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NEW IMPLICIT KIRK-TYPE SCHEMES FOR GENERAL CLASS OF QUASI-CONTRACTIVE OPERATORS IN GENERALIZED CONVEX METRIC SPACES

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ABSTRACT. In this paper, we introduce some new implicit Kirk-type iterative schemes in generalized convex metric spaces in order to approximate fixed points for general class of quasi-contractive type operators. The strong convergence, T-stability, equivalency, data dependence and convergence rate of these results were explored. The iterative schemes are faster and better, in term of speed of convergence, than their corresponding results in the literature. These results also improve and generalize several existing iterative schemes in the literature and they provide analogues of the corresponding results of other spaces, namely: normed spaces, CAT(0) spaces and so on.

Key words and phrases: Generalized convex metric space; Implicit Kirk-type schemes; Quasi-contractive operator.

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1. INTRODUCTION AND PRELIMINARIES

Numerous results have been proved on Kirk-type iterations to approximate fixed points for several classes of quasi-contractive operators in various spaces. See [1], [6], [12], [13], [16], [19], [30]. The *Kirk's iterative* procedure given by: for $x_0 \in E$

$$x_{n+1} = \sum_{i=0}^{k} \alpha_i T^i x_n, \ n \geq 0 \tag{1.1}$$

where k is a fixed integer, $k \geq 1$, $\alpha_i \geq 0$ for each i and $\sum_{i=1}^{k} \alpha_i = 1$ was defined in [16]. The sequence 1.1 generalized both Krasnoselskij and Picard iterations, in fact, 1.1 reduces to Krasnoselskij iteration when $k = 1$ as:

$$x_{n+1} = (1-\alpha_1)x_n + \alpha_1 T x_n, \ n \geq 0. \tag{1.2}$$

and reduces to Picard iteration [2] for $k = 1$, $\alpha_1 = 1$ as:

$$x_{n+1} = T x_n, \ n \geq 0. \tag{1.3}$$

Both 1.1 and 1.2 are mainly employed when 1.3 has no fixed point for a non-expansive operator. Other possible iterations which guarantee the existence of fixed points for non-expansive operators are categorized in what follows:
Let E be a normed linear space and $x_0 \in E$.

Label	Explicit scheme ($n \geq 1$)	Author(s)
Mann iteration	$x_n = (1-\alpha_n)x_{n-1} + \alpha_n T x_{n-1}$	Mann [17]
Ishikawa iteration	$x_n = (1-\alpha_n)x_{n-1} + \alpha_n T y_{n-1}$ $y_{n-1} = (1-\beta_n)x_{n-1} + \beta_n T x_{n-1}$	Ishikawa [14]
Noor iteration	$x_n = (1-\alpha_n)x_{n-1} + \alpha_n T y_{n-1}$ $y_{n-1} = (1-\beta_n)x_{n-1} + \beta_n T z_{n-1}$ $z_{n-1} = (1-\gamma_n)x_{n-1} + \gamma_n T x_{n-1}$	Noor [18]
Multistep iteration	$x_n = (1-\alpha_n)x_{n-1} + \alpha_n T y^{(1)}_{n-1}$ $y^{(l)}_{n-1} = (1-\beta^{(l)}_n)x_{n-1} + \beta^{(l)}_n T y^{l+1}_{n-1}$ $y^{(k-1)}_{n-1} = (1-\beta_n)x_{n-1} + \beta_n T x_{n-1}$ $l = 1, 2, \ldots, k-2, \ k \geq 2$	Rhoades and Soltuz [24]

The modified form of the above iterations are given as follows:

Label	Scheme ($n \geq 0$)	Author(s)
Thianwan iteration	$x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n$ $y_n = (1-\beta_n)x_n + \beta_n T x_n$	Thianwan [27]
SP-iteration	$x_{n+1} = (1-\alpha_n)y^1_n + \alpha_n T y^1_n$ $y^1_n = (1-\beta_n)y^2_n + \beta_n T y^2_n$ $y^2_n = (1-\gamma_n)x_n + \gamma_n T x_n$	Pheuengrattana and Suantai [22]
Multistep-SP iteration	$x_{n+1} = (1-\alpha_n)y^1_n + \alpha_n T y^1_n$ $y^l_n = (1-\beta^l_n)y^{l+1}_n + \beta^l_n T y^{l+1}_n$ $y^{k-1}_n = (1-\beta^{k-1}_n)x_n + \beta^{k-1}_n T x_n$ $l = 1, 2, 3, \ldots, k-2, \ k \geq 2$	G$\ddot{u}$rsoy et al. [11]

The implicit type of the above iterations are presented below:

Label	Implicit Scheme ($n \geq 1$)	Author(s)
Implicit Mann iteration	$x_n = (1-\alpha_n)x_{n-1} + \alpha_n T x_n$	Ciric et al. [8]
Implicit Ishikawa iteration	$x_n = (1-\alpha_n)y_n + \alpha_n T x_n$ $y_n = (1-\beta_n)x_{n-1} + \beta_n T y_n$	Xue and Zhang [28]
Implicit Noor iteration	$x_n = \alpha_n y_{n-1} + (1-\alpha_n)T x_n$ $y_{n-1} = \beta_n z_{n-1} + (1-\beta_n)T y_{n-1}$ $z_{n-1} = \gamma_n x_{n-1} + (1-\gamma_n)T z_{n-1}$	Chugh et al. [7]
Implicit Multistep-iteration or WR-iteration	$x_n = (1-\alpha_n)y_n^{(1)} + \alpha_n T x_n$ $y_n^{(l)} = (1-\beta_n^{(l)})y_n^{(l+1)} + \beta_n^{(l)} T y_n^{(l)}$ $y_n^{(k-1)} = (1-\beta_n^{(k-1)})x_{n-1} + \beta_n^{(k-1)} T y_n^{(k-1)}$ $l = 1, 2, 3, \dots, k-2,\ k \geq 2$	Wahab and Rauf [29]

where $\alpha_n, \beta_n, \gamma_n, \beta_n^l$ are sequences in $[0,1]$, $l = 1, 2, \dots, k-1$ with $\sum \alpha_n = \infty$.
In [19], Olatinwo introduced two hybrid schemes, namely, Kirk-Mann and Kirk-Ishikawa iterations in a normed space as: For $x_0 \in E$

$$x_{n+1} = \sum_{i=0}^{q} \alpha_{n,i} T^i x_n; \ \sum_{i=0}^{q} \alpha_{n,i} = 1, \ n \geq 0 \tag{1.4}$$

and

$$\begin{aligned} x_{n+1} &= \alpha_{n,0} x_n + \sum_{i=1}^{q} \alpha_{n,i} T^i y_n, \ \sum_{i=0}^{q} \alpha_{n,i} = 1, \\ y_n &= \sum_{i=0}^{r} \beta_{n,i} T^i x_n, \ \sum_{i=0}^{r} \beta_{n,i} = 1; \ n \geq 0 \end{aligned} \tag{1.5}$$

respectively, where q and r are fixed integers with $q \geq r$, $\{\alpha_{n,i}\}$ and $\{\beta_{n,i}\}$ are sequences in $[0,1]$ satisfying $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i} \geq 0$, $\beta_{n,0} \neq 0$.
The Kirk-Noor iteration was introduced in [6] as follows: For $x_0 \in E$

$$\begin{aligned} x_{n+1} &= \alpha_{n,0} x_n + \sum_{i=1}^{q} \alpha_{n,i} T^i y_n, \ \sum_{i=0}^{q} \alpha_{n,i} = 1, \\ y_n &= \beta_{n,0} x_n + \sum_{i=1}^{r} \beta_{n,i} T^i z_n, \ \sum_{i=0}^{r} \beta_{n,i} = 1, \\ z_n &= \sum_{i=0}^{s} \gamma_{n,i} T^i x_n, \ \sum_{i=0}^{s} \gamma_{n,i} = 1; \ n = 0, 1, 2 \dots \end{aligned} \tag{1.6}$$

where $q, \ r$ and s are fixed integers with $q \geq r \geq s$, $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$ and $\{\gamma_{n,i}\}$ are sequences in $[0,1]$ satisfying $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i} \geq 0$, $\beta_{n,0} \neq 0$, $\gamma_{n,i} \geq 0$, $\gamma_{n,0} \neq 0$.
In [12], the Kirk-Multistep iteration in Banach space E which generalized 1.4, 1.5 and 1.6. For

$x_0 \in E$,

$$x_{n+1} = \alpha_{n,0}x_n + \sum_{i=1}^{q_1} \alpha_{n,i}T^i y_n^{(1)},\ \sum_{i=0}^{q_1} \alpha_{n,i} = 1,$$
$$y_n^{(l)} = \beta_{n,0}^{(l)}x_n + \sum_{i=1}^{q_{l+1}} \beta_{n,i}^{(l)}T^i y_n^{(l+1)},\ \sum_{i=0}^{q_{l+1}} \beta_{n,i}^{(l)} = 1,\ l = 1(1)k-2 \tag{1.7}$$
$$y_n^{(k-1)} = \sum_{i=0}^{q_k} \beta_{n,i}^{(k-1)}T^i x_n,\ \sum_{i=0}^{q_k} \beta_{n,i}^{(k-1)} = 1;\ k \geq 2,\ n = 0,1,2\ldots$$

where $q_1, q_2, q_3, \ldots, q_k$ are fixed integers with $q_1 \geq q_2 \geq q_3 \geq \ldots \geq q_k$, $\{\alpha_{n,i}\}_{n=1}^{\infty}$ and $\left\{\beta_{n,i}^{(l)}\right\}_{n=1}^{\infty}$ are sequences in $[0,1]$ satisfying $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i}^{(l)} \geq 0$, $\beta_{n,0}^{(l)} \neq 0$ for each l.
Another hybrid iteration called Kirk-SP iteration was defined in [13] as follows: For $x_0 \in E$

$$x_{n+1} = \alpha_{n,0}y_n + \sum_{i=1}^{q} \alpha_{n,i}T^i y_n,\ \sum_{i=0}^{q} \alpha_{n,i} = 1,$$
$$y_n = \beta_{n,0}z_n + \sum_{i=1}^{r} \beta_{n,i}T^i z_n,\ \sum_{i=0}^{r} \beta_{n,i} = 1, \tag{1.8}$$
$$z_n = \sum_{i=0}^{s} \gamma_{n,i}T^i x_n,\ \sum_{i=0}^{s} \gamma_{n,i} = 1;\ n = 0,1,2\ldots$$

where $q,\ r$ and s are fixed integers with $q \geq r \geq s$, $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$ and $\{\gamma_{n,i}\}$ are sequences in $[0,1]$ satisfying $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i} \geq 0$, $\beta_{n,0} \neq 0$, $\gamma_{n,i} \geq 0$, $\gamma_{n,0} \neq 0$. The Kirk-Thianwan iteration can be deduced if $s = 0$ in 1.8.
The Kirk-multistep-SP, which generalized both Kirk-SP and Kirk-Thianwan schemes, was introduced in [1] and was defined as:

$$x_{n+1} = \alpha_{n,0}y_n^{(1)} + \sum_{i=1}^{q_1} \alpha_{n,i}T^i y_n^{(1)},\ \sum_{i=0}^{q_1} \alpha_{n,i} = 1,$$
$$y_n^{(l)} = \beta_{n,0}^{(l)}y_n^{(l+1)} + \sum_{i=1}^{q_{l+1}} \beta_{n,i}^{(l)}T^i y_n^{(l+1)},\ \sum_{i=0}^{q_{l+1}} \beta_{n,i}^{(l)} = 1,\ l = 1(1)k-2 \tag{1.9}$$
$$y_n^{(k-1)} = \sum_{i=0}^{q_k} \beta_{n,i}^{(k-1)}T^i x_n,\ \sum_{i=0}^{q_k} \beta_{n,i}^{(k-1)} = 1;\ k \geq 2,\ n = 0,1,2\ldots$$

where $q_1, q_2, q_3, \ldots, q_k$ are fixed integers with $q_1 \geq q_2 \geq q_3 \geq \ldots \geq q_k$, $\{\alpha_{n,i}\}_{n=1}^{\infty}$ and $\left\{\beta_{n,i}^{(l)}\right\}_{n=1}^{\infty}$ are sequences in $[0,1]$ satisfying $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i}^{(l)} \geq 0$, $\beta_{n,0}^{(l)} \neq 0$ for each l.

Zamfirescu operator has been the most generalized quasi-nonexpansive operator used by several authors to approximate fixed points. Zamfirescu [31] stated that:
Let X be a complete metric space and T be a self map of X. The operator T is Zamfirescu operator if for each pair of points $x, y \in X$, at least one of the following is true:

$$Z_1:\ d(Tx,Ty) \leq a d(x,y)$$
$$Z_2:\ d(Tx,Ty) \leq b\left[d(x,Tx) + d(y,Ty)\right] \tag{1.10}$$
$$Z_3:\ d(Tx,Ty) \leq c\left[d(x,Ty) + d(y,Tx)\right]$$

where a*,* b *and* c *are non-negative constants satisfying* $a \in [0,1)$*,* $b, c \leq \frac{1}{2}$*.*
The equivalence form of 1.10 is

$$d(Tx,Ty) \leq a \max\left\{d(x,y), \frac{1}{2}\left[d(x,Tx)+d(y,Ty)\right], \frac{1}{2}\left[d(x,Ty)+d(y,Tx)\right]\right\} \tag{1.11}$$

for $x, y \in X$ and $a \in [0,1)$.
It is observed in [3] that the condition 1.11 implies

$$d(Tx,Ty) \leq 2hd(x,Tx) + hd(x,y) \tag{1.12}$$

where $h = \max\left\{a, \frac{a}{2-a}\right\}$.
In [24], a more general contractive condition was defined as: For $x, y \in X$, there exists $a \in [0,1)$ such that:

$$d(Tx,Ty) \leq a \max\left\{d(x,y), \frac{1}{2}\left[d(x,Tx)+d(y,Ty)\right], d(x,Ty), d(y,Tx)\right\} \tag{1.13}$$

In [21], Osilike extended and generalized the contractive condition 1.13: For $x, y \in X$, there exists $a \in [0,1)$ and $L \geq 0$ such that

$$d(Tx,Ty) \leq Ld(x,Tx) + ad(x,y) \tag{1.14}$$

A general class of operator T satisfying the following quasi-contractive condition:

$$d(Tx,Ty) \leq ad(x,y) + \varphi\left(d(x,Tx)\right) \text{ for } x, y \in X \tag{1.15}$$

where $a \in (0,1)$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone increasing function with $\varphi(0) = 0$, see [15].
In [4], a class of quasi-contractive of the type 1.15 was introduced as follow:

$$d(Tx,Ty) \leq ad(x,y) + \epsilon d(x,Tx) \text{ for } x, y \in X \tag{1.16}$$

for any $x, y \in X$, $\epsilon \geq 0$ and $a \in (0,1)$.
A more general class of quasi-contractive operator T satisfying 1.15 is mentioned below:

Lemma 1. [19] *Let* (X,d) *be a metric space and* $T : X \to X$ *be a map satisfying 1.15. Let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a subadditive, monotone increasing function such that* $\varphi(0) = 0$*,* $\varphi(Lu) = L\varphi(u)$*, for* $u \in \mathbb{R}^+$*,* $L \geq 0$*. Then, for all* $i \in \mathbb{N}$*,* $L \geq 0$ *and for all* $x, y \in X$

$$d\left(T^i x, T^i y\right) \leq \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi^j\left(d\left(x, T^j x\right)\right) + a^i d\left(x,y\right),\ a \in (0,1) \tag{1.17}$$

Condition 1.15 and 1.16 are special cases of 1.17; for if $i = 1$ in 1.17, we have 1.15; and if $i = 1$ in 1.17 with $\varphi\left(d(x,Tx)\right) = \epsilon d(x,Tx)$, $\epsilon \geq 0$, we recover 1.16.
If we let $x = p \in F(T)$ in 1.17, then

$$d\left(p, T^i y\right) \leq a^i d\left(p,y\right),\ a \in (0,1) \tag{1.18}$$

Inequality 1.18 is a general class of operator and it is similar to the operator in [5].
The following Lemmas shall be useful in our main results.

Lemma 2. [3] *Let* δ *be a real number such that* $\delta \in [0,1)$ *and* $\{\epsilon_n\}_{n=0}^{\infty}$ *is a sequence of nonnegative numbers such that* $\lim_{n\to\infty} \epsilon_n = 0$*, then, for any sequence of positive numbers* $\{u_n\}_{n=0}^{\infty}$ *satisfying:*

$$u_{n+1} \leq \delta u_n + \epsilon_n, \text{ for all } n \in \mathbb{N} \tag{1.19}$$

we have $\lim_{n\to\infty} u_n = 0$*.*

We note here that if $\delta = 1$, then inequality 1.19 is a weaker condition and if $\delta < 1$, inequality 1.19 is a stronger condition. The case of $\delta = 0$ is obvious.

Lemma 3. [25] *Let* $\{a_n\}_{n=0}^{\infty}$ *be a nonnegative sequence for which there exists* $n_0 \in \mathbb{N}$ *such that, for all* $n \geq n_0$*, one has the following inequality:*

$$a_{n+1} \leq (1 - r_n)a_n + r_n t_n,$$

where $r_n \in (0,1)$*, for all* $n \in \mathbb{N}$*,* $\sum_{n=1}^{\infty} r_n = \infty$*, and* $t_n \geq 0$ *for* $n \in \mathbb{N}$*. Then,*

$$0 \leq \lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \sup t_n$$

We shall employ the following Definitions in the proof of our main results.

Definition 1. [19] *Let* (X, d, W) *be a convex metric space and* $T : X \to X$ *a self-mapping. Suppose that* $F(T) = \{p \in X : Tp = p\}$ *is the set of fixed points of* T*.*
Let $\{x_n\}_{n=0}^{\infty} \subset X$ *be the sequence generated by an iterative procedure involving* T *which is defined by*

$$x_{n+1} = f_{T,\alpha_n}^{x_n}, \ n \geq 0 \tag{1.20}$$

where $x_0 \in X$ *is the initial approximation and* $f_{T,\alpha_n}^{x_n}$ *is some function having convex structure such that* $\alpha_n \in [0,1]$*. Suppose that* $\{x_n\}$ *converges to a fixed point* p *of* T*. Let* $\{y_n\}_{n=0}^{\infty} \subset X$ *and set* $\epsilon_n = d\left(y_{n+1}, f_{T,\alpha_n}^{y_n}\right)$*,* $n = 0, 1, 2, \ldots$*. Then, the iterative procedure 1.20 is said to be* T*-stable or stable with respect to* T *if and only if* $\lim_{n\to\infty} \epsilon_n = 0$ *implies* $\lim_{n\to\infty} y_n = p$

Definition 2. [3] *Let* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *be two nonnegative real sequences which converge to* a *and* b*, respectively. Let*

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}$$

1. *if* $l = 0$*, then* $\{a_n\}_{n=0}^{\infty}$ *converges to* a *faster than* $\{b_n\}_{n=0}^{\infty}$ *to* b*.*
2. *if* $0 < l < \infty$*, then both* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *have the same convergence rate.*
3. *if* $l = \infty$*, then* $\{b_n\}_{n=0}^{\infty}$ *converges to* b *faster than* $\{a_n\}_{n=0}^{\infty}$ *to* a*.*

Let $\{x_n\}$ and $\{y_n\}$ be two iterative sequences converging to the same fixed point z of T such that

$$d(x_n, z) \leq a_n \text{ and } d(y_n, z) \leq b_n, \ n \geq 1$$

where a_n and b_n are sequences of positive real numbers (converging to zero). In view of Definition 2, if a_n converges faster than b_n, then we say that the sequence x_n converges faster than the sequence y_n.

Definition 3. [11] *Let* S *and* T *be two operators on a metric space* X*. One says* S *is approximate operator of* T *if, for all* $x \in X$ *and for a real number* $\epsilon > 0$*, one has* $d(Tx, Sx) \leq \epsilon$*.*

2. MAIN RESULTS

We begin this section by defining a generalized convex metric space as follow:

Definition 4. *Let* (X, d) *be a metric space. A mapping* $W : X \times X \times \ldots \times X \times [0,1] \times [0,1] \times \ldots \times [0,1] \to X$ *is called a generalized convex structure on* X *if for each* $x_i \in X$ *and* $\lambda_i \in [0,1]$

$$d(q, W(x_1, x_2, \ldots, x_r; \lambda_1, \lambda_2, \ldots, \lambda_r)) \leq \sum_{i=1}^{r} \lambda_i d(q, x_i) \tag{2.1}$$

holds for $q \in X$ *and* $\sum_{i=1}^{r} \lambda_i = 1$*. The metric space* (X, d) *together with a generalized convex structure* W *is called a generalized convex metric space.*

If $r = 3$ in inequality 2.1, we have a class of convex metric space defined in [23]. If $r = 2$ in inequality 2.1, we retrieve the convex metric space in [26].
Motivated by the above fact, we generate the following new sequences in a non-empty closed subset of generalized convex metric space (X, d, W).
Let C be a non-empty closed subset of a generalized convex metric space (X, d, W) and T be a self-map of C. For $x_0 \in C$, we define a sequence $\{x_n\}$, namely, implicit Kirk-Mann iteration as:

$$x_n = W(x_{n-1}, Tx_n, T^2x_n, \ldots, T^{q_1}x_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1});\ n \geq 1 \tag{2.2}$$

where $\sum_{i=0}^{q_1} \alpha_{n,i} = 1$; q_1 is a fixed integer with $q_1 \geq 1$, $\{\alpha_{n,i}\}$ is a sequence in $[0, 1]$ with $\alpha_{n,0} \neq 0$ and $\alpha_{n,i} \geq 0$ for each i.
For $x_0 \in C$, we define implicit Kirk-Ishikawa iteration as:

$$\begin{aligned} y_{n-1} &= W(x_{n-1}, Ty_{n-1}, T^2y_{n-1}, \ldots, T^{q_2}y_{n-1}; \beta_{n,0}, \beta_{n,1}, \beta_{n,2}, \ldots, \beta_{n,q_2}) \\ x_n &= W(y_{n-1}, Tx_n, T^2x_n, \ldots, T^{q_1}x_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1});\ n \geq 1 \end{aligned} \tag{2.3}$$

where $\sum_{i=0}^{q_1} \alpha_{n,i} = 1$, $\sum_{i=0}^{q_2} \beta_{n,i} = 1$; $q_1,\ q_2$ are fixed integers with $q_1 \geq q_2$, $\{\alpha_{n,i}\},\ \{\beta_{n,i}\}$ are sequence in $[0, 1]$ with $\alpha_{n,0} \neq 0$, $\beta_{n,0} \neq 0$, $\beta_{n,i} \geq 0$ and $\alpha_{n,i} \geq 0$ for each i.
The implicit Kirk-Noor iteration will be defined as: For $x_0 \in C$,

$$\begin{aligned} z_{n-1} &= W(x_{n-1}, Tz_{n-1}, T^2z_{n-1}, \ldots, T^{q_3}z_{n-1}; \gamma_{n,0}, \gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,q_3}) \\ y_{n-1} &= W(z_{n-1}, Ty_{n-1}, T^2y_{n-1}, \ldots, T^{q_2}y_{n-1}; \beta_{n,0}, \beta_{n,1}, \beta_{n,2}, \ldots, \beta_{n,q_2}) \\ x_n &= W(y_{n-1}, Tx_n, T^2x_n, \ldots, T^{q_1}x_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1});\ n \geq 1 \end{aligned} \tag{2.4}$$

where $\sum_{i=0}^{q_1} \alpha_{n,i} = 1$, $\sum_{i=0}^{q_2} \beta_{n,i} = 1$, $\sum_{i=0}^{q_3} \gamma_{n,i} = 1$; $q_1,\ q_2,\ q_3$ are fixed integers with $q_1 \geq q_2 \geq q_3$, $\{\alpha_{n,i}\},\ \{\beta_{n,i}\},\ \{\gamma_{n,i}\}$ are sequence in $[0, 1]$ with $\alpha_{n,0} \neq 0$, $\beta_{n,0} \neq 0$, $\gamma_{n,0} \neq 0$, $\alpha_{n,i} \geq 0$, $\beta_{n,i} \geq 0$, and $\gamma_{n,i} \geq 0$ for each i.
For $x_0 \in C$, we define implicit Kirk-Multistep iteration in a generalized convex metric space as:
(2.5)

$$\begin{aligned} x_{n-1}^{(k-1)} &= W(x_{n-1}, Tx_{n-1}^{(k-1)}, T^2x_{n-1}^{(k-1)}, \ldots, T^{q_k}x_{n-1}^{(k-1)}; \beta_{n,0}^{(k-1)}, \beta_{n,1}^{(k-1)}, \beta_{n,2}^{(k-1)}, \ldots, \beta_{n,q_k}^{(k-1)}) \\ x_{n-1}^{(l)} &= W(x_{n-1}^{(l+1)}, Tx_{n-1}^{(l)}, T^2x_{n-1}^{(l)}, \ldots, T^{q_{l+1}}x_{n-1}^{(l)}; \beta_{n,0}^{(l)}, \beta_{n,1}^{(l)}, \beta_{n,2}^{(l)}, \ldots, \beta_{n,q_{l+1}}^{(l)});\ l = 1(1)k-2 \\ x_n &= W(x_{n-1}^{(1)}, Tx_n, T^2x_n, \ldots, T^{q_1}x_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1});\ n \geq 1 \end{aligned}$$

where $\sum_{i=0}^{q_1} \alpha_{n,i} = 1$, $\sum_{i=0}^{q_{l+1}} \beta_{n,i}^{(l)} = 1$, $l = 1(1)k-1$; $q_1, q_2, \ldots, q_k$ are fixed integers with $q_1 \geq q_2 \geq \ldots \geq q_k$, $k \geq 2$, $\{\alpha_{n,i}\},\ \{\beta_{n,i}^{(l)}\}$ are sequence in $[0, 1]$ with $\alpha_{n,0} \neq 0$, $\beta_{n,0}^{(l)} \neq 0$, $\alpha_{n,i} \geq 0$, $\beta_{n,i}^{(l)} \geq 0$ for each i.
We present our main results as follows:

Theorem 1. *Let C be a nonempty closed subset of a generalized convex metric space (X, d, W). Let $T : C \to C$ be an operator satisfying the quasi-contractive operator 1.17 with $F(T) \neq \phi$. Then, for $x_0 \in C$, the sequence $\{x_n\}$ defined by 2.5 with $\sum_{n=1}^{\infty} (1 - \alpha_{n,0}) = \infty$ converges strongly to a unique fixed point $p \in F(T)$.*

Proof

Let $x_0 \in C$ and $p \in F(T)$, by applying condition 1.17 to the sequence $\{x_{n-1}^{(k-1)}\}$ in 2.5, we have

$$d\left(x_{n-1}^{(k-1)}, p\right) = d(W(x_{n-1}, Tx_{n-1}^{(k-1)}, T^2x_{n-1}^{(k-1)}, \ldots, T^{q_k}x_{n-1}^{(k-1)}; \beta_{n,0}^{(k-1)}, \beta_{n,1}^{(k-1)}, \beta_{n,2}^{(k-1)}, \ldots, \beta_{n,q_k}^{(k-1)}), p)$$
$$\leq \beta_{n,0}^{(k-1)} d(x_{n-1}, p) + \sum_{i=1}^{q_k} \beta_{n,i}^{(k-1)} d\left(T^i x_{n-1}^{(k-1)}, T^i p\right)$$
$$\leq \beta_{n,0}^{(k-1)} d(x_{n-1}, p) + \sum_{i=1}^{q_k} \beta_{n,i}^{(k-1)} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi^j \left(d\left(p, T^j p\right)\right) + a^i d\left(x_{n-1}^{(k-1)}, p\right)\right]$$
$$= \beta_{n,0}^{(k-1)} d(x_{n-1}, p) + \sum_{i=1}^{q_k} \beta_{n,i}^{(k-1)} a^i d\left(x_{n-1}^{(k-1)}, p\right)$$

implying that

$$d\left(x_{n-1}^{(k-1)}, p\right) \leq \frac{\beta_n^{(k-1)}}{1 - \sum_{i=1}^{q_k} \beta_{n,i}^{(k-1)} a^i} d(x_{n-1}, p) \tag{2.6}$$

Also, for $l = 1, 2, \ldots, k-2$ in 2.5, we have

$$d\left(x_{n-1}^{(l)}, p\right) = d\left(W(x_{n-1}^{(l+1)}, Tx_{n-1}^{(l)}, T^2x_{n-1}^{(l)}, \ldots, T^{q_{l+1}}x_{n-1}^{(l)}; \beta_{n,0}^{(l)}, \beta_{n,1}^{(l)}, \beta_{n,2}^{(l)}, \ldots, \beta_{n,q_{l+1}}^{(l)}), p\right)$$
$$\leq \beta_{n,0}^{(l)} d\left(x_{n-1}^{(l+1)}, p\right) + \sum_{i=1}^{q_{l+1}} \beta_{n,i}^{(l)} d\left(T^i x_{n-1}^{(l)}, T^i p\right)$$
$$\leq \beta_{n,0}^{(l)} d\left(x_{n-1}^{(l+1)}, p\right) + \sum_{i=1}^{q_{l+1}} \beta_{n,i}^{(l)} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi^j \left(d\left(p, T^j p\right)\right) + a^i d\left(x_{n-1}^{(l)}, p\right)\right]$$
$$= \beta_{n,0}^{(l)} d\left(x_{n-1}^{(l+1)}, p\right) + \sum_{i=1}^{q_{l+1}} \beta_{n,i}^{(l)} a^i d\left(x_{n-1}^{(l)}, p\right)$$

also implying

$$d\left(x_{n-1}^{(l)}, p\right) \leq \frac{\beta_{n,0}^{(l)}}{1 - \sum_{i=1}^{q_{l+1}} \beta_{n,i}^{(l)} a^i} d\left(x_{n-1}^{(l+1)}, p\right) \quad \text{for each } l \tag{2.7}$$

By substituting inequality 2.6 into 2.7 for $l = 1, 2, \ldots, k-2$, we obtain

$$d\left(x_{n-1}^{(1)}, p\right) \leq \prod_{l=1}^{k-1} \left(\frac{\beta_{n,0}^{(l)}}{1 - \sum_{i=1}^{q_{l+1}} \beta_{n,i}^{(l)} a^i}\right) d(x_{n-1}, p) \tag{2.8}$$

Using inequality 1.17 on sequence x_n in 2.5, we have

$$d(x_n, p) \leq \alpha_{n,0} d\left(x_{n-1}^{(1)}, p\right) + \sum_{i=1}^{q_1} \alpha_{n,i} d\left(T^i x_n, T^i p\right)$$
$$\leq \alpha_{n,0} d\left(x_{n-1}^{(1)}, p\right) + \sum_{i=1}^{q_1} \alpha_{n,i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi^j \left(d\left(p, T^j p\right)\right) + a^i d(x_n, p)\right]$$
$$= \alpha_{n,0} d\left(x_{n-1}^{(1)}, p\right) + \sum_{i=1}^{q_1} \alpha_{n,i} a^i d(x_n, p)$$

Therefore,

$$d(x_n,p)\le \frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}d\left(x_{n-1}^{(1)},p\right) \tag{2.9}$$

By inserting estimate 2.8 into 2.9, we have

$$d(x_n,p)\le \left(\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}\right)\prod_{l=1}^{k-1}\left(\frac{\beta_{n,0}^{(l)}}{1-\sum_{i=1}^{q_{l+1}}\beta_{n,i}^{(l)}a^i}\right)d(x_{n-1},p) \tag{2.10}$$

This further implies

$$d(x_n,p)\le \left(\alpha_{n,0}+\sum_{i=1}^{q_1}\alpha_{n,i}a^i\right)\prod_{l=1}^{k-1}\left(\sum_{i=0}^{q_{l+1}}\beta_{n,i}^{(l)}a^i\right)d(x_{n-1},p) \tag{2.11}$$

Since $a\in(0,1)$ implies $a^i\in(0,1)$ for $i=1,2,\ldots,q_k$, $k\ge 2$, then

$$\prod_{l=1}^{k-1}\left(\sum_{i=0}^{q_{l+1}}\beta_{n,i}^{(l)}a^i\right)\le \prod_{l=1}^{k-1}\left(\sum_{i=0}^{q_{l+1}}\beta_{n,i}^{(l)}\right)=1;\ \text{for } l=1,2,\ldots k-1 \tag{2.12}$$

Applying 2.12 in the estimate 2.11, we have

$$\begin{aligned}
d(x_n,p) &\le \left(\alpha_{n,0}+\sum_{i=1}^{q_1}\alpha_{n,i}a^i\right)d(x_{n-1},p)\\
&\le [\alpha_{n,0}+(1-\alpha_{n,0})a]\,d(x_{n-1},p)\\
&= [1-(1-\alpha_{n,0})(1-a)]\,d(x_{n-1},p)\\
&\le [1-(1-\alpha_{n,0})(1-a)]\,[1-(1-\alpha_{n-1,0})(1-a)]\,d(x_{n-2},p)\\
&\vdots\\
&\le \prod_{r=1}^{n}[1-(1-\alpha_{r,0})(1-a)]d(x_0,p)\\
&\le \frac{1}{\mathrm{e}^{(1-a)\sum_{r=1}^{n}(1-\alpha_{r,0})}}d(x_0,p)
\end{aligned}$$

Since $\frac{1}{\mathrm{e}^{(1-a)\sum_{r=1}^{n}(1-\alpha_{r,0})}}\longrightarrow 0$ as $n\to\infty$, then $\lim_{n\to\infty}d(x_n,p)\longrightarrow 0$.
Hence, the implicit Kirk-multistep iteration 2.5 converges strongly to $p\in F(T)$.
Furthermore, suppose $d(p_1,p_2)\neq 0$ for $p_1,p_2\in F(T)$, then by inequality 1.17, we have

$$d(p_1,p_2)=d(T^ip_1,T^ip_2)\le\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi^j\left(d(p_1,Tp_1)\right)+a^id(p_1,p_2)$$

This implies that

$$d(p_1,p_2)\le a^id(p_1,p_2)\le ad(p_1,p_2)$$

But $d(p_1,p_2)\le 0$ is a contradiction. Hence, $p_1=p_2=p\in F(T)$ is unique.

Corollary 1. *Let C be a nonempty closed subset of a generalized convex metric space (X,d,W). Let $T:C\to C$ be an operator satisfying the quasi-contractive operator 1.17 with $F(T)\neq\phi$. Then, for $x_0\in C$ and $\sum_{n=1}^{\infty}(1-\alpha_{n,0})=\infty$, the sequence $\{x_n\}$ defined by*

(i) *2.2 converges strongly to a unique fixed point $p\in F(T)$.*
(ii) *2.3 converges strongly to a unique fixed point $p\in F(T)$.*
(iii) *2.4 converges strongly to a unique fixed point $p\in F(T)$.*

Proof
The proof follows from Theorem 1 by letting $k = 2$ and $q_2 = 0$ in 2.5 for Corollary 1(i), $k = 2$ in 2.5 for Corollary 1(ii) and $k = 3$ in 2.5 for Corollary 1(iii).

Corollary 2. *Let C be a nonempty closed subset of a convex metric space (X, d, W). Let $T : C \to C$ be an operator satisfying the quasi-contractive operator 1.15 with $F(T) \neq \phi$. Then, for $x_0 \in C$ and $\sum_{n=1}^{\infty}(1-\alpha_{n,0}) = \infty$, the sequence $\{x_n\}$ given by*

(i) *implicit Mann iteration converges strongly to a unique fixed point $p \in F(T)$.*
(ii) *implicit Ishikawa iteration converges strongly to a unique fixed point $p \in F(T)$.*
(iii) *implicit Noor iteration converges strongly to a unique fixed point $p \in F(T)$.*
(iv) *implicit Multistep iteration converges strongly to a unique fixed point $p \in F(T)$.*

Proof
The proof of Corollary 2(i) follows from Corollary 1(i) if we let $q_1 = 1$, $q_2 = q_3 = q_4 = \ldots = q_k = 0$ in 2.2 with $\alpha_{n,1} = \alpha_n$, $r = 2$ in 2.1 and $i = 1$ in 1.17.
Corollary 2(ii) can be deduced from Corollary 1(ii) by letting $q_1 = q_2 = 1$, $q_3 = q_4 = \ldots = q_k = 0$ in 2.3 with $\alpha_{n,1} = \alpha_n$, $\beta_{n,1}^{(1)} = \beta_n^{(1)}$, $r = 2$ in 2.1 and $i = 1$ in 1.17.
Corollary 2(iii) can be deduced from Corollary 1(iii) if $q_1 = q_2 = q_3 = 1$, $q_4 = q_5 = \ldots = q_k = 0$ in 2.4 with $\alpha_{n,1} = \alpha_n$, $\beta_{n,1}^{(1)} = \beta_n^{(1)}$, $\beta_{n,1}^{(2)} = \beta_n^{(2)}$, $r = 2$ in 2.1 and $i = 1$ in 1.17.
If $q_1 = q_2 = q_3 = \ldots = q_k = 1$ in 2.5 of Theorem 1 with $\alpha_{n,1} = \alpha_n$, $\beta_{n,1}^{(l)} = \beta_n^{(l)}$, $r = 2$ in 2.1 and $i = 1$ in 1.17, then Corollary 2(iv) is obvious.

Remark 1. *Corollary 2(i) is the result of Ciric et al.* [8].
Corollary 2(ii) can be found in Xue and Zhang [28].
Corollary 2(iii) is the Theorem 9 of Chugh et al. [6].
Corollary 2(iv) is one of the main results of Wahab and Rauf [29].

We prove the equivalent results of our iterative schemes as follows:

Theorem 2. *Let $T : C \to C$ be a mapping satisfying condition 1.17 with $F(T) \neq \phi$. Then the following iterations are equivalent:*

(i) *For $u_0 \in C$, the implicit Kirk-Mann iteration 2.2 converges to p, i.e. $u_n \to p$.*
(ii) *For $v_0 \in C$, the implicit Kirk-Ishikawa iteration 2.3 converges to p, i.e. $v_n \to p$.*
(iii) *For $w_0 \in C$, the implicit Kirk-Noor iteration 2.4 converges to p, i.e. $w_n \to p$.*
(iv) *For $x_0 \in C$, the implicit Kirk-Multistep iteration 2.5 converges to p, i.e. $x_n \to p$.*

Proof
We first prove that $(i) \to (ii)$: Assume $u_n \to p$, we want to prove that $v_n \to p$. Employing 1.17, 2.2 and 2.3, we have the following estimates

$$\begin{aligned}
d(u_n, v_n) = d\big(&W(u_{n-1}, Tu_n, T^2u_n, \ldots, T^{q_1}u_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1}),\\
&W(v_{n-1}^{(1)}, Tv_n, T^2v_n, \ldots, T^{q_1}v_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1})\big)\\
&\leq \alpha_{n,0}d(u_{n-1}, v_{n-1}^{(1)}) + \sum_{i=1}^{q_1}\alpha_{n,i}d\left(T^iu_n, T^iv_n\right)\\
&\leq \alpha_{n,0}d(u_{n-1}, v_{n-1}^{(1)}) + \sum_{i=1}^{q_1}\alpha_{n,i}\left[\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d(u_n, T^iu_n)\right) + a^id\left(u_n, v_n\right)\right]
\end{aligned}$$

This gives:

$$d(u_n, v_n) \leq \frac{\alpha_{n,0} d(u_{n-1}, v_{n-1}^{(1)})}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} + \frac{\sum_{i=1}^{q_1} \alpha_{n,i}}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(u_n, T^i u_n\right)\right) \right] \tag{2.13}$$

and

$$\begin{aligned} d(u_{n-1}, v_{n-1}^{(1)}) &\leq \beta_{n,0} d(u_{n-1}, v_{n-1}) + \sum_{i=1}^{q_2} \beta_{n,i} d\left(u_{n-1}, T^i v_{n-1}^{(1)}\right) \\ &\leq \beta_{n,0} d(u_{n-1}, v_{n-1}) + \sum_{i=1}^{q_2} \beta_{n,i} d\left(u_{n-1}, T^i u_{n-1}\right) + \sum_{i=1}^{q_2} \beta_{n,i} d\left(T^i u_{n-1}, T^i v_{n-1}^{(1)}\right) \\ &\leq \beta_{n,0} d(u_{n-1}, v_{n-1}) + \sum_{i=1}^{q_2} \beta_{n,i} d\left(u_{n-1}, T^i u_{n-1}\right) \\ &\quad + \sum_{i=1}^{q_2} \beta_{n,i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(u_{n-1}, T^i u_{n-1}\right)\right) + a^i d(u_{n-1}, v_{n-1}^{(1)}) \right] \end{aligned}$$

which implies

$$\begin{aligned} d(u_{n-1}, v_{n-1}^{(1)}) \leq{}& \frac{\beta_{n,0}}{1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i} d(u_{n-1}, v_{n-1}) + \frac{\sum_{i=1}^{q_2} \beta_{n,i}}{1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i} d\left(u_{n-1}, T^i u_{n-1}\right) \\ &+ \frac{\sum_{i=1}^{q_2} \beta_{n,i}}{1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(u_{n-1}, T^i u_{n-1}\right)\right) \right] \end{aligned} \tag{2.14}$$

Combining 2.13 and 2.14, we have

$$\begin{aligned} d(u_n, v_n) \leq{}& \frac{\alpha_{n,0}}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} \left\{ \frac{\beta_{n,0} d(u_{n-1}, v_{n-1})}{1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i} + \frac{\sum_{i=1}^{q_2} \beta_{n,i}}{1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i} d\left(u_{n-1}, T^i u_{n-1}\right) \right. \\ &\left. + \frac{\sum_{i=1}^{q_2} \beta_{n,i}}{1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(u_{n-1}, T^i u_{n-1}\right)\right) \right] \right\} \\ &+ \frac{\sum_{i=1}^{q_1} \alpha_{n,i}}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(u_n, T^i u_n\right)\right) \right] \end{aligned}$$

This further implies

$$\begin{aligned} d(u_n, v_n) \leq{}& \frac{\alpha_{n,0} \beta_{n,0}}{(1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i)(1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i)} d(u_{n-1}, v_{n-1}) + \frac{\alpha_{n,0} \sum_{i=1}^{q_2} \beta_{n,i} d\left(u_{n-1}, T^i u_{n-1}\right)}{(1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i)(1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i)} \\ &+ \frac{\alpha_{n,0} \sum_{i=1}^{q_2} \beta_{n,i}}{(1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i)(1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i)} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(u_{n-1}, T^i u_{n-1}\right)\right) \right] \\ &+ \frac{\sum_{i=1}^{q_1} \alpha_{n,i}}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(u_n, T^i u_n\right)\right) \right] \end{aligned} \tag{2.15}$$

Since $\{\alpha_{n,i}\}, \{\beta_{n,i}\} \subset [0,1]$ and $a \in (0,1)$, then

$$\frac{\alpha_{n,0} \beta_{n,0}}{(1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i)(1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i)} \leq [1 - (1-a)(1-\alpha_{n,0})] \tag{2.16}$$

Moreso, $u_n \to p$ and $F(T) \neq \phi$, it follows that

$$\begin{aligned} d\left(u_n, T^i u_n\right) &\leq d\left(u_n, p\right) + d\left(T^i p, T^i u_n\right) \\ &\leq d\left(u_n, p\right) + \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(p, T^i p\right)\right) + a^i d\left(p, u_n\right) \\ &= (1+a^i) d\left(u_n, p\right) \longrightarrow 0 \end{aligned}$$

which implies $\lim_{n\to\infty} d(u_n, T^i u_n) = 0$, by the property of φ, we have $\varphi\left(d\left(u_n, T^i u_n\right)\right) = 0$ and $\varphi\left(d(u_{n-1}, T^i u_{n-1})\right) = 0$. Applying lemma 2 to inequality 2.15, putting 2.16 in mind, we get $\lim_{n\to\infty} d\left(u_n, v_n\right) = 0$.
Furthermore, since $\lim_{n\to\infty} u_n = p$, then

$$d(v_n, p) \leq d(v_n, u_n) + d(u_n, p) \longrightarrow 0$$

Hence, $\lim_{n\to\infty} v_n = p$.
For $(ii) \to (iii)$: Assume that $v_n \to p$, then, with the aid of 1.17, 2.3 and 2.4, we can deduce the following:

$$\begin{aligned} d(v_n, w_n) &= d\Big(W(v_{n-1}^{(1)}, Tv_n, T^2 v_n, \ldots, T^{q_1} v_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1}), \\ &\quad W(w_{n-1}^{(1)}, Tw_n, T^2 w_n, \ldots, T^{q_1} w_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1})\Big) \\ &\leq \alpha_{n,0} d(v_{n-1}^{(1)}, w_{n-1}^{(1)}) + \sum_{i=1}^{q_1} \alpha_{n,i} d\left(T^i v_n, T^i w_n\right) \\ &\leq \alpha_{n,0} d(v_{n-1}^{(1)}, w_{n-1}^{(1)}) + \sum_{i=1}^{q_1} \alpha_{n,i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(v_n, T^i v_n\right)\right) + a^i d\left(v_n, w_n\right)\right] \end{aligned}$$

This gives

$$d(v_n, w_n) \leq \frac{\alpha_{n,0} d(v_{n-1}^{(1)}, w_{n-1}^{(1)})}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} + \frac{\sum_{i=1}^{q_1} \alpha_{n,i}}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(v_n, T^i v_n\right)\right)\right] \tag{2.17}$$

Again, we have

$$\begin{aligned} d(v_{n-1}^{(1)}, w_{n-1}^{(1)}) &= d\Big(W(v_{n-1}, Tv_{n-1}^{(1)}, T^2 v_{n-1}^{(1)}, \ldots, T^{q_2} v_{n-1}^{(1)}; \beta_{n,0}, \beta_{n,1}, \beta_{n,2}, \ldots, \beta_{n,q_2}), \\ &\quad W(w_{n-1}^{(2)}, Tw_{n-1}^{(1)}, T^2 w_{n-1}^{(1)}, \ldots, T^{q_1} w_{n-1}^{(1)}; \beta_{n,0}, \beta_{n,1}, \beta_{n,2}, \ldots, \beta_{n,q_1})\Big) \\ &\leq \beta_{n,0} d(v_{n-1}, w_{n-1}^{(2)}) + \sum_{i=1}^{q_2} \beta_{n,i} d\left(T^i v_{n-1}^{(1)}, T^i w_{n-1}^{(1)}\right) \\ &\leq \beta_{n,0} d(v_{n-1}, w_{n-1}^{(2)}) + \sum_{i=1}^{q_2} \beta_{n,i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(v_{n-1}^{(1)}, T^i v_{n-1}^{(1)}\right)\right)\right. \\ &\quad \left. + a^i d\left(v_{n-1}^{(1)}, w_{n-1}^{(1)}\right)\right] \end{aligned}$$

This implies:
(2.18)

$$d(v_{n-1}^{(1)}, w_{n-1}^{(1)}) \leq \frac{\beta_{n,0} d(v_{n-1}, w_{n-1}^{(2)})}{1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i} + \frac{\sum_{i=1}^{q_2} \beta_{n,i}}{1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(v_{n-1}^{(1)}, T^i v_{n-1}^{(1)}\right)\right)\right]$$

and

$$d(v_{n-1}, w_{n-1}^{(2)}) \leq \gamma_{n,0} d(v_{n-1}, w_{n-1}) + \sum_{i=1}^{q_3} \gamma_{n,i} d\left(v_{n-1}, T^i w_{n-1}^{(2)}\right)$$

$$\leq \gamma_{n,0} d(v_{n-1}, w_{n-1}) + \sum_{i=1}^{q_3} \gamma_{n,i} d\left(v_{n-1}, T^i v_{n-1}\right) + \sum_{i=1}^{q_3} \gamma_{n,i} d\left(T^i v_{n-1}, T^i w_{n-1}^{(2)}\right)$$

$$\leq \gamma_{n,0} d(v_{n-1}, w_{n-1}) + \sum_{i=1}^{q_3} \gamma_{n,i} d\left(v_{n-1}, T^i v_{n-1}\right)$$

$$+ \sum_{i=1}^{q_3} \gamma_{n,i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(v_{n-1}, T^i v_{n-1}\right)\right) + a^i d(v_{n-1}, w_{n-1}^{(2)})\right]$$

which implies

$$d(v_{n-1}, w_{n-1}^{(2)}) \leq \frac{\gamma_{n,0}}{1 - \sum_{i=1}^{q_3} \gamma_{n,i} a^i} d(v_{n-1}, w_{n-1}) + \frac{\sum_{i=1}^{q_3} \gamma_{n,i}}{1 - \sum_{i=1}^{q_3} \gamma_{n,i} a^i} d\left(v_{n-1}, T^i v_{n-1}\right)$$
$$+ \frac{\sum_{i=1}^{q_3} \gamma_{n,i}}{1 - \sum_{i=1}^{q_3} \gamma_{n,i} a^i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(v_{n-1}, T^i v_{n-1}\right)\right)\right] \quad (2.19)$$

Using 2.18 and 2.19 in 2.17, we have

$$d(v_n, w_n) \leq \frac{\alpha_{n,0}}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} \left\{ \frac{\beta_{n,0}}{1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i} \left[\frac{\gamma_{n,0}}{1 - \sum_{i=1}^{q_3} \gamma_{n,i} a^i} d(v_{n-1}, w_{n-1}) \right.\right.$$
$$\left.\left. + \frac{\sum_{i=1}^{q_3} \gamma_{n,i}}{1 - \sum_{i=1}^{q_3} \gamma_{n,i} a^i} d\left(v_{n-1}, T^i v_{n-1}\right) + \frac{\sum_{i=1}^{q_3} \gamma_{n,i}}{1 - \sum_{i=1}^{q_3} \gamma_{n,i} a^i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(v_{n-1}, T^i v_{n-1}\right)\right)\right]\right]\right.$$
$$\left. + \frac{\sum_{i=1}^{q_2} \beta_{n,i}}{1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i} d\left(v_{n-1}, T^i v_{n-1}\right) + \frac{\sum_{i=1}^{q_2} \beta_{n,i}}{1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(v_{n-1}, T^i v_{n-1}\right)\right)\right]\right\}$$
$$+ \frac{\sum_{i=1}^{q_1} \alpha_{n,i}}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(v_n, T^i v_n\right)\right)\right]$$

By simplifying further, we have

(2.20)

$$d(v_n, w_n) \leq \frac{\alpha_{n,0}\beta_{n,0}\gamma_{n,0}}{(1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i)(1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i)(1 - \sum_{i=1}^{q_3} \gamma_{n,i} a^i)} d(v_{n-1}, w_{n-1})$$
$$+ \frac{\alpha_{n,0}\beta_{n,0} \sum_{i=1}^{q_3} \gamma_{n,i}}{(1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i)(1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i)(1 - \sum_{i=1}^{q_3} \gamma_{n,i} a^i)} d\left(v_{n-1}, T^i v_{n-1}\right)$$
$$+ \frac{\alpha_{n,0}\beta_{n,0} \sum_{i=1}^{q_3} \gamma_{n,i}}{(1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i)(1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i)(1 - \sum_{i=1}^{q_3} \gamma_{n,i} a^i)} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(v_{n-1}, T^i v_{n-1}\right)\right)\right]$$
$$+ \frac{\alpha_{n,0} \sum_{i=1}^{q_2} \beta_{n,i}}{(1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i)(1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i)} d\left(v_{n-1}, T^i v_{n-1}\right) + \frac{\alpha_{n,0} \sum_{i=1}^{q_2} \beta_{n,i}}{(1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i)(1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i)} \times$$
$$\left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(v_{n-1}, T^i v_{n-1}\right)\right)\right] + \frac{\sum_{i=1}^{q_1} \alpha_{n,i}}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(v_n, T^i v_n\right)\right)\right]$$

Since $\{\alpha_{n,i}\}, \{\beta_{n,i}\}, \{\gamma_{n,i}\} \subset [0,1]$ and $a \in (0,1)$, then

$$\frac{\alpha_{n,0}\beta_{n,0}\gamma_{n,0}}{(1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i)(1-\sum_{i=1}^{q_2}\beta_{n,i}a^i)(1-\sum_{i=1}^{q_3}\gamma_{n,i}a^i)} \leq [1-(1-a)(1-\alpha_{n,0})] \tag{2.21}$$

As $v_n \to p$ and $F(T) \neq \phi$, we have that

$$\begin{aligned} d\left(v_n, T^i v_n\right) &\leq d\left(v_n, p\right) + d\left(T^i p, T^i v_n\right) \\ &\leq d\left(v_n, p\right) + \sum_{j=1}^{i}\binom{i}{j} a^{i-j}\varphi\left(d\left(p, T^i p\right)\right) + a^i d\left(p, v_n\right) \\ &= (1+a^i) d\left(v_n, p\right) \longrightarrow 0 \end{aligned}$$

This implies $\lim_{n\to\infty} d\left(v_n, T^i v_n\right) = 0$, hence, $\varphi\left(d\left(v_n, T^i v_n\right)\right) = 0$, $\varphi\left(d\left(u_{n-1}, T^i u_{n-1}\right)\right) = 0$.
Application of lemma 2 to inequality 2.20, using 2.21 gives $\lim_{n\to\infty} d\left(v_n, w_n\right) = 0$.
Also, for $\lim_{n\to\infty}(v_n, p) = 0$, then

$$\lim_{n\to\infty} d(w_n, p) \leq \lim_{n\to\infty} d(w_n, v_n) + \lim_{n\to\infty} d(v_n, p)$$

Therefore, $\lim_{n\to\infty} d(w_n, p) = 0$.
For $(iii) \to (iv)$: Assume that $\lim_{n\to\infty} w_n = p$, then, by using 1.17, 2.4 and 2.5, we can easily deduce that $\lim_{n\to\infty} x_n = p$.
Finally, for $(iv) \to (i)$: If $x_n \to p$, then by 1.17, 2.2 and 2.5, we have the following estimate:

$$\begin{aligned} d(x_n, u_n) = d\Big(&W(x_{n-1}^{(1)}, Tx_n, T^2x_n, \ldots, T^{q_1}x_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1}), \\ &W(u_{n-1}, Tu_n, T^2u_n, \ldots, T^{q_1}u_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1})\Big) \\ &\leq \alpha_{n,0} d(x_{n-1}^{(1)}, u_{n-1}) + \sum_{i=1}^{q_1}\alpha_{n,i} d\left(T^i x_n, T^i u_n\right) \\ &\leq \alpha_{n,0} d(x_{n-1}^{(1)}, u_{n-1}) + \sum_{i=1}^{q_1}\alpha_{n,i}\left[\sum_{j=1}^{i}\binom{i}{j} a^{i-j}\varphi\left(d(x_n, T^i x_n)\right) + a^i d\left(x_n, u_n\right)\right] \end{aligned}$$

This gives

$$d(x_n, u_n) \leq \frac{\alpha_{n,0} d(x_{n-1}^{(1)}, u_{n-1})}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i} + \frac{\sum_{i=1}^{q_1}\alpha_{n,i}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}\left[\sum_{j=1}^{i}\binom{i}{j} a^{i-j}\varphi\left(d\left(x_n, T^i x_n\right)\right)\right] \tag{2.22}$$

Likewise, by 1.17 and 2.5, we have

$$\begin{aligned} d(x_{n-1}^{(1)}, u_{n-1}) &= d\left(W(x_{n-1}^{(2)}, Tx_{n-1}^{(1)}, T^2x_{n-1}^{(1)}, \ldots, T^{q_1}x_{n-1}^{(1)}; \beta_{n,0}^{(1)}, \beta_{n,1}^{(1)}, \beta_{n,2}^{(1)}, \ldots, \beta_{n,q_2}^{(1)}), u_{n-1}\right) \\ &\leq \beta_{n,0}^{(1)} d(x_{n-1}^{(2)}, u_{n-1}) + \sum_{i=1}^{q_2}\beta_{n,i}^{(1)} d\left(T^i x_{n-1}^{(1)}, u_{n-1}\right) \\ &\leq \beta_{n,0}^{(1)} d(x_{n-1}^{(2)}, u_{n-1}) + \sum_{i=1}^{q_2}\beta_{n,i}^{(1)} d\left(T^i x_{n-1}^{(1)}, T^i u_{n-1}\right) + \sum_{i=1}^{q_2}\beta_{n,i}^{(1)} d\left(T^i u_{n-1}, u_{n-1}\right) \\ &\leq \beta_{n,0}^{(1)} d(x_{n-1}^{(2)}, u_{n-1}) + \sum_{i=1}^{q_2}\beta_{n,i}^{(1)}\left[\sum_{j=1}^{i}\binom{i}{j} a^{i-j}\varphi\left(d\left(x_{n-1}^{(1)}, T^i x_{n-1}^{(1)}\right)\right) + a^i d\left(x_{n-1}^{(1)}, u_{n-1}\right)\right] \end{aligned}$$

This becomes
(2.23)
$$d(x^{(1)}_{n-1},u_{n-1}) \leq \frac{\beta^{(1)}_{n,0}d(x^{(2)}_{n-1},u_{n-1})}{1-\sum_{i=1}^{q_2}\beta^{(1)}_{n,i}a^i} + \frac{\sum_{i=1}^{q_2}\beta^{(1)}_{n,i}}{1-\sum_{i=1}^{q_2}\beta^{(1)}_{n,i}a^i}\left[\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x^{(1)}_{n-1},T^ix^{(1)}_{n-1}\right)\right)\right]$$

Again, using 2.5, we have

$$\begin{aligned}
d(x^{(2)}_{n-1},u_{n-1}) &= d\left(W(x^{(3)}_{n-1},Tx^{(2)}_{n-1},T^2x^{(2)}_{n-1},\dots,T^{q_1}x^{(2)}_{n-1};\beta^{(2)}_{n,0},\beta^{(2)}_{n,1},\beta^{(2)}_{n,2},\dots,\beta^{(2)}_{n,q_3}),u_{n-1}\right)\\
&\leq \beta^{(2)}_{n,0}d(x^{(3)}_{n-1},u_{n-1}) + \sum_{i=1}^{q_3}\beta^{(2)}_{n,i}d\left(T^ix^{(2)}_{n-1},u_{n-1}\right)\\
&\leq \beta^{(2)}_{n,0}d(x^{(3)}_{n-1},u_{n-1}) + \sum_{i=1}^{q_3}\beta^{(2)}_{n,i}d\left(T^ix^{(2)}_{n-1},T^iu_{n-1}\right) + \sum_{i=1}^{q_3}\beta^{(2)}_{n,i}d\left(T^iu_{n-1},u_{n-1}\right)\\
&\leq \beta^{(2)}_{n,0}d(x^{(3)}_{n-1},u_{n-1}) + \sum_{i=1}^{q_3}\beta^{(2)}_{n,i}\left[\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x^{(2)}_{n-1},T^ix^{(2)}_{n-1}\right)\right) + a^id\left(x^{(2)}_{n-1},u_{n-1}\right)\right]
\end{aligned}$$

This becomes
(2.24)
$$d(x^{(2)}_{n-1},u_{n-1}) \leq \frac{\beta^{(2)}_{n,0}d(x^{(3)}_{n-1},u_{n-1})}{1-\sum_{i=1}^{q_3}\beta^{(2)}_{n,i}a^i} + \frac{\sum_{i=1}^{q_3}\beta^{(2)}_{n,i}}{1-\sum_{i=1}^{q_3}\beta^{(2)}_{n,i}a^i}\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi(d(x^{(2)}_{n-1},T^ix^{(2)}_{n-1}))$$

By continuing these process in 2.5 up to $k-1$, we have

$$\begin{aligned}
d(x^{(k-1)}_{n-1},u_{n-1}) &= d\Big(W(x_{n-1},Tx^{(k-1)}_{n-1},T^2x^{(k-1)}_{n-1},\dots,T^{q_1}x^{(k-1)}_{n-1};\\
&\quad \beta^{(k-1)}_{n,0},\beta^{(k-1)}_{n,1},\beta^{(k-1)}_{n,2},\dots,\beta^{(k-1)}_{n,q_k}),u_{n-1}\Big)\\
&\leq \beta^{(k-1)}_{n,0}d(x_{n-1},u_{n-1}) + \sum_{i=1}^{q_k}\beta^{(k-1)}_{n,i}d\left(T^ix^{(k-1)}_{n-1},u_{n-1}\right)\\
&\leq \beta^{(k-1)}_{n,0}d(x_{n-1},u_{n-1}) + \sum_{i=1}^{q_k}\beta^{(k-1)}_{n,i}d\left(T^ix^{(k-1)}_{n-1},T^iu_{n-1}\right) + \sum_{i=1}^{q_k}\beta^{(k-1)}_{n,i}d\left(T^iu_{n-1},u_{n-1}\right)\\
&\leq \beta^{(k-1)}_{n,0}d(x_{n-1},u_{n-1}) + \sum_{i=1}^{q_k}\beta^{(k-1)}_{n,i}\Bigg[\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x^{(k-1)}_{n-1},T^ix^{(k-1)}_{n-1}\right)\right)\\
&\quad + a^id\left(x^{(k-1)}_{n-1},u_{n-1}\right)\Bigg]
\end{aligned}$$

This further gives
(2.25)
$$d(x^{(k-1)}_{n-1},u_{n-1}) \leq \frac{\beta^{(k-1)}_{n,0}d(x_{n-1},u_{n-1})}{1-\sum_{i=1}^{q_k}\beta^{(k-1)}_{n,i}a^i} + \frac{\sum_{i=1}^{q_k}\beta^{(k-1)}_{n,i}}{1-\sum_{i=1}^{q_k}\beta^{(k-1)}_{n,i}a^i}\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi(d(x^{(k-1)}_{n-1},T^ix^{(k-1)}_{n-1}))$$

Combining 2.22 and 2.23, we obtain

$$d(x_n,u_n) \le \frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}\left(\frac{\beta_{n,0}^{(1)}}{1-\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}a^i}d(x_{n-1}^{(2)},u_{n-1})\times\right. \\ \left.+\frac{\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}}{1-\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}a^i}\left[\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(1)},T^i x_{n-1}^{(1)}\right)\right)\right]\right) \\ +\frac{\sum_{i=1}^{q_1}\alpha_{n,i}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}\left[\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_n,T^i x_n\right)\right)\right] \tag{2.26}$$

Also, substitute 2.24 into 2.26, we have

$$d(x_n,u_n) \le \frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}\left(\frac{\beta_{n,0}^{(1)}}{1-\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}a^i}\left(\frac{\beta_{n,0}^{(2)}}{1-\sum_{i=1}^{q_3}\beta_{n,i}^{(2)}a^i}d(x_{n-1}^{(3)},u_{n-1})\right.\right. \\ \left.+\frac{\sum_{i=1}^{q_3}\beta_{n,i}^{(2)}}{1-\sum_{i=1}^{q_3}\beta_{n,i}^{(2)}a^i}\left[\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(2)},T^i x_{n-1}^{(2)}\right)\right)\right]\right) \\ \left.+\frac{\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}}{1-\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}a^i}\left[\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(1)},T^i x_{n-1}^{(1)}\right)\right)\right]\right) \\ +\frac{\sum_{i=1}^{q_1}\alpha_{n,i}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}\left[\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_n,T^i x_n\right)\right)\right] \tag{2.27}$$

By combining 2.25 and 2.27, we get

(2.28)

$$d(x_n,u_n) \le \frac{\alpha_{n,0}\beta_{n,0}^{(1)}\beta_{n,0}^{(2)}}{(1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i)(1-\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}a^i)(1-\sum_{i=1}^{q_3}\beta_{n,i}^{(2)}a^i)}\cdots\frac{\beta_{n,0}^{(k-1)}d(x_{n-1},u_{n-1})}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}a^i} \\ +\frac{\sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}a^i}\left[\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(k-1)},T^i x_{n-1}^{(k-1)}\right)\right)\right]+\cdots \\ +\frac{\sum_{i=1}^{q_3}\beta_{n,i}^{(2)}}{1-\sum_{i=1}^{q_3}\beta_{n,i}^{(2)}a^i}\left[\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(2)},T^i x_{n-1}^{(2)}\right)\right)\right] \\ +\frac{\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}}{1-\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}a^i}\left[\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(1)},T^i x_{n-1}^{(1)}\right)\right)\right] \\ +\frac{\sum_{i=1}^{q_1}\alpha_{n,i}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}\left[\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_n,T^i x_n\right)\right)\right]$$

Since $\{\alpha_{n,i}\}, \{\beta_{n,i}^{(l)}\} \subset [0,1]$ for $l = 1(1)k-1$ and $a \in (0,1)$, then

$$\frac{\alpha_{n,0}\beta_{n,0}^{(1)}\beta_{n,0}^{(2)}\cdots\beta_{n,0}^{(k-1)}}{(1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i)(1-\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}a^i)\cdots(1-\sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}a^i)} \le [1-(1-a)(1-\alpha_{n,0})] \tag{2.29}$$

While $x_n \to p$ and $F(T) \neq \phi$, we have that

$$\begin{aligned} d\left(x_n, T^i x_n\right) &\leq d\left(x_n, p\right) + d\left(T^i p, T^i x_n\right) \\ &\leq d\left(x_n, p\right) + \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(p, T^i p\right)\right) + a^i d\left(p, x_n\right) \\ &= (1 + a^i) d(x_n, p) \to 0 \end{aligned}$$

We conclude that $\lim_{n\to\infty} d\left(x_n, T^i x_n\right) = 0$ and $\varphi\left(d\left(x_n, T^i x_n\right)\right) = 0$, $\varphi\left(d\left(x_{n-1}, T^i x_{n-1}\right)\right) = 0$. By the application of lemma 2 to inequality 2.28, using 2.29 gives $\lim_{n\to\infty} d\left(x_n, u_n\right) = 0$. Furthermore, since $x_n \to p$, then

$$0 \leq \lim_{n\to\infty} d(u_n, p) \leq \lim_{n\to\infty} d(u_n, x_n) + \lim_{n\to\infty} d(x_n, p) = 0$$

Hence, $u_n \to p$. This completes the proof.

Corollary 3. *Let $T : C \to C$ be a mapping satisfying condition 1.17 with $F(T) \neq \phi$. Then the following iterations are equivalent:*

(i) *For $x_0 \in C$, the Kirk-Mann iteration 1.4 converges to $p \in F(T)$.*
(ii) *For $u_0 \in C$, the implicit Kirk-Mann iteration 2.2 converges to $p \in F(T)$.*

Proof

We prove that $(i) \to (ii)$: Assume $x_n \to p$, by employing 1.4, 1.17 and 2.2, we have the following estimates

$$\begin{aligned} d(x_n, u_n) = d\big(&W(x_{n-1}, Tx_{n-1}, T^2 x_{n-1}, \ldots, T^{q_1} x_{n-1}; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1}), \\ &W(u_{n-1}, Tu_n, T^2 u_n, \ldots, T^{q_1} u_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1})\big) \end{aligned}$$

$$\begin{aligned} d(x_n, u_n) &\leq \alpha_{n,0} d(x_{n-1}, u_{n-1}) + \sum_{i=1}^{q_1} \alpha_{n,i} d\left(T^i x_{n-1}, T^i u_n\right) \\ &\leq \alpha_{n,0} d(x_{n-1}, u_{n-1}) + \sum_{i=1}^{q_1} \alpha_{n,i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d(x_{n-1}, T^i x_{n-1})\right) + a^i d\left(x_{n-1}, u_n\right)\right] \\ &\leq \alpha_{n,0} d(x_{n-1}, u_{n-1}) + \sum_{i=1}^{q_1} \alpha_{n,i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d(x_{n-1}, T^i x_{n-1})\right) + a^i d(x_{n-1}, x_n)\right. \\ &\quad \left. + a^i d\left(x_n, u_n\right)\right] \end{aligned}$$

Since $d(p, x_n) = 0$ as $n \to \infty$ and $F(T) \neq \phi$, then

$$0 \leq d(x_{n-1}, T^i x_{n-1}) \leq d(x_{n-1}, p) + d(p, T^i x_{n-1}) \to 0$$

Hence, $\lim_{n\to\infty} d(x_{n-1}, T^i x_{n-1}) = 0$ implies $\varphi(d(x_{n-1}, T^i x_{n-1})) = 0$.
Therefore,

$$d(x_n, u_n) \leq \frac{\alpha_{n,0}}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} d(x_{n-1}, u_{n-1}) + \mu_n$$

where $\mu_n = \sum_{i=1}^{q_1} \alpha_{n,i} a^i d(x_{n-1}, x_n)$ and by lemma 2 and the fact that $\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1} \alpha_{n,i} a^i} \leq [1 - (1-a)(1-\alpha_{n,0})]$, we have that $\lim_{n\to\infty} d(x_n, u_n) = 0$.
Moreover, $d(p, x_n) = 0$, it follows that

$$0 \leq d(u_n, p) \leq d(u_n, x_n) + d(x_n, p) \to 0$$

and this implies that $\lim_{n\to\infty} d(u_n, p) = 0$.
$(ii) \to (i)$: Assume (ii), that is $\lim_{n\to\infty} u_n = p$, then by the condition 1.17 and the iterations 1.4, 2.2, we have

$$\begin{aligned}
d(u_n,x_n) &= d\left(W(u_{n-1}, Tu_n, T^2u_n, \dots, T^{q_1}u_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,q_1}),\right.\\
&\quad \left. W(x_{n-1}, Tx_{n-1}, T^2x_{n-1}, \dots, T^{q_1}x_{n-1}; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,q_1})\right)\\
&\le \alpha_{n,0} d(u_{n-1}, x_{n-1}) + \sum_{i=1}^{q_1} \alpha_{n,i} d\left(T^i u_n, T^i x_{n-1}\right)\\
&\le \alpha_{n,0} d(u_{n-1}, x_{n-1}) + \sum_{i=1}^{q_1} \alpha_{n,i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d(u_n, T^i u_n)\right) + a^i d\left(u_n, x_{n-1}\right)\right]\\
&\le \alpha_{n,0} d(u_{n-1}, x_{n-1}) + \sum_{i=1}^{q_1} \alpha_{n,i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d(u_n, T^i u_n)\right)\right.\\
&\quad \left. + a^i(d(u_n, u_{n-1}) + d\left(u_{n-1}, x_{n-1}\right))\right]
\end{aligned}$$

This implies

$$d(u_n, x_n) \le \sum_{i=0}^{q_1} \alpha_{n,i} a^i d(u_{n-1}, x_{n-1}) + \sum_{i=1}^{q_1} \alpha_{n,i} \left[\sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi(d(u_n, T^i u_n)) + a^i d(u_n, u_{n-1})\right]$$

While $u_n \to p$ and $F(T) \neq \phi$, it follows that

$$\begin{aligned}
d\left(u_n, T^i u_n\right) &\le d\left(u_n, p\right) + d\left(T^i p, T^i u_n\right)\\
&\le d\left(u_n, p\right) + \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(p, T^i p\right)\right) + a^i d\left(p, u_n\right)\\
&= (1 + a^i) d\left(u_n, p\right) \longrightarrow 0
\end{aligned}$$

which implies $\lim_{n\to\infty} d(u_n, T^i u_n) = 0$, $d(u_{n-1}, T^i u_{n-1}) = 0$ and $\varphi\left(d\left(u_n, T^i u_n\right)\right) = 0$. By lemma 2 and the fact $\alpha_{n,0} + \sum_{i=1}^{q_1} \alpha_{n,i} a^i = 1 - (1-a)(1-\alpha_{n,0}) < 1$, we can deduce that $\lim_{n\to\infty} d\left(u_n, x_n\right) = 0$.
More so, $\lim_{n\to\infty} u_n = p$, then

$$d(x_n, p) \le d(x_n, u_n) + d(u_n, p) \longrightarrow 0$$

Hence, $\lim_{n\to\infty} x_n = p$.

Corollary 4. *Let $T : C \to C$ be a mapping satisfying condition 1.15 with $F(T) \neq \phi$. Suppose all initial guess are the same, then the following iterative schemes are equivalent:*

(i) *The Kirk-Mann iteration converges to $p \in F(T)$.*
(ii) *The implicit Kirk-Mann iteration converges to $p \in F(T)$.*
(iii) *The Kirk-Ishikawa iteration converges to $p \in F(T)$.*
(iv) *The Kirk-Thianwan iteration converges to $p \in F(T)$.*
(v) *The implicit Kirk-Ishikawa iteration converges to $p \in F(T)$.*
(vi) *The Kirk-Noor iteration converges to $p \in F(T)$.*
(vii) *The Kirk-sp iteration converges to $p \in F(T)$.*
(viii) *The implicit Kirk-Noor iteration converges to $p \in F(T)$.*
(ix) *The Kirk-Multistep iteration converges to $p \in F(T)$.*
(x) *The Kirk-Multistep-SP iteration converges to $p \in F(T)$.*
(xi) *The implicit Kirk-Multistep iteration converges to $p \in F(T)$.*

Corollary 5. *Let $T : C \to C$ be a mapping satisfying condition 1.15 with $F(T) \neq \phi$. Suppose all initial guess are the same, then the following iterations are equivalent:*

(i) *The implicit Mann iteration converges to $p \in F(T)$.*
(ii) *The implicit Ishikawa iteration converges to $p \in F(T)$.*
(iii) *The implicit Noor iteration converges to $p \in F(T)$.*
(iv) *The implicit Multistep iteration converges to $p \in F(T)$.*

Theorem 3. *Let C be a non-empty closed subset of a generalized convex metric space (X, d, W) and T be a self map of C satisfying the quasi-contractive condition 1.17 with $F(T) \neq \phi$. Then, for $x_0 \in X$ and $p \in F(T)$, the sequence $\{x_n\}$ defined by*

(i) *2.2 is T-stable.*
(ii) *2.3 is T-stable.*
(iii) *2.4 is T-stable.*
(iv) *2.5 is T-stable.*

Proof
We prove Theorem 3(iii) as follow: Let $\{u_n\} \in C$ be an arbitrary sequence and let $\epsilon_n = d\left(u_n, W(v_{n-1}, Tu_n, T^2u_n, \ldots, T^{q_1}u_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1})\right)$, where

$$v_{n-1} = W(w_{n-1}, Tv_{n-1}, T^2v_{n-1}, \ldots, T^{q_1}v_{n-1}; \beta_{n,0}, \beta_{n,1}, \beta_{n,2}, \ldots, \beta_{n,q_2}),\ \sum_{i=1}^{q_2} \beta_{n,i} = 1$$

$$w_{n-1} = W(u_{n-1}, Tw_{n-1}, T^2w_{n-1}, \ldots, T^{q_1}w_{n-1}; \gamma_{n,0}, \gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,q_2}),\ \sum_{i=1}^{q_3} \gamma_{n,i} = 1$$

Let $\lim_{n\to\infty} \epsilon_n = 0$ and using condition 1.17 for $p \in F(T)$, we have

$$\begin{aligned} d(u_n, p) &\leq d\left(u_n, W(v_{n-1}, Tu_n, T^2u_n, \ldots, T^{q_1}u_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1})\right) \\ &\quad + d\left(W(v_{n-1}, Tu_n, T^2u_n, \ldots, T^{q_1}u_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1}), p\right) \\ &\leq \epsilon_n + \alpha_{n,0} d(v_{n-1}, p) + \sum_{i=1}^{q_1} \alpha_{n,i} d\left(T^i u_n, p\right) \\ &\leq \epsilon_n + \alpha_{n,0} d(v_{n-1}, p) + \sum_{i=1}^{q_1} \alpha_{n,i} a^i d(u_n, p) \end{aligned}$$

This becomes,

$$d(u_n, p) \leq \frac{\alpha_{n,0}}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} d(v_{n-1}, p) + \frac{\epsilon_n}{1 - \sum_{i=1}^{q_1} \alpha_{n,i} a^i} \tag{2.30}$$

But

$$d(v_{n-1}, p) \leq \frac{\beta_{n,0}}{1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i} d(w_{n-1}, p) + \frac{\epsilon_n}{1 - \sum_{i=1}^{q_2} \beta_{n,i} a^i} \tag{2.31}$$

and

$$d(w_{n-1}, p) \leq \frac{\gamma_{n,0}}{1 - \sum_{i=1}^{q_3} \gamma_{n,i} a^i} d(u_{n-1}, p) + \frac{\epsilon_n}{1 - \sum_{i=1}^{q_3} \gamma_{n,i} a^i} \tag{2.32}$$

Inserting 2.31 and 2.32 into 2.30 gives

(2.33)
$$\begin{aligned} d\left(u_n,p\right) \leq \frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}\left(\frac{\beta_{n,0}}{1-\sum_{i=1}^{q_2}\beta_{n,i}a^i}\left(\frac{\gamma_{n,0}}{1-\sum_{i=1}^{q_3}\gamma_{n,i}a^i}d\left(u_{n-1},p\right)\right.\right. \\ \left.\left.+\frac{\epsilon_n}{1-\sum_{i=1}^{q_3}\gamma_{n,i}a^i}\right)+\frac{\epsilon_n}{1-\sum_{i=1}^{q_2}\beta_{n,i}a^i}\right)+\frac{\epsilon_n}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}\end{aligned}$$

By applying 2.12 to 2.33, we have

(2.34)
$$d\left(u_n,p\right) \leq \left(\sum_{i=0}^{q_1}\alpha_{n,i}a^i\right)d\left(u_{n-1},p\right)+\frac{\epsilon_n}{\left(1-\sum_{i=1}^{q_3}\gamma_{n,i}a^i\right)\left(1-\sum_{i=1}^{q_2}\beta_{n,i}a^i\right)\left(1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i\right)}$$

Since $\lim_{n\to\infty}\epsilon_n=0$, then by lemma 2, we have that $\lim_{n\to\infty}d\left(u_n,p\right)=0$.
Conversely, if $\lim_{n\to\infty}d\left(u_n,p\right)=0$ for $p\in F(T)$, then, by quasi-contractive condition 1.17, we have that $\lim_{n\to\infty}\epsilon_n=0$.
Therefore, the iterative scheme 2.4 is T-stable.
The proof of Theorem 3(i) and (ii) is easily seen in Theorem 3(iii).

Remark 2. *The stability results for Theorem 3 (i), (ii) and (iv) are obvious since by Theorem 2, the implicit Kirk-type iterations are equivalent.*

The following Theorem discusses the data dependence results of our iterative schemes.

Theorem 4. *Let C be a closed subset of a generalized convex metric space (X,d,W) and T be a self map of C into itself satisfying 1.17 with $a^i\leq a\in(0,1)$, $i\in\mathbb{N}$ and let S be an approximate operator of T. Suppose $\{x_n\},\{u_n\}\subset C$ are two iterative sequences associated to T, S respectively, where $\{x_n\}$ is the iteration 2.5 and $\{u_n\}$ is given as:*

(2.35)
$$\begin{aligned} u_{n-1}^{(k-1)} &= W(u_{n-1},Tu_{n-1}^{(k-1)},T^2u_{n-1}^{(k-1)},\ldots,T^{q_k}u_{n-1}^{(k-1)};\beta_{n,0}^{(k-1)},\beta_{n,1}^{(k-1)},\beta_{n,2}^{(k-1)},\ldots,\beta_{n,q_k}^{(k-1)}); \\ u_{n-1}^{(l)} &= W(u_{n-1}^{(l+1)},Tu_{n-1}^{(l)},T^2u_{n-1}^{(l)},\ldots,T^{q_{l+1}}u_{n-1}^{(l)};\beta_{n,0}^{(l)},\beta_{n,1}^{(l)},\beta_{n,2}^{(l)},\ldots,\beta_{n,q_{l+1}}^{(l)}); \\ u_n &= W(u_{n-1}^{(1)},Tu_n,T^2u_n,\ldots,T^{q_1}u_n;\alpha_{n,0},\alpha_{n,1},\alpha_{n,2},\ldots,\alpha_{n,q_1});\ n\geq 1\end{aligned}$$

where $\sum_{i=0}^{q_1}\alpha_{n,i}=1$, $\sum_{i=0}^{q_{l+1}}\beta_{n,i}^{(l)}=1$, $\alpha_{n,i},\beta_{n,i}^{(l)}\subset[0,1]$, $l=1,2,\ldots,k-1$, with $\sum\left(1-\alpha_{n,0}\right)=\infty$. Then, for any given $\epsilon>0$, $p\in F(T)$, $q\in F(S)$ and $k\in\mathbb{N}$ we have

$$d(p,q)\leq\frac{k\epsilon}{(1-a)^2},\ a\in(0,1)$$

Proof

By Definition 3, Theorem 2, 3, iterative schemes 2.5, 2.35 and condition 1.17, we have

$$\begin{aligned} d(x_n,u_n) &= d(W(x_{n-1}^{(1)},Tx_n,T^2x_n,\ldots,T^{q_1}x_n;\alpha_{n,0},\alpha_{n,1},\alpha_{n,2},\ldots,\alpha_{n,q_1}), \\ &\quad W(u_{n-1}^{(1)},Tu_n,T^2u_n,\ldots,T^{q_1}u_n;\alpha_{n,0},\alpha_{n,1},\alpha_{n,2},\ldots,\alpha_{n,q_1})) \\ &\leq \alpha_{n,0}d(x_{n-1}^{(1)},u_{n-1}^{(1)})+\sum_{i=1}^{q_1}\alpha_{n,i}a^id(T^ix_n,S^iu_n) \\ &\leq \alpha_{n,0}d(x_{n-1}^{(1)},u_{n-1}^{(1)})+\sum_{i=1}^{q_1}\alpha_{n,i}\left(d(T^ix_n,S^ix_n)+d(S^ix_n,S^iu_n)\right) \\ &\leq \alpha_{n,0}d(x_{n-1}^{(1)},u_{n-1}^{(1)})+\sum_{i=1}^{q_1}\alpha_{n,i}\left[\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_n,S^ix_n\right)\right)+a^id(x_n,u_n)\right]\end{aligned}$$

which implies
(2.36)
$$d(x_n, u_n) \leq \frac{\alpha_{n,0} d(x_{n-1}^{(1)}, u_{n-1}^{(1)})}{1-\sum_{i=1}^{q_1} \alpha_{n,i} a^i} + \frac{\sum_{i=1}^{q_1} \alpha_{n,i}}{1-\sum_{i=1}^{q_1} \alpha_{n,i} a^i} \left(\epsilon + \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(x_n, S^i x_n\right)\right)\right)$$

Also, using 2.5 and 2.35, we can easily obtain the followings:
(2.37)
$$d(x_{n-1}^{(1)}, u_{n-1}^{(1)}) \leq \frac{\beta_{n,0}^{(1)} d(x_{n-1}^{(2)}, u_{n-1}^{(2)})}{1-\sum_{i=1}^{q_2} \beta_{n,i}^{(1)} a^i} + \frac{\sum_{i=1}^{q_2} \beta_{n,i}^{(1)}}{1-\sum_{i=1}^{q_2} \beta_{n,i}^{(1)} a^i} \left(\epsilon + \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(x_{n-1}^{(1)}, S^i x_{n-1}^{(1)}\right)\right)\right)$$

(2.38)
$$d(x_{n-1}^{(2)}, u_{n-1}^{(2)}) \leq \frac{\beta_{n,0}^{(2)} d(x_{n-1}^{(3)}, u_{n-1}^{(3)})}{1-\sum_{i=1}^{q_3} \beta_{n,i}^{(2)} a^i} + \frac{\sum_{i=1}^{q_3} \beta_{n,i}^{(2)}}{1-\sum_{i=1}^{q_3} \beta_{n,i}^{(2)} a^i} \left(\epsilon + \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(x_{n-1}^{(2)}, S^i x_{n-1}^{(2)}\right)\right)\right)$$

(2.39)
$$d(x_{n-1}^{(3)}, u_{n-1}^{(3)}) \leq \frac{\beta_{n,0}^{(3)} d(x_{n-1}^{(4)}, u_{n-1}^{(4)})}{1-\sum_{i=1}^{q_4} \beta_{n,i}^{(3)} a^i} + \frac{\sum_{i=1}^{q_4} \beta_{n,i}^{(3)}}{1-\sum_{i=1}^{q_4} \beta_{n,i}^{(3)} a^i} \left(\epsilon + \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(x_{n-1}^{(3)}, S^i x_{n-1}^{(3)}\right)\right)\right)$$

up to

(2.40)
$$\begin{aligned} d(x_{n-1}^{(k-2)}, u_{n-1}^{(k-2)}) \leq{}& \frac{\beta_{n,0}^{(k-2)} d(x_{n-1}^{(k-1)}, u_{n-1}^{(k-1)})}{1-\sum_{i=1}^{q_{k-1}} \beta_{n,i}^{(k-2)} a^i} + \frac{\sum_{i=1}^{q_{k-1}} \beta_{n,i}^{(k-2)}}{1-\sum_{i=1}^{q_{k-1}} \beta_{n,i}^{(k-2)} a^i} \times \\ & \left(\epsilon + \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(x_{n-1}^{(k-2)}, S^i x_{n-1}^{(k-2)}\right)\right)\right) \end{aligned}$$

(2.41)
$$\begin{aligned} d(x_{n-1}^{(k-1)}, u_{n-1}^{(k-1)}) \leq{}& \frac{\beta_{n,0}^{(k-1)} d(x_{n-1}, u_{n-1})}{1-\sum_{i=1}^{q_k} \beta_{n,i}^{(k-1)} a^i} + \frac{\sum_{i=1}^{q_k} \beta_{n,i}^{(k-1)}}{1-\sum_{i=1}^{q_k} \beta_{n,i}^{(k-1)} a^i} \times \\ & \left(\epsilon + \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(x_{n-1}^{(k-1)}, S^i x_{n-1}^{(k-1)}\right)\right)\right) \end{aligned}$$

Combining inequalities 2.36-2.41, we have
(2.42)
$$\begin{aligned} d(x_n, u_n) \leq{}& \frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1} \alpha_{n,i} a^i} \frac{\beta_{n,0}^{(1)}}{1-\sum_{i=1}^{q_2} \beta_{n,i}^{(1)} a^i} \frac{\beta_{n,0}^{(2)}}{1-\sum_{i=1}^{q_3} \beta_{n,i}^{(2)} a^i} \cdots \frac{\beta_{n,0}^{(k-1)} d(x_{n-1}, u_{n-1})}{1-\sum_{i=1}^{q_k} \beta_{n,i}^{(k-1)} a^i} \\ & + \frac{\sum_{i=1}^{q_k} \beta_{n,i}^{(k-1)}}{1-\sum_{i=1}^{q_k} \beta_{n,i}^{(k-1)} a^i} \left(\epsilon + \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(x_{n-1}^{(k-1)}, S^i x_{n-1}^{(k-1)}\right)\right)\right) + \cdots \\ & + \frac{\sum_{i=1}^{q_3} \beta_{n,i}^{(2)}}{1-\sum_{i=1}^{q_3} \beta_{n,i}^{(2)} a^i} \left(\epsilon + \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(x_{n-1}^{(2)}, S^i x_{n-1}^{(2)}\right)\right)\right) \\ & + \frac{\sum_{i=1}^{q_2} \beta_{n,i}^{(1)}}{1-\sum_{i=1}^{q_2} \beta_{n,i}^{(1)} a^i} \left(\epsilon + \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(x_{n-1}^{(1)}, S^i x_{n-1}^{(1)}\right)\right)\right) \\ & + \frac{\sum_{i=1}^{q_1} \alpha_{n,i}}{1-\sum_{i=1}^{q_1} \alpha_{n,i} a^i} \left(\epsilon + \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi\left(d\left(x_n, S^i x_n\right)\right)\right) \end{aligned}$$

By further simplifying 2.42, we get

(2.43)

$$\begin{aligned}
d(x_n,y_n) \leq{}& \frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}\frac{\beta_{n,0}^{(1)}}{1-\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}a^i}\frac{\beta_{n,0}^{(2)}}{1-\sum_{i=1}^{q_3}\beta_{n,i}^{(2)}a^i}\cdots\frac{\beta_{n,0}^{(k-1)}d(x_{n-1},u_{n-1})}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}a^i}\\
&+\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}\frac{\beta_{n,0}^{(1)}}{1-\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}a^i}\cdots\frac{\sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{(k-1)}a^i}\\
&\times\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(k-1)},S^ix_{n-1}^{(k-1)}\right)\right)\right)+\cdots\\
&+\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}.\frac{\beta_{n,0}^{(1)}}{1-\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}a^i}.\frac{\sum_{i=1}^{q_3}\beta_{n,i}^{(2)}}{1-\sum_{i=1}^{q_3}\beta_{n,i}^{(2)}a^i}\\
&\times\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(2)},S^ix_{n-1}^{(2)}\right)\right)\right)\\
&+\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}.\frac{\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}}{1-\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}a^i}\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(1)},S^ix_{n-1}^{(1)}\right)\right)\right)\\
&+\frac{\sum_{i=1}^{q_1}\alpha_{n,i}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}a^i}\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_n,S^ix_n\right)\right)\right)
\end{aligned}$$

Applying 2.29 to 2.43, we have

(2.44)

$$\begin{aligned}
d(x_n,u_n) \leq{}& \frac{\alpha_{n,0}}{1-(1-\alpha_{n,0})a}d(x_{n-1},u_{n-1})\\
&+\frac{1-\beta_{n,0}^{(k-1)}}{1-(1-\beta_{n,0}^{(k-1)})a}\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d(x_{n-1}^{(k-1)},S^ix_{n-1}^{(k-1)})\right)\right)+\cdots\\
&+\frac{1-\beta_{n,0}^{(2)}}{1-(1-\beta_{n,0}^{(2)})a}\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(2)},S^ix_{n-1}^{(2)}\right)\right)\right)\\
&+\frac{1-\beta_{n,0}^{(1)}}{1-(1-\beta_{n,0}^{(1)})a}\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(1)},S^ix_{n-1}^{(1)}\right)\right)\right)\\
&+\frac{1-\alpha_{n,0}}{1-(1-\alpha_{n,0})a}\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_n,S^ix_n\right)\right)\right)
\end{aligned}$$

This implies

(2.45)
$$\begin{aligned} d(x_n,u_n) &\leq \left[1-(1-a)(1-\alpha_{n,0})\right] d(x_{n-1},u_{n-1}) \\ &+ \frac{(1-a)(1-\beta_{n,0}^{(k-1)})}{(1-a)(1-a)}\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(k-1)},S^i x_{n-1}^{(k-1)}\right)\right)\right)+\cdots \\ &+ \frac{(1-a)(1-\beta_{n,0}^{(2)})}{(1-a)(1-a)}\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(2)},S^i x_{n-1}^{(2)}\right)\right)\right) \\ &+ \frac{(1-a)(1-\beta_{n,0}^{(1)})}{(1-a)(1-a)}\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(1)},S^i x_{n-1}^{(1)}\right)\right)\right) \\ &+ \frac{(1-a)(1-\alpha_{n,0})}{(1-a)(1-a)}\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_n,S^i x_n\right)\right)\right) \end{aligned}$$

This gives

$$\begin{aligned} d(x_n,u_n) &\leq \left[1-(1-a)(1-\alpha_{n,0})\right] d(x_{n-1},u_{n-1}) \\ &+ \frac{(1-a)(1-\alpha_{n,0})}{(1-a)(1-a)}\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(k-1)},S^i x_{n-1}^{(k-1)}\right)\right)\right)+\cdots \\ &+ \frac{(1-a)(1-\alpha_{n,0})}{(1-a)(1-a)}\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(2)},S^i x_{n-1}^{(2)}\right)\right)\right) \\ &+ \frac{(1-a)(1-\alpha_{n,0})}{(1-a)(1-a)}\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_{n-1}^{(1)},S^i x_{n-1}^{(1)}\right)\right)\right) \\ &+ \frac{(1-a)(1-\alpha_{n,0})}{(1-a)(1-a)}\left(\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\varphi\left(d\left(x_n,S^i x_n\right)\right)\right) \end{aligned} \tag{2.46}$$

This furthermore implies

(2.47)
$$\begin{aligned} d(x_n,u_n) &\leq \left[1-(1-a)(1-\alpha_{n,0})\right] d(x_{n-1},u_{n-1}) + \frac{(1-a)(1-\alpha_{n,0})}{(1-a)(1-a)}\times \\ &\left\{k\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\left[\varphi\left(d(x_{n-1}^{(k-1)},S^i x_{n-1}^{(k-1)})\right)+\cdots+\varphi\left(d(x_{n-1}^{(2)},S^i x_{n-1}^{(2)})\right)\right.\right. \\ &\left.\left.+\varphi\left(d(x_{n-1}^{(1)},S^i x_{n-1}^{(1)})\right)+\varphi\left(d(x_n,S^i x_n)\right)\right]\right\} \end{aligned}$$

If we let $a_n = d(x_n,u_n)$, $r_n = (1-a)(1-\alpha_{n,0})$ and
$t_n = \frac{1}{(1-a)^2}\left[\; k\epsilon+\sum_{j=1}^{i}\binom{i}{j}a^{i-j}\left[\sum_{l=1}^{k-1}\varphi\left(d(x_{n-1}^{(l)},S^i x_{n-1}^{(l)})\right)+\varphi\left(d(x_n,S^i x_n)\right)\right]\;\right]$ in 2.47.
Then, inequality 2.47 reduces to the form:

$$a_n \leq (1-r_n)a_{n-1}+r_n t_n$$

By the property of φ and Theorem 2, we have

$$\varphi\left(d(x_n, S^i x_n)\right) = 0 = \sum_{l=1}^{k-1} \varphi\left(d(x_{n-1}^{(l)}, S^i x_{n-1}^{(l)})\right)$$

Thus, by Lemma 3 and Theorem 1, we conclude that

$$d(p,q) \leq \frac{k\epsilon}{(1-a)^2}$$

Theorem 5. *Let C be a closed subset of a generalized convex metric space (X,d,W) and T be a self map of C into itself satisfying 1.17 with $a^i \leq a \in (0,1)$, $i \in \mathbb{N}$ and let S be an approximate operator of T. Suppose $\{x_n\}, \{u_n\} \subset C$ are two iterative sequences associated to T, S respectively, where $\{x_n\}$ is the iteration 2.4 and $\{u_n\}$ is given as:*

$$u_{n-1}^{(2)} = W(u_{n-1}, Tu_{n-1}^{(2)}, T^2 u_{n-1}^{(2)}, \ldots, T^{q_3} u_{n-1}^{(2)}; \gamma_{n,0}, \gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,q_3}); \ \sum_{i=0}^{q_3} \gamma_{n,i} = 1$$

$$u_{n-1}^{(1)} = W(u_{n-1}^{(2)}, Tu_{n-1}^{(1)}, T^2 u_{n-1}^{(1)}, \ldots, T^{q_2} u_{n-1}^{(1)}; \beta_{n,0}, \beta_{n,1}, \beta_{n,2}, \ldots, \beta_{n,q_2}); \ \sum_{i=0}^{q_2} \beta_{n,i} = 1$$

$$u_n = W(u_{n-1}^{(1)}, Tu_n, T^2 u_n, \ldots, T^{q_1} u_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,q_1}); \ \sum_{i=0}^{q_1} \alpha_{n,i} = 1, \ n \geq 1$$

where $\alpha_{n,i}$, $\beta_{n,i}$, $\gamma_{n,i} \subset [0,1]$, with $\sum (1-\alpha_{n,0}) = \infty$.
Then, for any given $\epsilon > 0$, $p \in F(T)$ and $q \in F(S)$, we have

$$d(p,q) \leq \frac{3\epsilon}{(1-a)^2}, \ a \in (0,1)$$

The proof of this Theorem follows from Theorem 4 if we let $k = 3$.

Remark 3. *(i) The data dependence results for implicit Kirk-Mann and implicit Kirk-Ishikawa iterations follow from Theorem 4. (ii) The data dependence results for implicit type iterations can be easily deduced from Theorem 4 and 5.*

3. COMPARISON OF FASTNESS AMONG ITERATIVE SCHEMES

We compare our iterative schemes with other iterations in the literature by using the following Example.

Example 1. *Let $T : [0,1] \to [0,1]$ and $Tx = \frac{x}{2}$ with initial guess $x_0 \neq 0$ and fixed point $p = 0$ using $\alpha_{n,0} = \beta_{n,0}^{(l)} = 1 - \frac{4}{\sqrt{n}}$, $\alpha_{n,i} = \beta_{n,i}^{(l)} = \frac{2}{\sqrt{n}}$, for each l, $n \geq 25$ and $q_1 = q_2 = q_3 = \ldots = q_k = 2$.*

For the implicit Kirk-Mann iteration (IKM), we have

$$x_n = W(x_{n-1}, Tx_n, T^2 x_n; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}) = \left(1 - \frac{4}{\sqrt{n}}\right) x_{n-1} + \frac{1}{\sqrt{n}} x_n + \frac{1}{2\sqrt{n}} x_n$$

This implies

$$x_n(\text{IKM}) = \frac{2\sqrt{n}-8}{2\sqrt{n}-3} x_{n-1} = \prod_{r=25}^{n} \left(\frac{2\sqrt{r}-8}{2\sqrt{r}-3}\right) x_0$$

Also, the implicit Kirk-Ishikawa iteration (IKI), we have

$$x_n = \frac{2\sqrt{n}-8}{2\sqrt{n}-3} x_{n-1}^{(1)}$$

with

$$x_{n-1}^{(1)} = \frac{2\sqrt{n}-8}{2\sqrt{n}-3} x_{n-1}$$

Hence,

$$x_n(\text{IKI}) = \left(\frac{2\sqrt{n}-8}{2\sqrt{n}-3}\right)^2 x_{n-1} = \prod_{r=25}^{n} \left(\frac{2\sqrt{r}-8}{2\sqrt{r}-3}\right)^2 x_0$$

Similarly, implicit Kirk Noor iteration (IKN) implies

$$x_n(\text{IKN}) = \left(\frac{2\sqrt{n}-8}{2\sqrt{n}-3}\right)^3 x_{n-1} = \prod_{r=25}^{n} \left(\frac{2\sqrt{r}-8}{2\sqrt{r}-3}\right)^3 x_0$$

While the implicit multistep Kirk iteration (IMK) gives

$$x_n(\text{IMK}) = \prod_{r=25}^{n} \left(\frac{2\sqrt{r}-8}{2\sqrt{r}-3}\right)^k x_0$$

Now, using Definition 2, we compare the implicit Kirk type iterations as follows: For $k \geq 4$, we have

$$\frac{|x_n(\text{IMK}) - 0|}{|x_n(\text{IKN}) - 0|} = \prod_{r=25}^{n} \left(\frac{2\sqrt{r}-8}{2\sqrt{r}-3}\right)^{k-3} = \prod_{r=25}^{n} \left(1 - \frac{5}{2\sqrt{r}-3}\right)^{k-3}$$

with

$$0 \leq \lim_{n\to\infty} \prod_{r=25}^{n} \left(1 - \frac{5}{2\sqrt{r}-3}\right)^{k-3} \leq \lim_{n\to\infty} \prod_{r=25}^{n} \left(1 - \frac{1}{r}\right)^{k-3}$$
$$= \left(\lim_{n\to\infty} \frac{24}{25} \cdot \frac{25}{26} \cdots \frac{n-2}{n-1} \cdot \frac{n-1}{n}\right)^{k-3} = \left(\lim_{n\to\infty} \frac{24}{n}\right)^{k-3} = 0, \ \forall k \geq 4$$

Remark 4. *The implicit multistep Kirk iteration (IMK) converges faster than the implicit Kirk Noor iteration (IKN) for* $k = 4, 5, \ldots$.

Also,

$$\frac{|x_n(\text{IKN}) - 0|}{|x_n(\text{IKI}) - 0|} = \prod_{r=25}^{n} \left(\frac{2\sqrt{r}-8}{2\sqrt{r}-3}\right) = \prod_{r=25}^{n} \left(1 - \frac{5}{2\sqrt{r}-3}\right)$$

with

$$0 \leq \lim_{n\to\infty} \prod_{r=25}^{n} \left(1 - \frac{5}{2\sqrt{r}-3}\right) \leq \lim_{n\to\infty} \prod_{r=25}^{n} \left(1 - \frac{1}{r}\right)$$
$$= \lim_{n\to\infty} \frac{24}{25} \cdot \frac{25}{26} \cdots \frac{n-2}{n-1} \cdot \frac{n-1}{n} = \lim_{n\to\infty} \frac{24}{n} = 0$$

Remark 5. *The implicit Kirk Noor iteration* $x_n(IKN)$ *converges to* $p = 0$ *faster than the implicit Kirk-Ishikawa iteration* $x_n(IKI)$ *to* $p = 0$.

Similarly, using Definition 2, we have that

$$\lim_{n\to\infty} \frac{|x_n(\text{IKI}) - 0|}{|x_n(\text{IKM}) - 0|} = 0$$

which implies that the implicit Kirk-Ishikawa iteration $x_n(\text{IKI})$ converges faster than the implicit Kirk-Mann iteration $x_n(\text{IKM})$.
For the Kirk-Mann iteration (KM), we have the following estimate:

$$x_n = W(x_{n-1}, Tx_{n-1}, T^2x_{n-1}; \alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}) = \left(1 - \frac{4}{\sqrt{n}}\right) x_{n-1} + \frac{1}{\sqrt{n}} x_{n-1} + \frac{1}{2\sqrt{n}} x_{n-1}$$

This implies

$$x_n(\text{KM}) = \left(1 - \frac{5}{2\sqrt{n}}\right) x_{n-1} = \prod_{r=25}^{n} \left(1 - \frac{5}{2\sqrt{r}}\right) x_0$$

The estimates for Kirk-Thianwan (KT), Kirk-SP (KSP) and Kirk-Multistep-SP (KMSP) iterations are, respectively,

$$x_n(\text{KT}) = \prod_{r=25}^{n} \left(1 - \frac{5}{2\sqrt{r}}\right)^2 x_0,$$

$$x_n(\text{KSP}) = \prod_{r=25}^{n} \left(1 - \frac{5}{2\sqrt{r}}\right)^3 x_0$$

and

$$x_n(\text{KMSP}) = \prod_{r=25}^{n} \left(1 - \frac{5}{2\sqrt{r}}\right)^k x_0$$

We compare Kirk-Mann (KM), Kirk-Thianwan (KT), Kirk-SP (KSP) and Kirk-Multistep-SP (KMSP) iterations with our iterative schemes as follows:
Again, using Definition 2, we have

$$\begin{aligned}\frac{|x_n(\text{IKM}) - 0|}{|x_n(\text{KM}) - 0|} &= \prod_{r=25}^{n} \left(\frac{2\sqrt{r} - 8}{2\sqrt{r} - 3}\right) \left(\frac{2\sqrt{r}}{2\sqrt{r} - 5}\right) \\ &= \prod_{r=25}^{n} \left(\frac{4r - 16\sqrt{r}}{4r - 16\sqrt{r} + 15}\right) \\ &= \prod_{r=25}^{n} \left(1 - \frac{15}{4r - 16\sqrt{r} + 15}\right)\end{aligned}$$

with

$$\begin{aligned}0 &\leq \lim_{n\to\infty} \prod_{r=25}^{n} \left(1 - \frac{15}{4r - 16\sqrt{r} + 15}\right) \leq \lim_{n\to\infty} \prod_{r=25}^{n} \left(1 - \frac{1}{r}\right) \\ &= \lim_{n\to\infty} \frac{24}{25} \cdot \frac{25}{26} \cdots \frac{n-2}{n-1} \cdot \frac{n-1}{n} = \lim_{n\to\infty} \frac{24}{n} = 0\end{aligned}$$

Remark 6. *The implicit Kirk-Mann iteration $x_n(\mathit{IKM})$ converges to $p = 0$ faster than the Kirk-Mann iteration $x_n(\mathit{KM})$ to $p = 0$.*

For the comparison of implicit Kirk-Ishikawa iteration $x_n(\text{IKI})$ and Kirk-Thianwan iteration $x_n(\text{KT})$, we have

$$\begin{aligned}\frac{|x_n(\text{IKI}) - 0|}{|x_n(\text{KT}) - 0|} &= \prod_{r=25}^{n} \left[\left(\frac{2\sqrt{r} - 8}{2\sqrt{r} - 3}\right) \left(\frac{2\sqrt{r}}{2\sqrt{r} - 5}\right)\right]^2 \\ &= \left[\prod_{r=25}^{n} \left(1 - \frac{15}{4r - 16\sqrt{r} + 15}\right)\right]^2\end{aligned}$$

with

$$0 \leq \left[\lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{15}{4r-16\sqrt{r}+15}\right)\right]^2 \leq \left[\lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{1}{r}\right)\right]^2$$
$$= \left(\lim_{n\to\infty}\frac{24}{25}\cdot\frac{25}{26}\cdots\frac{n-2}{n-1}\cdot\frac{n-1}{n}\right)^2 = \left(\lim_{n\to\infty}\frac{24}{n}\right)^2 = 0$$

Remark 7. *The implicit Kirk-Ishikawa iteration x_n(IKI) converges faster than the Kirk-Thianwan iteration x_n(KT).*

For the comparison of implicit Kirk-Noor iteration $x_n(\text{IKN})$ and Kirk-SP iteration $x_n(\text{KSP})$, we have

$$\frac{|x_n(\text{IKN})-0|}{|x_n(\text{KSP})-0|} = \prod_{r=25}^{n}\left[\left(\frac{2\sqrt{r}-8}{2\sqrt{r}-3}\right)\left(\frac{2\sqrt{r}}{2\sqrt{r}-5}\right)\right]^3$$
$$= \left[\prod_{r=25}^{n}\left(1-\frac{15}{4r-16\sqrt{r}+15}\right)\right]^3$$

with

$$0 \leq \left[\lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{15}{4r-16\sqrt{r}+15}\right)\right]^3 \leq \left[\lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{1}{r}\right)\right]^3$$
$$= \left(\lim_{n\to\infty}\frac{24}{25}\cdot\frac{25}{26}\cdots\frac{n-2}{n-1}\cdot\frac{n-1}{n}\right)^3 = \left(\lim_{n\to\infty}\frac{24}{n}\right)^3 = 0$$

Remark 8. *The implicit Kirk-Noor iteration x_n(IKN) has better convergence rate than the Kirk-SP iteration x_n(KSP).*

For the comparison of implicit Kirk-Multistep iteration $x_n(\text{IKM})$ and Kirk Multistep-SP iteration $x_n(\text{KMSP})$, we have

$$\frac{|x_n(\text{IKM})-0|}{|x_n(\text{KMSP})-0|} = \prod_{r=25}^{n}\left[\left(\frac{2\sqrt{r}-8}{2\sqrt{r}-3}\right)\left(\frac{2\sqrt{r}}{2\sqrt{r}-5}\right)\right]^k$$
$$= \left[\prod_{r=25}^{n}\left(1-\frac{15}{4r-16\sqrt{r}+15}\right)\right]^k$$

with

$$0 \leq \left[\lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{15}{4r-16\sqrt{r}+15}\right)\right]^k \leq \left[\lim_{n\to\infty}\prod_{r=25}^{n}\left(1-\frac{1}{r}\right)\right]^k$$
$$= \left(\lim_{n\to\infty}\frac{24}{25}\cdot\frac{25}{26}\cdots\frac{n-2}{n-1}\cdot\frac{n-1}{n}\right)^k = \left(\lim_{n\to\infty}\frac{24}{n}\right)^k = 0$$

Remark 9. *The implicit Kirk-Multistep iteration x_n(IKM) has better convergence rate than the Kirk Multistep-SP iteration x_n(KMSP) for $k \geq 4$.*

4. CONCLUSION

We have established and proved strong convergence, equivalency, T-stability, data dependence and convergence rate results for the implicit Kirk-multistep iteration, implicit Kirk-Noor iteration, implicit Kirk-Ishikawa iteration and implicit Kirk-Mann iteration of fixed points for the general class of quasi-contractive operators in a generalized convex metric space (X, d, W).

These iterative schemes are valid as shown by the analytical results; and from the numerical point of view, they are faster in term of convergence rate than the other iterative schemes such as: multistep Kirk-SP iteration, Kirk multistep iteration, Kirk-SP iteration, Kirk-Thianwan iteration, Kirk-Noor iteration, Kirk-Ishikawa iteration, Kirk-Mann iteration, implicit Noor iteration, implicit Ishikawa iteration, implicit Mann iteration and many other iterative schemes of fixed point in the literature.

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