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# NEW REFINEMENTS OF HÖLDER'S INEQUALITY

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ABSTRACT. In this paper, we define two mappings, investigate their properties, obtain some new refinements of Hölder's inequality.

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## 1. INTRODUCTION

Let $a_i > 0, b_i > 0\,(i = 1, 2, \cdots, n; n \geq 2)$, $p$ and $q$ be two non-zero real numbers such that $p^{-1} + q^{-1} = 1$. If $p > 1$, then

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} {a_i}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} {b_i}^q\right)^{\frac{1}{q}}, \tag{1.1}$$

if $p < 1\,(p \neq 0)$, then the inequality in (1.1) is reversed.

The inequality (1.1) is called the Hölder's inequality (see [1]-[2]). For some recent results which generalize, improve and extend this classical inequality, see [1]-[4].

To go further into (1.1), we define two mappings $F_1$ and $F_2$ by

$$F_1 : \{(n,k)|\, n \geq 2, k = 1, 2, \cdots, n; n \in \mathbb{N}\} \to \mathbb{R},$$
$$F_1(n,k) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{\sum_{j=1}^{k} {b_{i_j}}^q}{C_{n-1}^{k-1} \sum_{i=1}^{n} {b_i}^q} \left(\frac{\sum_{j=1}^{k} a_{i_j} b_{i_j}}{\sum_{j=1}^{k} {b_{i_j}}^q}\right)^p;$$

$$F_2 : \{(n,k)|\, n \geq 2, k = 1, 2, \cdots; n \in \mathbb{N}\} \to \mathbb{R},$$
$$F_2(n,k) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \frac{\sum_{j=1}^{k} {b_{i_j}}^q}{C_{n+k-1}^{k-1} \sum_{i=1}^{n} {b_i}^q} \left(\frac{\sum_{j=1}^{k} a_{i_j} b_{i_j}}{\sum_{j=1}^{k} {b_{i_j}}^q}\right)^p,$$

where $p$ and $q$ be two non-zero real numbers such that $p^{-1} + q^{-1} = 1$ .

The aim of this paper is to study the properties of $F_1$ and $F_2$, thus obtaining some new refinements of (1.1).

## 2. MAIN RESULTS

**Theorem 2.1.** *Let $a_i > 0, b_i > 0\,(i = 1, 2, \cdots, n; n \geq 2)$, $p$ and $q$ be two non-zero real numbers such that $p^{-1} + q^{-1} = 1$, and $F_1$ be defined as in the first section. We write $G_1(n,k) = \left(\sum_{i=1}^{n} {b_i}^q\right)(F_1(n,k))^{\frac{1}{p}}$. We have*

*(1) When $p > 1$, we get the following finite refinements of (1.1)*

$$\sum_{i=1}^{n} a_i b_i = G_1(n,n) \leq G_1(n,n-1) \leq \cdots \leq G_1(n,1) = \left(\sum_{i=1}^{n} {a_i}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} {b_i}^q\right)^{\frac{1}{q}}. \tag{2.1}$$

*(2) When $p < 1\,(p \neq 0)$, the inequalities in (2.1) are reversed.*

**Remark 1.** Theorem 2.1 is the finite refinements of Hölder's inequality with non-repetitive sample.

**Theorem 2.2.** *Let $a_i, b_i\,(i = 1, 2, \cdots, n; n \geq 2)$, $p$ and $q$ be defined as in the Theorem 2.1, and $F_2$ be defined as in the first section. We write $G_2(n,k) = \left(\sum_{i=1}^{n} {b_i}^q\right)(F_2(n,k))^{\frac{1}{p}}$. We have*

*(1) When $p>1$, we get the following infinite refinements of (1.1)*

$$\begin{aligned}\sum_{i=1}^{n} a_i b_i &\le \cdots \le G_2(n,n) \le G_2(n,n-1) \\ &\le \cdots \le G_2(n,1) = \left(\sum_{i=1}^{n} a_i{}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i{}^q\right)^{\frac{1}{q}}.\end{aligned} \tag{2.2}$$

*(2) When $p<1\,(p\neq 0)$, the inequalities in (2.2) are reversed.*

**Remark 2.** Theorem 2.2 is the infinite refinements of Hölder's inequality with repetitive sample.

## 3. SEVERAL LEMMAS

In order to prove the above theorems, we need the following two lemmas.

**Lemma 3.1.** *Let $a_i>0, b_i>0\,(i=1,2,\cdots,n; n\ge 2)$, $p$ and $q$ be two non-zero real numbers such that $p^{-1}+q^{-1}=1$, and $F_1$ be defined as in the first section. We have*

*(1) When $p>1$ or $p<0$, we get*

$$\left(\frac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} b_i{}^q}\right)^p = F_1(n,n) \le F_1(n,n-1) \le \cdots \le F_1(n,1) = \frac{\sum_{i=1}^{n} a_i{}^p}{\sum_{i=1}^{n} b_i{}^q}. \tag{3.1}$$

*(2) When $0<p<1$, the inequalities in (3.1) are reversed.*

*Proof of Lemma 3.1.* From the definition of $F_1$, since a simple calculation shows that

$$F_1(n,n) = \left(\frac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} b_i{}^q}\right)^p, \tag{3.2}$$

$$F_1(n,1) = \frac{\sum_{i=1}^{n} a_i{}^p}{\sum_{i=1}^{n} b_i{}^q}. \tag{3.3}$$

For $k=2,3,\cdots,n$, using some elementary identity involved combinatorial numbers, from (3.1) and (3.6) in [6], we can get

$$\sum_{j=1}^{k} a_{i_j} b_{i_j} = \sum_{\{r_1,\cdots,r_{k-1}\}\subset\{i_1,\cdots,i_k\}} \frac{\sum_{l=1}^{k-1} a_{r_l} b_{r_l}}{k-1} \tag{3.4}$$

and

$$\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\ \sum_{\{r_1,\cdots,r_{k-1}\}\subset\{i_1,\cdots,i_k\}}\left(\sum_{l=1}^{k-1}a_{r_l}b_{r_l}\right) = (n-k+1)\sum_{1\leqslant j_1<\cdots<j_{k-1}\leqslant n}\left(\sum_{m=1}^{k-1}a_{j_m}b_{j_m}\right), \tag{3.5}$$

respectively.

(1) When $p>1$ or $p<0$, $x^p$ is convex function on $(0,+\infty)$ with $x$. Let $t_{r_l}={b_{r_l}}^q, x_{r_l}=a_{r_l}{b_{r_l}}^{(1-q)}$, using Jensen's inequality of convex function, from (3.3) in [6], we get

$$\begin{aligned}\left(\frac{\sum_{j=1}^{k}a_{i_j}b_{i_j}}{\sum_{j=1}^{k}{b_{i_j}}^q}\right)^p &= \left(\sum_{\{r_1,\cdots,r_{k-1}\}\subset\{i_1,\cdots,i_k\}}\frac{\sum_{l=1}^{k-1}{b_{r_l}}^q}{(k-1)\sum_{j=1}^{k}{b_{i_j}}^q}\cdot\frac{\sum_{l=1}^{k-1}a_{r_l}b_{r_l}}{\sum_{l=1}^{k-1}{b_{r_l}}^q}\right)^p\\ &\leqslant \sum_{\{r_1,\cdots,r_{k-1}\}\subset\{i_1,\cdots,i_k\}}\frac{\sum_{l=1}^{k-1}{b_{r_l}}^q}{(k-1)\sum_{j=1}^{k}{b_{i_j}}^q}\left(\frac{\sum_{l=1}^{k-1}a_{r_l}b_{r_l}}{\sum_{l=1}^{k-1}{b_{r_l}}^q}\right)^p.\end{aligned} \tag{3.6}$$

From the definition of $F_1$ and (3.5)-(3.6), we get

$$F_1(n,k)=\sum_{1\leq i_1<\cdots<i_k\leq n}\frac{\sum_{j=1}^{k}{b_{i_j}}^q}{C_{n-1}^{k-1}\sum_{i=1}^{n}{b_i}^q}\left(\frac{\sum_{j=1}^{k}a_{i_j}b_{i_j}}{\sum_{j=1}^{k}{b_{i_j}}^q}\right)^p$$

$$\begin{aligned}&\leqslant \sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\ \sum_{\{r_1,\cdots,r_{k-1}\}\subset\{i_1,\cdots,i_k\}}\frac{\sum_{l=1}^{k-1}{b_{r_l}}^q}{(k-1)C_{n-1}^{k-1}\sum_{i=1}^{n}{b_i}^q}\left(\frac{\sum_{l=1}^{k-1}a_{r_l}b_{r_l}}{\sum_{l=1}^{k-1}{b_{r_l}}^q}\right)^p\\ &= \sum_{1\leqslant j_1<\cdots<j_{k-1}\leqslant n}\frac{(n-k+1)\sum_{m=1}^{k-1}{b_{j_m}}^q}{(k-1)C_{n-1}^{k-1}\sum_{i=1}^{n}{b_i}^q}\left(\frac{\sum_{m=1}^{k-1}a_{j_m}b_{j_m}}{\sum_{m=1}^{k-1}{b_{j_m}}^q}\right)^p\\ &= \sum_{1\leq j_1<\cdots<j_{k-1}\leq n}\frac{\sum_{m=1}^{k-1}{b_{j_m}}^q}{C_{n-1}^{k-2}\sum_{i=1}^{n}{b_i}^q}\left(\frac{\sum_{m=1}^{k-1}a_{j_m}b_{j_m}}{\sum_{m=1}^{k-1}{b_{j_m}}^q}\right)^p = F_1(n,k-1).\end{aligned} \tag{3.7}$$

Combination of (3.2)-(3.3) and (3.7) yields (3.1).

(2) When $0<p<1$, $x^p$ is concave function on $(0,+\infty)$ with $x$. Using Jensen's inequality of concave function, we get the reverse of (3.6) and (3.7), which implies the reverse of (3.1).

This completes the proof of Lemma 3.1.

**Lemma 3.2.** *Let $a_i, b_i\,(i = 1, 2, \cdots, n; n \geq 2)$, $p$ and $q$ be defined as in the Lemma 3.1, and $F_2$ be defined as in the first section. We have*

*(1) When $p > 1$ or $p < 0$, we get*

$$\left(\frac{\sum\limits_{i=1}^{n} a_i b_i}{\sum\limits_{i=1}^{n} {b_i}^q}\right)^p \leq \cdots \leq F_2\,(n, n) \leq F_2\,(n, n-1) \leq \cdots \leq F_2\,(n, 1) = \frac{\sum\limits_{i=1}^{n} {a_i}^p}{\sum\limits_{i=1}^{n} {b_i}^q}. \tag{3.8}$$

*(2) When $0 < p < 1$, the inequalities in (3.8) are reversed.*

*Proof of Lemma 3.2.* From the definition of $F_2$, since a simple calculation shows that

$$F_2\,(n, 1) = \frac{\sum\limits_{i=1}^{n} {a_i}^p}{\sum\limits_{i=1}^{n} {b_i}^q}. \tag{3.9}$$

For $k = 2, 3, \cdots$, using some elementary identity involved combinatorial numbers, we have

$$\sum_{i=1}^{n} a_i b_i = \frac{1}{C_{n+k-1}^{k-1}} \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} \left(\sum_{j=1}^{k} a_{i_j} b_{i_j}\right) \tag{3.10}$$

and

$$\begin{aligned} &\sum_{1\leq i_1\leq\cdots\leq i_k\leq n}\ \sum_{\{r_1,\cdots,r_{k-1}\}\subset\{i_1,\cdots,i_k\}} \left(\sum_{l=1}^{k-1} a_{r_l} b_{r_l}\right) \\ &= (n+k-1) \sum_{1\leq j_1\leq\cdots\leq j_{k-1}\leq n} \left(\sum_{m=1}^{k-1} a_{j_m} b_{j_m}\right), \end{aligned} \tag{3.11}$$

where (3.11) can also be obtained from (2.1) in [8].

(1) When $p > 1$ or $p < 0$, $x^p$ is convex function on $(0, +\infty)$ with $x$. Let $t_{i_j} = {b_{i_j}}^q, x_{i_j} = a_{i_j} {b_{i_j}}^{(1-q)}$, using Jensen's inequality of convex function, we have

$$\begin{aligned} \left(\frac{\sum\limits_{i=1}^{n} a_i b_i}{\sum\limits_{i=1}^{n} {b_i}^q}\right)^p &= \left(\sum_{1\leq i_1\leq\cdots\leq i_k\leq n} \frac{\sum\limits_{j=1}^{k} {b_{i_j}}^q}{C_{n+k-1}^{k-1} \sum\limits_{i=1}^{n} {b_i}^q} \cdot \frac{\sum\limits_{j=1}^{k} a_{i_j} b_{i_j}}{\sum\limits_{j=1}^{k} {b_{i_j}}^q}\right)^p \\ &\leq \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} \frac{\sum\limits_{j=1}^{k} {b_{i_j}}^q}{C_{n+k-1}^{k-1} \sum\limits_{i=1}^{n} {b_i}^q} \left(\frac{\sum\limits_{j=1}^{k} a_{i_j} b_{i_j}}{\sum\limits_{j=1}^{k} {b_{i_j}}^q}\right)^p = F_2\,(n, k)\,. \end{aligned} \tag{3.12}$$

From the definition of $F_2$ and (3.6), (3.11), we get

$$
\begin{aligned}
F_2(n,k) &= \sum_{1\le i_1\le\cdots\le i_k\le n} \frac{\sum_{j=1}^{k} {b_{i_j}}^q}{C_{n+k-1}^{k-1}\sum_{i=1}^{n} {b_i}^q}\left(\frac{\sum_{j=1}^{k} a_{i_j}b_{i_j}}{\sum_{j=1}^{k} {b_{i_j}}^q}\right)^p \\
&\le \sum_{1\le i_1\le\cdots\le i_k\le n}\ \sum_{\{r_1,\cdots,r_{k-1}\}\subset\{i_1,\cdots,i_k\}} \frac{\sum_{l=1}^{k-1} {b_{r_l}}^q}{(k-1)\,C_{n+k-1}^{k-1}\sum_{i=1}^{n} {b_i}^q}\left(\frac{\sum_{l=1}^{k-1} a_{r_l}b_{r_l}}{\sum_{l=1}^{k-1} {b_{r_l}}^q}\right)^p \\
&= \sum_{1\le j_1\le\cdots\le j_{k-1}\le n} \frac{(n+k-1)\sum_{m=1}^{k-1} {b_{j_m}}^q}{(k-1)\,C_{n+k-1}^{k-1}\sum_{i=1}^{n} {b_i}^q}\left(\frac{\sum_{m=1}^{k-1} a_{j_m}b_{j_m}}{\sum_{m=1}^{k-1} {b_{j_m}}^q}\right)^p \\
&= \sum_{1\le j_1\le\cdots\le j_{k-1}\le n} \frac{\sum_{m=1}^{k-1} {b_{j_m}}^q}{C_{n+k-2}^{k-2}\sum_{i=1}^{n} {b_i}^q}\left(\frac{\sum_{m=1}^{k-1} a_{j_m}b_{j_m}}{\sum_{m=1}^{k-1} {b_{j_m}}^q}\right)^p = F_2(n,k-1).
\end{aligned} \tag{3.13}
$$

Combination of (3.9) and (3.12)-(3.13) yields (3.8).

(2) When $0 < p < 1$, $x^p$ is concave function on $(0, +\infty)$ with $x$. Using Jensen's inequality of concave function, we get the reverse of (3.12) and (3.13), which implies the reverse of (3.8).

This completes the proof of Lemma 3.2.

## 4. PROOF OF THEOREMS

*Proof of Theorem 2.1.* From the definitions of $F_1$ and $G_1$, since a simple calculation shows that

$$G_1(n,n) = \sum_{i=1}^{n} a_i b_i, \tag{4.1}$$

$$G_1(n,1) = \left(\sum_{i=1}^{n} {a_i}^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} {b_i}^q\right)^{\frac{1}{q}}. \tag{4.2}$$

From $a_i > 0, b_i > 0\,(i = 1, 2, \cdots, n)$, we have

$$\frac{\sum_{i=1}^{n} {a_i}^p}{\sum_{i=1}^{n} {b_i}^q} > 0,\ \left(\frac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} {b_i}^q}\right)^p > 0, \sum_{i=1}^{n} {b_i}^q > 0. \tag{4.3}$$

For $k = 2, 3, \cdots, n$, we have

(1) When $p > 1$, $x^{\frac{1}{p}}$ is monotonically increasing on $(0, +\infty)$ with $x$. From (3.1) and (4.3), with a simple calculation, we get

$$G_1(n,k) \le G_1(n,k-1). \tag{4.4}$$

Combination of (4.1)-(4.2) and (4.4) yields (2.1).

(2) When $p < 1\,(p \neq 0)$.

**Case 1:** $0 < p < 1$, $x^{\frac{1}{p}}$ is monotonically increasing on $(0, +\infty)$ with $x$. From the reverse of (3.1) and (4.3), since a simple calculation shows that the reverse of (4.4).

**Case 2:** $p < 0$, $x^{\frac{1}{p}}$ is monotonically decreasing on $(0, +\infty)$ with $x$. From (3.1) and (4.3), since a simple calculation shows that the reverse of (4.4).

From above two cases, when $p < 1\,(p \neq 0)$, we have

$$G_1(n, k) \geq G_1(n, k-1)\,. \tag{4.5}$$

Combination of (4.1)-(4.2) and (4.5) yields the reverse of (2.1).

The proof of Theorem 2.1 is completed.

*Proof of Theorem 2.2.* From the definitions of $F_2$ and $G_2$, we have

$$G_2(n, 1) = \left(\sum_{i=1}^{n} a_i{}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i{}^q\right)^{\frac{1}{q}}. \tag{4.6}$$

For $k = 2, 3, \cdots$, we have

(1) When $p > 1$, $x^{\frac{1}{p}}$ is monotonically increasing on $(0, +\infty)$ with $x$. From (3.8) and (4.3), with a simple calculation, we get

$$\sum_{i=1}^{n} a_i b_i \leq \cdots \leq G_2(n, k) \leq G_2(n, k-1)\,. \tag{4.7}$$

Combination of (4.6) and (4.7) yields (2.2).

(2) When $p < 1\,(p \neq 0)$.

**Case 1:** $0 < p < 1$, $x^{\frac{1}{p}}$ is monotonically increasing on $(0, +\infty)$ with $x$. From the reverse of (3.8) and (4.3), since a simple calculation shows that the reverse of (4.7).

**Case 2:** $p < 0$, $x^{\frac{1}{p}}$ is monotonically decreasing on $(0, +\infty)$ with $x$. From (3.8) and (4.3), since a simple calculation shows that the reverse of (4.7).

From above two cases, when $p < 1\,(p \neq 0)$, we have

$$\sum_{i=1}^{n} a_i b_i \geq \cdots \geq G_2(n, k) \geq G_2(n, k-1)\,. \tag{4.8}$$

Combination of (4.6) and (4.8) yields the reverse of (2.2).

The proof of Theorem 2.2 is completed.

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