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CONVERGENCE AND STABILITY RESULTS FOR NEW THREE STEP ITERATION PROCESS IN MODULAR SPACES

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ABSTRACT. The aim of this paper is to introduce a new iteration process (2.5) for ρ-contraction mappings in Modular spaces. We obtain some analytical proof for convergence and stability of our iteration process (2.5). We show that our iteration process (2.5) gives faster convergence results than the leading AK iteration process (2.4) for contraction mappings.

Moreover, a numerical example (using the Matlab Software) is presented to compare the rate of convergence for existing iteration processes with our new iteration process (2.5).

Key words and phrases: Modular Spaces; ρ-contraction; ρ-convergence.

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1. INTRODUCTION

The theory of modular space was introduced by Nakano [2] in 1950 in connection with the theory of order spaces and Musielak and Orlicz [3] regeneralised this theory in 1959. Kozolwaski [4] extended this work in modular function space in 1988. In 2006, V.V. Chistiyakov [5] presented this theory on a arbitrary set and gave the concept of metric modular space. Some fixed point problems in modular spaces, generalizing the classical Banach contraction principal in metric space have been studied extensively, one can see [6, 7, 17].

Approximation method play an important role to find the fixed point of a problem which cannot be solved exactly by well known methods. It is very difficult work to discuss all the iteration processes. Several authors used the iteration processes to find fixed point of a sequence like the Picard, Mann, Ishikawa, Karanoselski, Agarwal, Noor, Abbas, SP, CR, Picard-Mann, Picard-S, Thakur new, Vatan two step and AK three step iteration processes. For more information one can see [1, 8, 9].

In 2014, Gursoy and Karakaya [10] authors claimed that Picard-S iteration process converges faster than all Picard, Mann, Ishikawa, Noor, SP, CR, Agarwal, S* Abbas and Normal S iteration process for contraction mappings. In 2015, Karakya et al. [11] proved that Vatan two step iteration process is faster than Picard-S, CR, SP and Picard-Mann iteration process for weak contraction mappings. In 2016, Thakur et al. [12] gave some numerical examples and proved that Thakur new iteration process converge faster than Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iteration processes for the class of Suzuki generalized non expansive mappings and also in 2016, Ullah and Arshad [1] proved that their AK iteration process is faster than leading Vatan two step iteration process for contraction mappings.

In this paper we generalize the AK iteration process and prove that our iteration process (2.5) is converges faster than AK iteration process, Vatan two step iteration process, Thakur New and Picard-S iteration process. In support of our claim we present a numerical example using the MATLAB Software.

2. PRELIMINARIES

Now, we recall some basic definitions and lemmas which will be used in our further results. We refer [2, 3, 6].

Definition 2.1 (Modular Spaces)**.** Let X be an arbitrary vector space over a field $K = (R$ or $C)$. A functional $\rho : X \to [0, \infty]$ is called modular on X if it satisfy the following conditions:

(i) $\rho(x) = 0$ if and only if $x = 0$.
(ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, for all $x \in X.$
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, for all $x, y \in X.$

The modular ρ is called a convex modular if (iii) is replaced by
(iii)′ $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, for all $x, y \in X.$
A modular ρ defines the corresponding modular space, i.e., the space X_ρ given by

$$X_\rho = \{x \in X \mid \rho(\alpha x) \to 0 \; as \; \alpha \to 0\}$$

if ρ is a modular in X, then

$$|x|_\rho = \inf\{u > 0, \rho(x/u) \leq u\}$$

is a F-norm and if ρ is convex modular, the modular X_ρ can be equipped with a norm called the Luxemburg norm defined by

$$\|x\|_\rho = \inf\{\alpha > 0; \rho(x/\alpha) \leq 1\}\,.$$

Definition 2.2. Let X_ρ be a modular space.

(a) A sequence $\{x_n\}_{n\in N}$ in X_ρ is said to be ρ-convergent to $x \in X_\rho$ and written as $x_n \xrightarrow{\rho} x$, if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(b) A sequence $\{x_n\}$ is called ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(c) The modular space X_ρ is called ρ-complete if every ρ-Cauchy sequence in X_ρ is ρ-convergent in X_ρ.
(d) A subset $S \subset X_\rho$ is said to be ρ-closed if for any sequence $\{x_n\}_{n\in N} \subset S$ with $\rho(x_n - x) \to 0$, then $x \in S$.
(e) A subset $S \subset X_\rho$ is said to be ρ-bounded if $\operatorname{diam}_\rho(S) < \infty$, where

$$\operatorname{diam}_\rho(S) = \sup\{\rho(x - y); x, y \in S\}.$$

(f) A function $f : X_\rho \to X_\rho$ is called ρ-continuous if

$$\rho(f(x_n) - f(x)) \to 0 \ \text{ as } \ \rho(x_n - x) \to 0.$$

Definition 2.3 (Contraction Mapping)**.** Let X_ρ be a modular space and C be any subset of X_ρ. A mapping $T : C \to C$ is called ρ-contraction if there exist $r \in (0, 1)$ such that $\rho(T(x) - T(y)) \le r\rho(x - y)$, for all $x, y \in C$.

Definition 2.4 ([8])**.** Let $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ be two real convergent sequences with limits x and y, respectively. Then we say that $\{x_n\}_{n=0}^\infty$ converges faster than $\{y_n\}_{n=0}^\infty$ if

$$\lim_{n\to\infty} \frac{\|x_n - x\|}{\|y_n - y\|} = 0\,.$$

In sense of modular space, we have

$$\frac{\rho(x_n - x)}{\rho(y_n - y)} \to 0 \quad \text{as} \quad n \to \infty\,.$$

Definition 2.5 ([8])**.** Let $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ be two fixed point iteration procedure sequences that converge to the same fixed point p. If $\|x_n - p\| \le u_n$ and $\|y_n - y\| \le v_n$, for all $n \ge 0$, where $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ are two sequences of positive numbers (converging to zero). Then we say that $\{x_n\}_{n=0}^\infty$ converges faster than $\{y_n\}_{n=0}^\infty$ to p if $\{u_n\}_{n=0}^\infty$ converges faster than $\{v_n\}_{n=0}^\infty$.

Definition 2.6 ([13])**.** Let $\{t_n\}_{n=0}^\infty$ be an arbitrary sequence in C. Then, an iteration procedure $x_{n+1} = f(T, x_n)$, converging to fixed point p, is said to be T-stable or stable with respect to T, if for

$$\epsilon_n = \|t_{n+1} - f(T, t_n)\|, \quad n = 0, 1, 2, 3, \ldots,$$

we have

$$\lim_{n\to\infty} \epsilon_n = 0 \iff \lim_{n\to\infty} t_n = p.$$

Lemma 2.1 ([14, 15])**.** *Let $\{\Psi_n\}_{n=0}^\infty$ and $\{\varphi_n\}_{n=0}^\infty$ be nonnegative real sequences satisfying the following inequality:*

$$\psi_{n+1} \le (1 - \phi)\psi_n + \varphi_n,$$

where $\phi_n \in (0, 1)$ for all $n \in N$, $\sum_{n=0}^{\infty} \phi_n = \infty$ and $\frac{\varphi_n}{\phi_n} \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} \psi_n = 0$.

Lemma 2.2 ([16])**.** *Let $\{\Psi_n\}_{n=0}^\infty$ be non negative real sequences for which one assume that there exists $n_0 \in N$ s.t. for all $n \ge n_0$, the following inequality satisfies:*

$$\psi_{n+1} \le (1 - \phi_n)\psi_n + \phi_n\varphi_n,$$

where $\phi_n \in (0, 1)$ for all $n \in N$, $\sum_{n=0}^{\infty} \phi_n = \infty$ and $\varphi \ge 0$ for all $n \in N$, then

$$0 \le \lim_{n\to\infty} \sup \psi_n \le \lim_{n\to\infty} \sup \varphi_n.$$

Now, we recall some iterative processes which are used in this paper.

In all iteration processes given below, we have $n \geq 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$, C is any subset of Banach space X and $T : C \to C$ is any mapping.

In 2014, Gursoy and Karakaya [10] introduced the following three step iterative process known as Picard-S iteration process given by:

$$\begin{cases} u_0 \in C \\ w_n = (1-\beta_n)u_n + \beta_n T u_n \\ v_n = (1-\alpha_n)Tu_n + \alpha_n T w_n \\ u_{n+1} = Tv_n \end{cases} \tag{2.1}$$

With the help of numerical example they conclude that the Picard-S iteration process can be used to approximate the fixed point of contraction mappings and converges faster than all Picard, Mann, Ishikawa, Noor, SP, CR, S, S*, Abbas, Normal-S and Two-step Mann iteration process.

In 2015, Kurakaya et al. [11] new two step iteration process, known as Vatan two-step iteration process as follows:

$$\begin{cases} u_0 \in C \\ v_n = T(1-\beta_n)u_n + \beta_n T u_n) \\ u_{n+1} = T((1-\alpha_n)v_n + \alpha_n T v_n) \end{cases} \tag{2.2}$$

They claim that it is even faster than Picard-S iteration process.

In 2016, Thakur et al. [12] defined a new iteration process as follows:

$$\begin{cases} u_0 \in C \\ w_n = (1-\beta_n)u_n + \beta_n T u_n \\ v_n = T((1-\alpha_n)u_n + \alpha_n w_n) \\ u_{n+1} = Tv_n \end{cases} \tag{2.3}$$

They used some numerical examples and claim that their new iteration process is faster convergence than Picard, Mann, Ishikawa, Noor and Abbas iteration processes for some class of mappings. It will be called Thakur new iteration process.

Recently, Kifayat Ullah and Muhammad Arshad [1] introduced a new iteration process, with the claim that their iteration process is faster than Picard-S, Thakur new and Vatan two step, as follows:

$$\begin{cases} x_0 \in C \\ z_n = T((1-\beta_n)x_n + \beta_n T x_n) \\ y_n = T((1-\alpha_n)z_n + \alpha_n T z_n) \\ x_{n+1} = Ty_n \end{cases} \tag{2.4}$$

For faster rate of convergence than the iteration process (2.1), (2.2), (2.3) and (2.4), we introduce the following new iteration process:

In this iteration process given below, we have $n \geq 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$, C is any subset of ρ-complete modular space X_ρ and $T : C \to C$ is any ρ-contraction mapping.

$$\begin{cases} x_0 \in C \\ z_n = T((1-\beta_n)x_n + \beta_n T x_n) \\ y_n = T((1-\alpha_n)z_n + \alpha_n T z_n) \\ x_{n+1} = T^5 y_n \end{cases} \tag{2.5}$$

Remark 2.1. Here, we have taken $T(T(T(T(Ty_n))))) = T^5 y_n$. When we take $T^2 y_n$, $T^3 y_n$, $T^4 y_n$ and upto $T^5 y_n$ then the error decreases and after $T^5 y_n$ there is no effect on errors to increase the power of T. So we have taken $T^5 y_n$ in our iteration (2.5).

We have to show that our iteration process (2.5) approximate the same fixed point p as existing iteration process (2.1), (2.2), (2.3) and (2.4). And with the help of numerical example we show that our iteration process (2.5) is faster than (2.1), (2.2), (2.3) and (2.4). Moreover, we work in modular spaces which is more generalised spaces than Banach spaces and also show that our iteration process (2.5) is stable.

3. MAIN RESULTS

Theorem 3.1. *Let C be a nonempty closed convex subset of a ρ-complete space X_ρ, where ρ is a convex modular on X and $T : C \to C$ be a ρ-contraction mapping. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by* (2.5) *with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfying $\{\alpha_n\}_{n=0}^{\infty} = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converging strongly to a unique fixed point of T.*

Proof. As we know, Banach contraction principle guarantees the existence and uniqueness of fixed point p. We will show that $\rho(x_n - p) \to 0$ as $n \to \infty$. From (2.5) we have

$$\begin{aligned}
\rho(z_n - p) &= \rho(T((1-\beta_n)x_n + \beta_n T x_n) - p) \\
&\le r\rho((1-\beta_n)x_n + \beta_n T x_n - (1-\beta_n+\beta_n)p) \\
&\le r((1-\beta_n)\rho(x_n - p) + \beta_n \rho(T x_n - Tp)) \\
&\le r((1-\beta_n)\rho(x_n - p) + \beta_n r\rho(x_n - p)) \\
&= r(1-\beta_n(1-r))\rho(x_n - p). \qquad (3.1)
\end{aligned}$$

similarly,

$$\begin{aligned}
\rho(y_n - p) &= \rho(T((1-\alpha_n)z_n + \alpha_n T z_n) - Tp) \\
&\le r\rho((1-\alpha_n)z_n + \alpha_n T z_n - p) \\
&\le r((1-\alpha_n)\rho(z_n - p) + \alpha_n \rho(T z_n - p)) \\
&\le r((1-\alpha_n)\rho(z_n - p) + \alpha_n r\rho(z_n - p)) \\
&\le r(1-\alpha_n(1-r))\rho(z_n - p) \\
&\le r^2(1-\alpha_n(1-r)(1-\beta_n(1-r))\rho(x_n - p). \qquad (3.2)
\end{aligned}$$

Hence

$$\begin{aligned}
\rho(x_{n+1} - p) &= \rho(T^4(Ty_n) - p) \\
&= \rho(T^4(Ty_n) - Tp) \\
&\le r\rho(T^3(Ty_n) - p) \\
&= r\rho(T^3(Ty_n) - Tp) \\
&\le r^2\rho(T^2(Ty_n) - p) \\
&= r^2\rho(T^2(Ty_n) - Tp) \\
&\le r^3\rho(T(Ty_n) - p) \\
&= r^3\rho(T(Ty_n) - Tp) \\
&\le r^4\rho(Ty_n - p) \\
&= r^4\rho(Ty_n - Tp)
\end{aligned}$$

$$\leq r^5\rho(y_n - p)$$
$$\leq r^7(1-\alpha_n(1-r))(1-\beta_n(1-r))\rho(x_n - p)$$
$$\leq r^7(1-\alpha_n(1-r))\rho(x_n - p) \tag{3.3}$$

From (3.3), we have

$$\rho(x_{n+1} - p) \leq r^7(1-\alpha_n(1-r))\rho(x_n - p)$$
$$\rho(x_n - p) \leq r^7(1-\alpha_{n-1}(1-r))\rho(x_{n-1} - p)$$
$$\rho(x_{n-1} - p) \leq r^7(1-\alpha_{n-2}(1-r))\rho(x_{n-2} - p)$$
$$\vdots$$
$$\rho(x_1 - p) \leq r^7(1-\alpha_0(1-r))\rho(x_0 - p) \tag{3.4}$$

By using (3.4), we can obtain

$$\rho(x_{n+1} - p) \leq \rho(x_0 - p)r^{7(n+1)}\prod_{j=0}^{n}(1-\alpha_j(1-r)), \tag{3.5}$$

where $1-\alpha_j(1-r) \in (0,1)$ because $r \in (0,1)$ and $\alpha_n \in [0,1]$, for all $n \in N$. Since we have the fact that $1-x \leq e^{-x}$ for all $x \in [0,1]$, so from (3.5) we get

$$\rho(x_{n+1} - p)\frac{\rho(x_0 - p)r^{7(n+1)}}{e^{(1-r)}\sum_{j=0}^{n}\alpha_j}. \tag{3.6}$$

Thus, by taking $\lim_{n\to\infty}\rho(x_n - p) = 0$ i.e. $x_n \to p$ for $n \to \infty$, as required. ∎

Theorem 3.2. *Let C be a nonempty closed convex subset of a ρ-complete space X_ρ, where ρ is a convex modular on X. And $T : C \to C$ be a $\rho-$ contraction mapping. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by* (2.5) *with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the iteration process* (2.5) *is T-stable.*

Proof. Let $\{t_n\}_{n=0}^{n=\infty}$ be an arbitrary sequence in C. Let the sequence generated by (2.5) is $X_{n+1} = f(T, x_n)$ converging to unique fixed point p (by Theorem 3.1) and $\epsilon_n = \rho(t_{n+1} - f(T, t_n)$. We will prove that $\lim_{n\to\infty}\epsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} t_n = p$. Let $\lim_{n\to\infty}\epsilon_n = 0$. By using (3.3), we get

$$\begin{aligned}\rho(t_{n+1} - p) &\leq \rho(t_{n+1} - \rho(T, t_n) + \rho(f(T, t_n) - p)\\ &= \epsilon_n + \rho(T^4(T((1-\alpha_n))T((1-\beta_n)t_n + \beta_n Tt_n)\\ &\qquad + \alpha_n T(T((1-\beta_n)t_n + \beta_n Tt_n)))) - p)\\ &\leq r^7(1-\alpha_n(1-r))\rho(x_n - p) \qquad \text{(by (3.3))}\end{aligned}$$

Define $\varphi_n = \rho(t_n - p)$, $\phi_n = (1-r) \in (0,1)$ and $\varphi_n = \epsilon_n$. Since $\lim_{n\to\infty}\epsilon_n = 0$, which implies that $\frac{\varphi_n}{\phi_n} \to 0$ as $n \to \infty$. Thus all the conditions of Lemma 2.1 are fulfilled by above inequality. Hence by Lemma 2.1 we get, $\lim_{n\to\infty} t_n = p$.

Conversely, let $\lim_{n\to\infty} t_n = p$, we have

$$\begin{aligned}\epsilon_n &= \rho(t_{n+1} - f(T, t_n))\\ &\leq \rho(t_{n+1} - p) + \rho f(T, t_n - p)\end{aligned}$$

$$\leq \rho(t_{n+1}-p)+r^7(1-\alpha_n(1-r))\rho(t_n-p)\,.$$

This implies that $\lim\limits_{n\to\infty}\epsilon_n=0$. Hence (2.5) is stable with respect to T. ∎

Theorem 3.3. *Let C be a nonempty closed convex subset of a $\rho-$ complete space X_ρ, where ρ is a convex modular on X. And $T:C\to C$ be a $\rho-$ contraction mapping. Let $\{u_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by* (2.2) *and* (2.5) *respectively with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfying $\sum\limits_{n=0}^{\infty}\alpha_n=\infty$. Then the following are equivalent:*

(i) *the iteration process* (2.5) *converges to the fixed point p of T.*
(ii) *the Vatan two step iteration process* (2.2) *converges to the fixed point p of T.*
(iii) *the AK iteration process* (2.4) *converges to the fixed point p of T.*

Proof. First we prove (2.1)⇒(2.2). Let the iteration process (2.5) converges to the fixed point p of T that is $\rho(x_n-p)\to 0$ as $n\to\infty$. Now using (2.2) and (2.5), we have

$$\begin{aligned}\rho(z_n-v_n)&=\rho(T((1-z_n)x_n+\beta_nTx_n)-T((1-\beta_n)u_n+\beta_nTu_n)\\&\leq r\{\rho(1-\beta_n)x_n+\beta_nTx_n-(1-\beta_nu_n-\beta_nTu_n\}\\&\leq r\{(1-\beta_n)\rho(x_n-u_n)+\beta_n\rho(Tx_n-Tu_n)\}\\&\leq r\{(1-\beta_n)\rho(x_n-u_n)+r\beta_n\rho(x_n-u_n)\}\\&=r(1-\beta_n)(1-r))\rho(x_n-u_n)\end{aligned}\tag{3.7}$$

Similarly using (2.2) and (2.5) together with (3.7), we have

$$\begin{aligned}\rho(x_{n+1}-u_{n+1})&=\rho(T^4(Ty_n)-u_{n+1})\\&=\rho(T^4(Ty_n)-y_n+y_n-u_{n+1})\\&\leq\rho(T^4(Ty_n)-y_n)+\rho(T(1-\alpha_n)z_n+\alpha_nTz_n)-T(1-\alpha_n)v_n+\alpha_nTv_n))\\&\leq\rho(T^4(Ty_n)-y_n)+r\rho((1-\alpha_n)z_n+\alpha_nTz_n)-(1-\alpha_n)v_n-\alpha_nTv_n))\\&\leq\rho(T^4(Ty_n)-y_n)+r\{((1-\alpha_n)\rho(z_n-v_n)+\alpha_n\rho(Tz_n-Tv_n))\\&\leq\rho(T^4(Ty_n)-y_n)+r\{((1-\alpha_n)\rho(z_n-v_n)+r\alpha_n\rho(z_n-v_n))\}\\&\leq\rho(T^4(Ty_n)-y_n)+r(1-\alpha_n(1-r))\rho(z_n-v_n)\\&\leq\rho(T^4(Ty_n)-y_n)+r^2(1-\alpha_n(1-r))(1-\beta_n(1-r))\rho(z_n-u_n).\end{aligned}\tag{3.8}$$

For $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$ and $r\in(0,1)$, we have

$$(1-\beta_n(1-r))<1.\tag{3.9}$$

From equations (3.9) and (3.8) we get

$$\rho(x_{n+1}-u_{n+1})\leq(1-\alpha_n(1-r))\rho(x_n-u_n)+\rho(T^4(t(ty_n)-y_n))\,.\tag{3.10}$$

Define

$$\psi_n=\rho(x_n-u_n),\phi_n=\alpha_n(1-r)\in(0,1)\ \text{ and }\ \varphi_n=\rho(T^4(T(Ty_n)-y_n).$$

Since $\rho(x_n-p)\to 0$ as $n\to\infty$ and $Tp=p$, we get

$$\begin{aligned}\lim_{n\to\infty}\rho(T^4(Ty_n)-y_n)&=\lim_{n\to\infty}\rho(T^4(Ty_n)-Tp+Tp-y_n)\\&=\lim_{n\to\infty}\rho(T^4(Ty_n)-Tp+p-y_n)\\&\leq\lim_{n\to\infty}\rho(T^4(Ty_n)-Tp)+\rho(p-y_n)\\&\leq\lim_{n\to\infty}r\rho(T^3(Ty_n)-p)+\rho(p-y_n)\end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{n\to\infty} r\rho(T^3(Ty_n) - Tp) + \rho(p - y_n) \\
&\leq \lim_{n\to\infty} r^2\rho(T^2(Ty_n) - Tp) + \rho(p - y_n) \\
&\leq \lim_{n\to\infty} r^3\rho(T(Ty_n) - p) + \rho(p - y_n) \\
&\leq \lim_{n\to\infty} r^3\rho(T(Ty_n) - Tp) + \rho(p - y_n) \\
&\leq \lim_{n\to\infty} r^4\rho(T(Ty_n) - p) + \rho(p - y_n) \\
&\leq \lim_{n\to\infty} r^4\rho(T(Ty_n) - Tp) + \rho(p - y_n) \\
&\leq \lim_{n\to\infty} r^5\rho(y_n) - p) + \rho(y_n - p) \\
&= (1 + r^5) \lim_{n\to\infty} \rho(y_n - p) \\
&= 0
\end{aligned}
$$

which implies that $\frac{\psi_n}{\phi_n} \to 0$ as $n \to \infty$. Thus all the conditions Lemma 2.1 are satisfied by (3.10). Thus we get

$$\lim_{n\to\infty} \psi_n = \lim_{n\to\infty} \rho(y_n - u_n) = 0\,. \tag{3.11}$$

Using the above inequality (3.11), we get

$$\rho(u_n - p) \to 0 \;\text{ as }\; n \to \infty \tag{3.12}$$

i.e, the Vatan two step iterative process (2.2) converges to the fixed point p of T.
Now we will prove that (2.2) implies (2.1). Let $\lim_{n\to\infty} \rho(u_n - p) = 0$ and we have to show that $\rho(x_n - p) \to 0$ as $n \to \infty$. Similarly as (3.12) if it is given that $\rho(u_n - p) \to 0$ as $n \to \infty$ then by using (3.11) we have

$$\rho(x_n - p) \leq \rho(x_n - u_n) + \rho(u_n - p) \to 0 \;\text{ as }\; n \to \infty.$$

Hence $x_n \to p$ as $n \to \infty$ that is Vatan two step iteration converges to fixed point p. Thus iteration (2.5) and Vatan two step iteration (2.2) are equivalent and by Theorem 3 of [1], we obtain that (iii) is equivalent to (ii) and (i). ∎

Theorem 3.4. *Let C be a nonempty closed convex subset of a ρ-complete space X_ρ, where ρ is a convex modular on X. Let $T : C \to C$ be a ρ-contraction mapping with fixed point p. For given $x_0^* = x_0 \in C$, let $\{x_n\}_{n=0}^\infty$ and $\{x_n^*\}_{n=0}^\infty$ be an iterative sequences generated by* (2.4) *and* (2.5) *respectively with real sequences $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ in $[0,1]$ such that $\alpha \leq \alpha_n < 1$, for some $\alpha > 0$ and for all $n \in N$. Then $\{x_n^*\}_{n=0}^\infty$ converges to p faster than $\{x_n\}_{n=0}^\infty$ does.*

Proof. From (3.5) we have

$$\rho(x_{n+1}^* - p) \leq \rho(x_0^* - p) r^{7(n+1)} \prod_{j=0}^{n} (1 - \alpha_j(1 - r))\,. \tag{3.13}$$

Since the iteration process (2.4) also converges to unique fixed point p [1, Theorem 1, eq. (9)], we have

$$\|x_{n+1} - p\| \leq \|x_0 - p\| \theta^{3(n+1)} \prod_{k=0}^{n} (1 - \alpha_k(1 - \theta)), \tag{3.14}$$

where $\theta \in (0,1)$. Since modular space is convex, equation (3.14) can be written in the sense of modular space as,

$$\rho(x_{n+1}-p) \leq \rho(x_0-p)r^{3(n+1)}\prod_{j=0}^{n}(1-\alpha_j(1-r)) \tag{3.15}$$

where $r \in (0,1)$. Since $\alpha \leq \alpha_n < 1$ for some $\alpha > 0$ and for all $n \in N$ equation (3.13) implies

$$\begin{aligned}\rho(x^*_{n+1}-p) &\leq \rho(x^*_0-p)r^{7(n+1)}\prod_{j=0}^{n}(1-\alpha(1-r)) \\ &= \rho(x^*_0-p)r^{7(n+1)}(1-\alpha(1-r))^{n+1}\,. \end{aligned} \tag{3.16}$$

Similarly if we assume $\alpha \leq \alpha_n < 1$ for some $\alpha > 0$ and for all $n \in N$, (3.15) becomes

$$\begin{aligned}\rho(x_{n+1}-p) &\leq \rho(x_0-p)r^{3(n+1)}\prod_{j=0}^{n}(1-\alpha(1-r)) \\ &= \rho(x_0-p)r^{3(n+1)}(1-r(1-r))^{n+1}\,. \end{aligned} \tag{3.17}$$

Define

$$a_n = \rho(x^*_0-p)r^{7(n+1)}(1-\alpha(1-r))^{n+1} \text{ and } b_n = \rho(x_0-p)r^{3(n+1)}(1-r(1-r))^{n+1}.$$

Then

$$\begin{aligned}\psi_n &= \frac{a_n}{b_n} \\ &= \frac{\rho(x^*_0-p)r^{7(n+1)}(1-\alpha(1-r))^{n+1}}{\rho(x_0-p)r^{3(n+1)}(1-r(1-r))^{n+1}} \\ &= r^{4(n+1)} \end{aligned} \tag{3.18}$$

Since $\lim_{n\to\infty} = \frac{\psi_{n+1}}{\psi_n} = \frac{r^{n+2}}{r^{4(n+1)}} = r^4 < 1$. So by ratio test $\sum_{n=0}^{\infty}\psi_n < \infty$. Hence from (3.18) we have,

$$\lim_{n\to\infty}\frac{\rho(x^*_{n+1}-p)}{\rho(x_{n+1}-p)} = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\psi_n = 0$$

which implies that $\{x^*_n\}_{n=0}^{\infty}$ converges faster than $\{x_n\}_{n=0}^{\infty}$. ∎

We have following numerical example to support analytical proof of Theorem 3.4 and shows that our iteration process (2.5) gave fast convergence as compared to others given iteration processes.

Example 3.1. *Let $T : [0,2] \to [0,2]$ defined by $T(x) = \frac{x^2+2}{5}$, be any mapping it is easy to see that T is a contraction mapping, moreover in the sence of modular spaces ρ-contraction mapping. Hence T has a unique fixed point* 0.438447187191170.

Table 3.1: Iterative values of our iteration (2.5)*, AK, Vatan two step, Thakur New and Picard-S iterative processes for* $\alpha_n = \beta_n = \frac{1}{3}$*, for all* n *and mapping* $T(x) = \frac{x^2+2}{5}$.

S.no.	Our iteration process (2.5)	A.K.	Vatan two-step	Thakur New	Picard-S
1	3.900000000000000	3.900000000000000	3.900000000000000	3.900000000000000	3.900000000000000
2	0.440970058201144	1.325103726482700	2.150701892967386	2.662130386687799	2.663524110167947
3	0.438447193978863	0.444565152469603	0.580569313396343	0.991131269922821	0.994551007831859
4	0.438447187191188	0.438464652033062	0.441102176663445	0.467156589085184	0.467618231585718
5	**0.438447187191170**	0.438447236728409	0.438490243657771	0.439278210900441	0.439292550132459
6	0.438447187191170	0.438447187331674	0.438447883560470	0.438470429620299	0.438470831507846
7	0.438447187191170	0.438447187191568	0.438447198453332	0.438447836593721	0.438447847823275
8	0.438447187191170	0.438447187191171	0.438447187373309	0.438447205335222	0.438447205648971
9	0.438447187191170	**0.438447187191170**	0.438447187194115	0.438447187698107	0.438447187706873
10	0.438447187191170	0.438447187191170	0.438447187191217	0.438447187205333	0.438447187205578
11	0.438447187191170	0.438447187191170	**0.438447187191170**	0.438447187191566	0.438447187191572
12	0.438447187191170	0.438447187191170	0.438447187191170	0.438447187191181	0.438447187191181
13	0.438447187191170	0.438447187191170	0.438447187191170	**0.438447187191170**	**0.438447187191170**

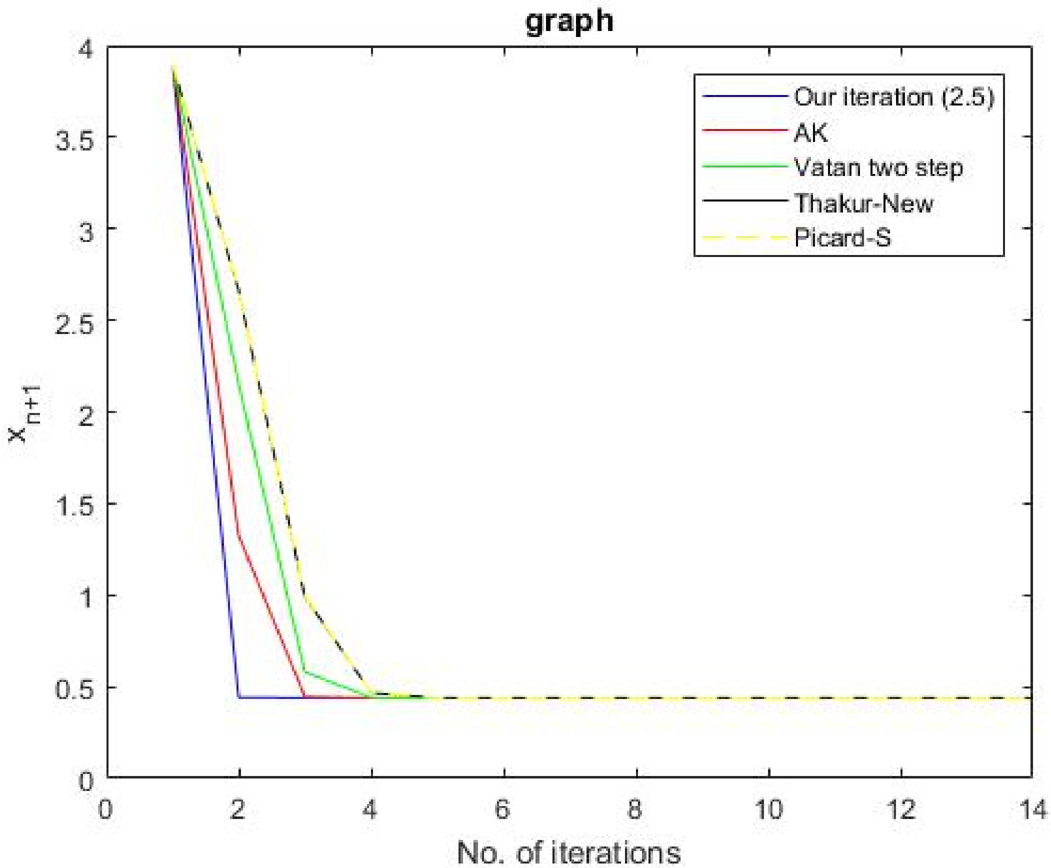

Figure 1: Converge of our iteration iteration process (2.5)*, AK, Vatan two-step, Thakur New and Picard-S iterations to the fixed point 0.43844718 of mapping* $T = \frac{x^2+2}{5}$

It is clear from the above graphical representation that our new ieteration (2.5) are the first covnerging one than the AK, Vatan two-step, Takur new and the Picard-S iterations.

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