The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 14, Issue 2, Article 15, pp. 1-11, 2017

COEFFICIENT ESTIMATES FOR CERTAIN SUBCLASSES OF BI-UNIVALENT SAKAGUCHI TYPE FUNCTIONS BY USING FABER POLYNOMIAL

P. MURUGABHARATHI, B. SRUTHA KEERTHI

Received 9 January, 2017; accepted 1 May, 2017; published 29 December, 2017.

MATHEMATICS DIVISION, SCHOOL OF ADVANCED SCIENCES, VIT CHENNAI, VANDALOOR, KELAMBAKKAM ROAD, CHENNAI - 600 127, INDIA.
bharathi.muhi@gmail.com
sruthilaya06@yahoo.co.in

ABSTRACT. In this work, considering a general subclass of bi-univalent Sakaguchi type functions, we determine estimates for the general Taylor-Maclaurin coefficients of the functions in these classes. For this purpose, we use the Faber polynomial expansions. In certain cases, our estimates improve some of those existing coefficient bounds.

Key words and phrases: Analytic univalent function; Faber polynomial; Coefficient estimate; Bi-univalent function; Sakaguchi type function.

2000 *Mathematics Subject Classification.* 30C45.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by $f^{-1}(f(z)) = z \; (z \in \mathbb{U})$ and $f(f^{-1}(w)) = w \; (|w| < r_0(f); r_0(f) \geq \frac{1}{4})$. In fact, the inverse function $g = f^{-1}$ is given by

$$\begin{aligned} g(w) = f^{-1}(w) &= w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \\ &=: w + \sum_{n=2}^{\infty} A_n w^n. \end{aligned} \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). The class of analytic bi-univalent functions was first introduced and studied by Lewin [22], where it was proved that $|a_2| < 1.51$. Brannan and Clunie [4] improved Lewin's result to $|a_2| \leq \sqrt{2}$ and later Netanyahu [24] proved that $|a_2| \leq \frac{4}{3}$. Brannan and Taha [5] and Taha [30] also investigated certain subclasses of bi-univalent functions and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For a brief history and interesting examples of functions in the class Σ, see [28] (see also [5]). In fact, the aforecited work of Srivastava et al. [28] essentially revived the the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Frasin and Aouf [14], Xu et al. [32, 33], Hayami and Owa [19], and others (see, for example, [2, 6, 7, 9, 10, 15, 23, 25, 27]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n > 3$. This is because the bi-univalency requirement makes the behaviour of the coefficients of the functions f and f^{-1} unpredictable. Here, in this paper, we use the Faber polynomial expansions for a general subclass of analytic bi-univalent functions to determine estimates for the general coefficient bounds $|a_n|$.

The Faber polynomials introduced by Faber [13] play an important role in various areas of mathematical sciences, especially in geometric function theory. The recent publications [16] and [18] applying the Faber polynomial expansions to meromorphic bi-univalent functions motivated us to apply this technique to classes of analytic bi-univalent functions.

In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions given by (1.1) using Faber polynomial expansions [17, 20, 21]. Hamidi and Jahangiri [17] considered the class of analytic bi-close-to-convex functions. Jahangiri and Hamidi [20] considered the class defined by Frasin and Aouf [14], and Jahangiri et al. [21] considered the class of analytic bi-univalent functions with positive real-part derivatives. By motivated the works of Serap Bulut we defined the following classes.

2. THE CLASS $M_\Sigma(\alpha, \rho, \mu, t)$

Definition 2.1. For $0 \leq \mu \leq \rho \leq 1$, $|t| \leq 1$ and $t \neq 1$, a function $f \in \Sigma$ given by (1.1) is said to be in class $M_\Sigma(\alpha, \rho, \mu, t)$ if the following conditions are satisfied

$$Re\left\{\frac{(1-t)[\rho\mu z^3 f'''(z) + (2\rho\mu + \rho - \mu)z^2 f''(z) + zf'(z)]}{\rho\mu z^2[f''(z) - t^2 f''(tz)] + (\rho - \mu)z[f'(z) - tf'(tz)] + (1 - \rho + \mu)[f(z) - f(tz)]}\right\} > \alpha \tag{2.1}$$

and

$$Re\left\{\frac{(1-t)[\rho\mu w^3 g'''(w)+(2\rho\mu+\rho-\mu)w^2g''(w)+wg'(w)]}{\rho\mu w^2[g''(w)-t^2g''(tw)]+(\rho-\mu)w[g'(w)-tg'(tw)]+(1-\rho+\mu)[g(w)-g(tw)]}\right\}>\alpha \tag{2.2}$$

where $0\leq\alpha<1$ and $z,w\in\mathbb{U}$ and $g=f^{-1}$ is defined by (1.2).

Definition 2.2. For $0\leq\rho\leq 1$, $|t|\leq 1$ and $t\neq 1$, a function $f\in\Sigma$ given by (1.1) is said to be in class $N_\Sigma(\alpha,\rho,t)$ if the following conditions are satisfied

$$Re\left\{\frac{(1-t)[\rho z^2 f''(z)+zf'(z)]}{\rho z[f'(z)-tf'(tz)]+(1-\rho)[f(z)-f(tz)]}\right\}>\alpha \tag{2.3}$$

and

$$Re\left\{\frac{(1-t)[\rho w^2 g''(w)+wg'(w)]}{\rho w[g'(w)-tg'(tw)]+(1-\rho)[g(w)-g(tw)]}\right\}>\alpha \tag{2.4}$$

where $0\leq\alpha<1$ and $z,w\in\mathbb{U}$ and $g=f^{-1}$ is defined by (1.2).

Remark 2.1. For $t=0$ and $\mu=0$, we get the well-known class $N_\Sigma(\alpha,\rho)$ of bi-starlike functions or order α. This class consists of functions $f\in\Sigma$ satisfying $Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha$ and $Re\left(\frac{wg'(w)}{g(w)}\right)>\alpha$ where $0\leq\alpha<1$ and $z,w\in\mathbb{U}$ and $g=f^{-1}$ is defined by (1.2).

3. COEFFICIENT ESTIMATES

Using the Faber polynomial expansion of functions $f\in\mathcal{A}$ of the form (1.1), the coefficients of its inverse map $g=f^{-1}$ may be expressed as [1]

$$g(w)=f^{-1}(w)=w+\sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\dots)w^n \tag{3.1}$$

where

$$\begin{aligned}K_{n-1}^{-n}&=\frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1}+\frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3\\&+\frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4+\frac{(-n)!}{2(-n+2))!(n-5)!}a_2^{n-5}[a_5+(-n+2)a_3^2]\\&+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6+(-2n+5)a_3a_4]+\sum_{j\geq 7}a_2^{n-j}v_j\end{aligned} \tag{3.2}$$

such that v_j with $7\leq j\leq n$ is a homogenous polynomial in the variables $a_2,a_3,\dots,a_n$ [3]. In particular, the first three terms of K_{n-1}^{-n} are

$$K_1^{-2}=-2a_2,\ \ K_2^{-3}=3(2a_2^2-a_3),\ \ K_3^{-4}=-4(5a_2^3-5a_2a_3+a_4). \tag{3.3}$$

In general, for any $p\in\mathbb{N}:=\{1,2,3,\dots\}$, an expansion of K_n^p is as, [1],

$$K_n^p=pa_n+\frac{p(p-1)}{2}D_n^2+\frac{p!}{(p-3)!3!}D_n^3+\cdots+\frac{p!}{(p-n)!n!}D_n^n, \tag{3.4}$$

where $D_n^p=D_n^p(a_2,a_3,\dots)$, and by [31],
$D_n^m(a_1,a_2,\dots,a_n)=\sum_{n=1}^{\infty}\frac{m!}{i_1!\cdots i_n!}a_1^{i_1}\dots a_n^{i_n}$ while $a_1=1$, and the sum is taken over all non-negative integers $i_1,\dots,i_n$ satisfying $i_1+i_2+\cdots+i_n=m$, $i_1+2i_2+\cdots+ni_n=n$, it is clear that $D_n^n(a_1,a_2,\dots,a_n)=a_1^n$.

Consequently, for functions $f \in N_\Sigma(\alpha,\rho,\mu,t)$ of the form (1.1), we can write:

$$\left\{\frac{(1-t)[\rho\mu z^3 f'''(z)+(2\rho\mu+\rho-\mu)z^2 f''(z)+zf'(z)}{\rho\mu z^2[f''(z)-t^2f''(tz)]+(\rho-\mu)z[f'(z)-tf'(tz)]+(1-\rho+\mu)[f(z)-f(tz)]}\right\}$$

$$= 1+\sum_{n=2}^{\infty} F_{n-1}(a_2,a_3,\dots,a_n)z^{n-1}, \tag{3.5}$$

which is the Faber polynomial of degree $(n-1)$ where

$$\begin{aligned}
F_1 &= \left\{\frac{[2[\rho\mu(1-t)+1]-[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2}{1+t(\rho-\mu)}\right\}a_2,\\
F_2 &= \{[6\rho\mu(2-t)+3(\rho-\mu)+3]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3\}\frac{a_3}{1+t(\rho-\mu)}\\
&\quad -\{4[\rho\mu(1-t)+1][\rho\mu(1+t)+(\rho-\mu)]-2[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]\\
&\quad [2\rho\mu t+\rho-\mu-1]u_2-[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]^2u_2^2\}\frac{a_2^2}{[1+t(\rho-\mu)]^2},\\
F_3 &= \{4[3\rho\mu(3-t)+2(\rho-\mu)+1]-[1+(\rho-\mu)(4t-1)+12\rho\mu t^2]u_4\}\frac{a_4}{1+t(\rho-\mu)}\\
&\quad -\{6[\rho\mu(1+t)+(\rho-\mu)][\rho\mu(3-2t)+2+(\rho-\mu)]+3u_2[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]\\
&\quad [\rho\mu(1-2t)+1]+2u_3[1+(\rho-\mu)(3t-1)+6\rho\mu t^2][-2\rho\mu t-\rho+\mu+1\\
&\quad -u_2[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]]\}\frac{a_2a_3}{[1+t(\rho-\mu)]^2}\\
&\quad +\{4[\rho\mu(1-t)+1][\rho\mu(1+t)+(\rho-\mu)]-2[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]\\
&\quad [2\rho\mu t+\rho-\mu-1]u_2-[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]^2u_2^2\}\{2[\rho\mu(1+t)+(\rho-\mu)]\\
&\quad +[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2\}\frac{a_2^3}{[1+t(\rho-\mu)]^3}, etc.
\end{aligned}$$

If $f \in N_\Sigma(\alpha,\rho,t)$ of the form 1.1, we can write

$$\frac{(1-t)[\rho z^2 f''(z)+zf'(z)]}{\rho z[f'(z)-tf'(tz)]+(1-\rho)[f(z)-f(tz)]} = 1+\sum_{n=2}^{\infty} F_{n-1}(a_2,a_3,\dots,a_n)z^{n-1}, \tag{3.6}$$

which is the Faber polynomial of degree $(n-1)$ where

$$\begin{aligned}
F_1 &= \frac{2-u_2(2\rho t-\rho+1)}{(1+\rho t)}a_2,\\
F_2 &= \frac{3\rho+3-u_3(3\rho t-\rho+1)}{(1+\rho t)}a_3-\frac{[2-u_2(2\rho t-\rho+1)][2\rho+u_2(2\rho t-\rho+1)]}{(1+\rho t)^2}a_2^2,\\
F_3 &= \frac{8\rho+4-u_4(4\rho t-4+1)}{(1+\rho t)}a_4+\frac{[2\rho+u_2(2\rho t-\rho+1)]^2[2-u_2(2\rho t-\rho+1)]}{(1+\rho t)^3}a_2^3\\
&\quad -\left\{\frac{\begin{array}{c}[2-u_2(2\rho t-\rho+1)][3\rho+u_3(3\rho t-\rho+1)]\\ +[2\rho+u_2(2\rho t-\rho+1)][3\rho+3-u_3(3\rho t-\rho+1)]\end{array}}{(1+\rho t)^2}\right\}a_2a_3,\ etc.
\end{aligned}$$

where $u_1=1$, $u_2=1+t$, $u_3=1+t+t^2$ and $u_4=1+t+t^2+t^3$.

Our first theorem introduces an upper bound for the coefficients $|a_n|$ of bi-univalent Sakaguchi type functions in the class $N_\Sigma(\alpha,\rho,\mu,t)$.

Theorem 3.1. *For $0 \leq \mu \leq \rho \leq 1$, $|t| \leq 1$, $t \neq 1$, let the function $f \in M_\Sigma(\alpha, \rho, \mu, t)$ be given by (1.1). If $a_k = 0$ $(2 \leq k \leq n-1)$, then*

$$|a_n| \leq \frac{2(1-\alpha)[1+t(\rho-\mu)]}{\begin{array}{l} n[(n-1)(n-1-t)\rho\mu + (n-2)(\rho-\mu)+1] \\ -[1+(\rho-\mu)(nt-1)+n(n-1)\rho\mu t^2]u_n \end{array}}$$

Proof. For the function $f \in M_\Sigma(\alpha, \rho, \mu, t)$ of the form (1.1), we have the expansion (3.5) and for the inverse map $g = f^{-1}$, considering (1.2), we obtain

$$\frac{(1-t)[\rho\mu w^3 g'''(w) + (2\rho\mu+\rho-\mu)w^2 g''(w) + wg'(w)]}{\rho\mu w^2[g''(w) - t^2 g''(tw)] + (\rho-\mu)w[g'(w) - tf'(tw)] + (1-\rho+\mu)[g(w)-g(tw)]}$$

$$= 1 + \sum_{n=2}^{\infty} F_{n-1}(A_2, A_3, \ldots, A_n) w^{n-1}, \tag{3.7}$$

with

$$A_n = \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n). \tag{3.8}$$

On the other hand, since $f \in M_\Sigma(\alpha, \rho, \mu, t)$ and $g = f^{-1} \in M_\Sigma(\alpha, \rho, \mu, t)$, by definition, there exist two positive real-part functions

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in A \text{ and } q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n \in A,$$

where $Re\{p(z)\} > 0$ and $Re\{q(w)\} > 0$ in $\mathbb{U}$ so that

$$\frac{(1-t)[\rho\mu z^3 f'''(z) + (2\rho\mu+\rho-\mu)z^2 f''(z) + zf'(z)]}{\rho\mu z^2[f''(z) - t^2 f''(tz)] + (\rho-\mu)z[f'(z) - tf'(tz)] + (1-\rho+\mu)[f(z)-f(tz)]}$$
$$= \alpha + (1-\alpha)p(z)$$
$$= 1 + (1-\alpha)\sum_{n=1}^{\infty} K_{n-1}^{1}(c_1, c_2, \ldots, c_n) z^n \tag{3.9}$$

and

$$\frac{(1-t)[\rho\mu w^3 g'''(w) + (2\rho\mu+\rho-\mu)w^2 g''(w) + wg'(w)]}{\rho\mu w^2[g''(w) - t^2 g''(tw)] + (\rho-\mu)w[g'(w) - tg'(tw)] + (1-\rho+\mu)[g(w)-g(tw)]}$$
$$= \alpha + (1-\alpha)p(w)$$
$$= 1 + (1-\alpha)\sum_{n=1}^{\infty} K_{n-1}^{1}(d_1, d_2, \ldots, d_n) w^n \tag{3.10}$$

Note that, by the Caratheodary lemma (e.g [12]), $|c_n| \leq 2$ and $|d_n| \leq 2$ $(n \in \mathbb{N})$.

Comparing the corresponding coefficients of (3.5)) and (3.9), for any $n \geq 2$, yields

$$F_{n-1}(a_2, a_3, \ldots, a_n) = (1-\alpha)K_{n-1}^{1}(c_1, c_2, \ldots, c_{n-1}) \tag{3.11}$$

and similarly, from (3.7) and (3.10) we find

$$F_{n-1}(A_2, A_3, \ldots, A_n) = (1-\alpha)K_{n-1}^{1}(d_1, d_2, \ldots, d_{n-1}). \tag{3.12}$$

Note that for $a_k = 0$ $(2 \leq k \leq n-1)$, we have $A_n = -a_n$ and so
$\frac{n[(n-1)(n-1-t)\rho\mu+(n-2)\rho\mu+1]+[1+(\rho-\mu)(nt-1)+n(n-1)\rho\mu t^2]u_n}{[1+t(\rho-\mu)]} a_n = (1-\alpha)c_{n-1},$

$-\frac{n[(n-1)(n-1-t)\rho\mu+(n-2)\rho\mu+1]+[1+(\rho-\mu)(nt-1)+n(n-1)\rho\mu t^2]u_n}{[1+t(\rho-\mu)]}a_n = (1-\alpha)d_{n-1}.$
Taking the absolute values of the above equalities, we obtain

$$\begin{aligned}|a_n| &= \frac{(1-\alpha)|c_{n-1}|[1+t(\rho-\mu)]}{n[(n-1)(n-1-t)\rho\mu+(n-2)\rho\mu+1]-[1+(\rho-\mu)(nt-1)+n(n-1)\rho\mu t^2]u_n}\\ &= \frac{(1-\alpha)|d_{n-1}|[1+t(\rho-\mu)]}{n[(n-1)(n-1-t)\rho\mu+(n-2)\rho\mu+1]-[1+(\rho-\mu)(nt-1)+n(n-1)\rho\mu t^2]u_n}\\ &= \frac{2(1-\alpha)[1+t(\rho-\mu)]}{n[(n-1)(n-1-t)\rho\mu+(n-2)\rho\mu+1]-[1+(\rho-\mu)(nt-1)+n(n-1)\rho\mu t^2]u_n}\end{aligned}$$

where $u_n = 1+t+t^2+\cdots+t^{n-1}$
which completes the proof of Theorem 3.1. ∎

The following corollary is an immediate consequence of the above theorem.

Corollary 3.2. *(see* [20]*) For $0 \le \rho \le 1$, $|t| \le 1$, $t \ne 1$ and $0 \le \alpha < 1$, let the function $f \in M_\Sigma(\alpha,\rho,t)$ be given by (1.1). If $a_k = 0$ $(2 \le k \le n-1)$, then*

$$|a_n| \le \frac{2(1-\alpha)(1+\rho t)}{n[(n-2)\rho+1]-u_n[n\rho t-\rho+1]}\ (n \ge 4).$$

where $u_n = 1+t+t^2+\cdots+t^{n-1}$

Theorem 3.3. *For $0 \le \mu \le \rho \le 1$, $|t| \le 1$, $t \ne 1$ and $0 \le \alpha < 1$, let the function $f \in M_\Sigma(\alpha,\rho,\mu,t)$ be given by (1.1). Then one has the following*

$$|a_2| \le \begin{cases} \sqrt{\frac{2(1-\alpha)[1+t(\rho-\mu)]^2}{B}}, 0 \le \alpha < \frac{A}{2B} \\ \\ \frac{2(1-\alpha)[1+t(\rho-\mu)]}{2[\rho\mu(1-t)+1]-[1+(\rho-\mu)(2t-1)-2\rho\mu t^2]u_2}, \frac{A}{2B} \le \alpha < 1 \end{cases} \tag{3.13}$$

$$|a_3| \le \begin{cases} \min\left\{\frac{4(1-\alpha)^2[1+t(\rho-\mu)]^2}{\{2[\rho\mu(1-t)+1]-[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2\}^2}\right. \\ \\ +\frac{2(1-\alpha)[1+t(\rho-\mu)]}{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3}, \\ \\ \left.\frac{2(1-\alpha)[1+t(\rho-\mu)]^2}{B}\right\}, 0 \le \rho \le \mu < 1 \\ \\ \frac{2(1-\alpha)[1+t(\rho-\mu)]}{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3}, \rho,\mu = 1. \end{cases} \tag{3.14}$$

$$\begin{aligned}&\left|a_3 - \frac{C}{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3[1+t(\rho-\mu)]}a_2^2\right|\\ &\le \frac{2(1-\alpha)[1+t(\rho-\mu)]}{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3}\end{aligned}$$

where

$$A = \left\{\begin{array}{l} 2\{1+(\rho-\mu)[(3t-1)+(\rho-\mu)3t-2\rho\mu(3t^2-8t+2)]+\rho\mu[2\rho\mu(t^2+2t-3) \\ +(4-6t)]\}+[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2\{4[\rho\mu(1+t)+(\rho-\mu)] \\ +[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2\}-2[1+(\rho-\mu)(3t-1)+6\rho\mu t^2][1+t(\rho-\mu)]u_3 \end{array}\right\}$$

$$B = \left\{ \begin{array}{l} [1+t(\rho-\mu)]\{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3\} \\ -4[\rho\mu(1-t)+1][\rho\mu(1+t)+(\rho-\mu)]+2[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2 \\ [2\rho\mu t+\rho-\mu-1]+[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2^2 \end{array} \right\}$$

and

$$C = \left\{ \begin{array}{l} 2\{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3\}[1+t(\rho-\mu)] \\ -4[\rho\mu(1-t)+1][\rho\mu(1+t)+(\rho-\mu)]+2[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2 \\ [2\rho\mu t+\rho-\mu-1]+[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]^2u_2^2 \end{array} \right\}$$

Proof. If we set $n=2$ and $n=3$ in (3.11) and (3.12), respectively, we get

$$\frac{\{2[\rho\mu(1-t)+1]-[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2\}a_2}{1+t(\rho-\mu)} = (1-\alpha)c_1 \tag{3.15}$$

$$\begin{aligned} &\{6\rho\mu(2-t)+3(\rho-\mu)+3-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_2\}\frac{a_3}{1+t(\rho-\mu)} \\ &+ \begin{array}{l}\{-4[\rho\mu(1-t)+1][\rho\mu(1+t)+(\rho-\mu)]+2[1+(\rho-\mu)(2t-1)+2\rho\mu t^2] \\ [2\rho\mu t+\rho-\mu-1]\}u_2+[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]^2u_2^2\end{array} \quad \frac{a_2^2}{[1+t(\rho-\mu)]^2} \\ &= (1-\alpha)c_2 \end{aligned} \tag{3.16}$$

$$\frac{-\{2[\rho\mu(1-t)+1]-[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2\}a_2}{1+t(\rho-\mu)} = (1-\alpha)d_1 \tag{3.17}$$

$$\begin{aligned} &\begin{array}{l} 2\{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3\}[1+t(\rho-\mu)] \\ -\{-4[\rho\mu(1-t)+1][\rho\mu(1+t)+(\rho-\mu)]+2[1+(\rho-\mu)(2t-1)+2\rho\mu t^2] \\ [2\rho\mu t+\rho-\mu-1]u_2+[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]^2u_2^2\} \end{array} \quad \frac{a_2^2}{[1+t(\rho-\mu)]^2} \\ &- \{-3[2\rho\mu(2-t)+(\rho-\mu)+1]+[1+(\rho-\mu)(2t-1)+6\rho\mu t^2]u_2\}\frac{a_3}{1+t(\rho-\mu)} = (1-\alpha)d_2 \end{aligned} \tag{3.18}$$

$$\begin{aligned} |a_2| &= \frac{(1-\alpha)|c_1|[1+t(\rho-\mu)]}{2[\rho\mu(1-t)+1]-[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2} \\ &= \frac{(1-\alpha)|d_1|[1+t(\rho-\mu)]}{2[\rho\mu(1-t)+1]-[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2} \\ &\leq \frac{2(1-\alpha)[1+t(\rho-\mu)]}{2[\rho\mu(1-t)+1]-[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2} \end{aligned} \tag{3.19}$$

Also from (3.16) and (3.18), we obtain

$$2B\frac{a_2^2}{[1+t(\rho-\mu)]^2} = (1-\alpha)(c_2+d_2) \tag{3.20}$$

Using the Caratheodory lemma, we get

$$|a_2| \leq \sqrt{\frac{2(1-\alpha)[1+t(\rho-\mu)]^2}{B}},$$

and combining this with inequality (3.19), we obtain the desired estimate on the coefficient $|a_2|$ as asserted in (3.13).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (3.18) from (3.16), we thus get

$$\begin{aligned} \{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3\}&\frac{-2a_2^2+2a_3}{1+t(\rho-\mu)} \\ &= (1-\alpha)(c_2-d_2) \end{aligned}$$

or

$$a_3 = a_2^2 + \frac{(1-\alpha)(c_2-d_2)[1+t(\rho-\mu)]}{2\{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3\}}, \tag{3.21}$$

Upon substituting the value of a_2^2 from (3.15) into (3.21), it follows that

$$\begin{aligned} a_3 = &\frac{(1-\alpha)^2 c_1^2[1+t(\rho-\mu)]^2}{\{2[\rho\mu(1-t)+1]-[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2\}^2} \\ &+ \frac{(1-\alpha)(c_2-d_2)[1+t(\rho-\mu)]}{2\{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3\}} \end{aligned}$$

We thus find (by the Caratheodary lemma) that

$$\begin{aligned} |a_3| \le &\frac{4(1-\alpha)^2[1+t(\rho-\mu)]^2}{\{2[\rho\mu(1-t)+1]-[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2\}^2} \\ &+ \frac{2(1-\alpha)[1+t(\rho-\mu)]}{\{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3\}} \end{aligned} \tag{3.22}$$

On the other hand, upon substituting the value of a_2^2 from (3.20) into (3.21), it follows that

$$a_3 = \frac{(1-\alpha)[1+t(\rho-\mu)]\left\{\begin{array}{l} [1+t(\rho-\mu)]c_2C - d_2[-4[\rho\mu(1-t)+1] \\ [\rho\mu(1+t)+(\rho-\mu)]+2[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2 \\ [2\rho\mu t+\rho-\mu-1]+[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]^2u_2^2] \end{array}\right\}}{2B\{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3\}} \tag{3.23}$$

Consequently (by the Caratheodary lemma), we have

$$|a_3| \le \frac{2(1-\alpha)[1+t(\rho-\mu)]^2}{B}$$

Combining (3.22) and (3.23), we get the desired estimate on the coefficient $|a_3|$ as asserted in (3.14). Finally, from (3.18), we deduce (by the Caratheodary lemma) that

$$\begin{aligned} &\left|a_3 - \frac{C}{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3[1+t(\rho-\mu)]}a_2^2\right| \\ &= \frac{(1-\alpha)|d_2|[1+t(\rho-\mu)]}{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3} \\ &\le \frac{2(1-\alpha)[1+t(\rho-\mu)]}{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3} \end{aligned}$$

where

$$A = \left\{\begin{array}{l} 2\{1+(\rho-\mu)[(3t-1)+(\rho-\mu)3t-2\rho\mu(3t^2-8t+2)]+\rho\mu[2\rho\mu(t^2+2t-3) \\ +(4-6t)]\}+[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2\{4[\rho\mu(1+t)+(\rho-\mu)] \\ +[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2\}-2[1+(\rho-\mu)(3t-1)+6\rho\mu t^2][1+t(\rho-\mu)]u_3 \end{array}\right\}$$

$$B = \left\{\begin{array}{l} [1+t(\rho-\mu)]\{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3\} \\ -4[\rho\mu(1-t)+1][\rho\mu(1+t)+(\rho-\mu)]+2[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2 \\ [2\rho\mu t+\rho-\mu-1]+[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2^2 \end{array}\right\}$$

and

$$C = \left\{\begin{array}{l} 2\{3[2\rho\mu(2-t)+(\rho-\mu)+1]-[1+(\rho-\mu)(3t-1)+6\rho\mu t^2]u_3\}[1+t(\rho-\mu)] \\ -4[\rho\mu(1-t)+1][\rho\mu(1+t)+(\rho-\mu)]+2[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]u_2 \\ [2\rho\mu t+\rho-\mu-1]+[1+(\rho-\mu)(2t-1)+2\rho\mu t^2]^2u_2^2 \end{array}\right\}$$

This evidently completes the proof of Theorem 3.3. ∎

Corollary 3.4. *(see* [20]*) For* $0 \leq \rho \leq 1$, $|t| \leq 1$, $t \neq 1$ *and* $0 \leq \alpha < 1$, *let the function* $f \in N_\Sigma(\alpha, \lambda, t)$ *be given by (1.1). Then one has the following*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\alpha)(1+\rho t)^2}{E}}, 0 \leq \alpha < \frac{D}{2E} \\ \\ \frac{2(1-\alpha)(1+\rho t)}{2-u_2(2\rho t-\rho+1)}, \frac{D}{2E} \leq \alpha < 1 \end{cases}$$

$$|a_3| \leq \begin{cases} \min\left\{\frac{4(1-\alpha)^2(1+\rho t)^2}{\{2-u_2(2\rho t-\rho+1)\}^2} + \frac{2(1-\alpha)(1+\rho t)}{3\rho+3-u_3(3\rho t-\rho+1)}\right\}, \\ \\ \frac{2(1-\alpha)(1+\rho t)^2}{E}, 0 \leq \rho < 1 \\ \\ \frac{2(1-\alpha)(1+\rho t)}{3\rho+3-u_3(3\rho t-\rho+1)}, \rho = 1 \end{cases}$$

$$\left|a_3 - \frac{2E}{[3\rho+3-u_3(3\rho t-\rho+1)](1+\rho t)}a_2^2\right| \leq \frac{2(1-\alpha)(1+\rho t)}{3\rho+3-u_3(3\rho t-\rho+1)}$$

where

$$D = 2[3\rho+3-u_3(3\rho t-\rho+1)](1+\rho t) - [4\rho+2+u_2(2\rho t-\rho+1)][2-u_2(2\rho t-\rho+1)]$$

and

$$E = [3\rho+3-u_3(3\rho t-\rho+1)](1+\rho t) - [2-u_2(2\rho t-\rho+1)][2\rho+u_2(2\rho t-\rho+1)]$$

REFERENCES

[1] H. AIRAULT and A. BOUALI, Differential calculus on the Faber polynomials, *Bull. Sci. Math.*, **130**(3) (2006), pp. 179–222.

[2] M.K. AOUF, R.M. EL-ASHWAH and A.M. ABD-ELTAWAB, New subclasses of biunivalent functions involving Dziok-Srivastava operator, *ISRN Math. Anal.*, (2013), Article ID 387178, 5 p.

[3] H. AIRAULT and J. REN, An algebra of differential operators and generating functions on the set of univalent functions, *Bull. Sci. Math.*, **126**(5) (2002), pp. 343–367.

[4] D.A. BRANNAN and J.G. CLUNIE (Eds.), Aspects of Contemporary Complex Analysis, *Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham*, July 1-20, 1979, Academic Press, New York and London, (1980).

[5] D.A. BRANNAN and T.S. TAHA, On some classes of bi-univalent functions, in: S.M. Mazhar, A. Hamoui, N.S. Faour (Eds.), Mathematical Analysis and Its Applications, Kuwait, February 18-21, 1985, in: KFAS Proc. Ser., vol. 3, Pergamon Press (Elsevier Science Limited), Oxford, 1988, pp. 53–60; see also D.A. Brannan, T.S. Taha, *Stud. Univ. Babes-Bolyai, Math.*, **31**(2) (1986), pp. 70–77.

[6] S. BULUT, Coefficient estimates for initial Taylor-Maclaurin coefficients for a subclass of analytic and bi-univalent functions defined by Al-Oboudi differential operator, *Sci. World J.*, (2013), Article ID 171039, 6 p.

[7] S. BULUT, Coefficient estimates for new subclasses of analytic and bi-univalent functions defined by Al-Oboudi differential operator, *J. Funct. Spaces Appl.*, (2013), Article ID 181932, 7 p.

[8] S. BULUT, Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions, *C. R. Math. Acad. Sci. Paris*, **352**(6) (2014), pp. 479–484.

[9] S. BULUT, Coefficient estimates for a new subclass of analytic and bi-univalent functions, *Ann. Alexandru Ioan Cuza Univ., Math.*, bf 62, No. 2, Vol. 1 (2016), pp. 305-311.

[10] S. BULUT, Coefficient estimates for a class of analytic and bi-univalent functions, Novi Sad J. Math., **43**(2) (2013), pp. 59–65.

[11] M. ÇAĞLAR, H. ORHAN and N. YAĞMUR, Coefficient bounds for new subclasses of bi-univalent functions, *Filomat*, **27**(7) (2013), pp. 1165–1171.

[12] P.L. DUREN, *Univalent Functions*, Grundlehren Math. Wiss., vol. 259, Springer, New York, 1983.

[13] G. FABER, Über polynomische Entwickelungen, *Math. Ann.*, **57**(3) (1903), pp. 389–408.

[14] B.A. FRASIN and M.K. AOUF, New subclasses of bi-univalent functions, *Appl. Math. Lett.*, **24** (2011), pp. 1569–1573.

[15] S.P. GOYAL and P. GOSWAMI, Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives, *J. Egypt. Math. Soc.*, **20** (2012), pp. 179–182.

[16] S.G. HAMIDI, S.A. HALIM and J.M. JAHANGIRI, Coefficient estimates for a class of meromorphic bi-univalent functions, *C. R. Acad. Sci. Paris, Ser. I*, **351**(9-10) (2013), pp. 349–352.

[17] S.G. HAMIDI and J.M. JAHANGIRI, Faber polynomial coefficient estimates for analytic bi-close-to-convex functions, *C. R. Acad. Sci. Paris, Ser. I*, **352**(1) (2014), pp. 17–20.

[18] S.G. HAMIDI, T. JANANI, G. MURUGUSUNDARAMOORTHY and J.M. JAHANGIRI, Coefficient estimates for certain classes of meromorphic bi-univalent functions, *C. R. Acad. Sci. Paris, Ser. I*, **352**(4) (2014), pp. 277–282.

[19] T. HAYAMI and S. OWA, Coefficient bounds for bi-univalent functions, *Panam. Math. J.*, **22**(4) (2012), pp. 15–26.

[20] J.M. JAHANGIRI and S.G. HAMIDI, Coefficient estimates for certain classes of bi-univalent functions, *Int. J. Math. Math. Sci.*, (2013), Article ID 190560, 4 p.

[21] J.M. JAHANGIRI, S.G. HAMIDI and S.A. HALIM, Coefficients of bi-univalent functions with positive real part derivatives, *Bull. Malays. Math. Soc.*, (2)bf 37, (2014), pp. 633-640.

[22] M. LEWIN, On a coefficient problem for bi-univalent functions, *Proc. Am. Math. Soc.*, **18** (1967), pp. 63–68.

[23] G. MURUGUSUNDARAMOORTHY, N. MAGESH and V. PRAMEELA, Coefficient bounds for certain subclasses of bi-univalent function, *Abstr. Appl. Anal.*, (2013), Article ID 573017, 3 p.

[24] E. NETANYAHU, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Arch. Ration. Mech. Anal.*, **32** (1969), pp. 100–112.

[25] S. PORWAL and M. DARUS, On a new subclass of bi-univalent functions, *J. Egypt. Math. Soc.*, **21**(3) (2013), pp. 190–193.

[26] S. PREMA and B.S. KEERTHI, Coefficient bounds for certain subclasses of analytic functions, *J. Math. Anal.*, **4**(1) (2013), pp. 22–27.

[27] H.M. SRIVASTAVA, S. BULUT, M. ÇAĞLAR and N. YAĞMUR, Coefficient estimates for a general subclass of analytic and bi-univalent functions, *Filomat*, **27**(5) (2013), pp. 831–842.

[28] H.M. SRIVASTAVA, A.K. MISHRA and P. GOCHHAYAT, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.*, **23** (2010) pp. 1188–1192.

[29] B. SRUTHA KEERTHI, Fekete-Szegö type equalities for certain subclasses of Sakaguchi type functions, *Romai J.*, **8**(2) (2012), pp. 119–127.

[30] T.S. TAHA, *Topics in Univalent Function Theory*, PhD thesis, University of London, 1981.

[31] P.G. TODOROV, On the Faber polynomials of the univalent functions of class Σ, *J. Math. Anal. Appl.*, **162**(1) (1991), pp. 268–276.

[32] Q.-H. XU, Y.-C. GUI and H.M. SRIVASTAVA, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, *Appl. Math. Lett.*, **25** (2012), pp. 990–994.

[33] Q.-H. XU, H.-G. XIAO and H.M. SRIVASTAVA, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, *Appl. Math. Comput.*, **218** (2012), pp. 11461–11465.