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BILINEAR REGULAR OPERATORS ON QUASI-NORMED FUNCTIONAL SPACES

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ABSTRACT. Positive and regular bilinear operators on quasi-normed functional spaces are introduced and theorems characterizing compactness of these operators are proved. Relations between bilinear operators and their adjoints in normed functional spaces are also studied.

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1. INTRODUCTION

Bilinear operators appear naturally in several branches of classical harmonic analysis and functional analysis. Several singular bilinear operators have been intensively studied and research on bilinear Hilbert transform (see [12]) have shown the need for new results for bilinear operators. In the paper by L. Grafakos and N. Kalton ([9]) more details about this subject may be found. Another important topic is the theory of ideals of operators and s-numbers in Banach spaces, where important definitions and results may not be adapted for the bilinear case from the linear case. The results in [8], [18] and [21] are references on this subject.

Positive and regular linear operators play a fundamental role in mathematics and their study form a very active research area. The bilinear counterpart of these operators were less studied. Some results giving connections between positive bilinear operators, function spaces and interpolation theory were explored in [7], [11] and [13].

Quasi-Banach spaces appear in a natural way as a generalization of Banach spaces, where the triangular inequality of the norm is changed by a weaker condition. From a geometrical point of view, the convex unitary ball of the Banach space case is replaced in the quasi-Banach case by a non convex unitary ball. Besides the classical works by Aoki ([4]), Rolewicz ([19] and [20]) and Kalton et al. ([10]), the study of geometrical aspects is one of the main issues for these spaces, with several results obtained recently, as may be seen in [1], [2] and [14].

On the other hand, connections between quasi-normed spaces, positive and regular bilinear operators were not properly studied in the literature. In current work positive and regular bilinear operators on quasi-normed functional spaces are introduced and their main properties and characterizations on lattices and quasi-normed lattices are proved. We also introduce a variant definition of functional quasi-norm (see [16]) and prove several theorems characterizing the compactness of bilinear operators. Finally, using a very interesting and powerful definition of adjoint of a bilinear mapping (see [18]), relations between compactness of bilinear operators and their adjoints in quasi-normed function spaces are also proved.

2. LATTICES TERMINOLOGY

The basic concepts and results about ordered sets and vector lattices are introduced in this section. The book [3] is a very good reference.

Definition 2.1. An ordered set is a set X endowed with a binary relation, denoted by $\leq$, which is supposed to be transitive ($x \leq y$ & $y \leq z \Longrightarrow x \leq y$), reflexive ($x \leq x$, for all $x \in X$) and anti-symmetric ($x \leq y$ &$y \leq x \Longrightarrow y = x$).

Let $(X, \leq)$ be an ordered set. We write $y \geq x$ to indicate $x \leq y$, and $x < y$ to express $x \leq y$ and $x \neq y$.

Definition 2.2. A subset $B \subset X$ is maximized (minimized) if there exists $x_0 \in X$ such that $b \leq x_0$, for all $b \in B$ (respectively, $x_0 \leq b$, for all $b \in B$); x_0 is called an upper bound (respectively, lower bound) of B in X.

Definition 2.3. For $x, y \in X$, the interval $[x, y]$ as the set of all $z \in X$ such that $x \leq z \leq y$; a set $B \subset X$ is bounded order if it is contained in a interval $[x, y]$.

Definition 2.4. Let B be a maximized bounded ordered subset of X. If there exists an upper bound of B which is a lower bound for all upper bounds of B (in X), such element is unique, and it is called supremum of B and denoted by $\sup B$. Analogously, we define the infimum of B ($\inf B$).

Definition 2.5. An ordered set $(R, \leq)$ is a lattice if, for all $x, y \in R$, the elements $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ exist in R.

Definition 2.6. A vector space E over $\mathbb{R}$, endowed with an order relation $\leq$, is a ordered vector space if, for all $x, y, z \in E$ and $\lambda \geq 0$, the conditions are verified:

(EVO1) $x \leq y \Longrightarrow x + z \leq y + z$,

(EVO2) $x \leq y \Longrightarrow \lambda x \leq \lambda y$,.

If E is an ordered vector space, the subset

$$E_+ := \{\, x \in E \;;\; x \geq 0 \,\}$$

is called the positive cone of E and the elements $x \in E_+$ are called positives.

Definition 2.7. A vector lattice is an ordered vector space E, such that $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ exist in E, for all $x, y \in E$.

Proposition 2.1. *In a vector lattice E one has $x + \sup A = \sup(x + A)$, $x + \inf A = \inf(x + A)$ and $\sup A = -\inf(-A)$ for all $x \in E$ and $A \subset E$.*

Definition 2.8. Let E be a vector lattice. For all $x \in E$, we define

$$x^+ := x \vee 0, \;\; x^- := -(x \wedge 0), \qquad |x| := x \vee (-x).$$

The elements x^+ and x^- are called the positive and negative parts of x, respectively, and $|x|$ the modulus of x.

Theorem 2.2. *Let E be a vector lattice. For all $x, y, x_1, y_1 \in E$ and $\lambda \geq 0$, one has*

$$x = x^+ - x^-; \tag{2.1}$$

$$|x| = x^+ + x^-; \tag{2.2}$$

$$|x| = 0 \Longleftrightarrow x = 0; \; |\lambda x| = |\lambda|\,|x|; \; |x + y| \leq |x| + |y|; \tag{2.3}$$

(2.1) is the unique representation of x as a difference of disjoint elements of E (that is, $|x| \wedge |y| = 0$).

3. POSITIVE OPERATORS

In this section it is defined the bilinear operators which are the subject of this work.

Definition 3.1. Let X, Y and Z be vector lattices. An bilinear operator $T : X \times Y \to Z$ is positive if given $x \in X_+$ and $y \in Y_+$, one has $T(x, y) \in Z_+$.

Remark 3.1. The Definition 3.1 above is not the same definition of positive bilinear operator given in [6].

Proposition 3.1. *Let X, Y and Z be vector lattices and $T : X \times Y \to Z$ a bilinear positive operator. Then,*

$$|T(x, y)| \leq T(|x|, |y|),$$

for all $(x, y) \in X \times Y$.

Proof.

$$\begin{aligned}|T(x,y)| &= |T(x^+ - x^-, y^+ - y^-)| \\ &\leq |T(x^+,y^+) - T(x^+,y^-)| + |T(x^-,y^+) - T(x^-,y^-)| \\ &\leq T(x^+,y^+) + T(x^+,y^-) + T(x^-,y^+) + T(x^-,y^-) \\ &= T(x^+ + x^-, y^+ + y^-) \\ &= T(|x|,|y|)\,.\end{aligned}$$

Thus,

$$|T(x,y)| \leq T(|x|,|y|)\,, \tag{3.1}$$

for positive T and $(x,y) \in X \times Y$. ∎

Definition 3.2. Let X, Y and Z be ordered vector spaces. An bilinear operator $T : X \times Y \to Z$ is regular if it may be written as

$$T = T_1 - T_2,$$

where T_1 and T_2 are positive bilinear operators.

Theorem 3.2. *Let X, Y and Z be vector lattices. A bilinear operator $T : X \times Y \to Z$ is regular if, and only if, there exists a positive bilinear operator $S : X \times Y \to Z$, such that*

$$|T(x,y)| \leq S(|x|,|y|), \tag{3.2}$$

for all $(x,y) \in X \times Y$. The bilinear operator S is called a positive upper bound of the operator T.

Proof. If T is regular, there exist A_1 and A_2 positive bilinear operators with $T = A_1 - A_2$ and

$$\begin{aligned}|T(x,y)| &= |(A_1 - A_2)(x,y)| = |A_1(x,y) - A_2(x,y)| \\ &\leq |A_1(x,y)| + |A_2(x,y)| \leq A_1(|x|,|y|) + A_2(|x|,|y|) \\ &= (A_1 + A_2)(|x|,|y|)\,.\end{aligned}$$

Taking $S = A_1 + A_2$, we obtain the desired operator.

On the other hand, if there is S satisfying (3.2), one has for $x \in X_+$ and $y \in Y_+$ that

$$T(x,y) \leq |T(x,y)| \leq S(|x|,|y|) = S(x,y).$$

Thus, $S(x,y) - T(x,y) = (S-T)(x,y) \geq 0$, soon $S - T$ is a positive operator. Finally, $T = S - (S - T)$. Therefore, T is regular. ∎

Definition 3.3. An ordered set X is Dedekind complete if every non-empty subset of X that is bounded above admits a supremum (in X).

Theorem 3.3. *Let X and Y be vector lattices and Z a Dedekind-complete vector lattice. If a bilinear operator $T : X \times Y \to Z$ is regular, then for each $(u,v) \in X_+ \times Y_+$, there exists $\omega \in Z_+$, such that*

$$T(x,y) \leq \omega,$$

for all $(x,y) \in X_+ \times Y_+$ with $0 \leq x \leq u$ and $0 \leq y \leq v$.

Proof. Since T is regular, there is a positive bilinear upper bound S of T. For each positive pair $(u,v) \in X_+ \times Y_+$ let us define $\omega(t) = S(u(t),v(t))$. Then, for each $0 \leq x \leq u$ and $0 \leq y \leq v$, by Theorem 3.2 one has

$$T(x,y) \leq |T(x,y)| \leq S(|x|,|y|) = S(x,y)\,. \tag{3.3}$$

Since $u - x \geq 0$, $S(u - x, y) \geq 0$, implying

$$S(u, y) \geq S(x, y). \tag{3.4}$$

In a similar way, since $v - y \geq 0$, one has $S(u, v - y) \geq 0$, thus

$$S(u, v) \geq S(u, y). \tag{3.5}$$

Putting (3.4) and (3.5) in (3.3),

$$T(x, y) \leq S(x, y) \leq S(u, y) \leq S(u, v) = \omega(t)\,,$$

proving the result.

∎

Remark 3.2. The converse of Theorem 3.3 is not true. A counterexample can be found in [15], where it is considered a bounded bilinear operator whose range is the real numbers and the domain is $\ell_2 \times \ell_2$.

4. QUASI-NORMED LATTICES

Definition 4.1. A **quasi-norm** in a vector space X is an application $||.||$ of X in $[0, \infty[$ such that, for $x, y \in X$ and $\lambda \in \mathbb{R}$, verifies the conditions

QN1) $||x|| = 0 \Longleftrightarrow x = 0$;

QN2) $||\lambda x|| = |\lambda|\,||x||$;

QN3) $||x + y|| \leq C\,(||x|| + ||y||)$,

for some $C \geq 1$.

A vector space X endowed with a quasi-norm is called a **quasi-normed space**.

A **quasi-Banach space** is a quasi-normed space which is complete in the topology generated by

$$d(x, y) = ||x - y||.$$

A classic result of Aoki-Rolewicz gives the following:

Theorem 4.1. *If X is a quasi-normed space endowed with a quasi-norm $||.||$, there exists a constant ρ, $0 < \rho \leq 1$, such that*

$$||x_1 + ... + x_n||^\rho \leq 4\,(||x_1||^\rho + ... + ||x_n||^\rho)\,,$$

for all finite sequence $x_1, ..., x_n$ em X.

Definition 4.2. If a quasi-normed space is also a vector lattice $(X, \leq)$, we say X is a **quasi-normed lattice** if

$$|x| \leq |y| \Longrightarrow ||x|| \leq ||y||.$$

Besides, if a quasi-normed lattice is complete, we say it is a **quasi-Banach lattice**.

Theorem 4.2. *Let X and Y be quasi-Banach lattices and Z a quasi-normed lattice. If a bilinear operator $T : X \times Y \to Z$ is positive, then it is bounded.*

Proof. Let us show that there exists a constant $M > 0$, such that

$$||T(x, y)|| \leq M||x||||y||\,, \tag{4.1}$$

for all positive $(x, y) \in X_+ \times Y_+$.

Suppose (4.1) is not true, that is, there exists a sequence of positive functions $(x_n, y_n) \in X_+ \times Y_+$ with $||(x_n, y_n)||_{X_+\times Y_+} \leq 1$, such that

$$||T(x_n, y_n)|| > n2^{2n}\,. \tag{4.2}$$

for all n. Let us consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n} x_n$ and $\sum_{n=1}^{\infty} \frac{1}{2^n} y_n$. We claim $\sum_{n=1}^{\infty} \frac{1}{2^n} x_n$ is convergent in X. Indeed, let ρ, $0 < \rho \leq 1$ be the constant given in Aoki-Rolewicz's Theorem. One has

$$|| \sum_{k=n+1}^{n+p} \frac{1}{2^k}\, x_k||_X^\rho \leq 4 \sum_{k=n+1}^{n+p} \frac{1}{2^{\rho k}}\, ||x_k||_X^\rho \leq 4 \sum_{k=n+1}^{n+p} \frac{1}{2^{\rho k}}.$$

Since $2^\rho > 1$, the series verifies the Cauchy criterium; and since X is complete it follows that it is convergent. In the same way, it may be proved $\sum_{n=1}^{\infty} \frac{1}{2^n} y_n$ is convergent in Y. We define $u_0 = \sum_{n=1}^{\infty} \frac{1}{2^n} x_n$ and $v_0 = \sum_{n=1}^{\infty} \frac{1}{2^n} y_n$ and then, $(u_0, v_0) \in X \times Y$.

Now, since $u_0 \geq \frac{x_n}{2^n}$ and $v_0 \geq \frac{y_n}{2^n}$ and T is positive, one has $T(u_0 - \frac{x_n}{2^n}, v_0) \geq 0$ which implies

$$T(u_0, v_0) \geq \frac{1}{2^n} T(x_n, v_0)\,, \tag{4.3}$$

and from $T(x_n, v_0 - \frac{y_n}{2^n}) \geq 0$,

$$T(x_n, v_0) \geq \frac{1}{2^n} T(x_n, y_n)\,. \tag{4.4}$$

Thus, (4.3) and (4.4) implies $T(u_0, v_0) \geq \frac{1}{2^{2n}} T(x_n, y_n)$ and, from the properties of the quasi-norm in Z, $||T(u_0, v_0)|| \geq \frac{1}{2^{2n}} ||T(x_n, y_n)||$, for all n. From (4.2) one has

$$||T(u_0, v_0)|| \geq \frac{1}{2^{2n}} ||T(x_n, y_n)|| \geq \frac{1}{2^{2n}} n2^{2n} = n\,,$$

for all n. But, this is a contradiction, since $T(u_0, v_0) \in Z$ and we obtain (4.1).

Finally, let $(x, y) \in X \times Y$ arbitrary. From Proposition (3.1) and (4.1),

$$\begin{aligned} ||T(x, y)|| &= ||\,|T(x, y)|\,|| \leq ||T(|x|, |y|)|| \\ &\leq M||\,|x|\,||||\,|y|\,|| \leq M||x||||y||\,. \end{aligned}$$

Thus, T is a bounded operator. ∎

Remark 4.1. For the case of quasi-Banach spaces, it must be noted that the operator may be trivial, since already for linear operators between quasi-Banach spaces (even not necessary positive) such triviality can occurs.

Corollary 4.3. *In the conditions of Theorem 4.2, if the bilinear operator $T : X \times Y \to Z$ is regular, it is also bounded.*

5. Operators and Function Spaces

In this section the function spaces which we will dealt with are introduced. In Definition 5.1 next, we define a variant general concept of functional quasi-norm which allows us to generalize several function spaces.

Let (Ω, μ) be a measure space. We denote by $L_+^0 = L_+^0(\Omega, \mu)$ the cone of real μ-measurable, non negative and finite μ-a.e. functions on Ω.

Definition 5.1. An application $\rho : L_+^0 \to [0, \infty]$ is a **functional quasi-norm** if, for all $f, g \in L_+^0$, for all $\lambda > 0$ and for all subset $D \subset \Omega$, with $\mu(D) < \infty$, the following conditions are verified:

C1) $\rho(f) = 0 \Longleftrightarrow f = 0,\ \mu - \text{q.s.};$

C2) $\rho(\lambda f) = \lambda\rho(f)$;

C3) $\rho(f+g) \leq C\,(\rho(f)+\rho(g)), \quad \text{for some} \quad C \geq 1.$

C4) $0 \leq g \leq f \quad \mu-\text{q.s.} \Longrightarrow \rho(g) \leq \rho(f)$;

C5) $\rho(\chi_D) < \infty$;

C6) $\lambda\,\mu(\{x \in D\,;\, |f(x)| \geq \lambda\})^{1/p} \leq C'\,\rho(f)$,

for some $p > 0$ and constant $C' > 0$, dependent of D and ρ, and independent of f.

We denote by $L^0 = L^0(\Omega,\mu)$ the class of real μ-measurable functions with extended scalar values and μ-a.e finite. Endowed with the topology of convergence in measure over finite measure sets, L^0 is a metric vector topological space.

The space $L^\infty = L^\infty(\Omega,\mu)$ is defined as the set of all measurable functions from Ω to $\mathbf{R}$ which are essentially bounded, i.e. bounded up to a set of measure zero. Two such functions are identified if they are equal almost everywhere. For $f \in L^\infty$, its norm is given by:

$$||f|| = \inf\{a \in \mathbf{R} : \mu(\{t : f(t) > a\}) = 0\}\,.$$

We denote by $S = S(\Omega,\mu)$ the subclass of simple functions.

Definition 5.2. Let ρ be a functional quasi-norm in $L^0_+(\Omega,\mu)$. The class of the functions $f \in L^0$ such that $\rho(|f|) < \infty$ is denoted by $X = X(\Omega,\mu,\rho)$.

Remark 5.1. Assumption (C5) means that $L^\infty \hookrightarrow X$ and assumption (C6) that $X \hookrightarrow L^{p,\infty}$ for some $p > 0$, where $L^{p,\infty}$ is a weak-L^p space. Therefore one has the following theorem.

Theorem 5.1. *Let ρ be a functional norm and $X = X(\Omega,\mu,\rho)$. For $f \in X$ let*

$$||f||_X = \rho(|f|).$$

Then, X is a quasi-normed vector subspace verifying the inclusions

$$S \subset X \hookrightarrow L^0\,.$$

In particular, if $f_n \to f$ in X, then $f_n \to f$ in measure over the sets with finite measure, thus there is a subsequence $f_{k(n)}$ converging μ-a.e.

Definition 5.3. Let $X = X(\Omega,\mu,\rho)$ be a quasi-normed functional space. A function $f \in X$ has **absolutely continuous quasi-norm** if, given $\varepsilon > 0$, there definitionexists $\delta > 0$, such that $\mu(D) < \delta$ implies

$$||f\chi_D|| < \varepsilon.$$

We denote by X_a the subspace of X of all absolutely continuous quasi-norms functions .

X has absolutely continuous quasi-norms if $X = X_a$.

Definition 5.4. A family $M \subset X$ has **equi-absolutely continuous quasi-norm** if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu(D) < \delta$ implies

$$||P_D f|| < \varepsilon,$$

for all $f \in M$, where $P_D f = f_{\chi_D}$.

6. COMPACTNESS THEOREMS

Next we will give several characterizations of compact bilinear operators on the quasi-normed functional spaces defined in Section 5 and we are assuming that $X = X(\Omega_1, \mu, \rho_1)$, $Y = Y(\Omega_2, \nu, \rho_2)$ and $Z = Z(\Omega_3, \upsilon, \rho_3)$ are quasi-normed functional spaces. We denote by $Bil(X \times Y, Z)$ the family of all bounded bilinear operators from $X \times Y$ to Z.

Definition 6.1. A bounded bilinear operator $T : X \times Y \to Z$ is compact in measure if the image $\{T(u_n, v_n)\}$, of any bounded sequence $\{(u_n, v_n)\}$ of $X \times Y$ contains a Cauchy subsequence with regard to the measure υ, that is, if $\max\{\|u_n\|_X, \|v_n\|_Y\} \leq C$, then there exists a subsequence $\{(u_{n_k}, v_{n_k})\}$, such that, given $\varepsilon > 0$ and $\delta > 0$, there exists $N = N(\varepsilon, \delta)$ with

$$\upsilon(\{s \in \Omega_3 : |T(u_{n_k}, v_{n_k})(s) - T(u_{m_k}, v_{m_k})(s)| > \varepsilon\}) < \delta$$

for all $n_k, m_k > N$.

Theorem 6.1. *Let X and Y be quasi-normed functional spaces and suppose that Z has absolutely continuous quasi-norms, i.e $Z = Z_a$. Let $T : X \times Y \to Z$ be a bounded bilinear operator. Then, T is compact if, and only if, T is compact in measure and the functions in the set $\{\, T(f, g) : ||f||_X \leq 1\,, ||g||_Y \leq 1\}$ have equi-absolutely continuous quasi-norms.*

Proof. Let T be compact. Since convergence implies convergence in measure, it is enough to verify the equi-absolutely continuity quasi-norms of the set $\{\, T(f, g) : ||f||_X \leq 1\,, ||g||_Y \leq 1\}$. Suppose this is not true: there exists a sequence $(f_n, g_n) \in X \times Y$, with $||f_n||_X \leq 1$, $||g_n||_Y \leq 1$ and a sequence of sets $E_n \subset Z$, such that $\upsilon(E_n) \to 0$, when $n \to \infty$, but

$$||P_{E_n} T(f_n, g_n)||_Z \geq \varepsilon_0,$$

for all $n \in \mathbb{N}$. By the compactness of T, there exists a subsequence $\{(f_{k(n)}, g_{k(n)}\}$ of $\{(f_n, g_n)\}$ and $h \in Z$, such that $||T(f_{k(n)}, g_{k(n)}) - h||_Z < \varepsilon_0/2C$, for $n > N_1$, and $||P_{E_{k(n)}} h||_Z < \varepsilon_0/2C$, for $n > N_2$. Thus, for $n > \max\{N_1, N_2\}$, one has

$$||P_{E_{k(n}} T(f_{k(n)}, g_{k(n)})||_Z \leq C\, ||P_{E_{k(n)}} h||_Z + C\, ||P_{k(n)} h||_Z < \varepsilon_0,$$

which is a contradiction.

Reciprocally, suppose T is compact in measure and $\{\, T(f, g) : ||f||_X \leq 1\,, ||g||_Y \leq 1\}$ has equi-absolutely continuous quasi-norms. Given $\varepsilon_0 > 0$, let $0 < \varepsilon < \varepsilon_0/(2C^2 + C||\chi_Z||_Z)$. For this given ε there exists $\delta = \delta(\varepsilon) > 0$, such that, for all set $E \subset Z$, with $\upsilon(E) < \delta$, one has $||P_E T(f, g)||_Z < \varepsilon$, for all $||f||_X \leq 1\,, ||g||_Y \leq 1$. Let

$$E_{m,n}(\varepsilon) = \{\, z \in Z\,;\ |T(f_m, g_m)(z) - T(f_n, g_n)(z)| > \varepsilon\,\}.$$

From the compactness in measure of T, there exists $\{(f_{k(n)}, g_{k(n)}\}$ and $N = N(\varepsilon, \delta)$ such that

$$\nu(E_{k(m),k(n)}(\varepsilon)) < \delta,$$

for $m, n > N$. Defining $E^c_{k(m),k(n)}(\varepsilon) = Y \setminus E_{k(m),k(n)}(\varepsilon)$, one has

$$\begin{aligned}
&||T(f_{k(m)}, g_{k(m)}) - T(f_{k(n)}, f_{k(n)})||_Z = \\
&= ||(P_{E_{k(m),k(n)}}(\varepsilon) + P_{E^c_{k(m),k(n)}(\varepsilon)}) T(f_{k(m)}, g_{k(m)}) - T(f_{k(n)}, f_{k(n)})||_Z \\
&\leq C\, ||P_{E_{k(m),k(n)}(\varepsilon)} T(f_{k(m)}, g_{k(m)}) - T(f_{k(n)}, f_{k(n)})||_Z \\
&+ C\, ||P_{E^c_{k(m),k(n)}(\varepsilon)} T(f_{k(m)}, g_{k(m)}) - T(f_{k(n)}, f_{k(n)})||_Z \\
&\leq C^2\, ||P_{E_{k(m),k(n)}(\varepsilon)} T(f_{k(m)}, g_{k(m)})||_Z + C^2\, ||P_{E_{k(m),k(n)}(\varepsilon)} T(f_{k(n)}, g_{k(n)})||_Z + \varepsilon\, C\, ||\chi_Z||_Z \\
&\leq \varepsilon\, (2C^2 + C\, ||\chi_Z||_Z) < \varepsilon_0,
\end{aligned}$$

and the theorem is proved. ∎

Theorem 6.2. *Let X, Y and Z be quasi-normed functional spaces. Moreover, suppose that Z has absolutely continuous quasi-norms, i.e $Z = Z_a$, and $\upsilon(\Omega_3) < \infty$. A bounded bilinear operator $T : X \times Y \to Z$ is compact if, and only if, T is compact in measure and satisfies*

$$\lim_{\upsilon(E)\to 0} \|P_E T\|_{Bil(X\times Y,Z)} = 0\,. \tag{6.1}$$

Proof. Equation (6.1) implies the image set

$$T(U_{X\times Y}) = \{T(x,y) : \|x\|_X \le 1\,, \|y\|_Y \le 1\}$$

has equi-absolutely continuous quasi-norms. Since T is compact in measure, from Theorem 6.2, $T(U_{X\times Y})$ is compact, then T is a compact operator.

Now, if T is compact, then $T(U_{X\times Y})$ is compact in Z and by Theorem 6.1, $T(U_{X\times Y})$ is compact in measure and it has equi-absolutely continuous quasi-norms. Suppose (6.1) is not true. Then, there exist functions $x_1, x_2, \dots \in X$, $y_1, y_2, \dots \in Y$ and a sequence of sets E_n such that, $\upsilon(E_n) \to 0$ for $n \to \infty$ and

$$\|P_{E_n} T(x_n, y_n)\| \ge \varepsilon_0 > 0\,,$$

for all n. But, this is a contradiction with the equi-absolutely continuous quasi-norms of $T(U_{X\times Y})$, implying (6.1) is valid, and the theorem follows. ∎

Theorem 6.3. *Let X, Y and Z be quasi-normed functional spaces, where $\mu(\Omega_1) < \infty$, $\nu(\Omega_2) < \infty$, $\upsilon(\Omega_3) < \infty$ and Z has absolutely continuous quasi-norms, i.e $Z = Z_a$. A bilinear regular operator $T : X \times Y \to Z$ is compact if, and only if, T is compact in measure and satisfies*

$$\lim_{\upsilon(E)+\mu(D_1)+\nu(D_2)\to 0} \|P_E T(P_{D_1}, P_{D_2})\|_{Bil(X\times Y,Z)} = 0\,. \tag{6.2}$$

Proof. The result will follow from Theorem 6.2 if (6.2) implies (6.1), for any regular operator. Let C and C' be the constants and p the parameter in **C3)** and **C6)** in the Definition 5.1.

From (6.2), given $\varepsilon_0 > 0$, let $0 < \delta_0 \le 1$, such that, for $\mu(D_1) + \nu(D_2) + \upsilon(E) < \delta_0$, where $D_1 \subset X$, $D_2 \subset Y$ and $E \subset Z$, one has

$$||P_E T(P_{D_1}, P_{D_2})||_{X\times Y\to Z} < \frac{\varepsilon_0}{2C^2}.$$

On the other hand, since the quasi-norms in Z are absolutely continuous, we define $u \equiv 1$ a.e on Ω_1 and $v \equiv 1$ a.e on Ω_2, such that $(u, v) \in X \times Y$. Then, given $\varepsilon > 0$, with $0 < \varepsilon < \varepsilon_0$, there exists $\delta > 0$, with $0 < \delta \le \delta_0$, such that $\upsilon(E) < \delta$ implies

$$||P_E S(u, v)||_Z < \frac{\varepsilon \delta_0^{2/p}}{2K^2C^2},$$

where S is a positive upper bound of T and $K > C'$.

Let $(f, g) \in X \times Y$, with $||f||_X \le 1$ and $||g||_Y \le 1$ and sets $D_1 \subset X$, $D_2 \subset Y$ and $E \subset Z$ fixed, such that $\mu(D_1) + \nu(D_2) + \upsilon(E) < \delta_0$. We denote by D_f the set of all elements $x \in D_1$ such that $|f(x)| > K\delta_0^{-1/p}$. Let D_f^c the complement of D_f, that is, the set of $x \in D_1$ such that $|f(x)| \le K\delta_0^{-1/p}$, and more $\chi_{D_f^c}|f| \le K\delta_0^{-1/p}\chi_{D_f^c}$. In a similar way, we define D_g. One has

$$\begin{aligned} ||P_E T(f, g)||_Z &\le C\,||P_E T(P_{D_f^c} f, g)||_Z + C\,||P_E T(P_{D_f} f, g)||_Z \\ &\le C^2\,||P_E T(P_{D_f^c} f, P_{D_g^c} g)||_Z + C^2\,||P_E T(P_{D_f^c} f, P_{D_g} g)||_Z \\ &\quad + C^2\,||P_E T(P_{D_f} f, P_{D_g^c} g)||_Z + C^2\,||P_E T(P_{D_f} f, P_{D_g} g)||_Z\,. \end{aligned} \tag{6.3}$$

Let us see each one of these norms:

$$\begin{aligned}
||P_E T(P_{D_f^c} f, P_{D_g^c} g)||_Z &= ||P_E |T(P_{D_f^c} f, P_{D_g^c} g)|||_Z \\
&\le ||P_E |S(P_{D_f^c} f, P_{D_g^c} g)|||_Z \\
&= \|P_E S(|\chi_{D_f^c} f|, |\chi_{D_g^c} g|)\|_Z \\
&= \|P_E S(|\chi_{D_f^c}||f|u, |\chi_{D_g^c}||g|v)\|_Z \\
&\le C^2 K^2 \delta_0^{-2/p} \, \|P_E S(|\chi_{D_f^c}|u, |\chi_{D_g^c}|v)\|_Z \\
&\le C^2 K^2 \delta_0^{-2/p} \, \|P_E S(u, v)\|_Z \\
&\le C^2 K^2 \delta_0^{-2/p} \frac{\varepsilon \delta_0^{2/p}}{2K^2C^2} = \frac{\varepsilon}{2} \,. \qquad (6.4)
\end{aligned}$$

Now, from **C6)** of Definition 5.1, we have

$$\delta_0^{-1/p} \mu(\{\, x\, ; \, |f(x)| \le K\delta_0^{-1/p} \,\})^{1/p} < ||f||_X \le 1,$$

that is, $\mu(D_f) < \delta_0$, and the same is obtained for $\nu(D_g)$. Thus, $||P_E T(P_{D_f}, P_{D_g})||_Z < \dfrac{\varepsilon}{2C}$ and

$$\begin{aligned}
||P_E T(P_{D_f^c} f, P_{D_g} g)||_Z &= ||P_E |T(P_{D_f^c} f, P_{D_g} g)|||_Z \\
&\le ||P_E S(|P_{D_f^c} f|, |P_{D_g} g|)||_Z \\
&= \|P_E S(|\chi_{D_f^c} f|, |P_{D_g} g|)\|_Z \\
&= \|P_E S(|\chi_{D_f^c}||f|u, |P_{D_g} g|v)\|_Z \\
&\le CK\delta_0^{-1/p} \, \|P_E S(|\chi_{D_f^c}|u, |P_{D_g} g|v)\|_Z \\
&\le CK\delta_0^{-1/p} \, \|P_E S(u, v)\|_Z \\
&\le CK\delta_0^{-1/p} \frac{\varepsilon \delta_0^{2/p}}{2K^2C^2} \\
&\le \frac{\varepsilon}{2} \,, \qquad (6.5)
\end{aligned}$$

and the same is obtained for $||P_E T(P_{D_f} f, P_{D_g^c} g)||_Z$. Therefore,

$$||P_E T(f, g)||_Z < \varepsilon,$$

and the theorem is proved. ∎

7. Compactness and adjoint bilinear operators

The present section is devoted to the relationships among the corresponding regular bilinear operators and their adjoint. Let us recall that Schauder's well-known result states that an operator T between Banach spaces is compact if, and only if, its adjoint, T^*, is compact.

Ramanujan and Schock studied in [18] ideals of bilinear operators between Banach spaces, including the ideal of bilinear compact operators, i.e., $T \in Bil(X \times Y, Z)$, such that $T(U_X \times U_Y)$ is relatively compact in Z. Given $T \in Bil(X \times Y, Z)$, the adjoint linear map $T^\times \colon Z^* \to Bil(X \times Y)$ is given by

$$T^\times z^*(x, y) = z^*(T(x, y)), \qquad (x, y) \in X \times Y.$$

Clearly $T^\times$ is a bounded operator, and $\|T\| = \|T^\times\|$. It must be noted that this notion of adjoint differs from Arens's definition of adjoint of a bilinear mapping (see [5]). In [18] it is proved that the analogues of Schauder's theorem which states that if $T \in Bil(X \times Y, Z)$, then T is compact if, and only if $T^\times$ is compact. And more, if $T \in Bil(X \times Y, Z)$ and $S \in L(Z, W)$, then

$$(ST)^\times = T^\times S^*$$

where S^* is the classical linear adjoint.

Theorem 7.1. *Let X, Y and Z be normed functional spaces, where Z has absolutely continuous norms, i.e $Z = Z_a$ and $\upsilon(\Omega_3) < \infty$. A bounded bilinear operator $T : X \times Y \to Z$ is compact if, and only if, $(T^\times)^*$ is compact in measure and satisfies*

$$\lim_{\mu(D)\to 0} \|T^\times P_D\| = 0\,.$$

Proof. If T is compact, then

$$(T^\times)^* : Bil(X \times Y)' \to Z''$$

is also compact, and by the linear version of Theorem 7.1 (see Theorem 3.4 from [16]), it follows $(T^\times)^*$ is compact in measure and $\lim_{\mu(E)\to 0} \|P_E(T^\times)^*\| = 0$. But,

$$\|P_E(T^\times)^*\| = \|(P_E(T^\times)^*)^*\| = \|((T^\times)^*)^* P_E\| = \|T^\times P_E\|\,,$$

and the theorem follows. ∎

Definition 7.1. Given $D \subset \Omega$, we define $\overline{P}_D : Bil(X \times Y) \to Bil(X \times Y)$, such that, for $b \in Bil(X \times Y)$, then $\overline{P}_D(b) \in Bil(X \times Y)$ and

$$\overline{P}_D(b)(x, y) = b(P_D x, P_D y)$$

for all $(x, y) \in X \times Y$.

Remark 7.1. If $T : X \times Y \to Z$, then $T^\times : Z' \longrightarrow Bil(X \times Y)$. Thus, for $D \subset \Omega$,

$$Z' \xrightarrow{T^\times} Bil(X \times Y) \xrightarrow{\overline{P}_D} Bil(X \times Y)\,,$$

which implies $Z' \xrightarrow{\overline{P}_D T^\times} Bil(X \times Y)$, where for $\phi \in Z'$ and $(x, y) \in X \times Y$,

$$\begin{aligned}(\overline{P}_D T^\times)(\phi)(x, y) &= \overline{P}_D(T^\times)(\phi)(x, y)) \\ &= \overline{P}_D(\phi(T(x, y))) = \overline{P}_D(\phi T(x, y)) \\ &= \phi T(P_D x, P_D y)\,.\end{aligned}$$

Proposition 7.2. $\overline{P}_D : Bil(X \times Y) \to Bil(X \times Y)$ *is a bounded linear operator.*

Proof. Given $b_1, b_2 \in Bil(X \times Y)$, one has

$$\begin{aligned}(\overline{P}_D(b_1 + b_2)(x, y) &= (b_1 + b_2)(P_D x, P_D y)) \\ &= b_1(P_D x, P_D y) + b_2(P_D x, P_D y) \\ &= (\overline{P}_D b_1)(x, y) + (\overline{P}_D b_2)(x, y)\,.\end{aligned}$$

Now,

$$\begin{aligned}&\|\overline{P}_D\|_{L(Bil(X\times Y), Bil(X\times Y))} \\ &= \sup\{\|\overline{P}_D b\|_{Bil(X\times Y)} : \|b\|_{Bil(X\times Y)} \leq 1\} \\ &= \sup\{\sup\{|(\overline{P}_D b)(x, y)| : \|x\|_X \leq 1\,, \|y\|_Y \leq 1\} : \|b\|_{Bil(X\times Y)} \leq 1\} \\ &= \sup\{\sup\{|b(P_D x, P_D y)| : \|x\|_X \leq 1\,, \|y\|_Y \leq 1\} : \|b\|_{Bil(X\times Y)} \leq 1\} \\ &\leq \sup\{\sup\{|b(x, y)| : \|x\|_X \leq 1\,, \|y\|_Y \leq 1\} : \|b\|_{Bil(X\times Y)} \leq 1\} \\ &= \sup\{\|b\|\|_{Bil(X\times Y)} : \|b\|\|_{Bil(X\times Y)} \leq 1\} \leq 1\,.\end{aligned}$$

∎

Remark 7.2. (i) Since $\overline{P}_D T^\times : Z' \to Bil(X \times Y)$, one has

$$\begin{aligned}
&\|\overline{P}_D T^\times\|_{L(Z',Bil(X\times Y)}\\
&= \sup\{\|\overline{P}_D T^\times g\|_{Bil(X\times Y)} : \|g\|_{Z'} \le 1\}\\
&= \sup\{\sup\{|(\overline{P}_D T^\times g)(x,y)| : \|x\|_X \le 1\,, \|y\|_Y \le 1\} : \|g\|_{Z'} \le 1\} \text{ (by (6.2))}\\
(7.1)\quad &= \sup\{\sup\{|gT(P_D x, P_D y)| : \|x\|_X \le 1\,, \|y\|_Y \le 1\} : \|g\|_{Z'} \le 1\}\\
&= \sup\{\sup\{|g(T(P_D x, P_D y))| : \|x\|_X \le 1\,, \|y\|_Y \le 1\} : \|g\|_{Z'} \le 1\}\\
&= \sup\{\sup\{|g(T(P_D x, P_D y))| : \|g\|_{Z'} \le 1\} : \|x\|_X \le 1\,, \|y\|_Y \le 1\}\\
&= \sup\{\|(T(P_D, P_D))(x,y)\|_Z : \|x\|_X \le 1\,, \|y\|_Y \le 1\}\\
&= \|T(P_D, P_D)\|_{Bil(X\times Y,Z)}
\end{aligned}$$

(ii) Since $T : X \times Y \to Z$ is bilinear and $(P_D, P_D) : X \times Y \to X \times Y$, it follows that $T \circ (P_D, P_D) : X \times Y \to Z$ is bilinear and

$$T \circ (P_D, P_D)(x,y) = T(P_D, P_D)(x,y) = T(P_D x, P_D y)\,.$$

(iii) Considering the sequence of operators

$$Z' \xrightarrow{P_E} Z' \xrightarrow{T^\times} Bil(X \times Y) \xrightarrow{\overline{P}_D} Bil(X \times Y)\,,$$

then, $Z' \overset{\overline{P}_D T^\times P_E}{\longrightarrow} Bil(X \times Y)$. For $\phi \in Z'$ and $(x,y) \in X \times Y$ one has

$$\begin{aligned}
(\overline{P}_D T^\times P_E)(\phi)(x,y) &= \overline{P}_D((T^\times(P_E\phi))(x,y))\\
&= \overline{P}_D((P_E\phi)(T(x,y)) = \overline{P}_D((\phi(P_D T)(x,y))\\
&= \phi P_D T(P_D x, P_D y)\,.
\end{aligned}$$

Thus, following the calculations from remarks (iii) and (iv), we obtain

$$\|\overline{P}_D T^\times P_E\|_{L(Z',Bil(X\times Y))} = \|P_D T(P_D, P_D)\|_{Bil(X\times Y,Z)}\,. \tag{7.2}$$

Theorem 7.3. *Let X, Y and Z be quasi-normed functional spaces, where $\mu(\Omega_1) < \infty$, $\nu(\Omega_2) < \infty$, $\upsilon(\Omega_3) < \infty$ and Z has absolutely continuous quasi-norms, i.e $Z = Z_a$. A bounded bilinear regular operator $T : X \times Y \to Z$ is compact if, and only if, $(T^\times)^*$ is compact in measure and satisfies*

$$\lim_{\upsilon(E)+\mu(D_1)+\nu(D_2)\to 0} \|P_E T^\times(P_{D_1}, P_{D_2})\|_{X\times Y,Z)} = 0\,. \tag{7.3}$$

Proof. If T is compact, from Theorem 6.3, T is compact in measure and

$$\lim_{\upsilon(E)+\mu(D_1)+\nu(D_2)} \|P_E A(P_{D_1}, P_{D_2})\|_{Bil(X\times Y,Z)} = 0\,.$$

From (7.3) the theorem follows. ∎

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