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STRONG CONVERGENCE THEOREMS FOR A COMMON ZERO OF AN INFINITE FAMILY OF GAMMA-INVERSE STRONGLY MONOTONE MAPS WITH APPLICATIONS

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ABSTRACT. Let E be a uniformly convex and uniformly smooth real Banach space with dual space E^* and let $A_k : E \to E^*, k = 1, 2, 3, ...$ be a family of inverse strongly monotone maps such that $\cap_{k=1}^{\infty} A_k^{-1}(0) \neq \emptyset$. A new iterative algorithm is constructed and proved to converge strongly to a common zero of the family. As a consequence of this result, a strong convergence theorem for approximating a common J-fixed point for an infinite family of gamma-strictly J-pseudocontractive maps is proved. These results are new and improve recent results obtained for these classes of nonlinear maps. Furthermore, the technique of proof is of independent interest.

Key words and phrases: J-Fixed points, J-Pseudocontractive mapping, Family of γ-inverse strongly monotone mappings, strong convergence.

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1. INTRODUCTION

Let E be a real normed space with dual space E^*. *The normalized duality map* is the map $J : E \to 2^{E^*}$ defined for all $x \in E$ by

$$Jx := \{g^* \in E^* : \langle x, g^* \rangle = \|x\|.\|g^*\|,\ \|x\| = \|g^*\|\},$$

where $\langle .,. \rangle$ denotes the generalized duality pairing between elements of E and E^*. It known that if E is strictly convex, J is injective. If, in addition E is reflexive and smooth, then the inverse of $J,\ J^{-1} : E^* \to E$ exists. Several other properties of the normalized duality map abound in the literature (see e.g., Alber [1], Cioranescu [26]). A map $A : E \to E^*$ is called *monotone* if for each $x, y \in E$, the following inequality holds:

$$\langle Ax - Ay, x - y \rangle \geq 0. \tag{1.1}$$

It is called *maximal monotone* if, in addition, the graph of A is not properly contained in the graph of any other monotone map. Also, A is called γ*-inverse strongly monotone* if for all $x, y \in E$, there exists $\gamma > 0$ such that the following inequality holds:

$$\langle Ax - Ay, x - y \rangle \geq \gamma ||Ax - Ay||^2.$$

It is easy to see that every γ*-inverse strongly monotone map* is *Lipschitz* with Lipschitz constant $\dfrac{1}{\gamma}$, where a map T with domain $D(T)$ in a normed space X, and range $R(T)$ in a normed space Y is called *Lipschitz* with *Lipschitz constant* L if for all $x, y \in D(T)$, there exists $L > 0$ such that the following inequality holds:

$$||Tx - Ty|| \leq L||x - y||.$$

Monotone maps were first studied in Hilbert spaces by Zarantonello [22], Minty [17], Kačurovskii [12] and a host of other authors. Interest in such maps stems mainly from their usefulness in several applications. In particular, monotone maps appear in convex optimization theory. Consider, for example, the following.

Let H be a real Hilbert space and $g : H \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The *subdifferential* of g, $\partial g : H \to 2^H$, is defined for each $x \in H$ by

$$\partial g(x) = \{x^* \in H : g(y) - g(x) \geq \langle y - x, x^* \rangle \ \forall\ y \in H\}.$$

It is easy to check that ∂g is a *monotone operator* on H, and that $0 \in \partial g(u)$ *if and only if* u *is a minimizer of* g. Setting $\partial g \equiv A$, it follows that solving the inclusion $0 \in Au$, in this case, is solving for a minimizer of g. A map $A : E \to E$ is called *accretive* if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$, such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0. \tag{1.2}$$

A is called *m-accretive* if, in addition, the graph of A is not properly contained in the graph of any other accretive operator. It is known that A is m*-accretive* if and only if it is accretive and $R(I + tA) = E$ for all $t > 0$, where $R(I + tA)$ denotes the range of $(I + tA)$. In a real Hilbert space, the normalized duality map is the identity map, and so, in this case, inequality (1.2) and inequality (1.1) coincide. Hence, in *Hilbert spaces, accretivity and monotonicity are equivalent*.

Accretive maps were introduced independently in 1967 by Browder [6] and Kato [13]. Interest in such maps stems mainly from their firm connection with the existence theory for nonlinear equations of evolution in real Banach spaces. Furthermore, it is known (see e.g., Zeidler [23]) that many physically significant problems can be modelled in terms of an initial-value problem of the form

$$\frac{du}{dt} + Au = 0, \quad u(0) = u_0, \tag{1.3}$$

where A is an accretive map on an appropriate real Banach space. Typical examples of such evolution equations are found in models involving the heat, wave or Schrödinger equations (see e.g., Browder [7], Zeidler [23]). Observe that in the model (1.3), if the solution u is independent of time (i.e., at the equilibrium state of the system), then $\frac{du}{dt} = 0$ and (1.3) reduces to

$$Au = 0, \tag{1.4}$$

whose solutions then correspond to the equilibrium state of the system described by (1.3). Solutions of equation (1.4) when A is accretive can also represent solutions of partial differential equations (see e.g., Benilan, Crandall and Pazy [4], Khatibzadeh and Morosanu [14], Khatibzadeh and Shokri [15], Showalter [19], Volpert [20], and so on). In studying the equation $Au = 0$ where A is an accretive operator on a Hilbert space H, Browder [6] introduced an operator T defined by $T := I - A$ where I is the identity map on H. He called such an operator *pseudocontractive*. It is clear that solutions of $Au = 0$, if they exist, correspond to fixed points of T. Examples of pseudocontractive maps include *nonexpansive maps*, where a map $T : D(T) \subset E \to E$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\| \; \forall \, , x, y \in \; D(T)$.

Within the past 40 years or so, methods for approximating solutions of equation (1.4) when A is an *accretive-type* operator have become a flourishing area of research for numerous mathematicians. Numerous convergence theorems have been published in various Banach spaces and under various continuity assumptions on the operator A. Many important theorems have been proved, thanks to geometric properties of Banach spaces developed from the mid 1980s to the early 1990s. The theory of approximation of solutions of equation (1.4) when A is of the *accretive-type* reached a level of maturity appropriate for an examination of its central themes. This resulted in the publication of several monographs which presented in-depth coverage of the main ideas, concepts and most important theorems on iterative algorithms for approximation of *fixed points of nonexpansive* and *pseudocontractive maps* and their *generalisations*, approximation of *zeros of accretive-type operators*; iterative algorithms for solutions of Hammerstein integral equations involving *accretive-type maps*; iterative approximation of *common fixed points* (and *common zeros*) of families of these maps; solutions of equilibrium problems; and so on (see e.g., Agarwal *et al.* [2], Berinde [5], Chidume [9], Kartsatos [18], Censor and Reich [8], William and Shahzad [21] and the references contained in them). Typical theorems published are the following.

Theorem 1.1 (Chidume, [11])**.** *Let E be a uniformly smooth real Banach space with modulus of smoothness ρ_E, and let $A : E \to 2^E$ be a multi-valued bounded m-accretive operator with $D(A) = E$ such that the inclusion $0 \in Au$ has a solution. For arbitrary $x_1 \in E$, define a sequence $\{x_n\}$ by,*

$$x_{n+1} = x_n - \lambda_n u_n - \lambda_n \theta_n (x_n - x_1), \; u_n \in Ax_n, \; n \geq 1,$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0, 1)$ satisfying appropriate conditions and there exists a constant $\gamma_0 > 0$ such that $\frac{\rho_E(\lambda_n)}{\lambda_n} \leq \gamma_0 \theta_n$. Then, the sequence $\{x_n\}$ converges strongly to a zero of A.

Theorem 1.2 (Ofoedu, [28])**.** *Let C be a closed convex nonempty subset of a reflexive and strictly convex real Banach space E which has a uniformly Gâteaux differentiable norm. Let $A_k : C \to E, \; k \in \mathbb{N}$, be a countable infinite family of m-accretive maps such that $\cap_{k=1}^{\infty} A_k^{-1}(0) \neq$*

$\emptyset$. Suppose that every bounded closed convex nonempty subset of C has the fixed point property for nonexpansive maps. For arbitrary $u, x_1 \in C$, let $\{x_n\}$ be iteratively generated by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S x_n, \ n \geq 1,$$

where $S = \sum_{k=1}^{\infty} \varepsilon_k J_{A_k}$; $J_{A_k} = (I + A_k)^{-1}, k \in \mathbb{N}$. Then, $\{x_n\}_{n \geq 1}$ converges strongly to a common zero of $\{A_k\}_{k \geq 1}$.

From the foregoing, it is clear that in real Banach spaces more general than Hilbert spaces, much has been done on the approximation of solutions of equation (1.4) when A is of *accretive-type*. However, little has been done in the case where the operator A is of the *monotone-type*. This is perhaps, because of the following two challenges.

First, most of the *inequalities* developed for proving convergence results for iterative schemes for zero of *accretive-type maps* are not directly applicable in the case of *monotone-type maps* as they involve the generalized duality maps, whereas the definition of monotone-type maps does not involve the generalized duality maps. Secondly, *fixed point technique* introduced by Browder is not readily applicable here because A maps a Banach space E into *another* Banach space E^*; thus the usual notion of fixed point does not make sense here.

However, with intensive research efforts, these challenges are gradually being overcome. Recently, Alber [1] (see also, Alber and Ryazantseva [27]) introduced a Lyapunov functional defined on real normed spaces which turns out to be very useful in developing inequalities that are applicable in iterative approximation of solutions of equation (1.4) when A is of *monotone-type* (see e.g., Aoyama *et al.* [31], Kamimura *et al.* [32], Kamimura and Takahashi [33], Zegeye and Shahzad [34], Chidume *et al.* ([29] , [30]), Zegeye [35]). A typical example of these results is contained in the following theorem of Chidume *et al.*[29].

Theorem 1.3 (Chidume *et al.*[29])**.** *Let E be a uniformly convex and uniformly smooth real Banach space and let E^* be its dual. Let $A : E \to E^*$ be a generalized Φ-strongly monotone and bounded map with $A^{-1}(0) \neq \emptyset$. For arbitrary $u_1 \in E$, define a sequence $\{u_n\}$ iteratively by:*

$$u_{n+1} = J^{-1}(Ju_n - \lambda_n A u_n), \ n \geq 1,$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$ satisfying certain conditions. Then, the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to u^*, a solution of $Au = 0$.

Furthermore, a new notion of J-*fixed points* (see Chidume and Idu [24]) also called *duality fixed point* (see Liu [25]) or *semi-fixed point* (see Zegeye [34]) and J-*pseudocontractions* (see Chidume and Idu [24]) recently introduced, turns out to be very useful for approximating solutions of equation (1.4) when A is of *monotone-type*. For instance, Chidume and Idu [24] showed that a map $T : E \to E^*$ is J-pseudocontractive if and only if $A := J - T$ is monotone and hat $x^* \in E$ is a J-fixed point of T if and only if $Ax^* = 0$, where E is a smooth real Banach space with dual space E^*. They employed this technique and proved the following strong convergence theorems.

Theorem 1.4 (Chidume and Idu [24])**.** *Let E be a uniformly convex and uniformly smooth real Banach space and let E^* be its dual. Let $T : E \to 2^{E^*}$ be a J-pseudocontractive and bounded map such that $(J - T)$ is maximal monotone. Suppose $F_E^J(T) := \{v \in E : Jv \in Tv\} \neq \emptyset$. For arbitrary $x_1, \ u \in E$, define a sequence $\{x_n\}$ iteratively by:*

$$x_{n+1} = J^{-1}\left[(1 - \lambda_n)Jx_n + \lambda_n \eta_n - \lambda_n \theta_n (Jx_n - Ju)\right], \ \eta_n \in Tx_n, \ n \geq 1, \quad (1.5)$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying appropriate conditions. Then, the sequence $\{x_n\}$ converges strongly to a J-fixed point of T.

Theorem 1.5 (Chidume and Idu [24])**.** *Let E be a uniformly convex and uniformly smooth real Banach space and let E^* be its dual. Let $A : E \to 2^{E^*}$ be a multi-valued maximal monotone and bounded map such that $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1,\, u \in E$, define a sequence $\{x_n\}$ iteratively by:*

$$x_{n+1} = J^{-1}\left[Jx_n - \lambda_n\eta_n - \lambda_n\theta_n(Jx_n - Ju)\right],\ \eta_n \in Ax_n,\ n \geq 1, \tag{1.6}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are in $(0,1)$ and satisfying certain conditions. Then, the sequence $\{x_n\}$ converges strongly to a zero of A.

In this paper, it is our purpose to apply theorem1.4 and approximate a common zero of an infinite family of an important class of monotone maps; the class of γ-inverse strongly monotone maps defined on a uniformly smooth and uniformly convex real Banach space. Our results are new and improve some recent important results for this class of nonlinear operators.

2. PRELIMINARIES

Let E be a real normed space of dimension ≥ 2. The *modulus of smoothness* of E , $\rho_E : [0,\infty) \to [0,\infty)$ is defined by:

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau,\ \tau > 0\right\}.$$

It known that in a smooth space E, $\rho_E(\tau) \leq \tau$ for all $\tau \geq 0$. A normed space E is called *uniformly smooth* if $\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0$. It is well known (see *e.g.*, Chidume [9] p. 16, also Lindenstrauss and Tzafriri[16]) that ρ_E is nondecreasing. If there exist a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be q-*uniformly smooth*. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p\ (or\ l_p)\ or\ W_p^m \text{ is } \left\{\begin{array}{ll} 2-\text{uniformly smooth} & \text{if } 2 \leq p < \infty; \\ p-\text{uniformly smooth} & \text{if } 1 < p < 2. \end{array}\right.$$

A Banach space E is said to be *strictly convex* if $\|x\| = \|y\| = 1, \quad x \neq y \implies \left\|\frac{x+y}{2}\right\| < 1$. The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1; \epsilon = \|x-y\|\right\}.$$

The space E is *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0,2]$. It is also well known (see *e.g.*, Chidume [9] p. 34, Lindenstrauss and Tzafriri [16]) that δ_E is nondecreasing. If there exist a constant $c > 0$ and a real number $p > 1$ such that $\delta_E(\epsilon) \geq c\epsilon^p$, then E is said to be p-*uniformly convex*. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p\ (or\ l_p)\ or\ W_p^m \text{ is } \left\{\begin{array}{ll} p-\text{uniformly convex} & \text{if } 2 \leq p < \infty; \\ 2-\text{uniformly convex} & \text{if } 1 < p < 2. \end{array}\right.$$

We now present the following definitions and remarks which will be used in the sequel.

Definition 2.1. Let E be a normed space. A map $T : E \to E$ is called *strictly pseudocontractive in the terminology of Browder and Petryshyn* if there exists $\gamma > 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq ||x-y||^2 - \gamma||x - y - (Tx - Ty)||^2 \tag{2.1}$$

for all $x, y \in E$ and for some $j(x-y) \in J(x-y)$. If $E = H$, a real Hilbert space, $T : E \to E$ is called *strictly pseudocontractive* if

$$||Tx - Ty||^2 \leq ||x-y||^2 + k||x - y - (Tx - Ty)||^2 \tag{2.2}$$

holds for all $x, y \in E$ and for some $k \in (0, 1)$.

Definition 2.2 (Chidume and Idu,[24])**.** Let E be a normed space. A map $T : E \to 2^{E^*}$ is called J*-pseudocontractive* if for every $x,\ y \in E$,

$$\langle \tau - \zeta, x - y\rangle \leq \langle \eta - \nu, x - y\rangle \ \text{ for all } \tau \in Tx,\ \zeta \in Ty,\ \eta \in Jx,\ \nu \in Jy.$$

If E is a smooth real Banach space, then $T : E \to E^*$ is J-pseudocontractive if for every $x, y \in E$ the following inequality holds:

$$\langle Tx - Ty, x - y\rangle \leq \langle Jx - Jy, x - y\rangle.$$

Definition 2.3 (Chidume *et al.* [10])**.** Let E be a real normed space and E^* be its dual. A map $T : E \to 2^{E^*}$ will be called *strictly* J*-pseudocontractive* if for every $x, y \in E,\ \tau_x \in Tx,\ \tau_y \in Ty,\ jx \in Jx,\ jy \in Jy$, there exists $\gamma > 0$ such that the following inequality is satisfied:

$$\langle \tau_x - \tau_y, x - y\rangle \leq \langle jx - jy, x - y\rangle - \gamma||(jx - \tau_x) - (jy - \tau_y)||^2.$$

Also, if E is a smooth real Banach space, $T : E \to E^*$ is strictly J-pseudocontraction if for every $x, y \in E$, there exists $\gamma > 0$ such that the following inequality holds:

$$\langle Tx - Ty, x - y\rangle \leq \langle Jx - Jy, x - y\rangle - \gamma||(J - T)x - (J - T)y||^2. \tag{2.3}$$

In this paper, strictly J-pseudocontractive maps shall be called γ-strictly J-pseudocontractive maps.

Definition 2.4 (Chidume and Idu [24])**.** Let E be an arbitrary normed space and E^* be its dual. Let $T : E \to 2^{E^*}$ be any map. A point $x \in E$ is called a J*-fixed point* of T if and only if there exists $\eta \in Tx$ such that $\eta \in Jx$

From the above definitions, we have that if E is a smooth real Banach space, $x \in E$ is a J-fixed point of $T : E \to E^*$ if and only if $Tx = Jx$. Observe that if $E = H$, a real Hilbert space, then the notions of fixed point and J-fixed point are equivalent, where a point x in the domain of a self-map T in a normed space is a fixed point of T if and only if $Tx = x$.

Remark 2.1. Let E be a smooth real Banach space, J be the normalized duality map on E and $T : E \to E^*$ be any map. It is easy to see that T is J-pseudocontractive if and only if $A := (J-T)$ is monotone. Also, T is γ-strictly J-pseudocontractive if and only if $A := (J-T)$ s γ-inverse strongly monotone. In both cases, it is also easy to see that $x \in E$ is a J-fixed point of T if and only if x is a zero of A; where $p \in E$ is a zero of A if and only if $Ap = 0$.

Remark 2.2. Observe that the class of γ-strictly J-pseudocontractive maps is a proper subclass of the class of J-pseudocontractive maps.

We shall denote the set of fixed point of $T : E \to E$ by $F(T)$, the set of J-fixed points of $T : E \to E^*$ by $F_J(T)$ and the set of zeros of $A : E \to E^*$ by $A^{-1}(0)$. We shall also denote the domain and range of a map A, by $D(A)$ and $R(A)$, respectively.

In the sequel, the following important lemma will be used.

Lemma 2.1 (Cioranescu [26], corrollary 2.7 pg 156)**.** *Let E be a real Banach space and E^* be its dual space. Let $A : E \to E^*$ be a monotone and semicontinuous map with $D(A) = E$. Then, A is maximal monotone.*

3. MAIN RESULTS

We shall make use of the following lemmas in what follows.

Lemma 3.1. *Let E be a smooth real Banach space with dual space E^* and let J be the normalized duality map of E. Let $A_k : E \to E^*, k = 1, 2, 3, ...$ be an infinite family of γ_k-inverse srtongly monotone maps such that $\gamma := \inf_{k\geq 1} \gamma_k > 0$ and $\cap_{k=1}^{\infty} A_k^{-1}(0) \neq \emptyset$. Define $T_k := J - A_k$ and $S := \sum\limits_{k=1}^{\infty} \varepsilon_k T_k$, where $\{\varepsilon_k\}_{k=1}^{\infty}$ is a sequence of positive real numbers satisfying $\sum_{n=1}^{\infty} \varepsilon_k = 1$. Then,*
(i) T_k is J-pseudocontractive for each $k = 1, 2, ...$,
(ii) S is well defined,
(iii) S is J-pseudocontractive,
(iv) S is bounded on bounded subsets of E,
(v) $F_J(S) = \cap_{k=1}^{\infty} F_J(T_k) = \cap_{k=1}^{\infty} A_k^{-1}(0)$.

Proof. (i) This follows from the fact that A_k is monotone for each $k = 1, 2, ...$. (ii) Let $x \in E$ and $x^* \in \cap_{k=1}^{\infty} A_k^{-1}(0)$. Then, by the Lipschitz property of A_k for each $k \geq 1$, we have

$$||\varepsilon_k T_k x|| = ||\varepsilon_k (J - A_k)x|| \leq |x|| + ||A_k x - A_k x^*|| \leq ||x|| + \gamma^{-1} ||x - x^*||. \tag{3.1}$$

This implies that $\sum\limits_{k=1}^{\infty} \varepsilon_k T_k x$ is absolutely convergent for each $x \in E$. Thus, S is well defined.

(iii) Let $x, y \in E$. Since T_k is J-pseudocontractive for each $k \geq 1$ and $\sum\limits_{k=1}^{\infty} \varepsilon_k = 1$, we have that

$$\begin{aligned} \langle Sx - Sy, x - y \rangle &= \sum_{k=1}^{\infty} \varepsilon_k \langle T_k x - T_k y, x - y \rangle \\ &\leq \langle Jx - Jy, x - y \rangle. \end{aligned}$$

Therefore, S is J-pseudocontractive.
(iv) Let B be any bounded subset of E and let $x \in B$, $p \in \cap_{k=1}^{\infty} A_k^{-1}(0)$. Then, using the fact that A_k is $\frac{1}{\gamma_k}$-Lipschitz, there exists $M > 0$ such that

$$\begin{aligned} ||Sx|| &= \Big|\Big| \sum_{k=1}^{\infty} \varepsilon_k (J - A_k)x \Big|\Big| \\ &\leq ||Jx|| + \sum_{k=1}^{\infty} \varepsilon_k ||A_k x - A_k p|| \\ &\leq ||x|| + \frac{1}{\gamma} ||x - p|| \leq M. \end{aligned}$$

Therefore, S is bounded on bounded sets. (v) It is trivial to see that $\cap_{k=1}^{\infty} F_J(T_k) = \cap_{k=1}^{\infty} A_k^{-1}(0)$ and that $\cap_{k=1}^{\infty} F_J(T_k) \subset F_J(S)$. Now, let $x^* \in F_J(S)$. Let $p \in \cap_{k=1}^{\infty} A_k^{-1}(0)$. Then, $Sx^* = Jx^*$. Using the definition of S and T_k, we have

$$0 = \langle Jx^* - Sx^*, x^* - p \rangle = \sum_{k=1}^{\infty} \varepsilon_k \langle A_k x^* - A_k p, x^* - p \rangle. \tag{3.2}$$

Since A_k is monotone for each $k \geq 1$ and $\sum\limits_{k=1}^{\infty} \varepsilon_k = 1$, we obtain from equation (3.2) that

$$\langle A_k x^* - A_k p, x^* - p \rangle = 0 \quad \forall\, k \geq 1.$$

Since A_k is γ_k-inverse strongly monotone for each $k \geq 1$, we obtain that $\forall\ k \geq 1$,

$$0 = \langle A_k x^* - A_k p, x^* - p\rangle \geq \gamma_k ||A_k x^* - A_k p||^2 \geq \gamma ||A_k x^*||^2.$$

This implies $A_k x^* = 0 \quad \forall\ k \geq 1$. Thus, $x^* \in \cap_{k=1}^{\infty} A_k^{-1}(0) = \cap_{k=1}^{\infty} F_J(T_k)$ and this completes the proof of the lemma. ∎

Lemma 3.2. *Let E be a smooth real Banach space and E^* be its dual. Let $A_k : D(A_k) = E \to E^*, k = 1, 2, 3, ...$ be an infinite family of γ_k-inverse strongly monotone maps. Then, $(J - S)$ is maximal monotone, where $S = \sum_{k=1}^{\infty} \varepsilon_k T_k,\ T_k = J - A_k$ and $\{\varepsilon_k\}$ is a positive sequence of real numbers satisfying $\sum_{k=1}^{\infty} \varepsilon_k = 1$.*

Proof. By (iii) of lemma 3.1, S is J-pseudocontractive. Thus, by remark 2.1, $(J - S)$ is monotone. Clearly, $(J - S)$ is continuous and is defined on the whole of E. Therefore, by lemma 2.1, $(J - S)$ is maximal monotone. ∎

In what follows, the sequences $\{\lambda_n\}$ and $\{\theta_n\}$ are in $(0, 1)$ and satisfy the same conditions as in theorem 1.4. That is:

(i) $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$;

(ii) $\lambda_n M_0^* \leq \gamma_0 \theta_n$; $\delta_E^{-1}(\lambda_n M_0^*) \leq \gamma_0 \theta_n$,

(iii) $\dfrac{\delta_E^{-1}\left(\frac{\theta_{n-1} - \theta_n}{\theta_n} K\right)}{\lambda_n \theta_n} \to 0,\ \dfrac{\delta_{E^*}^{-1}\left(\frac{\theta_{n-1} - \theta_n}{\theta_n} K\right)}{\lambda_n \theta_n} \to 0$, as $n \to \infty$,

(iv) $\dfrac{1}{2}\left(\frac{\theta_{n-1} - \theta_n}{\theta_n} K\right) \in (0, 1)$,

for some constants $M_0^* > 0$, $K > 0$ and $\gamma_0 > 0$; where $\delta_E : (0, \infty) \to (0, \infty)$ is the modulus of convexity of E. Prototypes of $\{\lambda_n\}$ and $\{\theta_n\}$ satisfying these conditions are: $\lambda_n = \frac{1}{(n+1)^a}$, $\quad \theta_n = \frac{1}{(n+1)^b}$, for some positive constants a and b. Verifications that these prototypes satisfy conditions $(i) - (iv)$ can be found in the paper of Chidume and Idu [24]. We now prove the following theorem.

Theorem 3.3. *Let E be a uniformly convex and uniformly smooth real Banach space with dual space E^* and let J be the normalized duality map on E. Let $A_k : D(A_k) = E \to E^*, k = 1, 2, 3, ...$ be an infinite family of γ_k-inverse strongly monotone maps such that $\gamma := \inf_{k \geq 1} \gamma_k > 0$ and $\cap_{n=1}^{\infty} A_k^{-1}(0) \neq \emptyset$. For arbitrary $x_1,\ u \in E$, define a sequence $\{x_n\}$ iteratively by:*

$$x_{n+1} = J^{-1}\left[(1 - \lambda_n)Jx_n + \lambda_n S x_n - \lambda_n \theta_n (Jx_n - Ju)\right],\ n \geq 1, \tag{3.3}$$

where $S := \sum_{k=1}^{\infty} \varepsilon_k T_k$, $T_k := J - A_k$ and $\{\varepsilon_k\}_{k=1}^{\infty}$ is a positive sequence of real numbers atisfying $\sum_{k=1}^{\infty} \varepsilon_k = 1$. Then, $\{x_n\}$ converges strongly to some $x^* \in \cap_{k=1}^{\infty} A_k^{-1}(0)$.

Proof. From (iii) and (iv) of lemma 3.1, we obtain that S is J-pseudocontractive and bounded on bounded subsets of E. From lemma 3.2, $(J - S)$ is maximal monotone. By theorem 1.4, we obtain that $\{x_n\}$ converges strongly to some $x^* \in F_J(S)$. Hence, by (v) of lemma 3.1, we have that $\{x_n\}$ converges strongly to some $x^* \in \cap_{k=1}^{\infty} A_k^{-1}(0)$. This completes the proof. ∎

4. SOME APPLICATIONS

In this section, we apply our theorem to approximate a common J-fixed point of an infinite family of strictly J-pseudocontractive maps in uniformly convex and uniformly smooth real Banach spaces and also to approximate a common fixed point of an infinite family of strictly pseudocontractive maps in real Hilbert spaces.

Corollary 4.1. *Let E be a uniformly convex and uniformly smooth real Banach space with dual space E^* and let J be the normalized duality map on E. Let $T_k : D(T_k) = E \to E^*, k = 1, 2, 3, ...$ be an infinite family of γ_k-strictly J-pseudocontractive maps such that $\gamma := \inf_{k\geq 1} \gamma_k > 0$ and $\cap_{n=1}^{\infty} F_J(T_k) \neq \emptyset$. For arbitrary $x_1,\, u \in E$, define a sequence $\{x_n\}$ iteratively by:*

$$x_{n+1} = J^{-1}\left[(1-\lambda_n)Jx_n + \lambda_n Sx_n - \lambda_n\theta_n(Jx_n - Ju)\right],\ n \geq 1, \tag{4.1}$$

where $S := \sum_{k=1}^{\infty} \varepsilon_k T_k$ and $\{\varepsilon_k\}_{k=1}^{\infty}$ is a positive sequence of real numbers satisfying $\sum_{k=1}^{\infty} \varepsilon_k = 1$. Then, $\{x_n\}$ converges strongly to some $x^* \in \cap_{k=1}^{\infty} F_J(T_k)$.

Proof. Clearly, S is J-pseudocontractive and bounded. Since T_k is γ_k-strictly J-pseudocontractive, then by remark 2.1, we have that $A_k := J - T_k$ is γ_k-inverse strongly monotone for each $k \geq 1$ and $\cap_{n=1}^{\infty} A_k^{-1}(0) \neq \emptyset$ (since $A_k^{-1}(0) = F_J(T_k)$ for each $k \geq 1$ and $\cap_{k=1}^{\infty} F_J(T_k) \neq \emptyset$). Therefore, the proof follows from theorem 3.3. ∎

Corollary 4.2. *Let H be a real Hilbert space. Let $T_k : D(T_k) = H \to H, k = 1, 2, 3, ...$ be an infinite family of γ_k-strictly pseudocontractive maps such that $\cap_{k=1}^{\infty} F(T_k) \neq \emptyset$. For arbitrary $x_1,\, u \in E$, define a sequence $\{x_n\}$ iteratively by:*

$$x_{n+1} = \left[(1-\lambda_n)x_n + \lambda_n Sx_n - \lambda_n\theta_n(x_n - u)\right],\ n \geq 1, \tag{4.2}$$

where $S := \sum_{k=1}^{\infty} \varepsilon_k T_k$ and $\{\varepsilon_k\}_{k=1}^{\infty}$ is a positive sequence of real numbers satisfying $\sum_{k=1}^{\infty} \varepsilon_k = 1$. Assume $\gamma := \inf_{k\geq 1} \beta_k > 0$, where $\beta_k = \frac{1-\gamma_k}{2}$, then $\{x_n\}$ converges strongly to some $x^* \in \cap_{k=1}^{\infty} F(T_k)$.

Proof. Since for each $k \geq 1$, T_k is γ_k-strictly pseudocontractive, then it is easy to see that for each $x, y \in E$,

$$\langle Tx - Ty, x - y\rangle \leq ||x - y||^2 - \beta_k||(I - T)x - (I - T)y||^2.$$

Thus, for each $k \geq 1$, T_k is β_k-strictly I-pseudocontractive. Hence, the proof follows from corollary 4.1. ∎

5. CONCLUSION

In this paper, a new iterative algorithm is constructed and used to approximate a common zero of an infinite family of gamma-inverse strongly monotone maps defined on uniformly convex and uniformly smooth real Banach spaces. As a consequence of this result, a strong convergence theorem for approximating a common J-fixed point for an infinite family of gamma-strictly J-pseudocontractive maps is proved. Furthermore, in particular, all results obtained are applicable in the important l^p, $L^p(G)$ and Sobolev spaces $W_m^p(G)$, $p \in (1, \infty)$, $p^{-1} + q^{-1} = 1$.

Remark 5.1. (see e.g., Alber and Ryazantseva, [27]; page 36) The analytical representations of duality maps are known in a number of Banach spaces. For instance, in the spaces l^p, $L^p(G)$ and Sobolev spaces $W_m^p(G)$, $p \in (1, \infty)$, $p^{-1} + q^{-1} = 1$, respectively,

$$Jx = ||x||_{l^p}^{2-p} y \in l^q,\ y = \{|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, ...\},\ x = \{x_1, x_2, ...\},$$
$$J^{-1}x = ||x||_{l^q}^{2-q} y \in l^p,\ y = \{|x_1|^{q-2}x_1, |x_2|^{q-2}x_2, ...\},\ x = \{x_1, x_2, ...\},$$
$$Jx = ||x||_{L^p}^{2-p} |x(s)|^{p-2}x(s) \in L^q(G),\ s \in G,$$
$$J^{-1}x = ||x||_{L^q}^{2-q} |x(s)|^{q-2}x(s) \in L^p(G),\ s \in G,$$

and

$$Jx = ||x||_{W_m^p}^{2-p} \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^{\alpha}(|D^{\alpha}x(s)|^{p-2}D^{\alpha}x(s)) \in W_{-m}^q(G), m > 0, s \in G.$$

REFERENCES

[1] Y. A. ALBER, *Metric and generalized projection operators in Banach spaces: properties and applications*. In *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type* (A. G. Kartsatos, Ed.), Marcel Dekker, New York (1996), pp. 15-50.

[2] R. P. AGARWAL, M. MEEHAN and D. O'REGAN, *Fixed Point Theory and Applications*, Vol. 141, Cambridge university press, 2001.

[3] Y. ALBER and I. RYAZANTSEVA, *Nonlinear Ill Posed Problems of Monotone Type*, Springer, London, UK, 2006.

[4] P. BENILAN, M. G. CRANDALL and A. PAZY, *Nonlinear Evolution Equations in Banach Spaces [preprint]*, Besançon 1994.

[5] V. BERINDE, *Iterative Approximation of Fixed points*, Lecture Notes in Mathematics, Springer, London, UK, 2007.

[6] F. E. BROWDER, Nonlinear mappings of nonexpansive and accretive-type in Banach spaces, *Bull. Amer. Math. Soc.*, Vol. 73 (1967), pp. 875-882.

[7] F. E. BROWDER, Nonlinear equations of evolution and nonlinear accretive operators in Banach spaces, *Bull. Amer. Math. Soc.*, Vol. 73 (1967), pp. 875-882.

[8] Y. CENSOR and S. RIECH, *Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization*, *Optimization*, Vol. 37, no. 4 (1996), pp. 323-339.

[9] C. E. CHIDUME. Geometric Properties of Banach Spaces and Nonlinear iterations, Vol. 1965 of *Lectures Notes in Mathematics*, Springer, London, UK, 2009.

[10] C. E. CHIDUME, U. V. NNYABA and O. M. ROMANUS, Convergence theorems for strictly J-pseudocontractions with application to zeros of gamma-inverse strongly monotone maps, *Panamerican Mathematical Journal*, (*accepted*).

[11] C. E. CHIDUME, Strong convergence theorems for bounded accretive operators in uniformly smooth Banach spaces, *Contemporary Mathematics*, Vol. 659, *Nonlinear Analysis and Optimization*, (B. S. Mordukhovich, S. Reich, A. J. Zaslavski), AMS, Providence, RI, 2016.

[12] R. I. KAČUROVSKII, On monotone operators and convex functionals, *Uspekhi Mathematicheskikh Nauk*, Vol. 15 (1960), no. 4, pp. 213-215.

[13] T. KATO, Nonlinear semigroups and evolution equations, *J. Math. Soc. Japan*, Vol. 19 (1967), pp. 508-520.

[14] H. KHATIBZADEH and G. MOROSANU, Strong and weak solutions to second order differential inclusions governed by monotone Operators, *Set-Valued and Variational Analysis*, Vol. 22 (2014), Issue 2, pp. 521-531.

[15] H. KHATIBZADEH and A. SHOKRI, On the first- and second-order strongly monotone dynamical systems and minimization problems, *Optimization Methods and Software*, Vol. 30 (2015) Issue 6, p. 1303-1309.

[16] J. LINDENSTRAUSS and L. TZAFRIRI, *Classical Banach spaces II: Function Spaces*, Ergebnisse Math. Grenzgebiete Bd. **97**, Springer-Verlag, Berlin, 1979.

[17] G. J. MINTY, Monotone (nonlinear) operators in Hilbert space, *Duke Math. J*, Vol. 29 (1962), no. 4, pp. 341-346.

[18] A. G. KARTSATOS (Ed.), *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, **Vol. 178** (1996) of *Lecture Notes in Pure and Appl. Math.*, Dekker, New York.
.

[19] R. E. SHOWALTER, Monotone operators in Banach spaces and nonlinear partial differential equations, *Mathematical Surveys and Monographs*, Vol. 49 (1997), AMS.

[20] V. VOLPERT, Elliptic Partial Differential Equations: Volume 2: Reaction-Diffusion Equations, *Monographs in Mathematics*, Vol. 104 Springer, 2014.

[21] K. WILLIAM and N. SHAHZAD, *Fixed Point Theory in Distance Spaces*, Springer Verlag, 2014.

[22] E. H. ZARANTONELLO, Solving functional equations by contractive averaging, *Tech. Rep. 160*, U. S. Army Math. Research Center, Madison, Wisconsin, 1960.

[23] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications Part II: Monotone Operators*, Springer-Verlag, Berlin, 1985.

[24] C. E. CHIDUME and K. O. IDU, Approximation of zeros of bounded maximal monotone mappings, solutions of Hammerstien integral equations and convex minimization problems, *Journal of Fixed Point Theory and Appl.*, 2016, **2016**:97, DOI: 10.1186/s13663-016-0582-8.

[25] B. LIU, Fixed point of strong duality pseudocontractive mappings and applications, *Abs. App. Anal.*, Vol 2012, doi:10.1155/2012/623625.

[26] I. CIORANESCU, *Geometry of Banach spaces, Duality Mappings and Nonlinear problems*, Kluwer Academic Publishers, Dordrecht, The Netherlands.

[27] YA. ALBER and I. RYAZANTSEVA, *Nonlinear Ill Posed Problems of Monotone-Type*, Springer, London, UK, 2006.

[28] E. U. OFOEDU, Iterative Approximation of a common zero of a countable infinite family of m-accretive operators in Banach spaces, *Fixed Point Theory and Applications*, Vol. 2008, DOI:10.1155/2008/325792.

[29] C. E. CHIDUME, O. M. ROMANUS and U. V. NNYABA, A new iterative algorithm for zeros of generalized Phi-strongly monotone and bounded maps with application, *British Journal of Mathematics and Computer Science*, Vol. 18(1), Art. no. BJMCS.25884, DOI: 10.9734/BJMCS/2016/25884.

[30] C. E. CHIDUME, A. U. BELLO and B. USMAN; Krasnoselskii-type algorithm for zeros of strongly monotone Lipschitz maps in classical Banach spaces, *SpringerPlus* (2015) 4:297, DOI 10.1186/s40064-015-1044-1.

[31] K. AOYAMA, F. KOHSAKA, and W. TAKAHASHI; Proximal point methods for monotone operators in Banach spaces, *Taiwanese Journal of Mathematics*, Vol. 15, No. 1 (2011), pp. 259-281.

[32] S. KAMIMURA, F. KOHSAKA and W. TAKAHASHI; Weak and strong convergence theorems for maximal monotone operators in a Banach space, *Set-Valued Analysis*, Vol. 12, No. 4 (2004) pp. 417-429.

[33] S. KAMIMURA and W. TAKAHASHI; Strong convergence of a proximal-type algorithm in a Banach space, *SIAM J. Optim.*, Vol. 13 (2002), No. 3, pp. 938-945.

[34] H. ZEGEYE and N. SHAHZAD; Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings, *Nonlinear Analysis*, 70 (2009), pp. 2707-2716.

[35] H. ZEGEYE, Strong convergence theorems for maximal monotone mappings in Banach spaces, *J. Math. Anal. Appl.*, 343 (2008), pp. 663-671.

[36] H. ZEGEYE and N. SHAHZAD; Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings, *Nonlinear Analysis*, 70 (2009), pp. 2707-2716.