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A NOTE ON DIVERGENT FOURIER SERIES AND λ-PERMUTATIONS

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ABSTRACT. We present a continuous function on $[-\pi, \pi]$ whose Fourier series diverges and it cannot be rearranged to converge by a λ-permutation.

Key words and phrases: Fourier series, Rearrangements, λ-permutations.

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1.1. **Fourier series.** The Fourier series associated with a continuous function f on $[-\pi, \pi]$ is defined by

$$\widetilde{f}(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta},$$

where

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta \ .$$

Here a_n's are called the Fourier coefficients of f and we denote by $\widetilde{f}$ the Fourier series associated with f.

In the eighteenth century mathematicians experimentally showed that for some functions (such as differentiable ones) pointwise convergence holds, i.e.,

$$\sum_{k=-n}^{n} a_k e^{ik\theta} \to f(\theta) \text{ as } n \to \infty \ . \tag{1.1}$$

Fourier claimed that this should always hold and Dirichlet rigorously formulated this convergence under some general conditions. Dirichlet's proof left an open question; whether the Fourier series of every Riemann integrable function (or at least every continuous function) converges back to the function. Dirichlet and many other mathematicians believed that the answer was affirmative. However Du Bois–Reymond surprisingly introduced a counterexample in 1876 [1], showing that the Fourier series of a continuous function may diverge.

After this and more elaborate counterexamples the following question raised naturally. If the Fourier series of a function diverges, is there anyway to fix this divergence, in other words is it possible to reconstruct f from the sequence of Fourier coefficients? One of the classic approaches to accomplish this is by the use of *Cesàro means*. The Cesàro means of a sequence $\{x_n\}$ is the sequence $\{y_n\}$ defined by

$$y_1 = \frac{x_1}{1},\, y_2 = \frac{x_1 + x_2}{2}, \ldots,\, y_n = \frac{\sum_{j=1}^{n} x_j}{n}.$$

Note that if $\{x_n\}$ converges to some number L, then so does $\{y_n\}$ to the same number. But there are instances where $\{x_n\}$ diverges but the Cesàro means $\{y_n\}$ converges, e.g. $x_n = (-1)^n$. When this process of Cesàro means is applied to the sequence of partial sums of a Fourier series of a continuous function, we guarantee the pointwise convergence to the associated function, see [5, page 97]. One drawback of this approach is the computational efficiency. Indeed adding so many terms together is not always the most efficient approach in computational problems.

1.2. λ**-permutations.** An alternative way of fixing divergent Fourier series is by λ-permutations. The notions of convergence-preserving rearrangements and λ-permutations were introduced first in the context of real series, see [3, 6] and references therein. It was later noticed that these permutations can be applied to divergent Fourier series [4].

A λ-permutation (see [3] for examples) is defined to be a permutation σ of the natural numbers with the following two properties:

(a) If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges.

(b) There is $\sum_{n=1}^{\infty} a_n$ that diverges but $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges.

In other words, σ is a λ-permutation if it is convergence-preserving and if there is at least one divergent series that it renders convergent. We denote the set of all such permutations by Λ. In

[6], Velleman furthered the discussion of λ-permutations with the notion of the block number and obtained a characterization for convergence-preserving permutations.

First, we define a block as a set of consecutive integers $[c,d]_{\mathbb{Z}} = \{x \in \mathbb{Z} : c \leq x \leq d\}$. Then for a permutation σ and a positive integer n, we can write

$$\{\sigma(1), \sigma(2), \ldots, \sigma(n)\} = [c_1, d_1]_{\mathbb{Z}} \cup [c_2, d_2]_{\mathbb{Z}} \cup \cdots \cup [c_{b_n}, d_{b_n}]_{\mathbb{Z}}$$

as a union of disjoint blocks of consecutive integers. Here b_n keeps track of number of disjoint blocks and $\{b_n\}_{n=1}^{\infty}$ is called the block number sequence of σ.

Theorem 1.1 (Velleman 2006)**.** *The following statements concerning a permutation σ of the natural numbers are equivalent:*

(a) *σ is convergence-preserving.*
(b) *The block number sequence of σ is bounded.*
(c) *σ is sum-preserving.*

With this characterization, a permutation σ is a λ-permutation if and only if the block number sequence of σ is bounded but the block number sequence of σ^{-1} is unbounded.

1.3. **Fixing Fourier series by λ-permutations.** In the early 1900s, Fejér constructed a function $F(x)$ that is continuous on $[-\pi, \pi]$ but whose Fourier series diverges at $x = 0$. We present the following rendition of $F(x)$ that turns out to be computationally more efficient. For details of the claims (for a similar example) we refer to [7, page 298].

Let $\alpha_k = (k(\log(k))^{2.1})^{-1}$ and $N_k = \left\lceil (1.1)^{k(\log(k))^{2.1}} \right\rceil$, for $k \in \mathbb{Z}$. The key properties of α_k and N_k are:

(1) $\sum \alpha_k < \infty$,
(2) $N_{k+1} > 3N_k$ if $k \geq 1$,
(3) $\lim_{k\to\infty} \alpha_k \log N_k = \infty$.

Now for any $N_k \in \mathbb{N}$, let

$$Q_{N_k}(x) = e^{2iN_k x} \sum_{\substack{j=-N_k \\ j \neq 0}}^{N_k} \frac{e^{ijx}}{j}.$$

Then one can show that $Q_{N_k}(x)$ is uniformly bounded on $[-\pi, \pi]$, see also Lemma 2.3 below. Now we define $F(x)$ as,

$$\begin{aligned} F(x) &= \sum_{k=2}^{\infty} \alpha_k Q_{N_k}(x) \\ &= \sum_{k=2}^{\infty} (k(\log(k))^{2.1})^{-1} \left(e^{2iN_k x} \sum_{\substack{j=-N_k \\ j \neq 0}}^{N_k} \frac{e^{ijx}}{j} \right), \end{aligned}$$

the series uniformly converges and therefore $F(x)$ is a continuous function. Let $S_{2N_k}F(x)$ denote the first $2N_k$ partial sums of the Fourier series of $F(x)$. We have that

$$S_{2N_k}F(x) = \sum_{m=2}^{2N_k} (m(\log(m))^{2.1})^{-1} \cdot \sum_{j=N_m}^{2N_m - 1} \frac{e^{ijx}}{j - 2N_m}.$$

From this, one can deduce that

$$\lim_{k\to\infty} S_{2N_k}F(0) = \infty.$$

A less rigorous (but quicker) way of seeing the divergence of the Fourier series of $F(x)$ is an implementation of these functions in MATLAB©. Figure 1 below shows that even at an early stage of the functions, it is clear that $S_{2N_k}F(x)$ does not behave very well at $x = 0$. With only five terms, we see that $S_{2N_5}F(x)$ does not closely approximate $F(x)$.

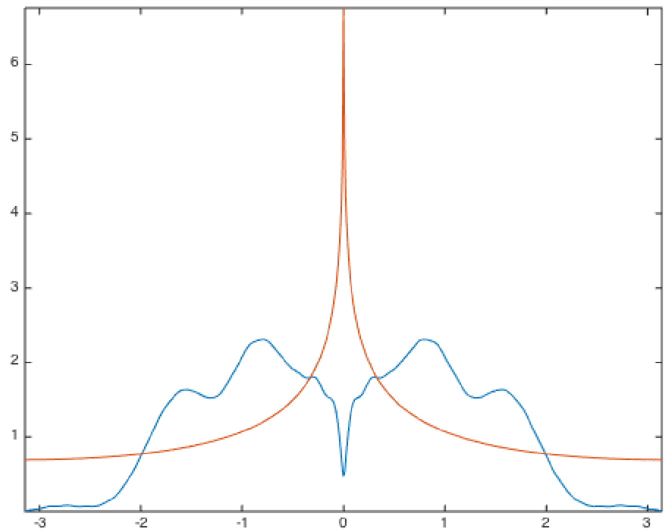

Figure 1: [Color online] Graph of $F(x)$ (blue) and its partial Fourier series (red) when $k = 5$.

It was noted in [4] that using a λ-permutation one can then fix the divergence of the Fourier series of $F(x)$. More precisely, there exists a λ-permutation σ such that the rearranged Fourier series by σ, denoted by $S_N^\sigma F(x)$, converges pointwise to $F(x)$.

This result leads to the following natural question. Is it possible to fix any divergent Fourier series by a λ-permutation? The answer would give a good comparison between fixing divergent series by Cesàro means and by λ-permutations. Below we answer the question in negative.

Theorem 1.2. *There exists a continuous function $F(x)$ on $[-\pi, \pi]$ with the following properties:*

(1) $\lim_{N\to\infty} S_N F(0)$ *does not exist,*
(2) $\lim_{N\to\infty} S_N^\sigma F(0)$ *does not exist for any $\sigma \in \Lambda$,*
(3) $\lim_{N\to\infty} S_N^\rho F(0)$ *does exist for some $\rho \notin \Lambda$.*

2. PROOF

Our construction of the desired function $F(x)$ is a composition of Fejer's idea (see [2]), and ideas in [4, 6]. We start with an explicit example of a conditionally convergent series that cannot be rearranged to converge by a λ-permutation. The similar example in [6] does not lead to a function $F(x)$ with the desired properties so we modify this example carefully. The rest of the construction is similar to the example in [4].

2.1. **The hopping sequence.** The following permutation of integers is an essential piece in our construction. For any $N \in \mathbb{N}$, permute the integers $\{1, 2, \cdots, 2N\}$ as

$$\{2N, 1, 2N-1, 2, 2N-2, 3, \cdots, N+2, N-1, N+1, N\}. \tag{2.1}$$

We label this permutation by η.

Let $\beta_k = \frac{1}{k^2}$ and $N_k = 2^{k^3}$ and consider the following infinite sum

$$\begin{aligned}\Sigma_1 = {} & \frac{\beta_1}{1} + \frac{\beta_1}{2} + \cdots + \frac{\beta_1}{N_1} - \frac{\beta_1}{1} - \frac{\beta_1}{2} - \cdots - \frac{\beta_1}{N_1} \\ & + \cdots \\ & + \frac{\beta_k}{1} + \frac{\beta_k}{2} + \cdots + \frac{\beta_k}{N_k} - \frac{\beta_k}{1} - \frac{\beta_k}{2} - \cdots - \frac{\beta_k}{N_k} \\ & + \cdots\end{aligned}$$

Lemma 2.1. *Σ_1 is divergent. However, there exists a λ-permutation that rearranges Σ_1 into a convergent series.*

Proof. We see the divergence by showing that the sequence of partial sums of Σ_1 is not Cauchy. Indeed, the stretch of terms

$$\frac{\beta_k}{1} + \frac{\beta_k}{2} + \cdots + \frac{\beta_k}{N_k}$$

can be estimated from below. For any $k \geq 1$,

$$\begin{aligned}\frac{\beta_k}{1} + \frac{\beta_k}{2} + \cdots + \frac{\beta_k}{N_k} &\geq \frac{1}{k^2} \ln N_{k+1} \\ &= \frac{1}{k^2}(k+1)^3 \ln 2 > k \ln 2.\end{aligned}$$

On the other hand, we can rearrange the terms of Σ_1 so that a nice cancellation happens. Indeed, the rearranged series

$$\begin{aligned}& \frac{\beta_1}{1} - \frac{\beta_1}{1} + \frac{\beta_1}{2} - \frac{\beta_1}{2} + \cdots + \frac{\beta_1}{N_1} - \frac{\beta_1}{N_1} \\ & + \cdots \\ & + \frac{\beta_k}{1} - \frac{\beta_k}{1} + \frac{\beta_k}{2} - \frac{\beta_k}{2} + \cdots + \frac{\beta_k}{N_k} - \frac{\beta_k}{N_k} \\ & + \cdots\end{aligned}$$

is an alternating series with general term approaching 0. Furthermore, this simple rearrangement has a bounded block number sequence, actually it is bounded by 2. Hence this permutation is convergence-preserving. Additionally, it fixes Σ_1 therefore it is indeed a λ-permutation. ∎

Next, we consider the following series,

$$\begin{aligned}\Sigma_2 = {} & \frac{\beta_1}{1} + \frac{\beta_1}{2} + \cdots + \frac{\beta_1}{N_1} - \frac{\beta_1}{\eta(1)} - \frac{\beta_1}{\eta(2)} - \cdots - \frac{\beta_1}{\eta(N_1)} \\ & + \cdots \\ & + \frac{\beta_k}{1} + \frac{\beta_k}{2} + \cdots + \frac{\beta_k}{N_k} - \frac{\beta_k}{\eta(1)} - \frac{\beta_k}{\eta(2)} - \cdots - \frac{\beta_k}{\eta(N_k)} \\ & + \cdots\end{aligned}$$

Lemma 2.2. *Σ_2 is divergent. However, no λ-permutation rearranges Σ_2 into a convergent series.*

Proof. Σ_2 diverges for the same reason as in Lemma 2.1. Suppose that there is a λ-permutation σ that fixes Σ_2. Let's denote the rearranged (convergent series) by $\widetilde{\Sigma}_2$ and it is partial sums by $(\widetilde{\Sigma}_2)_l$. Then there exists M such that for all $l_1 > l_2 \geq M$,

$$\left|(\widetilde{\Sigma}_2)_{l_1} - (\widetilde{\Sigma}_2)_{l_2}\right| < 1. \tag{2.2}$$

Now let's choose a sufficiently large k such that all the terms,

$$\frac{\beta_k}{1}, \frac{\beta_k}{2}, \cdots, \frac{\beta_k}{N_k}$$

appear after the first M terms of $\widetilde{\Sigma}_2$. Let $m_1 + 1$ be the first position where one of these terms shows up and m_2 be the last position where the last of these terms shows up in $\widetilde{\Sigma}_2$. In other words, the difference

$$(\widetilde{\Sigma}_2)_{m_2} - (\widetilde{\Sigma}_2)_{m_1}$$

contains all the terms

$$\frac{\beta_k}{1}, \frac{\beta_k}{2}, \cdots, \frac{\beta_k}{N_k}.$$

In the sum of all the terms $\frac{\beta_k}{1}, \frac{\beta_k}{2}, \cdots, \frac{\beta_k}{N_k}$, we will have the smallest block number when the difference $(\widetilde{\Sigma}_2)_{m_2} - (\widetilde{\Sigma}_2)_{m_1}$ keeps the order of these terms $\frac{\beta_k}{1}, \frac{\beta_k}{2}, \cdots, \frac{\beta_k}{N_k}$ as follows:

$$\frac{\beta_k}{1} + \frac{\beta_k}{2} + \frac{\beta_k}{3} + \cdots + \frac{\beta_k}{N_{k-1}} + \frac{\beta_k}{N_k}. \tag{2.3}$$

The block number keeps increasing whenever we switch the order of these terms $\frac{\beta_k}{1}, \frac{\beta_k}{2}, \cdots, \frac{\beta_k}{N_k}$. Now, let's consider the best possible scenario for the finite block number which has the smallest block number and note that this difference $(\widetilde{\Sigma}_2)_{m_2} - (\widetilde{\Sigma}_2)_{m_1}$ may contain some more of positive terms but it should contain certain amount of negative terms, otherwise we get a contradiction with (2.2). That is, the difference $(\widetilde{\Sigma}_2)_{m_2} - (\widetilde{\Sigma}_2)_{m_1}$ looks like

$$\frac{\beta_k}{1} + \frac{\beta_k}{2} + \cdots + \oplus \cdots \ominus + \ominus + \frac{\beta_k}{3} + \ominus + \cdots + \oplus + \ominus + \frac{\beta_k}{N_{k-1}} + \cdots + \frac{\beta_k}{N_k},$$

where $\oplus$ is a positive term and $\ominus$ is a negative term. Also, notice that these negative terms should appear frequently often between the positive terms in (2.3), otherwise we can again find a stretch of positive terms that would violate (2.2).

When the permutation σ inserts the negative terms of Σ_2 in between the positive terms in (2.3), it has to insert at least two consecutive ones together. Otherwise, the hopping sequence would increase the block number arbitrarily large. For example, if we insert a negative term one by one to fix the sum $\sum_{n=1}^{N_k} \beta_k/n$ to be convergent, the permutation σ will be

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 1 & N_k+2 & 2 & N_k+4 & 3 & N_k+6 & \cdots \end{pmatrix}.$$

Then the block number of this permutation is unbounded because

$$\begin{aligned} &\{\sigma(1), \cdots, \sigma(N_k)\} = \\ &[1, \frac{N_k}{2}] \cup [N_k+2, N_k+2] \cup [N_k+4, N_k+4] \cup [N_k+6, N_k+6] \cup \cdots, \end{aligned}$$

where N_k is 2^{k^3}.

Since the permutation σ inserts at least two consecutive negative terms in (2.3), the negative terms can separate N_k many positive terms at most into $N_k/2$ blocks. However, by pigeon-hole principle there has to be a block of positive terms with sum greater than 1 because once we use all negative terms to fix the sum (2.3), there will still be a block in where the sum of all terms is $\mathcal{O}(k)$. This again contradicts with (2.2). To meet Cauchy's criterion (2.2) we will have infinitely many blocks, so we conclude that no λ-permutation can fix Σ_2. ∎

Remark 2.1. Although no λ-permutation fixes Σ_2, the rearrangement similar to the one in Lemma 2.1 turns Σ_2 into a convergent series. Indeed, the rearranged series

$$\begin{aligned}
&\frac{\beta_1}{1} - \frac{\beta_1}{\eta(2)} + \frac{\beta_1}{2} - \frac{\beta_1}{\eta(4)} + \cdots + \frac{\beta_1}{N_1} - \frac{\beta_1}{\eta(1)} \\
&+ \cdots \\
&+ \frac{\beta_k}{1} - \frac{\beta_k}{\eta(2)} + \frac{\beta_k}{2} - \frac{\beta_k}{\eta(4)} + \cdots + \frac{\beta_k}{N_k} - \frac{\beta_k}{\eta(1)} \\
&+ \cdots
\end{aligned}$$

is an alternating series with general term approaching 0. However, the permutation has an unbounded block number sequence. Hence, it is not a λ-permutation.

2.2. **Function.** In this part, we construct the desired function in Theorem 1.2 by following ideas in [2, page 8], [7, page 298] and [4].

For any even positive integer N, we define

$$Q_N(x) = \left(\sum_{j=1}^{N} \frac{\exp(i(2N-j)x)}{j} - \sum_{j=1}^{N} \frac{\exp(i(2N+j)x)}{\eta(j)} \right).$$

Lemma 2.3. *$Q_N(x)$ is uniformly bounded on $[-\pi, \pi]$.*

Proof. Let's set $N = 2n$. Then, we have

$$\begin{aligned}
Q_N(x) &= \left(\sum_{j=1}^{N} \frac{\exp(i(2N-j)x)}{j} - \sum_{j=1}^{N} \frac{\exp(i(2N+j)x)}{\eta(j)} \right) \\
&= e^{i4nx} \left(\frac{e^{-ix}}{1} + \frac{e^{-2ix}}{2} + \cdots + \frac{e^{-2nix}}{2n} \right) \\
&\quad - e^{i4nx} \left(\frac{e^{ix}}{2n} + \frac{e^{2ix}}{1} + \frac{e^{3ix}}{2n-1} + \cdots + \frac{e^{(2n-1)ix}}{n+1} + \frac{e^{2nix}}{n} \right) \\
&= e^{i4nx} \left(\frac{e^{-ix}}{1} + \frac{e^{-2ix}}{2} + \cdots + \frac{e^{-2nix}}{2n} \right) \\
&\quad - e^{i4nx} \left(\frac{e^{2ix}}{1} + \frac{e^{4ix}}{2} + \cdots + \frac{e^{2nix}}{n} \right) \\
&\quad - e^{i4nx} \left(\frac{e^{ix}}{2n} + \frac{e^{3ix}}{2n-1} + \cdots + \frac{e^{(2n-1)ix}}{n+1} \right) \\
&= e^{i4nx} \left(\frac{e^{-ix}}{1} + \frac{e^{-2ix}}{2} + \cdots + \frac{e^{-2nix}}{2n} \right) \\
&\quad - e^{i4nx} \left(\frac{e^{2ix}}{1} + \frac{e^{4ix}}{2} + \cdots + \frac{e^{2nix}}{n} \right) \\
&\quad - e^{i6nx} \left(\frac{e^{-ix}}{n+1} + \frac{e^{-3ix}}{n+2} + \cdots + \frac{e^{-(2n-1)ix}}{2n} \right).
\end{aligned}$$

Each piece of the three expressions inside parentheses are uniformly bounded by [7, page 299, Equation 1.7]. Therefore $Q_N(x)$ is a uniformly bounded exponential polynomial. ∎

Again, let $\beta_k = \frac{1}{k^2}$ and $N_k = 2^{k^3}$, now we define $F(x)$ as

$$F(x) = \sum_{k=1}^{\infty} \beta_k Q_{N_k}(x).$$

By Lemma 2.3 and Weierstrass M-test, it is clear that $F(x)$ is continuous on $[-\pi, \pi]$. Next, we verify three claims in Theorem 1.2. We start by computing the Fourier series of $F(x)$, which is immediate since $F(x)$ is sum of exponential terms:

$$\tilde{F}(x) \sim \sum_{m=1}^{\infty} \beta_m \left(\sum_{j=1}^{N_m} \frac{\exp\left(i\left(2N_m - j\right)x\right)}{j} - \sum_{j=1}^{N_m} \frac{\exp\left(i\left(2N_m + j\right)x\right)}{\eta(j)} \right).$$

Here, it is not difficult to check $3N_{(k-1)} \leq N_k - 1$ for $k \geq 2$ and

$$\begin{aligned}
S_{2N_k}F(x) &= \sum_{m=1}^{k-1} \beta_m \left(\sum_{j=1}^{N_m} \frac{\exp\left(i\left(2N_m - j\right)x\right)}{j} - \sum_{j=1}^{N_m} \frac{\exp\left(i\left(2N_m + j\right)x\right)}{\eta(j)} \right) \\
&\quad + \beta_k \left(\frac{\exp(iN_k x)}{1} + \frac{\exp(i(N_k+1)x)}{2} + \cdots + \frac{\exp(i(2N_k-1)x)}{N_k} \right), \\
S_{N_k-1}F(x) &= \sum_{m=1}^{k-1} \beta_m \left(\sum_{j=1}^{N_m} \frac{\exp\left(i\left(2N_m - j\right)x\right)}{j} - \sum_{j=1}^{N_m} \frac{\exp\left(i\left(2N_m + j\right)x\right)}{\eta(j)} \right).
\end{aligned}$$

Thus, we observe, for $k \geq 2$,

$$S_{2N_k}F(x) - S_{N_k-1}F(x) = \beta_k \left(\frac{\exp(ix)}{1} + \frac{\exp(i2x)}{2} + \cdots + \frac{\exp(iN_k x)}{N_k} \right) \times e^{i(N_k-1)x}.$$

When we plug in $x = 0$, we get

$$\begin{aligned}
S_{2N_k}F(0) - S_{N_k-1}F(0) &= \beta_k \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{N_k} \right) \\
&\geq \frac{1}{k^2} \ln(N_{k+1}) \geq k \ln(2).
\end{aligned}$$

This shows that $\lim_{N\to\infty} S_N F(0)$ does not exist.

For the second claim, we look at the Fourier series of F at $x = 0$. We note that $S_N F(0)$ is exactly the partial sums of the series Σ_2 in Lemma 2.2. Therefore, we conclude that $\lim_{N\to\infty} S_N^{\sigma} F(0)$ does not exist for any $\sigma \in \Lambda$.

Finally by the Remark 2.1, we conclude that $\lim_{N\to\infty} S_N^{\rho} F(0)$ does exist for some $\rho \notin \Lambda$. This completes the proof of Theorem 1.2.

3. FURTHER DIRECTIONS

After this curious relation between divergent Fourier series and λ-permutations some natural questions arise. Here we list a few of them.

- One can investigate the computational efficiency of fixing divergent Fourier series by Cesàro means and by λ-permutations. In our calculations with the Fejér's example above the summation step was generally quite clumsy. However, the rearrangement approach performed faster. The study of numerical efficiency is an interesting direction to pursue.

- Here we presented two different functions, both with divergent Fourier series but one can be fixed by λ-permutations whereas the other cannot. Hence it is a natural question to search for necessary and/or sufficient conditions for functions whose Fourier series behave well under λ-permutations.
- Pointwise convergence is not the only mode of convergence and the other modes of convergence for Fourier series have been investigated in quite details. What is the effect of λ-permutations on other modes of convergence? In particular, can λ-permutations be used to produce more uniformly convergent Fourier series?

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