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WEAKLY COMPACT COMPOSITION OPERATORS ON REAL LIPSCHITZ SPACES OF COMPLEX-VALUED FUNCTIONS ON COMPACT METRIC SPACES WITH LIPSCHITZ INVOLUTIONS

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ABSTRACT. We first show that a bounded linear operator T on a real Banach space E is weakly compact if and only if the complex linear operator T' on the complex Banach space $E_{\mathbb{C}}$ is weakly compact, where $E_{\mathbb{C}}$ is a suitable complexification of E and T' is the complex linear operator on $E_{\mathbb{C}}$ associated with T. Next we show that every weakly compact composition operator on real Lipschitz spaces of complex-valued functions on compact metric spaces with Lipschitz involutions is compact.

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1. INTRODUCTION AND PRELIMINARIES

The Symbol $\mathbb{K}$ denotes a field that can be either $\mathbb{R}$ or $\mathbb{C}$. Let E and F be Banach spaces over $\mathbb{K}$. We denote by $B_{\mathbb{K}}(E,F)$ the Banach space over $\mathbb{K}$ consisting of all bounded linear operators from E into F with the operator norm $\|\cdot\|_{op}$. We write $B_{\mathbb{K}}(E)$ instead $B_{\mathbb{K}}(E,E)$. Let us recall that $T\in B_{\mathbb{K}}(E,F)$ is compact (weakly compact, respectively) if the closure of $T(U)$ in F is compact with the norm-topology (weak-topology, respectively), where U is the open unit ball in E.

It is known that if E,F and G are Banach spaces over $\mathbb{K}$, $S\in B_{\mathbb{K}}(E,F)$ and $T\in B_{\mathbb{K}}(F,G)$, then $T\circ S$ is compact (weakly compact, respectively) whenever T or S is compact (weakly compact, respectively).

Applying the Eberlein-Šmulian theorem [4, Theorem V.6.1] and the definition of weakly compact operators between Banach spaces over $\mathbb{K}$, we obtain the following result.

Theorem 1.1. *Let $(E,\|\cdot\|)$ and $(F,\|\cdot\|)$ be Banach spaces and $T:E\longrightarrow F$ be a linear operator from E into F over $\mathbb{K}$. Then T is weakly compact if and only if for each bounded sequence $\{a_n\}_{n=1}^{\infty}$ in $(E,\|\cdot\|)$ there exist a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ and an element b of F such that $\lim_{k\to\infty}Ta_{n_k}=b$ in F with the weak-topology.*

Let X be a nonempty set, $V_{\mathbb{K}}(X)$ be a vector space of $\mathbb{K}$-valued functions on X and $T: V_{\mathbb{K}}(X)\longrightarrow V_{\mathbb{K}}(X)$ be a linear operator on $V_{\mathbb{K}}(X)$ over $\mathbb{K}$. If there exists a self-map $\phi:X\longrightarrow X$ such that $Tf=f\circ\phi$ for all $f\in V_{\mathbb{K}}(X)$, then T is called the *composition operator on $V_{\mathbb{K}}(X)$ induced by ϕ.*

Let X be a topological space. We denote by $C_{\mathbb{K}}(X)$ and $C^b_{\mathbb{K}}(X)$ the set of all $\mathbb{K}$– valued continuous and bounded continuous functions on X, respectively. Then $C_{\mathbb{K}}(X)$ is a commutative algebra over $\mathbb{K}$ with unit 1_X, the constant function on X with value 1, and $C^b_{\mathbb{K}}(X)$ is a subalgebra of $C_{\mathbb{K}}(X)$ containing 1_X. Moreover, $C^b_{\mathbb{K}}(X)$ is a unital commutative Banach algebra over $\mathbb{K}$ with the uniform norm

$$\|f\|_X=\sup\{|f(x)|:x\in X\}\qquad (f\in C^b_{\mathbb{K}}(X)).$$

Clearly, $C^b_{\mathbb{K}}(X)=C_{\mathbb{K}}(X)$ whenever X is compact. We write $C(X)$ and $C^b(X)$ instead of $C_{\mathbb{C}}(X)$ and $C^b_{\mathbb{C}}(X)$, respectively.

Let (X,d) and (Y,ρ) be metric spaces. A map $\phi:X\longrightarrow Y$ is called a *Lipschitz mapping* from (X,d) into (Y,ρ) if there exists a constant $M\geqslant 0$ such that $\rho(\phi(s),\phi(t))\leqslant Md(s,t)$ for all $s,t\in X$. For a map $\phi:X\longrightarrow Y$, the *Lipschitz constant* of ϕ is denoted by $p(\phi)$ and defined by

$$p(\phi)=\sup\left\{\frac{\rho(\phi(s),\phi(t))}{d(s,t)}:s,t\in X, s\neq t\right\}.$$

Clearly a map $\phi:X\longrightarrow Y$ is a Lipschitz mapping if and only if $p(\phi)<\infty$. A map $\phi:X\longrightarrow Y$ is called a *supercontractive mapping* from (X,d) into (Y,ρ) if for each $\varepsilon>0$ there exists a $\delta>0$ such that $\rho(\phi(s),\phi(t))/d(s,t)<\varepsilon$ for all $s,t\in X$ with $0<d(s,t)<\delta$. It is clear that if $\phi:X\longrightarrow Y$ is a supeccontractive mapping from (X,d) into (Y,ρ) such that $\phi(X)$ is bounded set in (Y,ρ), then ϕ is a Lipschitz mapping.

Let (X,d) be a metric space. A function $f:X\longrightarrow\mathbb{K}$ is called a $\mathbb{K}$-valued *Lipschitz function* (*supercontractive function*, respectively) on (X,d) if f is a Lipschitz mapping (supercontractive mapping, respectively) from (X,d) into the Euclidean metric space $\mathbb{K}$. For a Lipschitz function f on (X,d), the *Lipschitz number* of f is denoted by $L_{(X,d)}(f)$ and defined by

$$L_{(X,d)}(f)=\sup\left\{\frac{|f(x)-f(y)|}{d(x,y)}:x,y\in X, x\neq y\right\}.$$

Let (X, d) be a pointed metric space with the base point $e \in X$. We denote by $\mathrm{Lip}_{0,\mathbb{K}}(X, d)$ the set of all $\mathbb{K}$– valued Lipschitz functions f on (X, d) for which $f(e) = 0$. Clearly, $\mathrm{Lip}_{0,\mathbb{K}}(X, d)$ is a linear subspace $C_{\mathbb{K}}(X)$ over $\mathbb{K}$. Moreover, $\mathrm{Lip}_{0,\mathbb{K}}(X, d)$ with the norm $L_{(X,d)}(\cdot)$ is a Banach space over $\mathbb{K}$. We denote by $\mathrm{lip}_{0,\mathbb{K}}(X, d)$ the set of all $f \in \mathrm{Lip}_{0,\mathbb{K}}(X, d)$ for which f is a supercontrative function on (X, d). It is easy to see that $\mathrm{lip}_{0,\mathbb{K}}(X, d)$ is a linear subspace of $\mathrm{Lip}_{0,\mathbb{K}}(X, d)$ and it is a closed set in the Banach space $(\mathrm{Lip}_{0,\mathbb{K}}(X, d), L_{(X,d)}(\cdot))$. Therefore, $(\mathrm{lip}_{0,\mathbb{K}}(X, d), L_{(X,d)}(\cdot))$ is a Banach space over $\mathbb{K}$. Note that if $\phi : X \longrightarrow X$ is a base point-preserving Lipschitz mapping on (X, d), then $f \circ \phi \in \mathrm{Lip}_{0,\mathbb{K}}(X, d)$ ($f \circ \phi \in \mathrm{lip}_{0,\mathbb{K}}(X, d)$, respectively) for all $f \in \mathrm{Lip}_{0,\mathbb{K}}(X, d)$ ($f \in \mathrm{lip}_{0,\mathbb{K}}(X, d)$, respectively). For further general facts about Lipschitz spaces $\mathrm{Lip}_{0,\mathbb{K}}(X, d)$ and little Lipschitz spaces $\mathrm{lip}_{0,\mathbb{K}}(X, d)$, we refer to [14]. We write $\mathrm{Lip}_0(X, d)$ ($\mathrm{lip}_0(X, d)$, respectively) instead of $\mathrm{Lip}_{0,\mathbb{C}}(X, d)$ ($\mathrm{lip}_{0,\mathbb{C}}(X, d)$, respectively). Note that there are $\mathrm{lip}_{0,\mathbb{K}}$ spaces containing only the zero function as for instance, $\mathrm{lip}_{0,\mathbb{K}}([0, 1], d)$ whenever d is the Eucleadian metric on $[0, 1]$.

Let (X, d) be a pointed compact metric space. It is said that $\mathrm{lip}_{0,\mathbb{K}}(X, d)$ *separates points uniformly* on X if there exists a constant $a > 1$ such that, for every $x, y \in X$ there exists $f \in \mathrm{Lip}_{0,\mathbb{K}}(X, d)$ with $L_{(X,d)}(f) \leqslant a$ such that $f(x) = d(x, y)$ and $f(y) = 0$. For instance, if X is the middle-thirds Cantor set in $[0, 1]$ and d is the Euclidcan metric on X, the $\mathrm{lip}_{0,\mathbb{K}}(X, d)$ separates points uniformly on X (See [14, Proposition 3.2.2(a)]).

Let (X, d) be a metric space and $\alpha \in (0, 1]$. We know that the map $d^\alpha : X \times X \longrightarrow \mathbb{R}$ defined by $d^\alpha(x, y) = (d(x, y))^\alpha$, is a metric on X and the generated topology on X by d^α coincides by the generated topology on X by d. It is known [14, Proposition 3.2.2(b)] that $\mathrm{lip}_0(X, d^\alpha)$ separates points uniformly on X whenever (X, d) is a pointed compact metric space and $\alpha \in (0, 1)$.

Let (X, d) be a metric space and $\alpha \in (0, 1]$. We denote by $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ the set of all $\mathbb{K}$-valued bounded Lipschitz functions on (X, d^α). Clearly, $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ is a subalgebra of $C^b(X)$ containing 1_X. Moreover, $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ is a Banach space under the norm

$$\|f\|_{X,L_{(X,d^\alpha)}} = \max\{\|f\|_X, L_{(X,d^\alpha)}(f)\} \quad (f \in \mathrm{Lip}(X, d^\alpha)).$$

Let $\mathrm{lip}_{\mathbb{K}}(X, d^\alpha)$ denote the set of all $\mathbb{K}$-valued supercontractive bounded functions on (X, d^α). Clearly, $\mathrm{lip}_{\mathbb{K}}(X, d^\alpha)$ is a subalgebra of $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ and it is a closed set in the Banach space $(\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha), \cdot\|_{X,L_{(X,d^\alpha)}})$. Hence, $(\mathrm{lip}(X, d^\alpha), \|\cdot\|_{X,L_{(X,d^\alpha)}})$ is a Banach space over $\mathbb{K}$. Moreover, $\mathrm{Lip}(X, d^\beta) \subseteq \mathrm{lip}(X, d^\alpha)$ whenever $0 < \alpha < \beta \leqslant 1$. It is known that $\mathrm{Lip}_{\mathbb{K}}(X, d^1)$ separates the points of X. Lipschitz algebras $\mathrm{Lip}(X, d^\alpha)$ and little Lipschitz algebras $\mathrm{lip}(X, d^\alpha)$ were first introduced by Sherbert in [12] and [13].

Komowitz and Scheinberg in [8] characterized compact composition operators on $\mathrm{Lip}(X, d^\alpha)$ for $\alpha \in (0, 1]$ and $\mathrm{lip}(X, d^\alpha)$ for $(0, 1)$ whenever (X, d) is a compact metric space.

Jim$\acute{e}$nez-Vargas and Villegas-Vallecillos in [7] studied and characterized compact composition operators on $\mathrm{Lip}_0(X, d)$ whenever (X, d) is a pointed metric space not necessarily compact, and on $\mathrm{Lip}(X, d)$ and $\mathrm{lip}(X, d)$ whenever (X, d) is a metric space not necessarily compact.

Jim$\acute{e}$nez-Vargas in [6] studied weakly compact composition operators on $\mathrm{Lip}_{0,\mathbb{K}}(X, d)$ and $\mathrm{lip}_{0,\mathbb{K}}(X, d)$ whenever (X, d) is a pointed compact metric space, and on $\mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ for $\alpha \in (0, 1]$ and $\mathrm{lip}_{\mathbb{K}}(X, d^\alpha)$ for $\alpha \in (0, 1)$ whenever (X, d) is a compact metric space and obtained the following results.

Theorem 1.2 (See [6, Thorem 2.3])**.** *Let (X, d) be a pointed compact metric space, the map $\phi : X \longrightarrow X$ be a base point-preserving Lipschitz mapping on (X, d) and $T : \mathrm{lip}_{0,\mathbb{K}}(X, d) \longrightarrow \mathrm{lip}_{0,\mathbb{K}}(X, d)$ be the composition operator on $\mathrm{lip}_{0,\mathbb{K}}(X, d)$ induced by ϕ. Suppose that the little Lipschitz space $\mathrm{lip}_{0,\mathbb{K}}(X, d)$ separates points uniformly on X. If T is weakly compact, then T is compact.*

Theorem 1.3 (See [6, Corollary 2.4]). *Let (X, d) be a pointed compact metric space, the map $\phi : X \longrightarrow X$ be a base point-preserving Lipschitz mapping on (X, d) and $T : \mathrm{Lip}_{0,\mathbb{K}}(X, d) \longrightarrow \mathrm{Lip}_{0,\mathbb{K}}(X, d)$ be the composition operator on $\mathrm{Lip}_{0,\mathbb{K}}(X, d)$ induced by ϕ. Suppose that the little Lipschitz space $\mathrm{lip}_{0,\mathbb{K}}(X, d)$ separates points uniformly on X. If T is weakly compact, then T is compact.*

Theorem 1.4 (See [6, Remark 2.1]). *Let (X, d) be a compact metric space, $E = \mathrm{Lip}_{\mathbb{K}}(X, d^\alpha)$ for $\alpha \in (0, 1]$ or $E = \mathrm{lip}_{\mathbb{K}}(X, d^\alpha)$ for $\alpha \in (0, 1)$, $\phi : X \longrightarrow X$ be a Lipschitz mapping on (X, d) and $T : E \longrightarrow E$ be the composition operator on E induced by ϕ. If T is weakly compact, then T is compact.*

Let X be a topological space. A self-map $\tau : X \longrightarrow X$ is called a *topological involution* on X if τ is continuous and $\tau(\tau(x)) = x$ for all $x \in X$. Clearly, such τ is a homeomorphism from X onto X.

Let X be a Hausdorff space and τ be a topological involution on X. Then the map $\tau^\star : C^b(X) \longrightarrow C^b(X)$ defined by $\tau^\star(f) = \bar{f} \circ \tau$ is an algebra involution on $C^b(X)$, which is called the *algebra involution on $C^b(X)$ induced by* τ. We now define

$$C(X, \tau) = \{f \in C(X) : \tau^\star(f) = f\},$$

$$C^b(X, \tau) = \{f \in C^b(X) : \tau^\star(f) = f\}.$$

Then $C(X, \tau)$ is a real subalgebra of $C(X), 1_X \in C(X, \tau), i1_X \notin C(X, \tau)$ and $C(X) = C(X, \tau) \oplus iC(X, \tau)$. Moreover $C^b(X, \tau)$ is a unital self– adjoint uniformly closed real subalgebra of $C^b(X), i1_X \notin C^b(X, \tau), C^b(X) = C^b(X, \tau) \oplus iC^b(X, \tau)$ and

$$\max\{\|f\|_X, \|g\|_X\} \leqslant \|f + ig\|_X \leqslant 2 \max\{\|f\|_X, \|g\|_X\}$$

for all $f, g \in C^b(X, \tau)$. Clearly, $C^b(X, \tau) = C(X, \tau)$ if X is compact.

Real Banach algebra $(C(X, \tau), \| \cdot \|_X)$ was defined explicitly by Kulkarni and Limaye in [9], where (X, d) is a compact Hausdorff space and τ is a topological involution on X. For further general facts about $C(X, \tau)$ and certain real subalgebras, we refer to [10].

Let (X, d) be a metric space. A self-map $\tau : X \longrightarrow X$ is called a *Lipschitz involution* on (X, d) if $\tau(\tau(x)) = x$ for all $x \in X$ and τ is a Lipschitz mapping on (X, d).

Note that if τ is a Lipschitz involution on (X, d), then τ is a topological involution on (X, d) and $1 \leqslant p(\tau) < \infty$.

Let (X, d) be a pointed metric space and τ be a base point-preserving Lipschitz involution on (X, d). Then $\tau^\star(\mathrm{Lip}_0(X, d)) = \mathrm{Lip}_0(X, d)$ and $\tau^\star(\mathrm{lip}_0(X, d)) = \mathrm{lip}_0(X, d)$. We now define

$$\mathrm{Lip}_0(X, d, \tau) = \{f \in \mathrm{Lip}_0(X, d) : \tau^\star(f) = f\},$$

$$\mathrm{lip}_0(X, d, \tau) = \{f \in \mathrm{lip}_0(X, d) : \tau^\star(f) = f\}.$$

In fact $\mathrm{Lip}_0(X, d, \tau) = \mathrm{Lip}_0(X, d) \bigcap C(X, \tau)$ and $\mathrm{lip}_0(X, d, \tau) = \mathrm{lip}_0(X, d) \bigcap C(X, \tau)$. The following result is a modification of [2, Theorem 1.3].

Theorem 1.5. *Let (X, d) be a pointed metric space and τ be a base point-preserving Lipschitz involution on (X, d). Suppose that $A = \mathrm{Lip}_0(X, d, \tau)$ and $B = \mathrm{Lip}_0(X, d)$, or, $A = \mathrm{lip}_0(X, d, \tau)$ and $B = \mathrm{lip}_0(X, d)$. Then:*

(i) *A is a self-adjoint real subspace of $C^b(X, \tau)$ and B, $1_X \notin A$ and $i1_X \notin A$.*
(ii) *$B = A \oplus iA$.*
(iii) *For all $f, g \in A$ we have*

$$\begin{aligned}\max\{L_{(X,d)}(f), L_{(X,d)}(g)\} &\leqslant p(\tau) L_{(X,d)}(f + ig) \\ &\leqslant 2p(\tau) \max\{L_{(X,d)}(f), L_{(X,d)}(g)\}.\end{aligned}$$

(iv) *A is closed in $(B, L_{(X,d)}(\cdot))$ and so $(A, L_{(X,d)}(\cdot))$ is a real Banach space.*

(v) *$f \circ \phi \in A$ for all $f \in A$ if $\phi : X \longrightarrow X$ is a base point–preserving Lipschitz mapping on (X, d) with $\tau \circ \phi = \phi \circ \tau$.*

(vi) *If τ is the identity map on X, then $\mathrm{Lip}_0(X, d, \tau) = \mathrm{Lip}_{0,\mathbb{R}}(X, d)$ and $\mathrm{lip}_0(X, d, \tau) = \mathrm{lip}_{0,\mathbb{R}}(X, d)$.*

Let (X, d) be a metric space and the map $\tau : X \longrightarrow X$ be a Lipschitz involution on (X, d). Then $\tau^\star(\mathrm{Lip}(X, d^\alpha)) = \mathrm{Lip}(X, d^\alpha)$ and $\tau^\star(\mathrm{lip}(X, d^\alpha)) = \mathrm{lip}(X, d^\alpha)$ for $\alpha \in (0, 1]$. We now define

$$\mathrm{Lip}(X, d^\alpha, \tau) = \{f \in \mathrm{Lip}(X, d^\alpha) : \tau^\star(f) = f\},$$
$$\mathrm{lip}(X, d^\alpha, \tau) = \{f \in \mathrm{lip}(X, d^\alpha) : \tau^\star(f) = f\}.$$

The following result is a modification of [2, Theorem 1.2].

Theorem 1.6. *Let (X, d) be a metric space and τ be a Lipschitz involution on (X, d). Suppose that $\alpha \in (0, 1]$ and $A = \mathrm{Lip}(X, d^\alpha, \tau)$ and $B = \mathrm{Lip}(X, d^\alpha)$, or, $A = \mathrm{lip}(X, d^\alpha, \tau)$ and $B = \mathrm{lip}(X, d^\alpha)$. Then :*

(i) *A is a real subalgebra of $C^b(X, \tau)$ and B, $1_X \in A$, $i1_X \notin A$.*

(ii) *$B = A \oplus iA$.*

(iii) *For all $f, g \in A$ we have*

$$\begin{aligned}\max\{\|f\|_{X,L_{(X,d^\alpha)}}, \|g\|_{X,L_{(X,d^\alpha)}}\} &\leqslant (p(\tau))^\alpha \, \|f + ig\|_{X,L_{(X,d^\alpha)}} \\ &\leqslant 2\,(p(\tau))^\alpha \max\{\|f\|_{X,L_{(X,d^\alpha)}}, \|g\|_{X,L_{(X,d^\alpha)}}\}.\end{aligned}$$

(iv) *A is closed in $(B, \|\cdot\|_{X,L_{(X,d^\alpha)}})$ and so $(A, \|\cdot\|_{X,L_{(X,d^\alpha)}})$ is a real Banach space.*

(v) *$f \circ \phi \in A$ for all $f \in A$ if $\phi : X \longrightarrow X$ is a Lipschitz mapping on (X, d) with $\phi \circ \tau = \tau \circ \phi$.*

(vi) *If τ is the identity map on X, then $\mathrm{Lip}(X, d^\alpha, \tau) = \mathrm{Lip}_{\mathbb{R}}(X, d^\alpha)$ and $\mathrm{lip}(X, d^\alpha, \tau) = \mathrm{lip}_{\mathbb{R}}(X, d^\alpha)$.*

Real Lipschitz algebras $\mathrm{Lip}(X, d^\alpha, \tau)$ and real little Lipschitz algebras $\mathrm{lip}(X, d^\alpha, \tau)$ were first introduced in [1], whenever (X, d) is a compact metric space. In this case, Ebadian and Ostadbashi characterized compact composition operators on these algebras in [5]. Compact composition operators on $\mathrm{Lip}_0(X, d, \tau)$, $\mathrm{Lip}(X, d, \tau)$ and $\mathrm{lip}(X, d, \tau)$ characterized in [2].

In Section 2, we first show that a bounded linear operator T on a real Banach space E is weakly compact if and only if the complex linear operator T' on the complex Banach space $E_{\mathbb{C}}$ is weakly compact, where $E_{\mathbb{C}}$ is a suitable complification of E and T' is the complex linear operator on $E_{\mathbb{C}}$ associated with T. Next we show that if T is a weakly compact composition operator on real Lipschitz spaces of complex-valued Lipschitz functions $\mathrm{Lip}_0(X, d, \tau)$ and $\mathrm{lip}_0(X, d, \tau)$ on pointed compact metric space (X, d) with Lipschitz involution τ or on real Lipschitz space of complex-valued Lipschitz functions $\mathrm{Lip}(X, d^\alpha, \tau)$ and $\mathrm{lip}(X, d^\alpha, \tau)$ on compact metric spaces (X, d) with Lipschitz involution τ for $\alpha \in (0, 1)$, then T is compact under certain conditions. Finally, we show that the class of weakly compact composition operator on real Lipschitz spaces of complex-valued Lipschitz functions on compact metric spaces with Lipschitz involutions is larger than the class of weakly compact composition operators on complex Lipschitz spaces of complex-valued functions on compact metric spaces.

2. RESULTS

Let E be a real vector space. A complex vector space $E_{\mathbb{C}}$ is called a *complexification* of E if there exists an injective real linear map $J : E \longrightarrow E_{\mathbb{C}}$ such that $E_{\mathbb{C}} = J(E) \oplus iJ(E)$. Clearly,

$E \times E$ with addition and scalar multiplication defined by

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2) \quad (a_1, b_1, a_2, b_2 \in E)$$

$$(\alpha + i\beta)(a, b) = (\alpha a - \beta b, \beta a + \alpha b) \quad (\alpha, \beta \in \mathbb{R}, a, b \in E),$$

is a complexification of E under the injective real linear map $J : E \longrightarrow E \times E$ defined by $J(a) = (a, 0), a \in X$.

Let $(E, \|\cdot\|)$ be a real Banach space. By a similar proof of [3, Proposition I.13.3], one can show that there is a norm $|||\cdot|||$ on $E \times E$ with $|||(a, 0)||| = \|a\|$ for all $a \in E$ such that

$$\max\{\|a\|, \|b\|\} \leqslant |||(a, b)||| \leqslant 2 \max\{\|a\|, \|b\|\}$$

for all $a, b \in E$. Clearly, $(E \times E, |||\cdot|||)$ is a complex Banach space.

Definition 2.1. Let E be a real linear space and $E_{\mathbb{C}}$ be a complexification of E under an injective real linear map $J : E \longrightarrow E_{\mathbb{C}}$. Suppose that $T : E \longrightarrow E$ is a real linear operator on E and the linear map $T' : E_{\mathbb{C}} \longrightarrow E_{\mathbb{C}}$ defines by

$$T'(J(a) + iJ(b)) = J(T(a)) + iJ(T(b)) \quad (a, b \in E).$$

Clearly, T' is a complex linear operator on $E_{\mathbb{C}}$. We say that T' is the *complex linear operator on $E_{\mathbb{C}}$ associated with T*.

For further general facts about the complexifications of real Banach spaces, we refer to [11].

The following result is a modification of [2, Theorem 2.1] and we use it in the sequel.

Theorem 2.1. *Let $(E, \|\cdot\|)$ be a real Banach space and $E_{\mathbb{C}}$ be a complexification of E under an injective real linear map $J : E \longrightarrow E_{\mathbb{C}}$. Suppose that $|||\cdot|||$ is a norm on $E_{\mathbb{C}}$ with $|||J(a)||| = \|a\|$ for all $a \in E$ and there exist positive constants k_1 and k_2 such that*

$$\max\{\|a\|, \|b\| \leqslant k_1 |||J(a) + iJ(b)||| \leqslant k_2 \max\{\|a\|, \|b\|\}\}$$

for all $a, b \in E$. Let $T : E \longrightarrow E$ be a bounded real linear operator on E and $T' : E_{\mathbb{C}} \longrightarrow E_{\mathbb{C}}$ be the complex linear operator on $E_{\mathbb{C}}$ associated with T. Then the following statements hold.

(i) *T' is bounded and $\|T'\|_{op} \leqslant k_1 k_2 \|T\|_{op}$.*
(ii) *T' is compact if and only if T is compact.*

For a Banach space E over $\mathbb{K}$, we denote by $E^\star$ the dual space of E.

The following lemma is a modification [11, Theorem 7] and its proof is straightforward. We will use this lemma in the next theorem.

Lemma 2.2. *Let $(E, \|\cdot\|)$ be a real linear Banach space and $E_{\mathbb{C}}$ be a complexification of E under an injective real linear map $J : E \longrightarrow E_{\mathbb{C}}$. Suppose that $|||\cdot|||$ is a norm on $E_{\mathbb{C}}$ with $|||J(a)||| = \|a\|$ for all $a \in E$ and there exist positive constant k_1 and k_2 such that*

$$\max\{\|a\|, \|b\|\} \leqslant k_1 |||J(a) + iJ(b)||| \leqslant k_2 \max\{\|a\|, \|b\|\}$$

for all $a, b \in E$.

(i) *If for $\lambda, \mu \in E^\star$ the map $\lambda \diamond \mu : E_{\mathbb{C}} \longrightarrow \mathbb{C}$ defines by*

$$(\lambda \diamond \mu)(J(a) + iJ(b)) = (\lambda(a) - \mu(b)) + i(\mu(a) + \lambda(b)) \quad (a, b \in E),$$

then $\lambda \diamond \mu \in (E_{\mathbb{C}})^\star$ and

$$\|\lambda \diamond \mu\|_{op} \leqslant 2k_1(\|\lambda\|_{op} + \|\mu\|_{op}).$$

(ii) *Let $\|\cdot\|_\star$ be a norm on $E^\star \times E^\star$, as a complexification of $E^\star$, with $\|(\lambda, 0)\|_\star = \|\lambda\|_{op}$ for all $\lambda \in E^\star$ and*

$$\max\{\|\lambda\|_{op}, \|\mu\|_{op}\} \leqslant \|(\lambda, \mu)\|_\star \leqslant 2\max\{\|\lambda\|_{op}, \|\mu\|_{op}\}$$

for all $\lambda, \mu \in E^\star$. Then the map $\Psi : E^\star \times E^\star \longrightarrow (E_{\mathbb{C}})^\star$ defined by

$$\Psi(\lambda, \mu) = \lambda \diamond \mu \qquad (\lambda, \mu \in E^\star),$$

is a bijective complex linear operator and continuous from the complex Banach space $(E^\star \times E^\star, \|\cdot\|_\star)$ onto the complex Banach space $((E_{\mathbb{C}})^\star, \|\cdot\|_{op})$. Moreover, Ψ^{-1} is a bounded linear operator from $((E_{\mathbb{C}})^\star, \|\cdot\|_{op})$ to $(E^\star \times E^\star, \|\cdot\|_\star)$.

Theorem 2.3. *Let $(E, \|\cdot\|)$ be a real linear Banach space and $E_{\mathbb{C}}$ be a Complexification of E under an injective real linear map $J : E \longrightarrow E_{\mathbb{C}}$. Suppose that $|||\cdot|||$ is a norm on $E_{\mathbb{C}}$ with $|||J(a)||| = \|a\|$ for all $a \in E$ and there exist positive constants k_1 and k_2 such that*

$$\max\{\|a\|, \|b\|\} \leqslant k_1|||J(a) + iJ(b)||| \leqslant k_2 \max\{\|a\|, \|b\|\}$$

for all $a, b \in E$. Let $T : E \longrightarrow E$ be a bounded real linear operator on E and $T' : E_{\mathbb{C}} \longrightarrow E_{\mathbb{C}}$ be the complex linear operator on $E_{\mathbb{C}}$ associated with T. Then T is a weakly compact operator on the real Banach space $(E, \|\cdot\|)$ if and only if T' is weakly compact operator on the complex Banach space $(E_{\mathbb{C}}, |||\cdot|||)$.

Proof. Let $\|\cdot\|_\star$ be a norm on $E^\star \times E^\star$, as a complexification of $E^\star$, with $|||(\lambda, 0)||| = \|\lambda\|_{op}$ for all $\lambda \in E^\star$ and

$$\max\{\|\lambda\|_{op}, \|\mu\|_{op}\} \leqslant \|(\lambda, \mu)\|_\star \leqslant 2\max\{\|\lambda\|_{op}, \|\mu\|_{op}\}$$

for all $\lambda, \mu \in E^\star$. Define the map $\Psi : E^\star \times E^\star \longrightarrow (E_{\mathbb{C}})^\star$ by

$$\Psi(\lambda, \mu) = \lambda \diamond \mu \qquad (\lambda, \mu \in E^\star),$$

where $\lambda \diamond \mu \in (E_{\mathbb{C}})^\star$ defines by

$$(\lambda \diamond \mu)(J(a) + iJ(b)) = (\lambda(a) - \mu(b)) + i(\mu(a) + \lambda(b)), \qquad (a, b \in E).$$

By Lemma 2.2, Ψ is a bijection complex linear operator and a homeomorphism from the complex Banach space $(E^\star \times E^\star, \|\cdot\|_\star)$ onto the complex Banach space $((E_{\mathbb{C}})^\star, \|\cdot\|_{op})$.

We first assume that T is weakly compact. To prove the weakly compactness of T', let $\{c_n\}_{n=1}^\infty$ be a bounded sequence in $(E_{\mathbb{C}}, |||\cdot|||)$. For each $n \in \mathbb{N}$ there exists $(a_n, b_n) \in E \times E$ such that $c_n = J(a_n) + iJ(b_n)$. It is clear that $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are bounded sequces in $(E, \|\cdot\|)$. Since T is a weakly compact linear operator on $(E, \|\cdot\|)$, by Theorem 1.1, there exist strictly increasing functions $q, r : \mathbb{N} \longrightarrow \mathbb{N}$ and elements $a, b \in E$ such that

$$\lim_{k\to\infty} Ta_{q(k)} = a \qquad (\text{in } E \text{ with the weak-topology}),$$

$$\lim_{k\to\infty} Tb_{r(k)} = b \qquad (\text{in } E \text{ with the weak-topology}).$$

For each $k \in \mathbb{N}$, set $n_k = r(q(k))$. Then $\{a_{n_k}\}_{k=1}^\infty$ is a subsequence of $\{a_n\}_{n=1}^\infty$, $\{b_{n_k}\}_{k=1}^\infty$ is a subsequence of $\{b_n\}_{n=1}^\infty$,

$$\lim_{k\to\infty} Ta_{n_k} = a \qquad (\text{in } E \text{ with the weak-topology}), \tag{2.1}$$

and,

$$\lim_{k\to\infty} Tb_{n_k} = b \qquad (\text{in } E \text{ with the weak-topology}). \tag{2.2}$$

Let $\Lambda \in (E_{\mathbb{C}})^\star$. Then there exist $\lambda, \mu \in E^\star$ such that

$$\Lambda = \Psi(\lambda, \mu) = \lambda \diamond \mu. \tag{2.3}$$

Now we have

$$\lim_{k\to\infty} \lambda(a_{n_k}) = \lambda(a), \tag{2.4}$$

and,

$$\lim_{k\to\infty} \mu(b_{n_k}) = \mu(b), \tag{2.5}$$

by (2.1) and (2.2), respectively. From (2.4), (2.5) and (2.3) we get

$$\lim_{k\to\infty} \Lambda(T'C_{n_k}) = \Lambda(J(a) + iJ(b)). \tag{2.6}$$

Since $(E_{\mathbb{C}})^\star$ separates the point of $E_{\mathbb{C}}$ and (2.6) holds for each $\Lambda \in (E_{\mathbb{C}})^\star$, we conclude that

$$\lim_{k\to\infty} T'(C_{n_k}) = J(a) + iJ(b) \quad \text{(in } E_{\mathbb{C}} \text{ with the weak-topology).} \tag{2.7}$$

This implies that T' is weakly compact operator on $(E_{\mathbb{C}}, |||\cdot|||)$ by Theorem 1.1.

We now assume that T' is a weakly compact operator on $(E_{\mathbb{C}}, |||\cdot|||)$. To prove the weakly compactness of T on $(E, \|\cdot\|)$, let $\{a_n\}_{n=1}^\infty$ be a bounded sequence in $(E, \|\cdot\|)$. It is clear that $\{J(a_n)\}_{n=1}^\infty$ is a bounded sequence in $(E_{\mathbb{C}}, |||\cdot|||)$. Since $T' : E_{\mathbb{C}} \longrightarrow E_{\mathbb{C}}$ is a weakly compact linear operator on $E_{\mathbb{C}}$, by Theorem 1.1, there exist a subsequence $\{a_{n_k}\}_{k=1}^\infty$ of $\{a_n\}_{k=1}^\infty$ and an element $c \in E_{\mathbb{C}}$ such that

$$\lim_{k\to\infty} T'(J(a_{n_k})) = c \quad \text{(in } E_{\mathbb{C}} \text{ with the weak-topology).} \tag{2.8}$$

Since $c \in E_{\mathbb{C}}$, there exists $(a, b) \in E \times E$ such that

$$c = J(a) + iJ(b). \tag{2.9}$$

We claim that

$$\lim_{k\to\infty} T(a_{n_k}) = a \quad \text{(in } E \text{ with the weak-topology).}$$

Let $\lambda \in E^\star$. Set $\Lambda = \Psi(\lambda, 0)$. Then $\Lambda \in (E_{\mathbb{C}})^\star$. Hence, by (2.8) we have

$$\lim_{k\to\infty} \Lambda(T'(J(a_{n_k}))) = \Lambda(c). \tag{2.10}$$

From (2.10) and (2.9), we get

$$\lim_{k\to\infty} (\lambda \diamond 0)(J(Ta_{n_k})) = (\lambda \diamond 0)(J(a) + iJ(b))$$

and so

$$\lim_{k\to\infty} \lambda(Ta_{n_k}) = \lambda(a). \tag{2.11}$$

Since $E^\star$ separates the points of E and (2.11) holds for each $\lambda \in E^\star$, we deduce that

$$\lim_{k\to\infty} T(a_{n_k}) = a \quad \text{(in } E \text{ with the weak-topology).}$$

Therefore, T is weakly compact by Theorem 1.1. ∎

Theorem 2.4. *Let (X, d) be a pointed compact metric space, τ be a base point-preserving Lipschitz involution on (X, d) and $A = \mathrm{Lip}_0(X, d, \tau)$ or $A = \mathrm{lip}_0(X, d, \tau)$. Suppose that the complex little Lipschitz space $\mathrm{lip}_0(X, d)$ separates points uniformly on X. Let $\phi : X \longrightarrow X$ be a base point-preserving Lipschitz mapping on (X, d) with $\tau \circ \phi = \phi \circ \tau$ and $T : A \longrightarrow A$ be the composition operator on A induced by ϕ. If T is weakly compact, then T is compact.*

Proof. We assume that $A_{\mathbb{C}} = \mathrm{Lip}_0(X,d)$ if $A = \mathrm{Lip}_0(X,d,\tau)$ and $A_{\mathbb{C}} = \mathrm{lip}_0(X,d,\tau)$ if $A = \mathrm{Lip}_0(X,d)$. By Theorem 1.5, $A_{\mathbb{C}}$ is a complexification of A under the injective real linear map $J : A \longrightarrow A_{\mathbb{C}}$ defined by $J(f) = f\ (f \in A)$, $(A, L_{(X,d)}(\cdot))$ is a real Banach space and $L_{(X,d)}(\cdot)$ is a norm on the complex vector space $A_{\mathbb{C}}$ with $L_{(X,d)}(J(f)) = L_{(X,d)}(f)$ for all $f \in A$ and

$$\begin{aligned}\max\{L_{(X,d)}(f), L_{(X,d)}(g)\} &\leqslant p(\tau)L_{(X,d)}(J(f)+iJ(g))\\ &\leqslant 2p(\tau)\max\{L_{(X,d)}(f), L_{(X,d)}(g)\}\end{aligned}$$

for all $f, g \in A$. Suppose that T is weakly compact. Let $T' : A_{\mathbb{C}} \longrightarrow A_{\mathbb{C}}$ be the complex linear operator on $A_{\mathbb{C}}$ associated with T. Then T' is weakly compact by Theorem 2.3. It is easy to see that T' is the composition operator on $A_{\mathbb{C}}$ induced by ϕ. Since $\mathrm{lip}_0(X,d)$ separates points uniformly on X, we deduce that T' is compact by Theorem 1.2. This implies that $T : A \longrightarrow A$ is compact by Theorem 2.1. ∎

By part (vi) of Theorem 1.5, it is clear that Theorem 2.4 extends [6, Theorem 2.3] and [6, Corollary 2.4] whenever $\mathbb{K} = \mathbb{R}$.

Theorem 2.5. *Let (X,d) be a compact metric space, τ be a Lipschitz involution on (X,d), $\alpha \in (0,1)$ and $A = \mathrm{Lip}(X,d^\alpha,\tau)$ or $A = \mathrm{lip}(X,d^\alpha,\tau)$. Let $\phi : X \longrightarrow X$ be a Lipschitz mapping on (X,d) with $\tau \circ \phi = \phi \circ \tau$ and $T : A \longrightarrow A$ be the composition operator on A induced by ϕ. If T is weakly compact, then T is compact.*

Proof. We assume that $A_{\mathbb{C}} = \mathrm{Lip(X,d^\alpha)}$ if $A = \mathrm{Lip}(X,d^\alpha,\tau)$ and $A_{\mathbb{C}} = \mathrm{lip(X,d^\alpha)}$ if $A = \mathrm{lip}(X,d^\alpha,\tau)$. By Theorem 1.6, $A_{\mathbb{C}}$ is a complexification of A under the injective real linear map $J : A \longrightarrow A_{\mathbb{C}}$ defined by $J(f) = f \quad (f \in A)$, $(A, \|\cdot\|_{X,L_{(X,d^\alpha)}})$ is a real Banach space and $\|\cdot\|_{X,L_{(X,d^\alpha)}}$ is a norm on the complex vector space $A_{\mathbb{C}}$ with $\|J(f)\|_{X,L_{(X,d^\alpha)}} = \|f\|_{X,L_{(X,d^\alpha)}}$ for all $f \in A$ and

$$\begin{aligned}\max\{\|f\|_{X,L_{(X,d^\alpha)}}, \|g\|_{X,L_{(X,d^\alpha)}}\} &\leqslant (p(\tau))^\alpha \|J(f) + (J(g))\|_{X,L_{(X,d^\alpha)}}\\ &\leqslant 2(p(\tau))^\alpha \max\{\|f\|_{X,L_{(X,d^\alpha)}}, \|g\|_{X,L_{(X,d^\alpha)}}\}\end{aligned}$$

for all $f, g \in A$. Suppose that T is weakly compact. Let $T' : A_{\mathbb{C}} \longrightarrow A_{\mathbb{C}}$ be the complex linear operator on $A_{\mathbb{C}}$ associated with T.Then T' is weakly compact by Theorem 2.3. It is easy to see that T' is the composition operator on $A_{\mathbb{C}}$ induced by ϕ. By Theorem 1.4, T' is compact. This implies that T is compact by Theorem 2.1. ∎

By part (vi) of Theorem 1.6, it is clear that Theorem 2.5 extends Theorem 1.4 in the case $\mathbb{K} = \mathbb{R}$.

Now, we show that the class of weakly compact composition operators on real Lipschitz spaces of complex-valued functions on compact metric spaces with Lipschitz involutions is larger than the class of complex linear operators on complex Lipschitz spaces of complex-valued functions on compact metric spaces.

Theorem 2.6. *Let (X,d) be a compact metric space, $B = \mathrm{Lip}(X,d^\alpha)$ for $\alpha \in (0,1]$ or $B = \mathrm{lip}(X,d^\alpha)$ for $\alpha \in (0,1]$ and $T : B \longrightarrow B$ be a unital complex endomorphism of B induced by the Lipschitz mapping ϕ on (X,d). Let $Y = X \times \{0,1\}$, ρ be the metric on Y defined by $\rho((x_1,y_1),(x_2,y_2)) = \max\{d(x_1,x_2), |y_1 - y_2|\}$ and $\tau : Y \longrightarrow Y$ be the Lipschitz involution on (Y,ρ) defined by*

$$\tau(x,0) = (x,1), \quad \tau(x,1) = (x,0) \quad (x \in X).$$

Suppose that $A = \mathrm{Lip}(Y,\rho^\alpha,\tau)$ if $B = \mathrm{Lip}(X,d^\alpha)$ and $A = \mathrm{lip}(Y,\rho^\star,\tau)$ if $B = \mathrm{lip}(X,d^\alpha)$. Let $\phi : Y \longrightarrow Y$ be the self-map of Y defined by

$$\psi(x,0) = (\phi(x),0), \quad \psi(x,1) = (\phi(x),1) \quad (x \in X).$$

Then the following statements hold.

(i) ψ *is a Lipschitz involution on* (Y, ρ) *and* $\psi \circ \tau = \tau \circ \psi$.

(ii) *If* $S : A \longrightarrow A$ *is the composition endomorphism of* A *induced by* ψ, *then* S *is weakly compact if and only if* T *is weakly compact.*

Proof. Clearly, (i) holds. We prove (ii) in the case $B = \mathrm{Lip}(X, d^\alpha)$ and $A = \mathrm{Lip}(Y, \rho^\alpha, \tau)$ for $\alpha \in (0, 1]$. Define the map $\Lambda : B \longrightarrow A$ by

$$(\Lambda f)(x, 0) = f(x) \quad (f \in B, x \in X),$$
$$(\Lambda f)(x, 1) = \overline{f(x)} \quad (f \in B, x \in X).$$

Then Λ is an injective bounded real linear operator from $(B, \|\cdot\|_{L_{(X,d^\alpha)}})$, regarded as a real Banach algebra, onto $(A, \|\cdot\|_{Y,L_{(Y,\rho^\alpha)}})$. By open mapping theorem for real Banach spaces, Λ^{-1} is a bounded linear operator from $(A, \|\cdot\|_{Y,L_{(Y,\rho^\alpha)}})$ into $(B, \|\cdot\|_{X,L_{(X,d^\alpha)}})$. We can easily show that $\Lambda \circ T \circ \Lambda^{-1} = S$. Therefore, S is weakly compact if only if T is weakly compact.

To prove (ii) in the case $B = \mathrm{lip}(X, d^\alpha)$ and $A = \mathrm{lip}(Y, \rho^\alpha, \tau)$ for $\alpha \in (0, 1)$, it is sufficient that we apply $\Gamma = \Lambda|_{\mathrm{lip}(X,d^\alpha)}$ instead of Λ. ∎

According to Theorem 2.5, we deduce that Theorem 2.6 extends Theorem 1.4 whenever $K = \mathbb{C}$.

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