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EULER SERIES SOLUTIONS FOR LINEAR INTEGRAL EQUATIONS

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ABSTRACT. In this work, we seek the approximate solution of linear integral equations by truncation Euler series approximation. After substituting the Euler expansions for the given functions of the equation and the unknown one, the equation reduces to a linear system, the solution of this latter gives the Euler coefficients and thereafter the solution of the equation. The convergence and the error analysis of this method are discussed. Finally, we compare our numerical results by others.

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1. INTRODUCTION

The integral equation arises naturally in many physical applications related to wave propagation and vibration phenomena. It is often used to describe the acoustic cavity problem, the scattering of a wave and the radiation wave. Numerical solution of integral equations of Fredholm and Volterra have been investigated by many authors [1, 3, 4]. For methods that use quadrature rule, collocation and interpolation, degenerate kernels, Chebyshev series, Haar series and so on [5, 7, 8]. In this paper, we consider the Euler series approximation for the solution of such integral equations, certainly this technical leads us to the best approximations

$$\varphi(s)-\int_{\Omega}k(s,t)\varphi(t)dt=f(s),\qquad s\in\Omega \tag{1.1}$$

where the functions $f(s)$, and $k(s,t)$ are given and continuous functions in Ω and $\Omega\times\Omega$ respectively, the function $\varphi(t)$ is to be determined as continuous function in Ω.Depending on the domain $\Omega=[a,t]$ or $[a,b]$ the equation (1.1) describes the Volterra integral equation or Fredholm integral equation, respectively.

The equation (1.1) can be put in the form of a linear functional equation

$$\varphi(s)-A\varphi(s)=f(s),\qquad s\in\Omega,$$

with the linear mapping A given by

$$A\varphi(s)=\int_{\Omega}k(s,t)\varphi(t)dt.$$

For the solution of the equation (1.1) in the complete function spaces, usually take it $C(\Omega)$, we choose a sequence of finite dimensional subspaces $V_n,\ n\geq 1$, having n basis functions $\{E_1,\ E_2,\ ...,E_n\}$ with dimension of $V_n=n$.

Seeking the approximate function $\varphi_n\in V_n$ of the function φ given by

$$\varphi_n(s)=\sum_{k=1}^{n}\alpha_kE_k(s), \tag{1.2}$$

where the expression $(refe2)$ describes the truncated Euler series of the solution of the equation (1.1), with the functions $\{E_k\}_{0\leq k\leq n}$ represent the Euler polynomials and $\{\alpha_k\}_{0\leq k\leq n}$the coefficients to be determined. In other words, we can write

$$\begin{aligned}
r_n(s) &= \varphi_n(s)-A_n\varphi(s)-f(s),\\
r_n(s) &= \varphi_n(s)-\int_{\Omega}k(s,t)\varphi_n(t)dt-f(s),\\
&= \sum_{k=1}^{n}\alpha_kE_k(s)-\sum_{k=1}^{n}\alpha_k\int_{\Omega}k(s,t)E_k(t)dt-f(s)\\
&= \sum_{k=1}^{n}\alpha_k\left(E_k(s)-\int_{\Omega}k(s,t)E_k(t)dt\right)-f(s),\quad s\in\Omega.
\end{aligned}$$

2. SOLUTION WITH COLLOCATION METHODS

Choose a selection of distinct points $s_1,\ s_2,.....s_n\in\Omega$ and require

$$r_n(s_j)=0,\quad j=1,2,...,n. \tag{2.1}$$

The condition (2.1) leads us to determine the coefficients $\{\alpha_1,\ \alpha_2,\ ...,\alpha_n\}$ solution of the linear system

$$\sum_{k=1}^{n}\alpha_k\left(E_k(s_j)-\int_{\Omega}k(s_j,t)E_k(t)dt\right)=f(s_j),\ \ j=1,2,...,n. \tag{2.2}$$

Define the matrices

$$E=(E_{kj})=E_k(s_j)$$

and

$$K=(K_{kj})=\int_{\Omega}k(s_j,t)E_k(t)dt.$$

If the $\det(E-K)\neq 0$, we can ensure that, there exists a solution of the linear system (2.2) and consequently the approximate solution $\varphi_n(s)$ as a linear combination

$$\varphi_n(s)=\sum_{k=1}^{n}\alpha_kE_k(s),$$

for which

$$\varphi_n(s_j)-\int_{\Omega}k(s_j,t)\varphi_n(t)dt=f(s_j),\quad j=1,2,...,n.$$

In fact, The linear system may be written in matrix

$$(E-K)\alpha=F, \tag{2.3}$$

where $\alpha=(\alpha_1,\ \alpha_2,\ ...,\alpha_n)^T$ and $F=(f(s_1),\ f(s_2),...,f(s_n))^T$. For the determinant of the system (2.3) is different from zero $\det(E-K)\neq 0$, then it has a unique solution

$$\alpha=(\alpha_1,\ \alpha_2,\ ...,\alpha_n)^T=(E-K)^{-1}F.$$

The corresponding approximate solution

$$\varphi_n(s)=\sum_{k=1}^{n}\alpha_kE_k(s),$$

has the property that its residual $r_n(s)$vanishes at the selected nodes s_j.

Euler polynomials

The nth Euler polynomials $E_n(t)$ is defined by $E_0(t)=1$ and the following recursion

$$E_n(t)=2t^n-\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)E_k(t).$$

Noting that, the Euler polynomial $E_n(t)$ is polynomials with rational coefficients

$$\begin{aligned}
E_0(t) &= 1\\
E_1(t) &= t-\frac{1}{2}\\
E_2(t) &= t^2-t\\
E_3(t) &= t^3-\frac{3}{2}t^2+\frac{1}{4}\\
E_4(t) &= t^4-2t^3+\frac{2}{3}t\\
E_5(t) &= t^5-\frac{5}{2}t^4+\frac{5}{3}t^2-\frac{1}{2}
\end{aligned}$$

and also admits the equalities

$$E_n(t+s) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) E_k(t) s^{n-k} \quad \text{and} \quad E_n(t+1) + E_n(t) = 2t^n .$$

3. ERROR ANALYSIS

Theorem 3.1. *Let $A : C(\Omega) \to C(\Omega)$ be compact operator and the equation*

$$(I - A)\varphi = f, \tag{3.1}$$

admit a unique solution. Assume that the projections P_n :$C(\Omega) \to V_n$ satisfy to $\|P_n A - A\| \to 0,\ n \to \infty$. Then, for sufficiently large n, the approximate equation

$$\varphi_n - P_n A \varphi_n = P_n f, \tag{3.2}$$

has a unique solution for all $f \in C(\Omega)$ and there holds an error estimate

$$\|\varphi - \varphi_n\| \leq M \|\varphi - P_n \varphi\| , \tag{3.3}$$

with some positive constant M depending on A.

Proof. As it is known for all sufficiently large n the inverse operators $(I - P_n A)^{-1}$ exist and are uniformly bounded. To verify the error bound, we apply the projection operator P_n to the equation (3.1) and get

$$P_n \varphi - P_n A \varphi = P_n f,$$

or again

$$\varphi - P_n A \varphi = P_n f + \varphi - P_n \varphi.$$

Subtracting this from (3.2) we find

$$(I - P_n A)(\varphi - \varphi_n) = (I - P_n)\varphi.$$

Hence the estimate (3.3) follows. ∎

4. ILLUSTRATING EXAMPLES

Example 4.1. *Consider the linear integral equation of Volterra*

$$\varphi(s) - \int_0^s (t-s)\varphi(t)dt = 1, \quad 0 \leq x, t \leq 1,$$

where the function $f(s)$ is chosen so that the exact solution is given by

$$\varphi(t) = \cos(t).$$

The approximate solution $\varphi_n(t)$ of $\varphi(t)$ is obtained by the truncated Euler series method.

Values of t	Exact sol φ	Approx sol φ_n	Error	Error [5]
0.000	1.000e+000	1.000e+000	0.000e+000	0.0e+00
0.200	9.800e-001	9.800e-001	2.070e-009	1.8e-04
0.400	9.210e-001	9.210e-001	8.116e-009	8.3e-04
0.600	8.253e-001	8.253e-001	1.765e-008	2.3e-03
0.800	6.967e-001	6.967e-001	2.990e-008	7.8e-03
1.000	5.403e-001	5.403e-001	4.384e-008	5.7e-03

Table 4.1: We present the exact and the approximate solutions of the equation in the Example 4.1 in some arbitrary points, the error for $N = 10$ is calculated and compared with the ones treated in [5].

Example 4.2. *Consider the linear integral equation of Volterra*

$$\varphi(s) - \int_0^s (t-s)\varphi(t)dt = 1 - t - \frac{t^2}{2}, \quad 0 \le x, t \le 1,$$

where the function $f(s)$ is chosen so that the exact solution is given by

$$\varphi(t) = 1 - \sinh(x).$$

The approximate solution $\varphi_n(t)$ of $\varphi(t)$ is obtained by the truncated Euler series method.

Values of t	Exact sol φ	Approx sol φ_n	Error	Error [2]
0.000	0.000e+000	0.000e+000	0.000e+000	0.0e+00
0.200	7.986e-001	7.986e-001	2.809e-008	2.2e-05
0.400	5.892e-001	5.892e-001	5.815e-008	8.3e-05
0.600	3.633e-001	3.633e-001	9.226e-008	1.7e-04
0.800	1.118e-001	1.118e-001	1.327e-007	2.9e-04
1.000	-1.752e-001	-1.752e-001	1.823e-007	3.6e-04

Table 4.2: We present the exact and the approximate solutions of the equation in the Example 4.2 in some arbitrary points, the error for $N = 10$ is calculated and compared with the ones treated in [2].

Example 4.3. *Consider the linear integral equation of Fredholm*

$$\varphi(s) - \int_0^\pi (\cos s - \cos t)\varphi(t)dt = \sin t, \quad 0 \le x, t \le \pi,$$

where the function $f(s)$ is chosen so that the exact solution is given by

$$\varphi(t) = \sin t + \frac{4}{2-\pi^2}\cos t + \frac{2\pi}{2-\pi^2}.$$

The approximate solution $\varphi_n(t)$ of $\varphi(t)$ is obtained by the truncated Euler series method.

Values of t	Exact sol φ	Approx sol φ_n	Error	Error [3]
0.000	-1.306e+000	-1.306e+000	1.758e-004	5.2e-001
0.785	-4.507e-001	-4.508e-001	1.558e-004	2.1e-002
1.570	2.015e-001	2.014e-001	1.074e-004	3.6e-001
2.355	2.681e-001	2.680e-001	5.908e-005	1.1e-001
3.140	-2.901e-001	-2.901e-001	3.904e-005	7.5e-001

Table 4.3: We present the exact and the approximate solutions of the equation in the Example 4.3 in some arbitrary points, the error for $N = 8$ is calculated and compared with the ones treated in [3].

Example 4.4. *Consider the linear integral equation of Fredholm*

$$\varphi(s) - \int_0^1 (\sqrt{t} + \sqrt{s})\varphi(t)dt = 1 + s, \quad 0 \le x, t \le 1,$$

where the function $f(s)$ is chosen so that the exact solution is given by

$$\varphi(t) = -\frac{129}{70} - \frac{141}{35}\sqrt{s} + s.$$

The approximate solution $\varphi_n(t)$ of $\varphi(t)$ is obtained by the truncated Euler series method.

Values of t	Exact sol φ	Approx sol φ_n	Error	Error [3]
0.000	-1.842e+000	-1.851e+000	8.932e-003	6.9e-001
0.250	-3.607e+000	-3.620e+000	1.341e-002	8.2e-002
0.500	-4.191e+000	-4.207e+000	1.597e-002	3.7e-002
0.750	-4.581e+000	-4.599e+000	1.754e-002	6.4e-002
1.000	-4.871e+000	-4.890e+000	1.883e-002	9.5e-002

Table 4.4: We present the exact and the approximate solutions of the equation in the Example 4.4 in some arbitrary points, the error for $N = 10$ is calculated and compared with the ones treated in [3].

Example 4.5. *Consider the linear integral equation of Volterra*

$$\varphi(s) - \int_0^s \left(-16s + 0.1s(s^2 - 3t^2)\right) \varphi(t)dt = 1 + 16s^2 + s^3 + 4s^5 + 0.025s^7, \quad 0 \leq x, t \leq 1,$$

where the function $f(s)$ is chosen so that the exact solution is given by

$$\varphi(t) = 1 + t^3.$$

The approximate solution $\varphi_n(t)$ of $\varphi(t)$ is obtained by the truncated Euler series method.

Values of t	Exact sol φ	Approx sol φ_n	Error	Error [9]
0.000	1.000e+000	1.000e+000	4.884e-015	8.03e-002
0.200	1.008e+000	1.008e+000	7.142e-008	8.03e-002
0.400	1.064e+000	1.064e+000	3.613e-007	8.03e-002
0.600	1.216e+000	1.215e+000	7.083e-007	8.03e-002
0.800	1.512e+000	1.511e+000	9.874e-007	8.03e-002
1.000	2.000e+000	1.999e+000	1.252e-006	8.03e-002

Table 4.5: We present the exact and the approximate solutions of the equation in the Example 4.5 in some arbitrary points, the error for $N = 10$ is calculated and compared with the ones treated in [9].

Example 4.6. *Consider the linear integral equation of Volterra*

$$\varphi(s) - \int_0^s (s - t)\,\varphi(t)dt = 1 + s, \quad 0 \leq x, t \leq 1,$$

where the function $f(s)$ is chosen so that the exact solution is given by

$$\varphi(t) = \exp(t).$$

The approximate solution $\varphi_n(t)$ of $\varphi(t)$ is obtained by the truncated Euler series method.

5. CONCLUSION

A numerical method for solving linear integral equations, based on the truncated Euler series of the solution is presented. If the residual $r_n(s_j) = 0$ for all $j = 1, 2, ..., n$ then the approximate solution $\varphi_n(s)$ will be measurably close to the solution $\varphi(s)$ on the entire interval Ω. The efficiency of this method is tested by solving some examples for which the exact solution is known. This allows us to estimate the exactness with our numerical results and compare those with another results treated by other authors [2, 3, 5, 9].

Values of t	Exact sol φ	Approx sol φ_n	Error	Error [9]
0.000	1.000e+000	1.000e+000	2.220e-016	5.49e-006
0.200	1.221e+000	1.221e+000	3.815e-007	5.49e-006
0.400	1.491e+000	1.491e+000	7.923e-007	5.49e-006
0.600	1.822e+000	1.822e+000	1.344e-006	5.49e-006
0.800	2.225e+000	2.225e+000	2.081e-006	5.49e-006
1.000	2.718e+000	2.718e+000	2.990e-006	5.49e-006

Table 4.6: We present the exact and the approximate solutions of the equation in the Example 4.6 in some arbitrary points, the error for $N = 10$ is calculated.

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