2 function in general, we first need to characterize L^2 functions that admit a continuous version. To do this we have constructed a chain of Hilbert subspaces of a given Hilbert space $\mathbb{H}_0$ in such a way that it enables us to control the approximation error of elements of $\mathbb{H}_0$ by their expansion with respect to an orthonormal basis. More precisely, suppose we have an orthonormal basis $\{e_n : n \geq 0\}$ of $\mathbb{H}_0$. Any element φ of $\mathbb{H}_0$ can be written

$$\varphi = \sum_{n=0}^{+\infty} \varphi^{(n)} e_n$$

where $\varphi^{(n)} = \langle \varphi, e_n \rangle_0$ is the scalar product of φ against e_n and the convergence is in $\mathbb{H}_0$. This means, in particular, that for each positive integer N the partial sum

$$\varphi_N = \sum_{n=0}^{N} \varphi^{(n)} e_n \tag{1.1}$$

approximates φ as an element of $\mathbb{H}_0$. Our first result, in this context, gives an estimation of the approximation error

$$\|\varphi - \varphi_N\|_0^2 = \sum_{n=N+1}^{+\infty} |\varphi^{(n)}|^2 \tag{1.2}$$

using the chain of Sobolev like subspaces, see Lemma 2.2. Further we have characterized the belonging of an element of $\mathbb{H}_0$ to a given Sobolev subspace using a Laplace like transformation, see Theorem 2.4.

Now let us consider the case where $\mathbb{H}_0 = L^2(\mathbb{R}, \mu)$ for a given probability measure μ. We have proved, under standard conditions on μ, that there exists a rank q after which the Sobolev subspaces, $\mathbb{H}_p$ for $p \geq q$, contains functions that admit a continuous version. For those functions we have proved that the approximation (1.1) is still valid pointwisely and uniformly on compact subsets of $\mathbb{R}$. This leads to a polynomial interpolation algorithm of L^2 real functions. We claim that this new algorithm has at least two major advantages. First, the error estimation doesn't depend on the inter-distances between initial points or knots while in most all other algorithms the estimation error depends on strongly. Second, the error estimation in our case doesn't require regularity more that continuity, where in most all other algorithms it does. Finally we have implemented the new algorithm and compare it with the cubic spline one using Matlab.

Our paper is organized as follows: In section 2 we have constructed a chain of decreasing Sobolev subspaces, $\{\mathbb{H}_q\}_{q\geq 1}$, of $\mathbb{H}_0$ and we have controlled the approximation error (1.2). Then we have defined a Laplace like transformation of elements in $\mathbb{H}_0$ and use it to characterize the belonging of a given element into a given Sobolev subspace $\mathbb{H}_q$ for some $q \geq 1$.

The third section is devoted to the study of the L^2 type Hilbert spaces, $\mathbb{H}_0 = L^2(\mathbb{R}, d\mu)$, where μ is a probability measure. In particular we have characterized probability measures for which there exists a rank $q \geq 1$, such that, elements of $\mathbb{H}_p$ for $p \geq q$ admit a continuous version and such that, for each function, the partial sum (1.1) converges pointwisely on $\mathbb{R}$ and uniformly on compact subsets.

In the fourth section we have implemented a new polynomial interpolation algorithm and compared with the cubic spline one.

2. Abstract Hilbert space case

$\mathbb{H}_0$ still denotes a Hilbert space endowed with an Hilbertian norm $\| \ \|_0$ and with an orthonormal basis $\{e_n : n \geq 0\}$. For any integer $q \geq 1$ we define the Hilbert subspace of $\mathbb{H}_0$, $\mathbb{H}_q$ by the completion of the following subspace:

$$\mathbb{H}_q = \{\varphi \in \mathbb{H}_0 : \ \|\varphi\|_q^2 = \sum_{n=0}^{+\infty} 2^{nq} |\varphi^{(n)}|^2 < \infty\}.$$

Remark 2.1. Actually the choice of the weight 2^q could be replaced by any sequence $\alpha_q > 1$ that increases to infinity when q goes to infinity.

Proposition 2.1. *The sequence of Hilbert subspaces $\{\mathbb{H}_q\}_{q\geq 1}$ satisfies the following properties:*

(i) It is decreasing:

$$\cdots \subsetneq \mathbb{H}_{q+1} \subsetneq \mathbb{H}_q \subsetneq \cdots \subsetneq \mathbb{H}_0.$$

(ii) For any $q \geq 1$, the subspace $\mathbb{H}_q$ is dense in the space $\mathbb{H}_0$:

$$\overline{(\mathbb{H}_q, \| \ \|_0)} = \mathbb{H}_0.$$

(iii) For any $p > q$, the natural injection $i_{pq} : \mathbb{H}_p \hookrightarrow \mathbb{H}_q$ is continuous, actually, it is of Hilbert-Shmidt type.

(iv) The space

$$(\mathbb{H}) = \cap_{q\geq 0} \mathbb{H}_q$$

is a nontrivial topological nuclear countably Hilbert space.

Proof:

(i) Since the sequence of norms $(\| \ \|_q)_{q\in\mathbb{N}}$ is increasing the sequence of subspaces $(\mathbb{H}_q)_{q\in\mathbb{N}}$ is decreasing.

(ii) The orthonormal basis $\{e_n : n \geq 0\}$ forms by definitions a dense subset of $\mathbb{H}_0$ and since it is included in all $\mathbb{H}_q$ for $q \in \mathbb{N}$, then each $\mathbb{H}_q$ forms in its turn a dense subspace of $\mathbb{H}_0$.

(iii) Let $p > q$ two integers and define $e_n^p = 2^{-np/2} e_n$ then it is easy to see that the family $\{e_n^p : n \geq 0\}$ forms an orthonormal basis of $\mathbb{H}_p$.
To check that the injection i_{pq} is of Hilbert-Shmidt type we just need to check that the series below is convergent

$$\sum_{n\in\mathbb{N}} \|e_n^p\|_q^2 = \sum_{n\in\mathbb{N}} 2^{(q-p)} \ < \ \infty$$

which is the case.

(iv) It is a direct consequence of the Hilbert-Shmidt property of the injections i_{pq} for $p > q$. $\square$

Lemma 2.2. *Let $\varphi \in \mathbb{H}_q$ for some $q \geq 1$. Then for any $N \geq 0$, the partial sum in (1.1) satisfies:*

$$\|\varphi - \varphi_N\|_0 \leq 2^{-\frac{q(N+1)}{2}} \|\varphi\|_q \tag{2.1}$$

Proof: Let $\varphi \in \mathbb{H}_q$ for some $q \geq 1$ and φ_N the partial sum in (1.1) for some integer $N \geq 1$, then:

$$\begin{aligned} \|\varphi - \varphi_N\|_0^2 &= \sum_{n \geq N+1} |\varphi^{(n)}|^2 \\ &= \sum_{n \geq N+1} 2^{-qn} 2^{qn} |\varphi^{(n)}|^2 \\ &\leq 2^{-q(N+1)} \|\varphi\|_q^2. \end{aligned}$$ $\square$

Remark 2.2. (i) It is worth mentioning that the error estimation in (2.2) is very important and actually cannot be obtained without a similar construction of a Sobolev subspaces chain.

(ii) To check the belonging of a given φ in $\mathbb{H}_q$ for some q we need a good estimation of all coefficients $\{\varphi^{(n)} : n \geq 0\}$, which could be very hard, in particular in numerical problems. In the next paragraph we introduce a Laplace like transformation to give a direct characterization of the belonging of a given function φ to a certain $\mathbb{H}_q$.

For any complex number $\xi \in \mathbb{C}$, denote by $G(\cdot, \xi)$ the element of $\mathbb{H}_0$ defined by

$$G(\cdot, \xi) = \sum_{n=0}^{+\infty} \frac{\xi^n}{n!} e_n. \tag{2.2}$$

It is easy to check that for any $q \geq 0$, $G(\cdot, \xi)$ belongs to $\mathbb{H}_q$ and

$$\begin{aligned} \|G(\cdot, \xi)\|_q^2 = \sum_{n=0}^{+\infty} 2^{nq} \frac{|\xi|^{2n}}{(n!)^2} &\leq C \cosh\left(2^{\frac{q}{2}+1} |\xi|\right) \\ &\leq C \exp\left(2^{\frac{q}{2}+1} |\xi|\right), \end{aligned}$$

where $C > 0$ is a constant.

We define the Laplace like transform of a given $\varphi \in \mathbb{H}_0$ as the complex function:

$$\begin{aligned} \mathcal{L}[\varphi] : \quad \mathbb{C} &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \mathcal{L}[\varphi](\xi) = \langle G(\cdot, \xi), \varphi \rangle_0 = \sum_{n=0}^{+\infty} \frac{\varphi^{(n)}}{n!} \xi^n \end{aligned} \tag{2.3}$$

Proposition 2.3.

(i) If φ belongs to $\mathbb{H}_q$ for some $q \geq 0$, its Laplace transform $\mathcal{L}[\varphi]$ satisfies the following exponential growth:

$$|\mathcal{L}[\varphi](\xi)| \leq \|\varphi\|_q \exp\left\{2^{-q/2}|\xi|\right\}. \tag{2.4}$$

(ii) For any $\varphi \in \mathbb{H}_0$, $\mathcal{L}[\varphi]$ is an entire function.

(iii) The Laplace transform $\mathcal{L}$ is linear and one to one.

Proof:

(i) Let $\varphi \in \mathbb{H}_q$ for a given $q \geq 0$, we have

$$\begin{aligned} |\mathcal{L}[\varphi](\xi)| &= \left|\sum_{n=0}^{+\infty} \frac{\varphi^{(n)}}{n!}\xi^n\right| \\ &\leq \sum_{n=0}^{+\infty} \frac{|\varphi^{(n)}|}{n!}|\xi|^n \\ &\leq \sum_{n=0}^{+\infty} 2^{nq/2}|\varphi^{(n)}|\frac{2^{-nq/2}|\xi|^n}{n!} \\ &\leq \|\varphi\|_q \sum_{n=0}^{+\infty}\left[\frac{2^{-nq}|\xi|^{2n}}{(n!)^2}\right]^{1/2}. \end{aligned}$$

By the Stirling formula $(n!)^2 \sim (2n)!2^{-2n}\sqrt{\pi n}$,
we then obtain

$$\begin{aligned} |\mathcal{L}[\varphi](\xi)| &\leq \|\varphi\|_q \sum_{n=0}^{+\infty}\left[\frac{2^{-nq}|\xi|^{2n}}{(2n)!2^{-2n}\sqrt{\pi n}}\right]^{1/2} \\ &\leq \|\varphi\|_q \sum_{n=0}^{+\infty}\left[\frac{2^{2n(1-q/2)}|\xi|^{2n}}{(2n)!}\right]^{1/2} \\ &= \|\varphi\|_q\left[\cosh 2\left(2^{-q/2}|\xi|\right)\right]^{1/2}, \end{aligned}$$

which, with the elementary inequality: $\cosh(x) \leq e^x$ for $x \geq 0$ yields to the desired estimation

$$|\mathcal{L}[\varphi](\xi)| \leq \|\varphi\|_q \exp\left(2^{-q/2}|\xi|\right).$$

(ii) For any $\varphi \in \mathbb{H}_0$, $\mathcal{L}[\varphi]$ is a complex power series, due to 1. it converges in the disk of radius R for any $R > 0$ so it is entire.

(iii) The Laplace like transform is linear so we need to check that if $\mathcal{L}[\varphi] = 0$ then $\varphi = 0$. This is true since $\mathcal{L}[\varphi] = 0$ means that all its derivatives at $\xi = 0$ vanish i.e $\varphi^{(n)} = 0$ for all $n \geq 0$ so φ. □

Now we are in a position to state the folowing Theorem which characterizes, via the Laplace like transform, the belonging of any φ to the Sobolev subspace $\mathbb{H}_q$ for some $q \geq 1$.

Theorem 2.4. *Let φ be an element of $\mathbb{H}_0$ and suppose that there exit $p \geq 1$ and $C > 0$ such that its Laplace transform $\mathcal{L}[\varphi]$ satisfies:*

$$|\mathcal{L}[\varphi](\xi)| \leq C \exp\left(2^{-p/2}|\xi|\right) \tag{2.5}$$

for any $\xi \in \mathbb{C}$. Then φ belongs to $\mathbb{H}_q$ for any $q < p$.

Proof: Let φ be an element of $\mathbb{H}_0$ and suppose that its Laplace like transform satisfies (2.5). Since it is an entire function we have

$$\frac{d^n\mathcal{L}[\varphi]}{d\xi^n}(0) = \varphi^{(n)}$$

and so the Cauchy estimation gives

$$\left|\frac{\varphi^{(n)}}{n!}\right| \leq \frac{\sup_{\xi=r}|\mathcal{L}[\varphi](\xi)|}{r^n},$$

$\forall r > 0$, which implies

$$\left|\frac{\varphi^{(n)}}{n!}\right| \leq C\frac{e^{2^{\frac{-p}{2}}r}}{r^n},$$

$\forall r > 0$, where $C > 0$ is the same constant as in (2.5).
The right hand side of the above inequality reaches its minimum $\dfrac{e^n}{n^n}2^{\frac{-np}{2}}$ at $r = \frac{n}{2^{\frac{-p}{2}}}$. Then by the use of the Stirling formula we get

$$|\varphi^{(n)}| \leq C2^{\frac{-np}{2}}\sqrt{2n\pi}.$$

Now for $q \geq 0$ we have

$$\begin{aligned}\|\varphi\|_q^2 &= \sum_{n=0}^{+\infty} 2^{nq}|\varphi^{(n)}|^2 \\ &\leq C^2\sum_{n=0}^{+\infty} 2^{n(q-p)}(2\pi n).\end{aligned}$$

The latter series converges for any $q < p$ which implies that φ belongs to $\mathbb{H}_q$ for any $q < p$. The proof of the Theorem is complete. $\square$

Definition 2.1. Let $\mathcal{F}_1(\mathbb{C})$ be the set of entire functions with exponential growth of order one and of type as small as possible i.e $f \in \mathcal{F}_1(\mathbb{C})$ if and only if for any $q \geq 0$

$$\sup_{\xi\in\mathbb{C}}|f(\xi)|\exp\left(-2^{-q/2}|\xi|\right) < \infty$$

These spaces of entire functions with exponential growth has been introduced in a different context in [4].

Corollary 2.5. *The Laplace like transform realizes the following topological isomorphism:*

$$\bigcap_{q\geq 0}\mathbb{H}_q = (\mathbb{H}) \xrightarrow{\mathcal{L}} \mathcal{F}_1(\mathbb{C})$$

Proof: The proof derives immediately from the proof of Theorem 2.4. $\square$

3. The L^2 case

Suppose from now on that $\mathbb{H}_0 = L^2(\mathbb{R}, \mu)$ where μ is a probability measure.
We distinguish two cases:

- **Case 1:** The probability measure μ has a compact support.
- **Case 2:** The probability measure μ has an unbounded support and satisfies the hypothesis:
  (H1): The Laplace transform of μ, $\mathcal{L}[\mu](\xi) = \displaystyle\int_{-\infty}^{+\infty} e^{\xi x} d\mu(x)$ is holomorphic in a neighborhood of zero.

Remarks 1.

(a) The hypothesis (H1) on the probability measure μ is equivalent to the following one:
(H1'): There exists $\epsilon > 0$ such that:

$$\int_{-\infty}^{+\infty} e^{\epsilon|x|} d\mu(x) < \infty.$$

(b) Note that in both cases above, one can prove that the sequence of polynomials $\{x^n : n \geq 0\}$ is total in $\mathbb{H}_0 = L^2(\mathbb{R}, \mu)$, see for instance [1].

Let $\{e_n : n \geq 0\}$ be the orthonormal basis of $\mathbb{H}_0$ obtained by the Gramm-Shmidt procedure applied on the total set $\{x^n : n \geq 0\}$. This means in particular that any vector e_n is a polynomial function of degree n and that the partial sum

$$\varphi_N(x) = \sum_{n=0}^{N} \varphi^{(n)} e_n(x) \tag{3.1}$$

is a polynomial function of degree at most $N \in \mathbb{N}$.
The next Theorem treats the case where μ has a compact support. Denote by

$$\gamma_n = \sup_{x \in supp(\mu)} |e_n(x)| \tag{3.2}$$

Theorem 3.1. *Suppose that μ has a compact support.*

(i) If $\lim\limits_{n\to+\infty} \gamma_n = 0$ then for any $\varphi \in \mathbb{H}_1$ the partial sum φ_N converges uniformly on $supp(\mu)$ and its limit constitutes a continuous version of φ. Moreover there exists a constant $C > 0$ such that

$$|\varphi(x) - \varphi_N(x)| \leq C\|\varphi\|_1 2^{-N/2} \tag{3.3}$$

for any $x \in supp(\mu)$.

(ii) If $\lim\limits_{n\to+\infty} \gamma_n = +\infty$ and there exists $q \geq 1$ such that $\lim\limits_{n\to+\infty} \left[\gamma_n 2^{-\frac{nq}{2}}\right] = 0$.
Then for any $\varphi \in \mathbb{H}_{q+1}$ the partial sum φ_N converges uniformly on $supp(\mu)$ and its limit constitutes a continuous version of φ. Moreover there exists a constant $C > 0$ such that

$$|\varphi(x) - \varphi_N(x)| \leq C\|\varphi\|_{q+1} 2^{-N/2} \tag{3.4}$$

for any $x \in supp(\mu)$.

Proof:

(i) Let φ be in $\mathbb{H}_1$, then

$$\begin{aligned}\sum_{n\geq 0}|\varphi^{(n)}||e_n(x)| &\leq \sum_{n\geq 0}|\varphi^{(n)}|\gamma_n \\ &\leq \sum_{n\geq 0}2^{n/2}|\varphi^{(n)}|2^{-n/2}\gamma_n \\ &\leq \|\varphi\|_1\left[\sum_{n\geq 0}2^{-n}\gamma_n^2\right]^{1/2} \\ &\leq C\|\varphi\|_1.\end{aligned}$$

The last inequality shows that for any $\varphi\in\mathbb{H}_1$ the series $\sum_{n\geq 0}\varphi^{(n)}e_n(x)$ converges uniformly on $supp(\mu)$. A similar calculation implies the pointwise convergence estimation (3.3).

(ii) Let $\varphi\in\mathbb{H}_{q+1}$, then

$$\begin{aligned}\sum_{n\geq 0}|\varphi^{(n)}||e_n(x)| &\leq \sum_{n\geq 0}|\varphi^{(n)}|\gamma_n \\ &\leq \sum_{n\geq 0}2^{n(q+1)/2}|\varphi^{(n)}|2^{-n(q+1)/2}\gamma_n \\ &\leq \|\varphi\|_{q+1}\left[\sum_{n\geq 0}2^{-n(q+1)}\gamma_n^2\right]^{1/2} \\ &\leq C\|\varphi\|_{q+1}\left[\sum_{n\geq 0}2^{-n}\right]^{1/2} \\ &\leq C\|\varphi\|_{q+1}.\end{aligned}$$

We then deduce that, for any $\varphi\in\mathbb{H}_{q+1}$, the series $\sum_{n\geq 0}\varphi^{(n)}e_n(x)$ converges uniformly on $supp(\mu)$. The pointwise convergence estimation (3.4) follows by a similar arguments. □

Example: The uniform case

Suppose that μ is the uniform probability measure on $[-1,1]$. The polynomials family $\{\sqrt{\frac{1}{2}(2n+1)}P_n(x)\}$ constitutes an orthonormal basis of $L^2([-1,1],\frac{dx}{2})$, where P_n are the Legendre polynomials defined via their generating function

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{+\infty}P_n(x)t^n.$$

It is known, see [2], that Legendre polynomials satisfy the following upper bound

$$|P_n(x)|\leq 1,\ \text{for any } n\geq 0 \text{ and } x\in[-1,1]$$

and so the polynomials $e_n(x)$ satisfy

$$|e_n(x)| = \sqrt{\frac{1}{2}(2n+1)}|P_n(x)| \leq \sqrt{\frac{1}{2}(2n+1)}$$

which implies that

$$2^{-n}\gamma_n = 2^{-n}\sup_{|x|\leq 1}|e_n(x)| \xrightarrow[n\to+\infty]{} 0.$$

The second assertion of Theorem 3.1 implies that any $\varphi \in \mathbb{H}_2$ admits a continuous version and satisfies the error estimation (3.4). $\square$

Suppose now that the probability measure μ has an unbounded support and satisfies either (H1) or (H1') and denote by

$$\gamma_n(K) = sup_{x\in K}|e_n(x)|$$

for any given compact subset K of $\mathbb{R}$.

Corollary 3.2.

(i) Suppose that there exists a sequence of compact subsets of $\mathbb{R}$, $\{K_j\}_{j\in\mathbb{N}}$ that covers $\mathbb{R}$ and such that for any $j \in \mathbb{N}$, $\lim\limits_{n\to+\infty}\gamma_n(K_j) = 0$.
Then for any $\varphi \in \mathbb{H}_1$ the partial sum φ_N converges absolutely on $\mathbb{R}$ and uniformly on each compact subset and its limit constitutes a continuous version of φ. Moreover for any compact subset K there exists a constant $C > 0$ such that

$$|\varphi(x) - \varphi_N(x)| \leq C\|\varphi\|_1 2^{-N/2} \tag{3.5}$$

for any $x \in K$.

(ii) Suppose that there exists a sequence of compact subsets of $\mathbb{R}$, $\{K_j\}_{j\in\mathbb{N}}$ that covers $\mathbb{R}$ and such that for any $j \in \mathbb{N}$, $\lim\limits_{n\to+\infty}\gamma_n(K_j) = +\infty$ and there exists $q_j \geq 1$ such that $\lim\limits_{n\to+\infty}\left[\gamma_n(K_j)2^{-\frac{nq_j}{2}}\right] = 0$. Then for any $\varphi \in (\mathbb{H})$ the partial sum φ_N converges absolutely on $\mathbb{R}$ and uniformly on each compact subset of $\mathbb{R}$ and its limit constitutes a continuous version of φ. Moreover for any compact subset K of $\mathbb{R}$, there exists a constant $C > 0$ and an integer $q \geq 1$ such that

$$|\varphi(x) - \varphi_N(x)| \leq C\|\varphi\|_{q+1} 2^{-N/2} \tag{3.6}$$

for any $x \in K$.

Proof: The proof is quite similar to that of Theorem 3.1. $\square$

Example: The Gaussian case

Suppose that μ is the standard Gaussian probability measure $\mu = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$. The polynomials family $\{\frac{H_n}{\sqrt{n!}} : n \geq 0\}$ constitutes an orthonormal basis for $L^2(\mathbb{R},\mu)$, where $\{H_n\}_{n\geq 0}$ are the Hermite polynomials defined via their generating function

$$e^{\xi x - \frac{\xi^2}{2}} = \sum_{n=0}^{+\infty}\frac{\xi^n}{n!}H_n(x).$$

It is known that Hermite polynomials satisfy the following integral representation, see [5]:

$$H_n(x) = \begin{cases} (-1)^{[n/2]} e^{x^2/2} \dfrac{2^{\frac{n}{2}+1}}{\sqrt{\pi}} \displaystyle\int_0^{+\infty} t^n e^{-t^2} \cos(\sqrt{2}xt)dt & n \text{ even} \\ \\ (-1)^{[(n+1)/2]} e^{x^2/2} \dfrac{2^{\frac{n}{2}+1}}{\sqrt{\pi}} \displaystyle\int_0^{+\infty} t^n e^{-t^2} \sin(\sqrt{2}xt)dt & n \text{ odd} \end{cases} \tag{3.7}$$

This implies in particular that for any $n \in \mathbb{N}$

$$|e_n(x)| = \left| \frac{H_n(x)}{\sqrt{n!}} \right| \leq e^{x^2/2} \frac{2^{\frac{n}{2}}}{\sqrt{\pi}} \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n!}}.$$

Using the Stirling formula we have

$$\frac{\Gamma(\frac{n+1}{2})}{\sqrt{n!}} \approx_{+\infty} 2^{-n/2} (2\pi)^{1/4} \frac{1}{n^{1/4}}$$

which implies that for any compact subset of $\mathbb{R}$, K

$$\lim_{n \to +\infty} \gamma_n(K) = 0$$

and so, by assertion (i) of Corollary 3.2, any function $\varphi \in \mathbb{H}_1$ has a continuous version and could be approximated by the partial sum φ_N in (1.1). □

Remark 3.1. Actually it was not known which elements of $L^2(\mathbb{R}, \mu)$ admits a continuous versions, while Corollary 3.2 shows that all elements of the Hilbert subspace $\mathbb{H}_1 \subset L^2(\mathbb{R}, \mu)$ admit a continuous version.

Example: The Gamma case

Suppose that μ is the Gamma probability measure on $\mathbb{R}_+ = [0, +\infty[$, $\mu = \dfrac{1}{\Gamma(\alpha)} x^\alpha e^{-x} dx$ for $\alpha > 0$. The polynomials family $\{\sqrt{\dfrac{n!}{\Gamma(\alpha+1)\Gamma(n+\alpha)}} L_n^\alpha(x) \,:\, n \geq 0\}$ constitutes an orthonormal basis for the space $L^2(\mathbb{R}_+, \mu)$ where $\{L_n^\alpha(x) \,:\, n \geq 0\}$ are the Laguerre polynomials defined, for $\alpha > 0$, by

$$L_n^\alpha(x) = e^x \frac{x^{-\alpha}}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}).$$

The Laguerre polynomials satisfy the following integral representation, see [5]

$$e^{-x} x^{\alpha/2} L_n^\alpha(x) = \frac{1}{n!} \int_0^{+\infty} e^{-t} t^{n+\alpha/2} J_\alpha\{2\sqrt{tx}\} dt \tag{3.8}$$

where J_α are Bessel functions of first kind of order α, which satisfy, in their turn:

$$J_\alpha(x) = \frac{(x/2)^\alpha}{\Gamma(\alpha + \frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\alpha - \frac{1}{2}} e^{ixt} dt, \tag{3.9}$$

see also [5]. Substituting (3.9) in (3.8) gives the following

$$|L_\alpha(x)| \leq \frac{e^x}{\Gamma(\alpha + \frac{1}{2})\Gamma(\frac{1}{2})} \frac{\Gamma(n+1+\alpha)}{n!}$$

for $x \in \mathbb{R}_+$ and $\alpha > 0$. The above inequality implies that for any compact subset, K, of $\mathbb{R}$

$$\gamma_n(K) = \sup_{x\geq 0} |e_n(x)| \leq \sup_{x\geq 0}(e^x)\frac{\Gamma(n+1+\alpha)}{n!\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}\left(\frac{n!}{\Gamma(n+\alpha)\Gamma(\alpha+1)}\right)^{1/2}$$

by using the Stirling formula we obtain

$$\gamma_n(K) \leq C(n+1)^{\alpha+1}$$

where $C > 0$ is a constant that depends of the compact subset K. This implies that

$$\lim_{n\to+\infty} 2^{-n}\gamma_n(K) = 0.$$

Assertion (ii) of Corollary 3.2 tells us that any function $\varphi \in (\mathbb{H})$ has a continuous version and could be approximated by the partial sum φ_N in (1.1). □

Remark 3.2. The next Lemma shows that in the L^2 case, the sequence of Hilbert subspaces $\{\mathbb{H}_q\}_{q\in\mathbb{N}}$ constitutes a new regularity chain which has nothing to do with the standard differentiability chain. We note that this point is very important since it shows that the error estimations in (3.3)-(3.6) are more adapted for non-differentiable functions.

Lemma 3.3. *For any $q \in \mathbb{N}$ there exists a continuous and nowhere differentiable function φ that belongs to $\mathbb{H}_q$.*

Proof:
Let M be the so called McCarthy function defined by the infinite series

$$M(x) = \sum_{n=0}^{+\infty} \frac{1}{2^n} g(2^{2^n}x) \tag{3.10}$$

where g is the 4-periodic function defined of $[-2, 2]$ by

$$g(x) = \begin{cases} 1+x, & x \in [-2,0] \\ 1-x, & x \in [0,2]. \end{cases} \tag{3.11}$$

It is known that the McCarthy function is continuous and nowhere differential on $\mathbb{R}$, see [3]. Since the function g is bounded it belongs to $L^2(\mathbb{R}, \mu)$ and then its laplace like transform satisfies

$$|\mathcal{L}[g](\xi)| \leq Ce^{|\xi|}.$$

It is easy to check that for any φ in $L^2(\mathbb{R}, \mu)$ and any $a > 0$

$$\mathcal{L}[\varphi_a](\xi) = \frac{1}{a}\mathcal{L}[\varphi](\frac{\xi}{a}) \tag{3.12}$$

where $\varphi_a(x) = \varphi(ax)$.
The latter shows that for any $n \in \mathbb{N}$ the function $g_{2^{2^n}}$ is of exponential growth of order 1 and of type $\frac{1}{2^{2^n}}$

$$|\mathcal{L}[g_{2^{2^n}}](\xi)| \leq \frac{C}{2^{2^n}} e^{\frac{|\xi|}{2^{2^n}}}. \tag{3.13}$$

Now consider for any $k \in \mathbb{N}$ the truncation of the McCarthy function

$$M_k(x) = \sum_{n\geq k} \frac{1}{2^n} g(2^{2^n}x),$$

it is also continuous and nowhere differentiable and its Laplace like transform is of exponential growth of order 1 and type $\frac{1}{2^{2^k}}$. Using Theorem 2.4 we see that for any $q \in \mathbb{N}$ there exists a continuous and nowhere differentiable function that belongs to $\mathbb{H}_q$. $\square$

4. A numerical illustration

In this section we shall use the approximation results of the previous section to set up a new algorithm of interpolation of real functions. Suppose for instance that μ is the standard Gaussian probability measure and let φ be a function in $\mathbb{H}_1$, then by Corollary 3.2, it has a continuous version and it can be pointwisely approximated by the partial sum (3.1). Suppose now that φ is known at a finite number of points (called knots)

$$\big(r_0, y_0 = \varphi(r_0)\big), \cdots, \big(r_N, y_N = \varphi(r_N)\big)$$

and that the interpolation function $\varphi_N = \sum_{n=0}^{N} \varphi^{(n)} e_n$ coincides with φ at the points $\{r_0, \cdots, r_N\}$ i.e

$$\left| \begin{array}{l} \varphi(r_0) \\ \vdots \\ \varphi(r_N) \end{array} \right. = \left| \begin{array}{l} \varphi_N(r_0) \\ \vdots \\ \varphi_N(r_N). \end{array} \right.$$

This leads to the following linear system

$$\begin{bmatrix} e_0(r_0) & \cdots & e_N(r_0) \\ \vdots & & \vdots \\ e_0(r_N) & \cdots & e_N(r_N) \end{bmatrix} \begin{bmatrix} \varphi^{(0)} \\ \vdots \\ \varphi^{(N)} \end{bmatrix} = \begin{bmatrix} \varphi(r_0) \\ \vdots \\ \varphi(r_N) \end{bmatrix}. \tag{4.1}$$

By solving the linear system above we obtain an explicit formula of the interpolation polynomial of φ:

$$\varphi_N(x) = \sum_{n=0}^{N} \varphi^{(n)} e_n.$$

Remark 4.1. If we consider the polynomial function with several variables:

$$(x_0, \cdots, x_N) \longmapsto \Delta(x_0, \cdots, x_N) = \det \begin{bmatrix} e_0(x_0) & \cdots & e_N(x_0) \\ \vdots & & \vdots \\ e_0(x_N) & \cdots & e_N(x_N) \end{bmatrix}$$

we see that the subset of $\mathbb{R}^N$

$$\{(x_0, \cdots, x_N) \in \mathbb{R}^N \, : \, \Delta(x_0, \cdots, x_N) \neq 0\}$$

is everywhere dense, which means that the matrix in (4.1) is invertible in "most of the cases".

Returning back to our algorithm, we consider that it has at least two major advantages compared to the standard algorithms:

(1) In our case the accuracy doesn't depend on the inter-distances between the initial points of the knot's set

$$|\varphi(x) - \varphi_N(x)| \leq C \|\varphi\|_1 2^{-N/2}$$

while in the standard polynomial interpolation algorithm, the accuracy depends strongly on the inter-distances:

$$|\varphi(x) - p_N(x)| \leq \frac{\left(\max_{1\leq k\leq N} |r_k - r_{k-1}|\right)^{N+1}}{(N+1)!} \sup_{r_0\leq u\leq r_1} \left|\frac{d^{N+1}\varphi}{du^{N+1}}(u)\right|, \tag{4.2}$$

where p_N denotes the interpolation polynomials in the standard cases. We believe that this point is very important, since in practice we don't know the initial function φ unless at a few points which we couldn't choose.

(2) In our case the accuracy depends on the norm $\mathbb{H}_1$ of the function φ, while in the standard interpolation algorithms the accuracy depends strongly on the regularity of the function φ (the boundness of the N^{th} derivative of φ).

We have written down a code for the "new" algorithm using Matlab. To compare with the well known cubic spline algorithm, we have considered two examples of functions, one infinitely many differentiable and one continuous and nowhere differentiable.

Example 1: Consider the function defined on $[-1, 1]$ by $\varphi(x) = \frac{1}{1+x^2}$. Let $x = \{x_1, \cdots, x_{100}\}$ be a series of 100 points uniformly spread out on the interval $[-1, 1]$ and $y = \{y_1 = \varphi(x_1), \cdots, y_{100} = \varphi(x_{100})\}$ the initial values series. Chose $N = 5, 10, 15$ equidistant points from the initial values set of φ to constitute a set of knots and then apply our "new" algorithm and the cubic spline one. Denote the 100 output points of both algorithm respectively by y^H and y^C. Below we have a graphical illustration for $N = 5$:

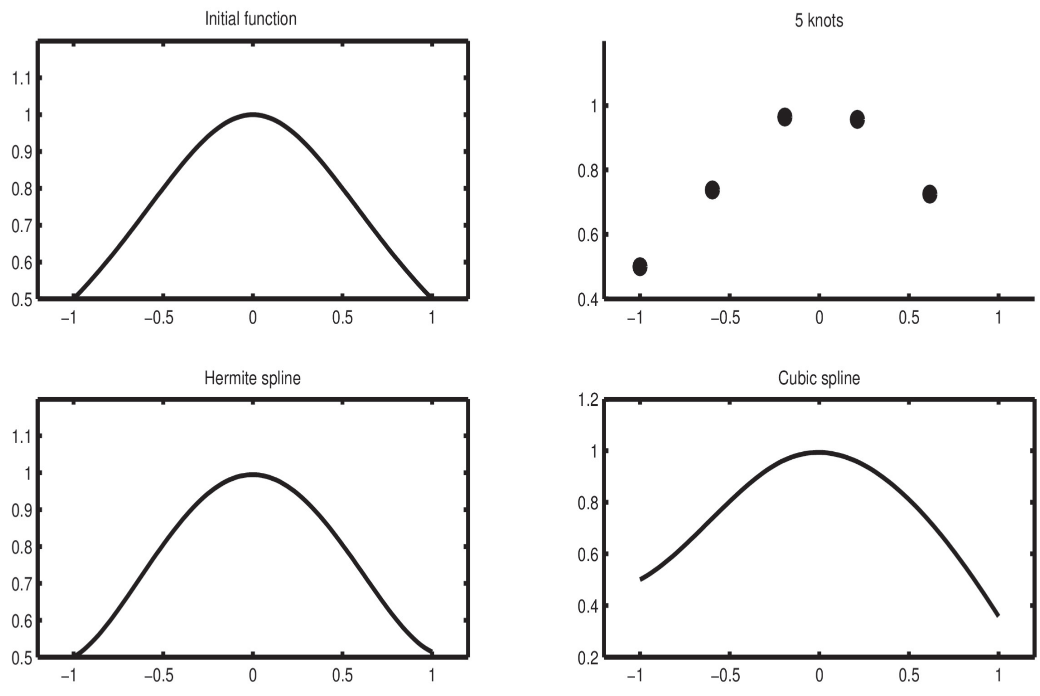

Fig 1. New Hermite vs Cubic spline: $\varphi(x) = \dfrac{1}{1+x^2}$.

We have computed the standard deviations of the differences between the initial values and the output values of the two algorithms:

$$\sigma_H^2 = \frac{1}{99}\sum_{i=1}^{100}(y_i - y_i^H)^2, \quad \sigma_C^2 = \frac{1}{99}\sum_{i=1}^{100}(y_i - y_i^C)^2$$

N	σ_H	σ_C
5	0.0061	0.0309
10	0.0078	0.0015
15	$2.8480e-005$	$2.1405e-005$

We note cautiously that our algorithm is more accurate than the cubic spline one for small knots set but we couldn't understand why it is not for larger knots set.
Example 2: Consider the so called McCarthy function defined by the infinite series

$$M(x) = \sum_{n=0}^{+\infty} \frac{1}{2^k} g(2^{2^k} x)$$

where

$$g(x) = \begin{cases} 1 + x, & x \in [-2, 0] \\ 1 - x, & x \in [0, 2] \end{cases}$$

and $g(x+4) = g(x)$ for any real x. It is known that the McCarthy function is continuous and nowhere differential on $\mathbb{R}$. We keep the same notations as for the regular function and we execute the same computations, we obtain:

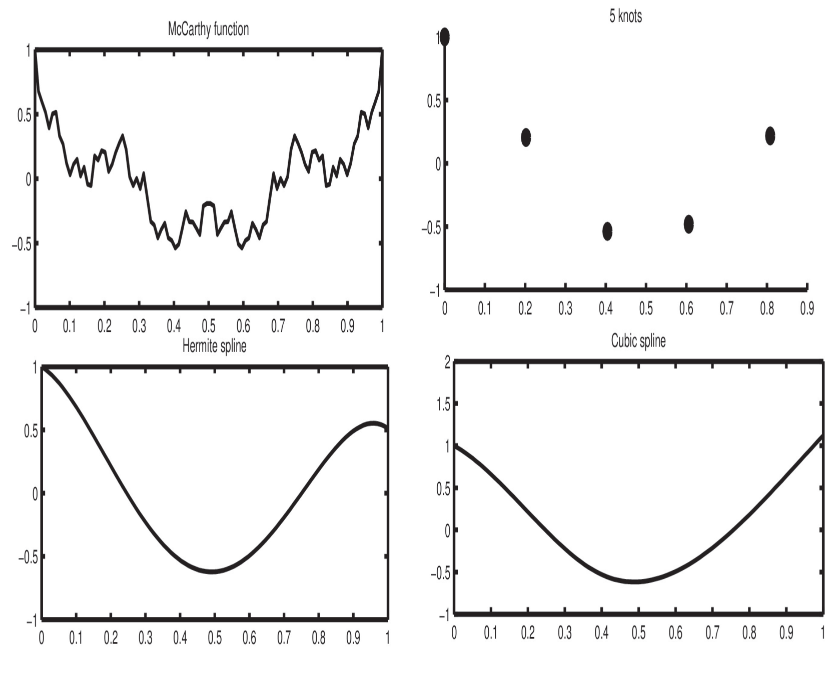

Fig 2. New Hermite vs Cubic spline: McCarthy function

and

N	σ_H	σ_C
5	0.6450	0.5594
10	0.5532	0.4312
15	0.7966	0.4777

Actually we have to say that there is no numerical evidence that our algorithm is better than the cubic spline one. However we note that the approximation errors of both algorithms are of the same order.

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