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EXTREME CURVATURE OF POLYNOMIALS AND LEVEL SETS

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ABSTRACT. Let f be a real polynomial of degree n. Determining the maximum number of zeros of kappa, the curvature of f, is an easy problem: since the zeros of kappa are the zeros of f”, the curvature of f is 0 at most n-2 times. A much more intriguing problem is to determine the maximum number of relative extreme values for the function kappa. Since kappa’=0 at each extreme point of kappa, we are interested in the maximum number of zeros of kappa’. In 2004, the first author and R. Gordon showed that if all the zeros of f” are real, then f has at most n-1 points of extreme curvature. We use level curves and auxiliary functions to study the zeros of the derivatives of these functions. We provide a partial solution to this problem, showing that f has at most n-1 points of extreme curvature, given certain geometrical conditions. The conjecture that f has at most n-1 points of extreme curvature remains open.

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1. INTRODUCTION

Let f be a real polynomial of degree n, where $n \geq 1$. Then

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

where $a_i \in \mathbb{R}$. The curvature of f is defined to be

$$\kappa = \frac{f''}{(1 + f'^2)^{\frac{3}{2}}}.$$

Determining the maximum number of zeros of κ is an easy problem: since the zeros of κ are the zeros of f'', the curvature of f is 0 at most $n-2$ times. A much more intriguing problem is to determine the maximum number of relative extreme values for the function κ. A point c is an extreme point of κ if κ has either a relative maximum or a relative minimum value at c. In this case the value of κ at c is an *extreme value* of κ. Since $\kappa' = 0$ at each extreme point of κ, we are interested in the maximum number of zeros of κ'.

In 2004, the first author and R. Gordon [3] explored the notion of extreme curvature and posed the following conjecture:

Conjecture 1.1 (Edwards-Gordon (2004)). *If f is a real polynomial of degree n greater than 1, then the curvature κ of f has at most $n-1$ extreme points.*

In their paper, they verified the conjecture for $n = 1, n = 2$, and $n = 3$ and went further to establish the following partial result:

Theorem 1.2. [3] *If f is a real polynomial of degree $n > 1$ and f'' has only real zeros, then the curvature κ of f has at most $n-1$ extreme points.*

In this paper, we remove the hypothesis that f'' has only real zeros, however, we add a hypothesis on the geometry of the level sets that we will be studying.

Theorem 1.3. *Let f be a real polynomial of degree n such that f' has only simple zeros and $K = \{z \in H^+ : \text{Im}Q(z) > 0\}$ has a only one unbounded component with boundary intersecting the real axis where $Q(z) = z - h(z)/h'(z)$ and $h = f'/\sqrt{1 + (f')^2}$. Then f has at most $n-1$ points of extreme curvature.*

2. BACKGROUND

The proof of Theorem 1.2 was inspired by a solution to a Pólya and Szegö exercise [10]; Let P be a real polynomials of degree $n > 1$ such that P has only real zeros. Find the number of real and non-real zeros of $P^2 + P'$. If one is to remove the hypothesis on the zeros of P, the problem becomes quite difficult. In fact, it was not until the late 1980's when T. B. Sheil-Small [11] was able to avoid the hypothesis on P to prove this $P^2 + P'$ problem by using auxiliary functions,

$$f(z) = e^{\int P(z)\,dz} \qquad L(z) = \frac{f'(z)}{f(z)} \qquad Q(z) = z - \frac{f(z)}{f'(z)}$$

and studying the zero level curves of the imaginary part of these functions. In addition he studied the components

$$\Lambda = \{z \in H^+ : \text{Im}L(z) > 0\} \qquad K = \{z \in H^+ : \text{Im}Q(z) > 0\} \tag{2.1}$$

where $H^+ = \{z \in \mathbb{C} : \text{Im}z > 0\}$ and counted the number of non-real zeros of Q' inside K, where

$$Q'(z) = \frac{f(z)f''(z)}{(f'(z))^2} \qquad f''(z) = \left((P(z))^2 + P'(z)\right) e^{\int P(z)dz}$$

A. Hinkkanen ([8], [9], [7], [6]), while working with a certain class of meromorphic functions, studied the same auxiliary functions, their zero level curves, and the same components as Sheil-Small. In addition, he also studied the components of

(2.2) $$\Lambda^- = \{z \in H^+ : \text{Im}L(z) < 0\} \qquad K^- = \{z \in H^+ : \text{Im}Q(z) < 0\}.$$

Further, the first author and S. Hellerstein expounded on the Sheil-Small and Hinkkanen techniques in [4]. A survey of the use of level curves can be found in [2].

We begin by defining our auxiliary functions. Let f be a real polynomial of degree n and f' have only simple zeros. Define $f' = g$ and let h be defined as

(2.3) $$h(z) = \frac{g(z)}{\sqrt{1 + (g(z))^2}}.$$

Then, we define the auxiliary functions, L and Q, as follows:

(2.4) $$L(z) = \frac{h'(z)}{h(z)} = \frac{g'(z)}{g(z)(1 + (g(z))^2)} \qquad Q(z) = z - \frac{h(z)}{h'(z)}.$$

We will be interested in $Q'(z)$ because $Q' = \dfrac{hh''}{(h')^2}$ and $h'' = \dfrac{(1+g^2)g'' - 3g(g')^2}{(1+g^2)^{5/2}}$. It should be noted that the numerator of h'' is exactly equal to the numerator of κ' when g is replaced by f'. So, finding the real and non-real zeros of h'' will find the real and non-real zeros of κ', and the real zeros of h'' are the points of extreme curvature of f.

With these observations, we can rephrase our conjecture.

Conjecture 2.1. *If g is a real polynomial of degree $n > 1$, then the polynomial $3gg'^2 - g''(1+g^2)$ has at most n real roots.*

Noting that the degree of h'' is $3n - 2$, we want to show that there are at most n real zeros of h''. We can again restate our conjecture in an equivalent form.

Conjecture 2.2. *If f is a real polynomial of degree n, then there exist at least $n - 1$ zeros of $h'' \in H^+$, where $H^+ = \{z \in \mathbb{C} : \text{Im}\,(z) > 0\}$.*

3. THE GEOMETRY OF THE COMPONENTS OF Λ AND K

In this section, we define the level sets K, K^-, Λ, Λ^-, and the various geometrical characteristics of these sets. We will characterize the locations of the zeros of g and g' (and equivantly, h and h') with regard to these level sets, as well as define some characteristics of the level sets themselves. We note that the zeros of g (and h) are the poles of L; the zeros of g' (and h') are zeros of L and poles of Q.

We begin by stating a fundamental result for polynomials. It states that each real value cannot be taken on infinitely many times by our two functions, Q and L.

Lemma 3.1. *Suppose $g(z)$ is a real polynomial of degree n. Then both $Q(z) = z - \dfrac{g(z)(1+g(z)^2)}{g'(z)}$ and $L(z) = \dfrac{g'(z)}{g(z)(1+g(z)^2)}$ take on each real value finitely often in H^+ and $3n$ times in $\mathbb{C}$.*

Proof. See Beardon[1] page 31. ∎

3.1. **Zeros.** We now discuss the relationship between the zeros of g, g' and h'' and the components $K, K^-, \Lambda,$ and Λ^- where h is as in equation (2.3), $K, \Lambda, K^-, \Lambda^-$ are as in equations (2.1), (2.2). We note that the zeros of a polynomial g are simple poles of $L = g'/g$ and that the zeros of g (all simple) are also zeros of h and $Q' = hh''/(h')^2$ where Q is as in equation (2.4). Further, if z_0 is a zero of multiplicity $m > 1$, then Q' has a removable singularity at z_0 and $Q'(z_0) = (m-1)/m$.

The following results can be found in [4] in section 3.1.

Lemma 3.2. *Let $g(z_0) = 0$ where $z_0 \in \mathbb{R} \cup H^+$.*

(1) *If $z_0 \in \mathbb{R}$ and z_0 is a simple zero of g, then $z_0 \in \partial K \cap \partial K^-$.*
(2) *If $z_0 \in H^+$ then $z_0 \in K$ and $z_0 \in \partial \Lambda^- \cap \partial \Lambda$.*

Lemma 3.3. *Let $g'(z_0) = 0$ where $z_0 \in \mathbb{R} \cup H^+$.*

(1) *If $z_0 \in H^+$, then $z_0 \in \partial K \cap \partial K^-$*
(2) *If $z_0 \in \mathbb{R}$ and z_0 is a simple zero of g':*
    (a) *If $\left(\frac{g''}{g}\right)(z_0) > 0$, then $z_0 \in \partial \Lambda$*
    (b) *If $\left(\frac{g''}{g}\right)(z_0) < 0$, then $z_0 \in \partial \Lambda^-$*
(3) *$z_0 \in \partial \Lambda$ if and only if $z_0 \in \partial K$*
(4) *$z_0 \in \partial \Lambda^-$ if and only if $z_0 \in \partial K^-$*

Proof. This proof is similar to that in [4]. It is essential to note that

$$sgn(L'(z_0)) = sgn(\frac{g''}{g}(z_0))$$

when $z_0 \in \mathbb{R}$. ∎

We now discuss multiple zeros of L (or equivantly, multiple zeros of g' and h') on the boundaries of both Λ and K. To so this, begin by assigning a weight to each zero of L.

Let $x \in \mathbb{R}$ where $L(x) = \dfrac{g'(x)}{g(x)} = 0$. The **weight** ω, of the point x, a real zero of L of multiplicity m is defined to be $\omega = \dfrac{m+\xi}{2}$ where $\xi = 0$ if m is even and $\xi = \text{sgn}\big((\dfrac{g^{(m+1)}}{g})(x)\big) \in \{1, -1\}$ if m is odd. Let $z \in H^+$ where $L(z) = 0$. The **weight** ω, of the point z, a non-real zero of L of multiplicity m is defined to be $\omega = m$.
The following lemma is due to Hinkkanen (Lemma 5.1 (5) in [8]).

Lemma 3.4. *Let $x_0 \in \partial K \cap \partial K^- \cap \mathbb{R}$ where x_0 is a multiple pole of the function $Q(z) = z - \dfrac{g(z)}{g'(z)}$. Then $g'(x_0) = 0$ and $g(x_0) \neq 0$.*
In addition, in a neighborhood of x_0 in H^+ there are m sector-like slices with vertex at x_0, lying alternating in K and K^-, where $m = ord(g', x_0)$.
It follows that if the number of slices in K is equal to the weight assigned to the zero of L.

Definition 3.1. Let γ be a boundary component of a component $U(U^-)$ of $K(K^-)$. At each point $\zeta \in \gamma$ with $Q'(\zeta) \neq 0, \infty$ we choose a unique unit tangent vector T to γ defined by the requirement that iT is a unit inner normal vector N for $U(U^-)$ at ζ. Then $T(\zeta)$ defines the "positive direction" of γ relative to $U(U^-)$. Similarly for λ, a boundary component of $V(V^-)$ of $\Lambda(\Lambda^-)$.

Lemma 3.5. [4] *Let γ be a boundary component of a component $U(U^-)$ of $K(K^-)$. If γ is traversed in the positive direction relative to $U(U^-)$ then Q is monotone increasing (or decreasing) along any arc $\gamma' \subset \gamma$ with $Q'(\zeta) \neq 0, \infty$ for $\zeta \in \gamma'$. Similarly, let λ be a boundary component of a component $V(V^-)$ of $\Lambda(\Lambda^-)$. If λ is traversed in the positive direction relative to $V(V^-)$ then L is monotone increasing (or decreasing) along any arc $\lambda' \subset \lambda$ with $L'(\zeta) \neq 0, \infty$ for $\zeta \in \lambda'$.*

Remarks

(1) If $Q'(\zeta_0) = 0, \infty$ and $\zeta_0 \in \gamma$, then as we leave ζ_0 and transverse γ in a positive direction relative to $U(U^-)$ we may return to ζ_0 k times, with $0 \leq k \leq k_0 - 1$ and k_0 the multiplicity of the zero ζ_0 of Q' or respectively, the multiplicity of the pole of Q at ζ_0. This follows immediately from the sector like configuration of the components of K and K^- in a neighborhood of ζ_0 (Lemma 3.4). Each departure from ζ_0 followed by a first subsequent return to ζ_0 defines a closed curve, "a loop", $\gamma' \subset \gamma$, (which may not be simple). It is tacitly assumed that every loop γ' is traversed exactly once along γ.

(2) Since Q is monotone increasing (decreasing) along a loop λ at ζ_0 we must encounter at least one pole of Q along λ at some point η on λ.

(3) Since there are finitely many poles of Q (g is a polynomial), we have at most finitely many poles on γ, a boundary component of U (Lemma 3.4). The number of times a pole ζ_0 of Q is encountered as we traverse γ will be referred to as the **"U multiplicity"** of the pole ζ_0, which exceeds by one the number of loops of γ at ζ_0. The total number of the U multiplicities of poles of Q on γ will determine the "U-multiplicity of the poles on γ". If γ is a boundary component of U^-, then we still have only finitely many poles on γ, since Q has only finitely many poles on H^+. In analogy with the above we will use the terms the **"U^- multiplicity"** of the pole $\zeta_0 \in \gamma$ and the "U-multiplicity of the poles on γ" with their obvious meanings.

(4) We note that the $U(U^-)$-*multiplicity of a pole, $\zeta_0 \in \gamma$ is the number of sectorial regions in all local neighborhoods of ζ_0 contained in $U(U^-)$*. We now have an obvious meaning for the terms the $K(K^-)$-multiplicity of one (or more) pole(s) of Q on a boundary component or particular boundary components of $K(K^-)$.

(5) Each of the above remarks hold for L and λ, a boundary component of $V(V^-)$ of $\Lambda(\Lambda^-)$.

3.2. **Components of Λ and K.** We will now move on to components of Λ and Λ^-, K and K^-. We first turn our attention to unbounded components of Λ and K. We let g be a polynomial of degree n with only simple zeros. Since L has a zero at infinity of order $2n+1$ and Q has a pole at infinity of the same order, in a neighborhood of infinity, there are sector like slices of K and K^-, Λ and Λ^-. Whether the boundaries of the components of the sector-like slices touch the real axis is not entirely clear from the function, however, empirical evidence suggest that we can find a small positive constant so that there is one unbounded component of K and exactly two unbounded components of K^- whose boundaries intersect the real axis. If there is one unbounded component of K and exactly two unbounded components of K^- whose boundaries intersect the real axis, we call that unbounded component of K the **" main artery"**.

In the rest of this paper, we will assume that the function g is a polynomial of degree n with only zeros and is such that the geometry of the components of K has a main artery.

We now introduce a topological definition which we will find convenient.

Definition 3.2. Let $U(U^-)$ be a component of $K(K^-)$ in H^+ and Γ a component of the boundary of $U(U^-)$ in H^+. Then Γ is the **"exterior boundary component"** of $U(U^-)$ if the component of the complement of Γ in $\mathbb{C}$ containing H^- contains no points of $U(U^-)$. The component of the complement of Γ containing $U(U^-)$ we term the **"interior"** of Γ, and denote it by Int(Γ). We may now talk about a component being in the interior of another component.

We also require the following elementary and useful observation.

Lemma 3.6. *Suppose that g is a real polynomial of degree n. Then $\Lambda \subset K$.*

Proof. This is Lemma 2 of Sheil-Small's paper [11], and follows from the observation that for $z \in \Lambda$, $\text{Im}Q(z) = \text{Im}z + \dfrac{\text{Im}L(z)}{|L(z)|^2} > \dfrac{\text{Im}L(z)}{|L(z)|^2} > 0$. ∎

3.3. **The Degree of Q.** We now turn our attention to finding the degree of the function Q in components of K and K^-.

Lemma 3.7. *Let U be a component of K. Then Q maps U onto H^+. Similarly, let U^- be a component of K^-. Then Q maps U^- onto H^-.*

Proof. Let U be a component of K. Since $U \subset K = \{z \in H^+ : \text{Im}Q(z) > 0\}$, it follows that $Q(U) \subset H^+$. Suppose that $Q(U) \neq H^+$. Then there is a point $z_0 \in H^+$ such that $z_0 \in \partial Q(U)$. There is also a neighborhood, N of z_0 such that $N \cap Q(U) \neq \emptyset$ and $N \cap Q(U)^c \cap \neq \emptyset$ (where $Q(U)^c = H^+ - Q(U)$). In order to consider the appropriate branch of Q^{-1}, choose $a \in U$ such that $Q'(a) \neq 0$. Then a branch of the inverse function exists at $b = Q(a)$ such that $Q^{-1}(b) = a$ because $Q(z) \to \infty$ as $z \to \infty$ in K. Using this branch, we have that $Q^{-1}(N) \cap U \neq \emptyset$ and $Q^{-1}(N) \cap U^c \neq \emptyset$ (where $U^c = H^+ - U$). This means that $\partial U \cap Q^{-1}(N) \neq \emptyset$. Now Q maps ∂U onto the real axis and therefore $z_0 \in \mathbb{R}$ which is a contradiction. Thus Q maps U onto H^+.

To show that Q maps a component U^- of K onto H^-, consider the auxiliary function $F(z) = -Q(z)$. It follows then that $F(U^-) \subset H^+$. The above argument shows that $F(U^-) = H^+$. ∎

The Counting Lemma is a form of the so called Riemann-Hurwitz Theorem. It can be found for rational functions in Beardon [1] and can be found for Riemann surfaces in Farkas and Kra [5]. It is also known as the McDonald Lemma found in Pólya and Szegö[10], page 297. Since the exact form needed was not found in the literature, we will present Sheil-Small's form.

Lemma 3.8 (The Counting Lemma)**.** *Suppose that U is a component of K such that $Q' = 0$ exactly k times in U. Then Q maps U onto H^+ exactly ν times where $1 \leq \nu \leq k+1$.*

To determine the degree of Q in a component of K or K^-, we will need to find the number of times Q takes on a certain value. As Q takes on each real value the same number of times, we will look for where Q takes on large real values, i.e. near points that are poles of Q.

Lemma 3.9. *Let g be a real polynomial of degree n with only simple zeros be such that the geometry of K has a main artery. Then there are exactly $n+2$ unbounded components of $K \cup K^-$.*

Proof. We will show that the number of unbounded components of $K \cup K^-$ in H^+ is equal to $n+2$ first looking at the case that K has a single unbounded component and then in the general case.

Since K has a main artery, K has exactly one component, U_0 that intersects the real axis. We assume that this is the only unbounded component of K. Since Q has a pole of order $2n+1$ at infinity, there are $2n+1$ sector like slices of K and K^- in H^+. In particular, there are n slices in K alternating with $n+1$ slices in K^-. To see that there must be exactly $n+1$ unbounded components of K^-, note that if there were more, there would be more than the $n+1$ slices at infinity in H^+. Suppose that the number of unbounded components is at most n. Then there must be one component for two of the slices. In order for this to happen, that component must have a component of K interior to it. This contradicts that U_0 is the one unbounded component.

Thus, there are $n+1$ unbounded components of K^- and a total of $n+2$ unbounded components of $K \cup K^-$ in H^+

Suppose that K^- has k unbounded components, $U_0^-, U_1^-, \ldots U_k^-$ and inside each component, U_i^-, there are m_i unbounded components of K. Choose $R > 0$ so that $\{z : |z| = R\}$ intersects each unbounded component of ∂K. Let $D = K^- \cap \{z : |z| > R\}$. Since Q has a pole of order $2n + 1$ at infinity, it must be there case that D consists of $n + 1$ disjoint components in K^- $(D_1, D_2, \ldots, D_{n+1})$ alternating with n disjoint components of $(H^+ - D) \cap \{z : |z| > R\}$. Thus there are $m_i + 1$ components of D in U_i^- for $1 \le i \le k$ and so $\sum_{i=1}^{k} m_i + k = n + 1$. Further the number of unbounded components of K is $\sum_{i=1}^{k} m_i + 1$ (the $+1$ is there because the one unbounded component whose boundary intersects the real axis can not be inside one of the $U_i's$) and the number of unbounded components of K^- is k. Thus, the number of unbounded components of $K \cup K^-$ in H^+ is equal to $\sum_{i=1}^{k} m_i + 1 + k = n + 2$.

∎

Lemma 3.10. *Let g be a real polynomial of degree n with only simple zeros such that the geometry of K has a main artery. The number of zeros of Q' in H^+ is at least $m + n - 1$, where m is the K-multiplicity of poles of Q on ∂U_0 and U_0 is the main artery.*

Proof. We first find the number of zeros of Q' in the main artery, U_0 of K and then we find a lower bound for the number of zeros of Q' in $H^+ - U_0$.

Choose $R > \max\{|a_1|, \ldots, |a_n|\}$ where $g(a_i) = 0$ for $1 \le i \le n$. Let $z_1, \ldots z_s$ be the distinct poles of Q on ∂U_0 and μ_j the U_0-multiplicity of the pole of Q at z_j for $1 \le j \le s$. Fix disjoint neighborhoods $N_1, \ldots N_s$ about $z_1, \ldots, z_s$ so that $N_j \cap U_0$ is the union of μ_j disjoint components on each of which Q is a univalent map onto a neighborhood of ∞ in H^+. Then $\bigcup_{j=1}^{s} N_j \cap U_0$ is the union of m_0 disjoint components. Let N_j' be those components for $1 \le j \le m_0$.

Choose $R_0 > R$ so that the disk $|z| < R_0$ contains all the bounded components of ∂U_0 as well as $\bigcup_{j=1}^{s} \overline{N}_j'$ and so that the circle $|z| = R_0$ intersects each of the U_0's unbounded boundary components. Suppose there are ν_0 unbounded components of K^- inside U_0 that share a boundary component with U_0. Put $D = U_0 \cap \{z : |z| > R_0\}$. Denote the components of D as $D_1, \ldots D_{\nu_0+1}$. Each $D_k (k = 1, \ldots, \nu_0 + 1)$ has exactly two unbounded boundary curves defined by $\partial D_k \cap \partial U_0$, say γ_k and γ_k', with Q monotone on γ_k and γ_k' and mapping γ_k and γ_k' one-to-one onto intervals $(-\infty, x_k]$ and $[x_k', \infty)$ respectively. Put $S = U_0 - \left(\bigcup_{j=1}^{s} N_j' \cup D\right)$. Then Q is analytic on the compact S and, for some $M > 0$, $|Q| \le M$ on S. For each k, $1 \le k \le \nu_0 + 1$ choose $\xi_k' \in \gamma_k'$ so that $Q(\xi_k') > M$ and $Q'(\xi_k') \ne 0$. Then $Q(\xi_k')$ is assumed exactly once on ∂D_k.

Similarly, each component of $\overline{N_j'} \cap \partial U_0$ consists of exactly two bounded boundary curves, say λ_j and λ_j', with Q monotone on λ_j and λ_j' and mapping λ_j and λ_j' one-to-one onto intervals $(-\infty, y_j]$ and $[y_j', \infty)$ respectively. For each j, $1 \le j \le s$ choose $\zeta_j' \in \lambda_j'$ so that $Q(\zeta_j') > M$ and $Q'(\zeta_j') \ne 0$. Then $Q(\zeta_j')$ is assumed exactly once on $\partial N_j'$.

We choose a neighborhood V_k' of ξ_k' on which Q is univalent. Then $F_k = \overline{D_k} - V_k'$ is a closed set with $Q \to \infty$ as $z \to \infty$ in F_k and $Q(z) \ne \tilde{x}_k' = Q(\xi_k')$ for $z \in F_k$. It follows that there is a neighborhood W_k of $\tilde{x}_k'$ so that every $w_k \in W_k$ is assumed exactly once in V_k and never assumed in F_k.

Similarly, we choose a neighborhood G_j' of ζ_j' on which Q is univalent. Then $H_j = \overline{N_j'} - G_j'$ is a closed set with $Q \to \infty$ as $z \to z_j$ in G_j and $Q(z) \ne \tilde{y}_j' = Q(\zeta_j')$ for $z \in H_j$. It follows that there is a neighborhood Ω_j of $\tilde{y}'_j$ so that every $\omega_j \in W_j$ is assumed exactly once in G_j and never assumed in H_j.

Choose a point, $w_{0_k} \in W_k$ and $\omega_{0_j} \in \Omega_j$ and let $w = \max\{w_{0_1}, \ldots w_{0_{v_0+1}}, \omega_{0_1}, \ldots, \omega_{0_m}\}$.

It also follows that we have the value w assumed exactly $\nu_0 + 1$ in D and $\sum_{i=1}^{s} \mu_i$ times in $\bigcup_{j=1}^{m} N'_j$. Thus $\sum_{i=1}^{s} \mu_i + \nu_0 + 1 = m + \nu_0 + 1$ times in U_0, once in each of the N'_j and once in each of the D_k. So Q is an $m + \nu_0 + 1$-fold map from U_0 to H^+ and there are $m + \nu_0$ zeros of Q' in U_0.

Suppose there are k unbounded components of K, $U_1, \ldots, U_k$, whose boundaries do not intersect the real axis with $c_1, \ldots c_k$ unbounded components of K^- inside $U_1, \ldots U_k$ respectively and that m_i is the $U - i$-multiplicity of the non-real poles of Q. Suppose further that there are l unbounded components of K^- with $d_1, \ldots, d_l$ unbounded components of K inside $U_1^-, \ldots, U_l^-$ respectively. By the argument above, Q is at least a $c_i + 1 + m_i$-fold map from U_i to H^+ for $1 \le i \le k$ and Q is at least a $d_i + 1$-fold map from U_i^- to H^- for $1 \le i \le l$. So, by Lemma 3.8, we have that the number of zeros of Q' in U_i is at least $c_i + m_i$ and in U_i^- is at least d_i. Therefore, there are at least $m_0 + \nu_0 + \sum_{i=1}^{k}(c_i + m_i) + \sum_{i=1}^{l} d_i$ zeros of Q' in H^+.

Further, since Q has a pole at infinity of order $2n+1$, it follows that $\nu_0 + \sum_{i=1}^{k} c_i + k + 1 = n$ and $\sum_{i=1}^{l} d_i + l = n+1$. So, the number of zeros of Q' in H^+ is $\sum_{i=0}^{k} m_i + 2n + 1 - (k+1+l) = m + 2n - (k + l)$. By Lemma 3.9 we have that $k + l = n + 1$ and so our count is at least $m + 2n - (n + 1) = m + n - 1$. ∎

Lemma 3.11. *Let g be a real polynomial of degree n with only simple zeros and K has a main artery. The number of zeros of h in H^+ is equal to the "K-multiplicity" of poles of Q on the boundary of K.*

Proof. Let $R > \max\{|a_1|, \ldots, |a_n|, |b_1|, \ldots, |b_{n-1}|\}$ where a_i is a zero of g and b_i is zero of g'. Note, that by Lemma 3.3, it is the case that the K-multiplicity of poles of Q is equal to the Λ-multiplicity of zeros of L. Let m represent the Λ-multiplicity of zeros of L in H^+. Since $L = \dfrac{g'}{g(g^2 + 1)}$ has a zero of order $2n + 1$ at infinity, we have $2n + 1$ sector like slices of Λ and Λ^- in $\{|z| > R\} \cap H^+$, alternating in Λ and Λ^- beginning and ending in Λ^-. Thus, there are n sectors in Λ. Note that $L(z) \to 0$ as $z \to \infty$ in Λ and $L = 0$ precisely m times on the finite $\partial\Lambda$. Recall that $\Lambda \subset K$ by Lemma 3.6. On each boundary component of Λ, L increases (decreases) monotonically, so there must exist points, $\xi_1, \ldots, \xi_{n+m} \in \partial\Lambda$ such that $L(\xi_i) = \infty$ for $1 \le i \le m + n$. Therefore $g(\xi_i) = 0$ or $g^2(\xi_i) = -1$. Since $g^2(\xi_i) = -1$ precisely n times, it follows that $g(\xi_i) = 0$ precisely m times. ∎

4. EXTREME CURVATURE

We are now in a position to prove our theorem.

Theorem 4.1. *If g is a real polynomial of degree n with only simple zeros and the geometry of K has a main artery, then there exist at least $n - 1$ zeros of $h'' \in H^+$, with $H^+ = \{z \in \mathbb{C} : Im\,(z) > 0\}$ and $h = g/\sqrt{1 + (g)^2}$.*

Proof. Let g be a real polynomial of degree n with only simple zeros such that there is one unbounded component U_0 of K whose boundary intersects the real axis. By Lemma 3.10 there are at least $m + n - 1$ zeros of Q' in H^+, where m is the U_0-multiplicity of the zeros of L on K. Note that the number of zeros of h in K is equal to the K-multiplicity of the zeros of L on

K by Lemma 3.11. Since $Q' = hh''/(h')^2$, we have that the number of zeros of h'' in H^+ is at least $n-1$. ∎

Theorem 1.3. Let f be a real polynomial of degree n such that f' has only simple zeros and $K = \{z \in H^+ : \text{Im}Q(z) > 0\}$ has a main artery where $Q(z) = z - h(z)/h'(z)$ and $h = f'/\sqrt{1+(f')^2}$ Then f has at most $n-1$ points of extreme curvature.

Proof. Let f be a polynomial of degree n. Then, let $g = f'$. Applying Theorem 4.1 to g we have that h'' has at least $n-2$ zeros in H^+. Since h'' is of degree $3n-5$ and at least $2n-4$ of the zeros are non-real, we have that at most $3n-5-(2n-4) = n-1$ zeros of h'' are real. The real zeros of h'' are exactly the points of extreme curvature of f.

∎

5. OPEN QUESTIONS

The extreme curvature conjecture, is still open. It is easily stated so that even a calculus students can understand it and empirically verify it with a computer algebra system, but a proof aludes us. If a level curves technique is to work, much must be learned about the geometry.

We now list some conjectures:

(1) If f is a real polynomial of degree n greater than 1, then the curvature κ of f has at most $n-1$ extreme points.
(2) If f is a real polynomial of degree n greater than 1, there is a small positive constant, $a > 0$ so that the geometry of $K_a = \{z \in H^+ : \text{Im}Q_a(z) > 0\}$ has a main artery where $Q_a(z) = z - h(z)/h'(z)$ and $h = af'/\sqrt{1+(af')^2}$.
(3) If f is a real polynomial of degree n greater than 1, then the curvature κ of af has at most $n-1$ extreme points, where a is a suitably small positive constant.
(4) If f is a real polynomial of degree n greater than 1 and h is defined by $h(z) = f(z) - iz$ for $z \in \mathbb{C}$, then $\{z \in \mathbb{R} : \{h, z\} \in \mathbb{R}\}$ contains at most $n-1$ points, where $\{h, z\}$ is the Schwarzian derivative of an analytic function, h. [3]

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