The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 14, Issue 1, Article 1, pp. 1-287, 2017

OSTROWSKI TYPE INEQUALITIES FOR LEBESGUE INTEGRAL: A SURVEY OF RECENT RESULTS

SILVESTRU SEVER DRAGOMIR[1,2]

Received 1 November, 2016; accepted 4 January, 2017; published 20 February, 2017.

[1]MATHEMATICS, COLLEGE OF ENGINEERING & SCIENCE, VICTORIA UNIVERSITY, PO BOX 14428, MELBOURNE CITY, MC 8001, AUSTRALIA.

[2]DST-NRF CENTRE OF EXCELLENCE, IN THE MATHEMATICAL AND STATISTICAL SCIENCES, SCHOOL OF COMPUTER SCIENCE & APPLIED MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, JOHANNESBURG 2050, SOUTH AFRICA.

sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

ABSTRACT. The main aim of this survey is to present recent results concerning Ostrowski type inequalities for the Lebesgue integral of various classes of complex and real-valued functions. The survey is intended for use by both researchers in various fields of Classical and Modern Analysis and Mathematical Inequalities and their Applications, domains which have grown exponentially in the last decade, as well as by postgraduate students and scientists applying inequalities in their specific areas.

Key words and phrases: Ostrowski inequality, Midpoint inequality, Lebesgue integral, Absolutely continuous functions, Functions of Bounded variation, Convex functions, Logarithmic convex functions, Monotonic functions, Mean value theorem.

2010 *Mathematics Subject Classification.* 26D15, 26D10

ISSN (electronic): 1449-5910

Contents

1. INTRODUCTION

The main aim of this survey is to present recent results concerning Ostrowski type inequalities for the Lebesgue integral of various classes of complex and real-valued functions. It is a natural continuation of the edited monograph from 2002 [*Ostrowski Type Inequalities and Applications in Numerical Integration*. Edited by Sever S. Dragomir and Themistocles M. Rassias. Kluwer Academic Publishers, Dordrecht, 2002. xx+481 pp. ISBN: 1-4020-0562-8 26-06] whose preprint version is available at `http://rgmia.org/monographs/Ostrowski.html`.

As revealed by a simple search in the database *MathSciNet* with the key words *"Ostrowski"* and *"inequality"* in the title, an exponential evolution of research papers devoted to this result has been registered in the last decade. There are now at least 470 papers that can be found by performing the above search. Numerous extensions, generalizations in both the integral and discrete case have been discovered. More general versions for n-time differentiable functions, the corresponding versions on time scales, for vector valued functions or multiple integrals have been established as well. Numerous applications in Numerical Analysis, Approximation Theory, Probability Theory & Statistics, Information Theory and other fields have been also given.

In the first Chapter of the survey, some Ostrowski type inequalities for functions of bounded variation, monotonic and convex functions are given. The case of absolutely continuous functions and bounds in terms of the Lebesgue p-norms of the derivatives are presented in Chapter 2 . Then, in Chapter 3 , by applying Cauchy Mean Value Theorem and Pompeiu Mean Value Theorem some Ostrowski type inequalities for differentiable functions are provided. In Chapter 4 , more Pompeiu type inequalities for absolutely continuous functions, powers and exponential are given, while in Chapter 5 , Ostrowski type inequalities for convex derivatives, derivatives that are convex or h-convex in modulus are presented. In Chapter 6 we introduce the class of S and H-dominated functions and establish the corresponding Ostrowski type inequalities as well as some inequalities for products of two functions. Further, in Chapter 7 , we established various perturbed Ostrowski type inequalities while in Chapter 8 we present some companions of Ostrowski inequality for various classes of functions, including functions of bounded variation, Lipschitzian functions, convex functions and absolutely continuous functions whose derivatives satisfy certain properties. The last part, Chapter 9 , deals with Ostrowski-Jensen type inequalities in the general setting of abstract Lebesgue integral for composite functions in which one function is Lebesgue integrable on a measurable space while the second is either of bounded variation or absolutely continuous and with the derivative satisfying some usual properties such as boundedness or of Lipschitz type etc...

The survey also provides a chronological list of published papers, the corresponding versions on time scales, for vector valued functions or multiple integrals devoted to Ostrowski inequality in different settings including the integral and discrete case, the corresponding versions on time scales, fractional integrals, for vector valued functions or multiple integrals.

The survey is intended for use by both researchers in various fields of Classical and Modern Analysis and Mathematical Inequalities and their Applications, domains which have grown exponentially in the last decade, as well as by postgraduate students and scientists applying inequalities in their specific areas.

Each chapter contains the necessary references and therefore can be read independently.

CHAPTER 1

Inequalities for Functions of Bounded Variation

1. Ostrowski Inequality for Functions of Bounded Variation

The following inequality for functions of bounded variation holds:

THEOREM 1.1 (Dragomir, 1999 [**4**]). *Let $f : [a, b] \to \mathbb{C}$ be a function of bounded variation on $[a, b]$. Then for all $x \in [a, b]$, we have the inequality*

$$\left| \int_a^b f(t)\,dt - (b-a) f(x) \right| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f), \tag{1.1}$$

where $\bigvee_a^b (f)$ denotes the total variation of f. The constant $\frac{1}{2}$ is the best possible one.

PROOF. Using the integration by parts formula for Riemann-Stieltjes integrals we have

$$\int_a^x (t-a)\,df(t) = f(x)(x-a) - \int_a^x f(t)\,dt$$

and

$$\int_x^b (t-b) df(t) = f(x)(b-x) - \int_x^b f(t)\,dt.$$

If we add the above two equalities, we obtain the following equality of interest, see [**6**]

$$(b-a) f(x) - \int_a^b f(t)\,dt = \int_a^b p(x,t)\,df(t), \tag{1.2}$$

where

$$p(x,t) := \begin{cases} t-a \text{ if } t \in [a,x) \\ t-b \text{ if } x \in [x,b] \end{cases},$$

for all $x, t \in [a, b]$.
It is well known [**1**, p. 177] that if $p : [a, b] \to \mathbb{C}$ is continuous on $[a, b]$ and $v : [a, b] \to \mathbb{C}$ is of bounded variation on $[a, b]$, then

$$\left| \int_a^b p(x)\,dv(x) \right| \leq \sup_{x \in [a,b]} |p(x)| \bigvee_a^b (v). \tag{1.3}$$

Applying the inequality (1.3) for $p(x,t)$ as above and $v(x) = f(x)$, $x \in [a, b]$, we get

$$\begin{aligned} \left| \int_a^b p(x,t)\,df(t) \right| &\leq \sup_{t \in [a,b]} |p(x,t)| \bigvee_a^b (f) = \max\{x-a, b-x\} \bigvee_a^b (f) \\ &= \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f) \end{aligned} \tag{1.4}$$

and then by (1.4), via the identity (1.2), we deduce the desired inequality (1.1).
Now to prove that $\frac{1}{2}$ is the best possible constant assume that the inequality (1.1) holds with a constant $C > 0$. That is,

$$\left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \le \left[C(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \tag{1.5}$$

for all $x \in [a,b]$.
Consider the function $f : [a,b] \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 0 \text{ if } x \in [a,b] \setminus \{\frac{a+b}{2}\} \\ 1 \text{ if } x = \frac{a+b}{2} \end{cases}$$

in (1.5). Then f is of bounded variation on $[a,b]$, and $\bigvee_a^b (f) = 2, \int_a^b f(t)\,dt = 0$. For $x = \frac{a+b}{2}$, we get in (1.5)$1 \le 2C$, which implies that $C \ge \frac{1}{2}$ and the theorem is completely proved. ∎

COROLLARY 1.2 (Dragomir, 2000 [**5**]). *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation. Then we have the inequality:*

$$\left|\int_a^b f(t)\,dx - (b-a) f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{2}(b-a) \bigvee_a^b (f). \tag{1.6}$$

The constant $\frac{1}{2}$ is best possible.

For other Ostrowski type inequalities, see [**2**].

2. Refinements for Functions of Bounded Variation

2.1. A Hölder Type Inequality for Total Variation. The following lemma is of interest in itself as well.

LEMMA 2.1 (Dragomir, 2008 [**10**]). *Let $f, u : [a,b] \to \mathbb{C}$. If f is continuous on $[a,b]$ and u is of bounded variation on $[a,b]$, then*

$$\begin{aligned} \left|\int_a^b f(t)\,du(t)\right| &\le \int_a^b |f(t)|\, d\left(\bigvee_a^t (u)\right) \\ &\le \left[\bigvee_a^b (u)\right]^{\frac{1}{q}} \left\{\int_a^b |f(t)|^p\, d\left(\bigvee_a^t (u)\right)\right\}^{\frac{1}{p}} \quad \textit{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ &\le \max_{t \in [a,b]} |f(t)| \bigvee_a^b (u). \end{aligned} \tag{2.1}$$

PROOF. Since the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists, then for any division $I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ with the norm $v(I_n) := \max_{i \in \{0,\ldots,n-1\}} (t_{i+1} - t_i) \to 0$ and for any intermediate points $\xi_i \in [t_i, t_{i+1}]$, $i \in \{0, \ldots, n-1\}$ we have:

$$\begin{aligned} \left|\int_a^b f(t)\,du(t)\right| &= \left|\lim_{v(I_n) \to 0} \sum_{i=0}^{n-1} f(\xi_i)[u(t_{i+1}) - u(t_i)]\right| \\ &\le \lim_{v(I_n) \to 0} \sum_{i=0}^{n-1} |f(\xi_i)|\,|u(t_{i+1}) - u(t_i)|. \end{aligned} \tag{2.2}$$

However,

$$|u(t_{i+1}) - u(t_i)| \leq \bigvee_{t_i}^{t_{i+1}}(u) = \bigvee_{a}^{t_{i+1}}(u) - \bigvee_{a}^{t_i}(u), \tag{2.3}$$

for any $i \in \{0, \dots, n-1\}$, and by (2.3) we have

$$\left|\int_a^b f(t)\,du(t)\right| \leq \lim_{v(I_n)\to 0}\sum_{i=0}^{n-1}|f(\xi_i)|\left[\bigvee_{a}^{t_{i+1}}(u) - \bigvee_{a}^{t_i}(u)\right] = \int_a^b |f(t)|\,d\left(\bigvee_{a}^{t}(u)\right),$$

and the last Riemann-Stieltjes integral exists since $|f|$ is continuous and $\bigvee_a^{\cdot}$ is monotonic nondecreasing on $[a, b]$.

The last part follows from the following well known Hölder type inequality for the Riemann-Stieltjes integral with monotonic integrator, namely:

$$\begin{aligned} &\left|\int_a^b g(t)\,dv(t)\right| \\ &\leq [v(b) - v(a)]^{\frac{1}{q}}\left[\int_a^b |g(t)|^p\,dv(t)\right]^{\frac{1}{p}} \quad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ &\leq \max_{t\in[a,b]}|g(t)|\,[v(b) - v(a)], \end{aligned} \tag{2.4}$$

holding for any g continuous on $[a, b]$ and v monotonic nondecreasing on $[a, b]$.

The details are omitted. ∎

2.2. Refinement of the Ostrowski Inequality.

The following result that provides a refinements of Ostrowski inequality for functions of bounded variation may be stated:

THEOREM 2.2 (Dragomir, 2008 [**10**]). *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then*

$$\left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \leq Q(x) \tag{2.5}$$

for any $x \in [a, b]$, where

$$\begin{aligned} Q(x) &:= [2x - (a+b)]\bigvee_{a}^{x}(f) - \int_a^b \operatorname{sgn}(x-t)\bigvee_{a}^{t}(f)\,dt \\ &\leq (b-x)\bigvee_{x}^{b}(f) + (x-a)\bigvee_{a}^{x}(f) \\ &\leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\bigvee_{a}^{b}(f). \end{aligned} \tag{2.6}$$

We also have

$$Q(x) \le \left[\bigvee_a^b (f)\right]^{\frac{1}{q}} \left\{ [(x-a)^p - (b-x)^p] \bigvee_a^x (f) - p\int_a^b r_p(x,t)\,\mathrm{sgn}(x-t) \bigvee_a^t (f)\,dt \right\}^{\frac{1}{p}} \tag{2.7}$$
$$\le \left[\bigvee_a^b (f)\right]^{\frac{1}{q}} \left\{ (b-x)^p \bigvee_x^b (f) + (x-a)^p \bigvee_a^x (f) \right\}^{\frac{1}{p}}$$
$$\le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f),$$

for any $x \in [a,b]$, *where* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $r_p : [a,b]^2 \to \mathbb{R}$ *with*

$$r_p(x,t) := \begin{cases} (t-a)^{p-1} & \textit{if } t \in [a,x], \\ (b-t)^{p-1} & \textit{if } t \in (x,b]. \end{cases} \tag{2.8}$$

PROOF. We use the identity obtained in [**6**]

$$f(x)(b-a) - \int_a^b f(t)\,dt = \int_a^b p(x,t)\,df(t), \tag{2.9}$$

where $p : [a,b]^2 \to \mathbb{R}$ is given by

$$p(x,t) := \begin{cases} t-a & \text{if } t \in [a,x], \\ t-b & \text{if } t \in (x,b], \end{cases}$$

and $x \in [a,b]$.

Now, if f is of bounded variation on $[a,b]$, then on applying the first inequality in (2.1) we deduce

$$\begin{aligned} \left|\int_a^b p(x,t)\,df(t)\right| &\le \int_a^b |p(x,t)|\,d\left(\bigvee_a^t (f)\right) \\ &= \int_a^x (t-a)\,d\left(\bigvee_a^t (f)\right) + \int_x^b (b-t)\,d\left(\bigvee_a^t (f)\right) \\ &= (x-a)\bigvee_a^x (f) - \int_a^x \left(\bigvee_a^t (f)\right) dt \\ &\quad - (b-x)\bigvee_a^x (f) + \int_x^b \left(\bigvee_a^t (f)\right) dt \\ &= [2x - (a+b)]\bigvee_a^x (f) - \int_a^b \mathrm{sgn}(x-t)\bigvee_a^t (f)\,dt \\ &= Q(x), \end{aligned}$$

for each $x \in [a,b]$, and the inequality (2.5) is proved.

Now, since $\bigvee_a^{\cdot}$ is monotonic nondecreasing on $[a,b]$, then

$$Q(x) \le [2x-(a+b)] \bigvee_a^x (f) + (b-x) \bigvee_a^b (f)$$
$$= (b-x) \bigvee_x^b (f) + (x-a) \bigvee_a^x (f)$$

and the inequality (2.6) is proved.

Utilising the second part of the second inequality in (2.1) we deduce that

$$Q(x) \le \left[\bigvee_a^b (f)\right]^{\frac{1}{q}} \left\{\int_a^b |p(x,t)|^p \, d\left(\bigvee_a^t (f)\right)\right\}^{\frac{1}{p}}. \tag{2.10}$$

Now, observe that

$$\begin{aligned} R(x) &:= \int_a^b |p(x,t)|^p \, d\left(\bigvee_a^t (f)\right) \\ &= \int_a^x (t-a)^p \, d\left(\bigvee_a^t (f)\right) + \int_x^b (b-t)^p \, d\left(\bigvee_a^t (f)\right) \\ &= (x-a)^p \bigvee_a^x (f) - p\int_a^x (t-a)^{p-1} \bigvee_a^t (f)\, dt \\ &\quad - (b-x)^p \bigvee_a^x (f) + p\int_x^b (b-t)^{p-1} \bigvee_a^t (f)\, dt \\ &= [(x-a)^p - (b-x)^p] \bigvee_a^x (f) - p\int_a^b r_p(x,t)\operatorname{sgn}(x-t) \bigvee_a^t (f)\, dt \end{aligned} \tag{2.11}$$

where r_p is given in (2.8). Utilising (2.9), we deduce the first part of (2.7).

Since $\bigvee_a^{\cdot}(t)$ is increasing, we also have

$$R(x) \le [(x-a)^p - (b-x)^p] \bigvee_a^x (f) + p \bigvee_a^b (f) \cdot \int_x^b (b-t)^{p-1}\, dt$$
$$= [(x-a)^p - (b-x)^p] \bigvee_a^x (f) + (b-x)^p \cdot \bigvee_a^b (f)$$
$$= (x-a)^p \cdot \bigvee_a^x (f) + (b-x)^p \cdot \bigvee_x^b (f),$$

which proves the last part of (2.7). ∎

The following particular case that provides a refinement of the midpoint inequality is of interest.

COROLLARY 2.3. *Let* $f : [a, b] \to \mathbb{C}$ *be a function of bounded variation on* $[a, b]$ *. Then*

$$\left|\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a)\right| \tag{2.12}$$
$$\leq \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) \cdot \bigvee_a^t (f)\,dt$$
$$\leq p^{1/p}\left[\bigvee_a^b (f)\right]^{\frac{1}{q}}\left[\int_a^b r(t)\operatorname{sgn}\left(t - \frac{a+b}{2}\right) \cdot \bigvee_a^t (f)\,dt\right]^{\frac{1}{p}}$$
$$\leq \frac{1}{2}(b-a) \cdot \bigvee_a^b (f),$$

where

$$r(t) = \begin{cases} (t-a)^{p-1} & \text{if } t \in \left[a, \frac{a+b}{2}\right], \\ (b-t)^{p-1} & \text{if } t \in \left(\frac{a+b}{2}, b\right] \end{cases}$$

and $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

REMARK 2.1. The inequalities (2.5)–(2.7) provide refinements for the Ostrowski inequality

$$\left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \tag{2.13}$$

for any $x \in [a, b]$, obtained by the author in [**4**] and [**6**]. The inequalities (2.12) are refinements of the midpoint inequality for functions of bounded variation obtained in [**5**].

3. OSTROWSKI FOR CUMULATIVE VARIATION

3.1. Cumulative Variation Function. For a function of bounded variation $v : [a, b] \to \mathbb{C}$ we define the *Cumulative Variation Function* (CVF) $V : [a, b] \to [0, \infty)$ by

$$V(t) := \bigvee_a^t (v),$$

i.e., the total variation of the function v on the interval $[a, t]$ with $t \in [a, b]$.

It is know that the CVF is monotonic nondecreasing on $[a, b]$ and is continuous in a point $c \in [a, b]$ if and only if the generating function v is continuing in that point. If v is Lipschitzian with the constant $L > 0$, i.e.

$$|v(t) - v(s)| \leq L|t - s| \text{ for any } t, s \in [a, b],$$

then V is also Lipschitzian with the same constant.

The following lemma is of interest in itself as well [**1**, p. 177], see also [**10**] for a generalization.

LEMMA 3.1. *Let* $f, u : [a, b] \to \mathbb{C}$*. If* f *is continuous on* $[a, b]$ *and* u *is of bounded variation on* $[a, b]$ *, then the Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$ *exists and*

$$\left|\int_a^b f(t)\,du(t)\right| \leq \int_a^b |f(t)|\,d(V(t)) \leq \max_{t \in [a,b]} |f(t)| \bigvee_a^b (u). \tag{3.1}$$

3.2. Some Refinements. The following result may be stated.

THEOREM 3.2 (Dragomir, 2013 [**11**]). *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then*

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \\
&\leq \int_a^x \left(\bigvee_t^x (f)\right) dt + \int_x^b \left(\bigvee_x^t (f)\right) dt \\
&\leq (x-a)\bigvee_a^x (f) + (b-x)\bigvee_x^b (f) \\
&\leq \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f), \\ \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right](b-a), \end{cases}
\end{aligned} \tag{3.2}$$

for any $x\in[a,b]$.

PROOF. We start with the equality

$$f(x)(b-a) - \int_a^b f(t)\,dt = \int_a^x (t-a)\,df(t) + \int_x^b (t-b)\,df(t) \tag{3.3}$$

that holds for any $x\in[a,b]$ and $f:[a,b]\to\mathbb{C}$ a function of bounded variation on $[a,b]$ (see [**4**] or [**6**]).

Taking the modulus in (3.3) and using the property (3.1) we have

$$\begin{aligned}
&\left|f(x)(b-a) - \int_a^b f(t)\,dt\right| \\
&\leq \left|\int_a^x (t-a)\,df(t)\right| + \left|\int_x^b (t-b)\,df(t)\right| \\
&\leq \int_a^x (t-a)\,d\left(\bigvee_a^t (f)\right) + \int_x^b (b-t)\,d\left(\bigvee_x^t (f)\right)
\end{aligned} \tag{3.4}$$

for any $x\in[a,b]$.

Utilising the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\begin{aligned}
\int_a^x (t-a)\,d\left(\bigvee_a^t (f)\right) &= (t-a)\bigvee_a^t (f)\Big|_a^x - \int_a^x \left(\bigvee_a^t (f)\right) dt \\
&= (x-a)\bigvee_a^x (f) - \int_a^x \left(\bigvee_a^t (f)\right) dt \\
&= \int_a^x \left(\bigvee_a^x (f) - \bigvee_a^t (f)\right) dt = \int_a^x \left(\bigvee_t^x (f)\right) dt
\end{aligned} \tag{3.5}$$

and

$$\int_x^b (b-t)\, d\left(\bigvee_x^t (f)\right) = (b-t) \bigvee_x^t (f)\Big|_x^b + \int_x^b \left(\bigvee_x^t (f)\right) dt \tag{3.6}$$
$$= \int_x^b \left(\bigvee_x^t (f)\right) dt$$

for any $x \in [a,b]$.

Utilising (3.4)-(3.6) we deduce the first inequality in (3.2).

Since $\bigvee_t^x (f) \le \bigvee_a^x (f)$ for $t \in [a,x]$ and $\bigvee_x^t (f) \le \bigvee_x^b (f)$ for $t \in [x,b]$, then

$$\int_a^x \left(\bigvee_t^x (f)\right) dt + \int_x^b \left(\bigvee_x^t (f)\right) dt \le (x-a) \bigvee_a^x (f) + (b-x) \bigvee_x^b (f)$$

for any $x \in [a,b]$, which proves the second inequality in (3.2).

The last part is obvious by the max properties and the fact that for $c, d \in \mathbb{R}$ we have $\max\{c,d\} = \frac{c+d+|c-d|}{2}$. The details are omitted. ∎

The following midpoint inequality holds:

COROLLARY 3.3. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)\, dt - f\left(\frac{a+b}{2}\right)(b-a)\right| \tag{3.7}$$
$$\le \int_a^{\frac{a+b}{2}} \left(\bigvee_t^{\frac{a+b}{2}} (f)\right) dt + \int_{\frac{a+b}{2}}^b \left(\bigvee_{\frac{a+b}{2}}^t (f)\right) dt \le \frac{1}{2}(b-a) \bigvee_a^b (f).$$

The first inequality in (3.7) is sharp and the constant $\frac{1}{2}$ in the second, is best possible.

PROOF. We must prove only the sharpness of the inequalities in (3.7).

If we consider the function $f : [a,b] \to \mathbb{R}$ given by

$$f(t) := \begin{cases} 0, & t \in [a, \frac{a+b}{2}) \\ 1, & t = \frac{a+b}{2} \\ 0 & t \in (\frac{a+b}{2}, b], \end{cases}$$

then this function is of bounded variation, we observe that $\bigvee_{\frac{a+b}{2}}^t (f) = 1$ for any $t \in (\frac{a+b}{2}, b]$ and $\bigvee_t^{\frac{a+b}{2}} (f) = 1$ for any $t \in [a, \frac{a+b}{2})$. Also, we have $\bigvee_a^b (f) = 2$.

If we replace these values in (3.7) we obtain in all terms the same quantity $b-a$. ∎

COROLLARY 3.4. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$. If $p \in (a,b)$ is a median point in variation, namely $\bigvee_a^p (f) = \bigvee_p^b (f)$, then we have the inequality*

$$\left|\int_a^b f(t)\, dt - f(p)(b-a)\right| \tag{3.8}$$
$$\le \int_a^p \left(\bigvee_t^p (f)\right) dt + \int_p^b \left(\bigvee_p^t (f)\right) dt \le \frac{1}{2}(b-a) \bigvee_a^b (f).$$

The first inequality in (3.2) is useful when some properties for the CVF are available, like for instance, below:

COROLLARY 3.5. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$. If for $x \in (a,b)$ there exist $L_x > 0$ and $\alpha > -1$ such that*

$$\left| \bigvee_x^t (f) \right| \le L_x |t-x|^\alpha \text{ for any } t \in [a,b] \setminus \{x\}, \tag{3.9}$$

then

$$\left| \int_a^b f(t)\,dt - f(x)(b-a) \right| \le \frac{1}{\alpha+1} L_x \left[(x-a)^{\alpha+1} + (b-x)^{\alpha+1} \right]. \tag{3.10}$$

In particular, for $\alpha = 1$ in (3.9) we get from (3.10) that

$$\left| \int_a^b f(t)\,dt - f(x)(b-a) \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] L_x (b-a)^2. \tag{3.11}$$

REMARK 3.1. If the CVF $\bigvee_a^t (f)$ is K-Lipschitzian, i.e.,

$$\left| \bigvee_s^t (f) \right| \le K |t-s| \text{ for any } t, s \in [a,b],$$

then

$$\left| \int_a^b f(t)\,dt - f(x)(b-a) \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] L (b-a)^2. \tag{3.12}$$

for any $x \in [a,b]$.

COROLLARY 3.6. *If there exists a constant $L_{\frac{a+b}{2}} > 0$ and $\alpha > -1$ such that*

$$\left| \bigvee_{\frac{a+b}{2}}^t (f) \right| \le L_{\frac{a+b}{2}} \left| t - \frac{a+b}{2} \right|^\alpha \text{ for any } t \in [a,b] \setminus \left\{ \frac{a+b}{2} \right\}, \tag{3.13}$$

then we have the midpoint inequality

$$\left| \int_a^b f(t)\,dt - f\left(\frac{a+b}{2} \right)(b-a) \right| \le \frac{1}{2^\alpha (\alpha+1)} L_{\frac{a+b}{2}} (b-a)^{\alpha+1}. \tag{3.14}$$

In particular, if we take $\alpha = 1$ in (3.13), then we get from (3.14)

$$\left| \int_a^b f(t)\,dt - f\left(\frac{a+b}{2} \right)(b-a) \right| \le \frac{1}{4} L_{\frac{a+b}{2}} (b-a)^2. \tag{3.15}$$

The constant $\frac{1}{4}$ is best possible in (3.15).

PROOF. First, we notice that if $h : [a,b] \to \mathbb{C}$ is of bounded variation, then $|h| : [a,b] \to [0,\infty)$ is of bounded variation and

$$\bigvee_a^b (|h|) \le \bigvee_a^b (h). \tag{3.16}$$

Indeed, by the continuity property of the modulus, we have that

$$\sum_{j=0}^{n-1} \left| |h(t_{j+1})| - |h(t_j)| \right| \le \sum_{j=0}^{n-1} |h(t_{j+1}) - h(t_j)|$$

for any division $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$, which, by taking the supremum over all divisions of $[a,b]$, produces the desired inequality (3.16).

If we consider the function $f_0 : [a,b] \to \mathbb{R}$, $f_0(s) := \left|s - \frac{a+b}{2}\right|$ then, by denoting with $\mathbf{e}$ the identity function on $[a,b]$, i.e. $\mathbf{e}(t) = t$, $t \in [a,b]$, we have

$$\begin{aligned}\left|\bigvee_{\frac{a+b}{2}}^{t}(f_0)\right| &= \left|\bigvee_{\frac{a+b}{2}}^{t}\left(\left|\mathbf{e}-\frac{a+b}{2}\right|\right)\right| \\ &\le \left|\bigvee_{\frac{a+b}{2}}^{t}\left(\mathbf{e}-\frac{a+b}{2}\right)\right| = \left|\bigvee_{\frac{a+b}{2}}^{t}(\mathbf{e})\right| = \left|t-\frac{a+b}{2}\right|\end{aligned}$$

for any $t \in [a,b]$.

Therefore the function f_0 satisfies the condition (3.13) for $\alpha = 1$ and with the constant $L_{\frac{a+b}{2}} = 1$. Since

$$\int_a^b f_0(t)\,dt = \int_a^b \left|t - \frac{a+b}{2}\right| dt = \frac{1}{4}(b-a)^2,$$

then we obtain in both sides of the inequality (3.15) the same quantity $\frac{1}{4}(b-a)^2$. ∎

REMARK 3.2. The inequalities (3.12) and (3.15) are known in the case of Lipschitzian functions with the constant $L > 0$. We obtained them here under weaker conditions for the function f. This show that the refinement in terms of the CVF for the Ostrowski inequality (3.2) is also useful to extend known results to larger classes of functions.

3.3. Bounds for Shifted Function. We have the following result as well:

THEOREM 3.7 (Dragomir, 2013 [**11**]). *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$. Then*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \\ &\le \int_a^x \left(\bigvee_t^x(|f(x)-f|)\right)dt + \int_x^b\left(\bigvee_x^t(|f(x)-f|)\right)dt \\ &\le (x-a)\bigvee_a^x(|f(x)-f|) + (b-x)\bigvee_x^b(|f(x)-f|) \\ &\le (x-a)\bigvee_a^x(f) + (b-x)\bigvee_x^b(f) \\ &\le \begin{cases}\left[\frac{1}{2}(b-a) + \left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f), \\ \\ \left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right](b-a),\end{cases}\end{aligned} \tag{3.17}$$

for any $x \in [a,b]$.

PROOF. Observe that

$$\begin{aligned}\left|f(x)(b-a) - \int_a^b f(t)\,dt\right| &= \left|\int_a^b [f(x)-f(t)]\,dt\right| \\ &\le \int_a^b |f(x)-f(t)|\,dt\end{aligned} \tag{3.18}$$

for any $x \in [a,b]$.

For a fixed $x \in [a,b]$, define the function $g_x : [a,b] \to [0,\infty)$ by $g_x(t) := |f(x) - f(t)|$. We observe that g_x is of bounded variation on $[a,b]$ and

$$\left| g_x(x)(b-a) - \int_a^b g_x(t)\,dt \right| = \int_a^b |f(x) - f(t)|\,dt. \tag{3.19}$$

Writing the inequality (3.2) for the function g_x we have

$$\begin{aligned} &\left| g_x(x)(b-a) - \int_a^b g_x(t)\,dt \right| \\ &\le \int_a^x \left(\bigvee_t^x (|f(x) - f|) \right) dt + \int_x^b \left(\bigvee_x^t (|f(x) - f|) \right) dt \\ &\le (x-a) \bigvee_a^x (|f(x) - f|) + (b-x) \bigvee_x^b (|f(x) - f|). \end{aligned} \tag{3.20}$$

Utilising (3.18)-(3.20) we deduce the first two inequalities in (3.17).

By the inequality (3.16) we have

$$\bigvee_a^x (|f(x) - f|) \le \bigvee_a^x (f(x) - f) = \bigvee_a^x (f)$$

and

$$\bigvee_x^b (|f(x) - f|) \le \bigvee_x^b (f(x) - f) = \bigvee_x^b (f),$$

which proves the third inequality in (3.17). ∎

COROLLARY 3.8. *If $f : [a,b] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$, then*

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a) \right| \\ &\le \int_a^{\frac{a+b}{2}} \left(\bigvee_t^{\frac{a+b}{2}} \left(\left| f\left(\frac{a+b}{2}\right) - f \right| \right) \right) dt \\ &+ \int_{\frac{a+b}{2}}^b \left(\bigvee_{\frac{a+b}{2}}^t \left(\left| f\left(\frac{a+b}{2}\right) - f \right| \right) \right) dt \\ &\le \frac{1}{2}(b-a) \bigvee_a^b \left(\left| f\left(\frac{a+b}{2}\right) - f \right| \right) \le \frac{1}{2}(b-a) \bigvee_a^b (f). \end{aligned} \tag{3.21}$$

All inequalities in (3.21) are sharp.

PROOF. If we consider the function $f : [a,b] \to \mathbb{R}$, with

$$f(t) := \begin{cases} 0, & t \in [a, \frac{a+b}{2}) \\ 1, & t = \frac{a+b}{2} \\ 0 & t \in (\frac{a+b}{2}, b], \end{cases}$$

then this function is of bounded variation, we observe that

$$\left|f\left(\frac{a+b}{2}\right)-f\left(t\right)\right|=\begin{cases}1, & t\in[a,\frac{a+b}{2})\\ 0, & t=\frac{a+b}{2}\\ 1 & t\in(\frac{a+b}{2},b],\end{cases}$$

$$\bigvee_{t}^{\frac{a+b}{2}}\left(\left|f\left(\frac{a+b}{2}\right)-f\right|\right)=1 \text{ for } t\in[a,\frac{a+b}{2}),$$

$$\bigvee_{\frac{a+b}{2}}^{t}\left(\left|f\left(\frac{a+b}{2}\right)-f\right|\right)=1 \text{ for } t\in(\frac{a+b}{2},b]$$

and

$$\bigvee_{a}^{b}\left(\left|f\left(\frac{a+b}{2}\right)-f\right|\right)=2.$$

Replacing these values in (3.21) we get in all terms of this inequality the same quantity $b-a$. ∎

COROLLARY 3.9. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. If $q\in(a,b)$ is a point for which*

$$\bigvee_{a}^{q}\left(\left|f\left(q\right)-f\right|\right)=\bigvee_{q}^{b}\left(\left|f\left(q\right)-f\right|\right),$$

then

$$\begin{aligned}&\left|\int_a^b f\left(t\right)dt-f\left(q\right)\left(b-a\right)\right|\\&\leq\int_a^q\left(\bigvee_{t}^{q}\left(\left|f\left(q\right)-f\right|\right)\right)dt+\int_q^b\left(\bigvee_{q}^{t}\left(\left|f\left(q\right)-f\right|\right)\right)dt\\&\leq\frac{1}{2}\bigvee_{a}^{b}\left(\left|f\left(q\right)-f\right|\right)\leq\frac{1}{2}\left(b-a\right)\bigvee_{a}^{b}\left(f\right).\end{aligned}\tag{3.22}$$

REMARK 3.3. Since

$$\begin{aligned}&\left(x-a\right)\bigvee_{a}^{x}\left(\left|f\left(x\right)-f\right|\right)+\left(b-x\right)\bigvee_{x}^{b}\left(\left|f\left(x\right)-f\right|\right)\\&\leq\begin{cases}\max\left\{x-a,b-x\right\}\bigvee_a^b\left|f\left(x\right)-f\right|\\ \\ \max\left\{\bigvee_a^x\left(\left|f\left(x\right)-f\right|\right),\bigvee_x^b\left(\left|f\left(x\right)-f\right|\right)\right\}\left(b-a\right)\end{cases}\\&=\begin{cases}\left[\frac{1}{2}\left(b-a\right)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b\left|f\left(x\right)-f\right|\\ \\ \left[\frac{1}{2}\bigvee_a^b\left|f\left(x\right)-f\right|+\frac{1}{2}\left|\bigvee_a^x\left(\left|f\left(x\right)-f\right|\right)-\bigvee_x^b\left(\left|f\left(x\right)-f\right|\right)\right|\right],\end{cases}\end{aligned}$$

then from (3.17) we also have the string of inequalities

$$\begin{aligned}
(3.23)\qquad & \left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \\
&\le \int_a^x \left(\bigvee_t^x (|f(x)-f|)\right) dt + \int_x^b \left(\bigvee_x^t (|f(x)-f|)\right) dt \\
&\le (x-a)\bigvee_a^x (|f(x)-f|) + (b-x)\bigvee_x^b (|f(x)-f|) \\
&\le \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b |f(x)-f| \\ \\ \left[\frac{1}{2}\bigvee_a^b |f(x)-f| + \frac{1}{2}\left|\bigvee_a^x (|f(x)-f|) - \bigvee_x^b (|f(x)-f|)\right|\right] \\ \times (b-a), \end{cases}
\end{aligned}$$

for any $x \in [a,b]$.

REMARK 3.4. If the function $f : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then

$$\begin{aligned}
(3.24)\qquad & \left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \\
&\le \int_a^b sgn(x-t)[f(x)-f(t)]\,dt \\
&\le (x-a)[f(x)-f(a)] + (b-x)[f(b)-f(x)] \\
&\le \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right][f(b)-f(a)], \\ \\ \left[\frac{1}{2}[f(b)-f(a)] + \frac{1}{2}\left|f(x) - \frac{f(a)+f(b)}{2}\right|\right](b-a), \end{cases}
\end{aligned}$$

for any $x \in [a,b]$.

In particular, for $x = \frac{a+b}{2}$, we get the midpoint inequality

$$\begin{aligned}
(3.25)\qquad & \left|\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a)\right| \\
&\le \int_a^b sgn\left(\frac{a+b}{2} - t\right)\left[f\left(\frac{a+b}{2}\right) - f(t)\right] dt \le \frac{1}{2}[f(b)-f(a)](b-a).
\end{aligned}$$

Moreover, if $p \in (a,b)$ is such that

$$f(p) = \frac{f(a)+f(b)}{2},$$

then we have the trapezoid inequality

$$\begin{aligned}
(3.26)\qquad & \left|\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt\right| \\
&\le \int_a^b sgn(p-t)\left[\frac{f(a)+f(b)}{2} - f(t)\right] dt \le \frac{1}{2}[f(b)-f(a)](b-a).
\end{aligned}$$

4. Ostrowski Inequality for Monotonic Functions

The following results of Ostrowski type holds.

THEOREM 4.1 (Dragomir, 1999 [**3**]). *Let $f:[a,b]\to\mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$. Then for all $x\in[a,b]$, we have the inequality*

$$\begin{aligned}(4.1)\qquad &\left|\int_a^b f(t)\,dt-(b-a)f(x)\right|\\ &\le [2x-(a+b)]f(x)+\int_a^b sgn(t-x)f(t)\,dt\\ &\le (x-a)[f(x)-f(a)]+(b-x)[f(b)-f(x)]\\ &\le \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right][f(b)-f(a)].\end{aligned}$$

All the inequalities in (4.1) are sharp and the constant $\frac{1}{2}$ is the best possible one.

PROOF. Using the integration by parts formula for Riemann-Stieltjes integrals we have

$$\int_a^x (t-a)\,df(t)=f(x)(x-a)-\int_a^x f(t)\,dt$$

and

$$\int_x^b (t-b)df(t)=f(x)(b-x)-\int_x^b f(t)\,dt.$$

If we add the above two equalities, we obtain the following equality of interest, see [**6**]

$$(b-a)f(x)-\int_a^b f(t)\,dt=\int_a^b p(x,t)\,df(t)\ ,\tag{4.2}$$

where

$$p(x,t):=\begin{cases}t-a \text{ if } t\in[a,x)\\ t-b \text{ if } x\in[x,b]\end{cases},$$

for all $x,t\in[a,b]$.

If $p:[a,b]\to\mathbb{C}$ is continuous on $[a,b]$ and $v:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then [**1**, p. 155]

$$\left|\int_a^b p(x)\,dv(x)\right|\le\int_a^b|p(x)|\,dv(x).\tag{4.3}$$

Now, by the property (4.3) we have

$$
\begin{aligned}
&\left|(b-a) f(x)-\int_a^b f(t)\, dt\right| \\
&=\left|\int_a^x (t-a)\, df(t)+\int_x^b (t-b)\, df(t)\right| \\
&\leq\left|\int_a^x (t-a)\, df(t)\right|+\left|\int_x^b (t-b)\, df(t)\right| \\
&\leq \int_a^x |t-a|\, df(t)+\int_x^b |t-b|\, df(t)=\int_a^x (t-a) df(t)+\int_x^b (b-t) df(t) \\
&=\ (t-a) f(t)|_a^x-\int_a^x f(t)\, dt-\ (b-t) f(t)|_x^b+\int_x^b f(t)\, dt \\
&=[2x-(a+b)] f(x)-\int_a^x f(t)\, dt+\int_x^b f(t)\, dt \\
&=[2x-(a+b)] f(x)+\int_a^b sgn(t-x) f(t)\, dt,
\end{aligned}
$$

which proves the first part of (4.1).

As f is monotonic nondecreasing on $[a,b]$, we can state that

$$\int_a^x f(t)\, dt \geq (x-a) f(a) \text{ and } \int_x^b f(t)\, dt \leq (b-x) f(b)$$

so that

$$\int_a^b sgn(t-x) f(t)\, dt \leq (b-x) f(b)-(x-a) f(a).$$

Consequently

$$
\begin{aligned}
&[2x-(a+b)] f(x)+\int_a^b sgn(t-x) f(t)\, dt \\
&\leq [2x-(a+b)] f(x)+(b-x) f(b)-(x-a) f(a) \\
&=(b-x)\,[f(b)-f(x)]+(x-a)\,[f(x)-f(a)]
\end{aligned}
$$

and the second part of (4.1) is proved.

Finally, let us observe that

$$
\begin{aligned}
&(b-x)\,[f(b)-f(x)]+(x-a)\,[f(x)-f(a)] \\
&\leq \max\{b-x, x-a\}[f(b)-f(x)+f(x)-f(a)] \\
&=\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right][f(b)-f(a)]
\end{aligned}
$$

and the inequality (4.1) is thus proved.

Now for the sharpness of the inequalities, assume that (4.1) holds with a constant $C > 0$ instead of $\frac{1}{2}$. That is,

$$\begin{aligned}(4.4)\qquad & \left|(b-a)\,f(x)-\int_a^b f(t)\,dt\right| \\ & \leq [2x-(a+b)]\,f(x)+\int_a^b sgn(t-x)f(t)\,dt \\ & \leq (x-a)\,[f(x)-f(a)]+(b-x)\,[f(b)-f(x)] \\ & \leq \left[C\,(b-a)+\left|x-\frac{a+b}{2}\right|\right][f(b)-f(a)]\,.\end{aligned}$$

Consider the mapping $f_0 : [a,b] \to \mathbb{R}$ given by

$$f_0(x) := \begin{cases} -1 & \text{if} \quad x = a \\ 0 & \text{if} \quad x \in (a,b] \end{cases}.$$

Putting in (4.4) $f = f_0$ and $x = a$, we have

$$\begin{aligned}& \left|(b-a)\,f(x)-\int_a^b f(t)\,dt\right| \\ & = [2x-(a+b)]f(x)+\int_a^b sgn(t-x)f(t)\,dt \\ & = (x-a)\,[f(x)-f(a)]+(b-x)\,[f(b)-f(x)] = b-a \\ & \leq \left[C\,(b-a)+\left|x-\frac{a+b}{2}\right|\right][f(b)-f(a)] = \left(C+\frac{1}{2}\right)(b-a)\,,\end{aligned}$$

which proves the sharpness of the first two inequalities and the fact that C cannot not be less than $\frac{1}{2}$. ∎

The following corollary is interesting.

COROLLARY 4.2. *Let f be as above. Then we have the midpoint inequality:*

$$\begin{aligned}(4.5)\qquad \left|(b-a)\,f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right| & \leq \int_a^b sgn\left(t-\frac{a+b}{2}\right)f(t)\,dt \\ & \leq \frac{1}{2}\,(b-a)\,[f(b)-f(a)]\,.\end{aligned}$$

For other Ostrowski type inequalities, see [**2**].

5. OSTROWSKI INEQUALITY FOR CONVEX FUNCTIONS

The following results of Ostrowski type holds.

THEOREM 5.1 (Dragomir, 2002 [**7**]). *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[a,b]$. Then for any $x \in [a,b]$ one has the inequality*

$$\begin{aligned}(5.1)\qquad & \frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right] \\ & \leq \int_a^b f(t)\,dt-(b-a)\,f(x) \\ & \leq \frac{1}{2}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_+(a)\right].\end{aligned}$$

The constant $\frac{1}{2}$ *is sharp in both inequalities. The second inequality also holds for* $x = a$ *or* $x = b$*.*

PROOF. It is easy to see that for any locally absolutely continuous function $f : (a, b) \to \mathbb{R}$, we have the identity

$$\int_a^x (t-a) f'(t)\, dt + \int_x^b (t-b) f'(t)\, dt = f(x) - \int_a^b f(t)\, dt \tag{5.2}$$

for any $x \in (a, b)$, where f' is the derivative of f which exists a.e. on (a, b).

Since f is convex, then it is locally Lipschitzian and thus (5.2) holds. Moreover, for any $x \in (a, b)$, we have the inequalities

$$f'(t) \le f'_-(x) \text{ for a.e. } t \in [a, x] \tag{5.3}$$

and

$$f'(t) \ge f'_+(x) \text{ for a.e. } t \in [x, b]. \tag{5.4}$$

If we multiply (5.3) by $t - a \ge 0$, $t \in [a, x]$ and integrate on $[a, x]$, we get

$$\int_a^x (t-a) f'(t)\, dt \le \frac{1}{2}(x-a)^2 f'_-(x) \tag{5.5}$$

and if we multiply (5.4) by $b - t \ge 0$, $t \in [x, b]$, and integrate on $[x, b]$, we also have

$$\int_x^b (b-t) f'(t)\, dt \ge \frac{1}{2}(b-x)^2 f'_+(x). \tag{5.6}$$

If we subtract (5.6) from (5.5) and use the representation (5.2), we deduce the first inequality in (5.1).

Now, assume that the first inequality (5.1) holds with $C > 0$ instead of $\frac{1}{2}$, i.e.,

$$C\left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x)\right] \le \int_a^b f(t)\, dt - (b-a) f(x). \tag{5.7}$$

Consider the convex function $f_0(t) := k\left|t - \frac{a+b}{2}\right|$, $k > 0$, $t \in [a, b]$. Then

$$f'_{0+}\left(\frac{a+b}{2}\right) = k, \quad f'_{0-}\left(\frac{a+b}{2}\right) = -k, \quad f_0\left(\frac{a+b}{2}\right) = 0$$

and

$$\int_a^b f_0(t)\, dt = \frac{1}{4}k(b-a)^2.$$

If in (5.7) we choose $f = f_0$, $x = \frac{a+b}{2}$, then we get

$$C\left[\frac{1}{4}(b-a)^2 k + \frac{1}{4}(b-a)^2 k\right] \le \frac{1}{4}k(b-a)^2$$

which gives $C \le \frac{1}{2}$ and the sharpness of the constant in the first part of (5.1) is proved.

If either $f'_+(a) = -\infty$ or $f'_-(b) = -\infty$, then the second inequality in (5.1) holds true.

Assume that $f'_+(a)$ and $f'_-(b)$ are finite. Since f is convex on $[a, b]$, we have

$$f'(t) \ge f'_+(a) \text{ for a.e. } t \in [a, x] \quad (x \text{ may be equal to } b) \tag{5.8}$$

and

$$f'(t) \le f'_-(b) \text{ for a.e. } t \in [x, b] \quad (x \text{ may be equal to } a). \tag{5.9}$$

If we multiply (5.8) by $t-a\geq 0$, $t\in [a,x]$ and integrate on $[a,x]$, then we deduce

$$\int_a^x (t-a) f'(t)\,dt \geq \frac{1}{2}(x-a)^2 f'_+(a) \tag{5.10}$$

and if we multiply (5.9) by $b-t\geq 0$, $t\in [x,b]$, and integrate on $[x,b]$, then we also have

$$\int_x^b (b-t) f'(t)\,dt \leq \frac{1}{2}(b-x)^2 f'_-(b). \tag{5.11}$$

Finally, if we subtract (5.10) from (5.11) and use the representation (5.2), we deduce the second inequality in (5.1). Now, assume that the second inequality in (5.1) holds with a constant $D>0$ instead of $\frac{1}{2}$, i.e.,

$$\int_a^b f(t)\,dt-(b-a)f(x)\leq D\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_+(a)\right]. \tag{5.12}$$

If we consider the convex function $f_0(t)=k\left|t-\frac{a+b}{2}\right|$, $k>0$, $t\in [a,b]$, then we have $f'_-(b)=k$, $f'_+(a)=-k$ and by (5.12) applied for f_0 in $x=\frac{a+b}{2}$ we get

$$\frac{1}{4}k(b-a)^2\leq D\left[\frac{1}{4}k(b-a)^2+\frac{1}{4}k(b-a)^2\right],$$

giving $D\geq\frac{1}{2}$ which proves the sharpness of the constant $\frac{1}{2}$ in the second inequality in (5.1). ∎

COROLLARY 5.2. *With the assumptions of Theorem 5.1 and if $x\in(a,b)$ is a point of differentiability for f, then*

$$\left(\frac{a+b}{2}-x\right)f'(x)\leq\frac{1}{b-a}\int_a^b f(t)\,dt-f(x). \tag{5.13}$$

Now, recall that the following inequality, which is well known in the literature as the Hermite-Hadamard inequality for convex functions, holds

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2}. \tag{5.14}$$

The following corollary provides both a sharper lower bound for the difference,

$$\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right),$$

which we know is nonnegative, and an upper bound.

COROLLARY 5.3. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then we have the inequality*

$$\begin{aligned}
0 &\leq \frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right](b-a) \\
&\leq \frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{8}\left[f'_-(b)-f'_+(a)\right](b-a).
\end{aligned} \tag{5.15}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

EXAMPLE 5.1. *Assume that* $-\infty < a < 0 < b < \infty$ *and consider the convex function* $f : [a, b] \to \mathbb{R}$, $f(x) = \exp|x|$. *We have*

$$f'(x) = \begin{cases} -e^{-x} & \text{if } x < 0, \\ \\ e^{x} & \text{if } x > 0; \end{cases}$$

and $f'_{-}(0) = -1$, $f'_{+}(0) = 1$. *Also,*

$$\int_a^b f(t)\,dt = \int_a^0 e^{-x}dx + \int_0^b e^{x}dx = \exp(b) + \exp(-a) - 2.$$

Now, if $\frac{a+b}{2} \neq 0$, *then by (5.15) we deduce the elementary inequality*

$$0 \le \frac{\exp(b) + \exp(-a) - 2}{b - a} - \exp\left|\frac{a+b}{2}\right| \le \frac{1}{8}\left[\exp(b) + \exp(-a)\right](b - a). \tag{5.16}$$

If $\frac{a+b}{2} = 0$ *and if we denote* $b = u$, $u > 0$, *thus* $a = -u$ *and by (5.15) we also have*

$$\frac{1}{2}u \le \frac{\exp(u) - 1}{u} - 1 \le \frac{1}{2}u\exp(u). \tag{5.17}$$

The reader may produce other elementary inequalities by choosing in an appropriate way the convex function f. We omit the details.

REFERENCES

[1] T. M. Apostol, *Mathematical Analysis,* Second Edition, Addison-Wesley Publishing Company, 1975.

[2] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view. *Handbook of analytic-computational methods in applied mathematics*, 135–200, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[3] S. S. Dragomir, Ostrowski's Inequality for monotonic mappings and applications, *J. KSIAM,* **3** (1999), 127-135.

[4] S. S. Dragomir, The Ostrowski integral inequality for mappings of bounded variation. *Bull. Austral. Math. Soc.* **60** (1999), No. 3, 495–508.

[5] S. S. Dragomir, On the midpoint quadrature formula for mappings with bounded variation and applications. *Kragujevac J. Math.* **22** (2000), 13–19.

[6] S. S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. Appl.* **4** (2001), No. 1, 59-66. Preprint: *RGMIA Res. Rep. Coll.* **2** (1999), Art. 7, [Online: http://rgmia.org/papers/v2n1/v2n1-7.pdf].

[7] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), No. 2, Article 31, 8 pp. [Online http://www.emis.de/journals/JIPAM/article183.html?sid=183].

[8] S. S. Dragomir, An Ostrowski like inequality for convex functions and applications. *Rev. Mat. Complut.* **16** (2003), No. 2, 373–382.

[9] S. S. Dragomir, An Ostrowski type inequality for convex functions. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* **16** (2005), 12–25.

[10] S. S. Dragomir, Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation. *Arch. Math.* (Basel) **91** (2008), No. 5, 450–460.

[11] S. S. Dragomir, Refinements of the Ostrowski inequality in terms of the cumulative variation and applications, *Analysis* (Berlin) **34** (2014), No. 2, 223–240. Preprint: *RGMIA Res. Rep. Coll.* **16** (2013), Art. 29 [Online:http://rgmia.org/papers/v16/v16a29.pdf].

CHAPTER 2

Inequalities for Absolutely Continuous Functions

1. Ostrowski for L_∞-norm

1.1. Ostrowski's Inequality. In 1938, A. Ostrowski [**10**], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x \in [a,b]$.

THEOREM 1.1 (Ostrowski, 1938 [**10**]). *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a), \tag{1.1}$$

for all $x\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

In [**7**], S.S. Dragomir and S. Wang, by the use of the Montgomery integral identity [**9**, p. 565],

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b p(x,t) f'(t)\,dt, \quad x\in[a,b],$$

where $p:[a,b]^2\to\mathbb{R}$ is given by

$$p(x,t) := \begin{cases} t-a & \text{if} \quad t\in[a,x], \\ t-b & \text{if} \quad t\in(x,b], \end{cases}$$

gave a simple proof of Ostrowski's inequality and applied it for special means (identric mean, logarithmic mean, etc.) and to the problem of estimating the error bound in approximating the Riemann integral $\int_a^b f(t)\,dt$ by one arbitrary Riemann sum (see [**7**], Section 3).

1.2. A Refinement for L_∞-norm. The following result, which is an improvement on Ostrowski's inequality, holds.

THEOREM 1.2 (Dragomir, 2002 [**2**]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ whose derivative $f'\in L_\infty[a,b]$. Then*

$$\begin{aligned}
(1.2)\qquad & \left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{2(b-a)}\left[\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2\right] \\
&\leq \begin{cases}
\|f'\|_{[a,b],\infty}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a); \\[2ex]
\frac{1}{2}\left[\|f'\|_{[a,x],\infty}^\alpha+\|f'\|_{[x,b],\infty}^\alpha\right]^{\frac{1}{\alpha}}\left[\left(\frac{x-a}{b-a}\right)^{2\beta}+\left(\frac{b-x}{b-a}\right)^{2\beta}\right]^{\frac{1}{\beta}}(b-a), \\[1ex]
\text{where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex]
\frac{1}{2}\left[\|f'\|_{[a,x],\infty}+\|f'\|_{[x,b],\infty}\right]\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^2(b-a)
\end{cases}
\end{aligned}$$

for all $x\in[a,b]$, where $\|\cdot\|_{[m,n],\infty}$ denotes the usual norm on $L_\infty[m,n]$, i.e., we recall that

$$\|g\|_{[m,n],\infty}=ess\sup_{t\in[m,n]}|g(t)|<\infty.$$

PROOF. Using the integration by parts formula for absolutely continuous functions on $[a,b]$, we have

$$(1.3)\qquad \int_a^x (t-a)f'(t)\,dt=(x-a)f(x)-\int_a^x f(t)\,dt$$

and

$$(1.4)\qquad \int_x^b (t-b)f'(t)\,dt=(b-x)f(x)-\int_x^b f(t)\,dt,$$

for all $x\in[a,b]$.

Adding these two equalities, we obtain the Montgomery identity (see for example [**9**, p. 565]):

$$(1.5)\qquad (b-a)f(x)-\int_a^b f(t)\,dt=\int_a^x (t-a)f'(t)\,dt+\int_x^b (t-b)f'(t)\,dt$$

for all $x\in[a,b]$.

Taking the modulus, we deduce

$$\begin{aligned}
(1.6)\qquad & \left|(b-a)f(x)-\int_a^b f(t)\,dt\right| \\
&\leq \left|\int_a^x (t-a)f'(t)\,dt\right|+\left|\int_x^b (t-b)f'(t)\,dt\right| \\
&\leq \int_a^x (t-a)|f'(t)|\,dt+\int_x^b (b-t)|f'(t)|\,dt \\
&\leq \|f'\|_{[a,x],\infty}\int_a^x (t-a)\,dt+\|f'\|_{[x,b],\infty}\int_x^b (b-t)\,dt \\
&=\frac{1}{2}\left[\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2\right]
\end{aligned}$$

and the first inequality in (1.2) is proved.

Now, let us observe that

$$\begin{aligned}
&\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2 \\
&\le \max\left\{\|f'\|_{[a,x],\infty},\|f'\|_{[x,b],\infty}\right\}\left[(x-a)^2+(b-x)^2\right] \\
&= \max\left\{\|f'\|_{[a,x],\infty},\|f'\|_{[x,b],\infty}\right\}\left[\frac{1}{2}(b-a)^2+2\left(x-\frac{a+b}{2}\right)^2\right] \\
&= (b-a)^2\max\left\{\|f'\|_{[a,x],\infty},\|f'\|_{[x,b],\infty}\right\}\left[\frac{1}{2}+2\cdot\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right] \\
&= (b-a)^2\|f'\|_{[a,b],\infty}\left[\frac{1}{2}+2\cdot\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right],
\end{aligned}$$

and the first part of the second inequality in (1.2) is proved.

For the second inequality, we employ the elementary inequality for real numbers which can be derived from Hölder's discrete inequality

$$0\le ms+nt\le(m^\alpha+n^\alpha)^{\frac{1}{\alpha}}\times\left(s^\beta+t^\beta\right)^{\frac{1}{\beta}}, \tag{1.7}$$

provided that $m,s,n,t\ge 0$, $\alpha>1$ and $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Using (1.7), we obtain

$$\begin{aligned}
&\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2 \\
&\le\left(\|f'\|^\alpha_{[a,x],\infty}+\|f'\|^\alpha_{[x,b],\infty}\right)^{\frac{1}{\alpha}}\left[(x-a)^{2\beta}+(b-x)^{2\beta}\right]^{\frac{1}{\beta}}
\end{aligned}$$

and the second part of the second inequality in (1.2) is also obtained.

Finally, we observe that

$$\begin{aligned}
&\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2 \\
&\le\max\left\{(x-a)^2,(b-x)^2\right\}\left[\|f'\|_{[a,x],\infty}+\|f'\|_{[x,b],\infty}\right] \\
&=\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^2\left[\|f'\|_{[a,x],\infty}+\|f'\|_{[x,b],\infty}\right]
\end{aligned}$$

and the last part of the second inequality in (1.2) is proved. ∎

The following corollary is also natural.

COROLLARY 1.3. *Under the above assumptions, we have the midpoint inequality*

$$\begin{aligned}(1.8)\qquad &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{8}(b-a)\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}+\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right] \\ &\leq \begin{cases} \frac{1}{4}(b-a)\|f'\|_{[a,b],\infty}; \\ \\ \frac{1}{2^{\frac{3\beta-1}{2}}}(b-a)\left[\|f'\|^{\alpha}_{\left[a,\frac{a+b}{2}\right],\infty}+\|f'\|^{\alpha}_{\left[\frac{a+b}{2},b\right],\infty}\right]^{\frac{1}{\alpha}}, \\ \\ \text{where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1. \end{cases}\end{aligned}$$

2. OSTROWSKI FOR L_1-NORM

2.1. L_1-norm Inequality. In 1997, Dragomir and Wang proved the following Ostrowski type inequality [**5**].

THEOREM 2.1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then we have the inequality*

$$(2.1)\qquad \left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_{[a,b],1},$$

for all $x\in[a,b]$, where $\|\cdot\|_1$ is the Lebesgue norm on $L_1[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],1}:=\int_a^b|g(t)|\,dt.$$

The constant $\frac{1}{2}$ is best possible.

Note that the fact that $\frac{1}{2}$ is the best constant for differentiable functions was proved in [**11**] and (2.1) can also be obtained from a more general result obtained by A. M. Fink in [**8**] choosing $n=1$ and doing some appropriate computation. However the inequality (2.1) was not stated explicitly in [**8**].

2.2. A Refinement for L_1-norm. The following result, which is an improvement on the inequality (2.1), holds.

THEOREM 2.2 (Dragomir, 2002 [**1**]). *Let* $f : [a,b] \to \mathbb{C}$ *be an absolutely continuous function on* $[a,b]$. *Then*

$$\begin{aligned}
&\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{x-a}{b-a}\|f'\|_{[a,x],1} + \frac{b-x}{b-a}\|f'\|_{[x,b],1} \\
&\leq \begin{cases} \frac{1}{2}\left[\|f'\|_{[a,b],1} + \left|\|f'\|_{[a,x],1} - \|f'\|_{[x,b],1}\right|\right] \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{\beta} + \left(\frac{b-x}{b-a}\right)^{\beta}\right]^{\frac{1}{\beta}}\left(\left|\|f'\|^{\alpha}_{[a,x],1}\right| + \|f'\|^{\alpha}_{[x,b],1}\right)^{\frac{1}{\alpha}} \\ \textit{where } \alpha > 1 \textit{ and } \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \left[\frac{1}{2} + \frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_{[a,b],1} \end{cases}
\end{aligned} \tag{2.2}$$

for all $x \in [a,b]$, *where* $\|\cdot\|_{[m,n],1}$ *denotes the usual norm on* $L_1[m,n]$ *with* $m < n$, *i.e., we recall that*

$$\|g\|_{[m,n],1} := \int_m^n |g(t)|\,dt < \infty.$$

PROOF. Using the integration by parts formula for absolutely continuous functions on $[a,b]$, we have

$$\int_a^x (t-a) f'(t)\,dt = (x-a) f(x) - \int_a^x f(t)\,dt \tag{2.3}$$

and

$$\int_x^b (t-b) f'(t)\,dt = (b-x) f(x) - \int_x^b f(t)\,dt \tag{2.4}$$

for all $x \in [a,b]$.

Adding the two inequalities, we obtain the Montgomery identity for absolutely continuous functions (see for example [**9**, p. 565])

$$(b-a) f(x) - \int_a^b f(t)\,dt = \int_a^x (t-a) f'(t)\,dt + \int_x^b (t-b) f'(t)\,dt \tag{2.5}$$

for all $x \in [a,b]$.

Taking the modulus, we deduce

$$\begin{aligned}
&\left|(b-a) f(x) - \int_a^b f(t)\,dt\right| \\
&\leq \left|\int_a^x (t-a) f'(t)\,dt\right| + \left|\int_x^b (t-b) f'(t)\,dt\right| \\
&\leq \int_a^x (t-a)|f'(t)|\,dt + \int_x^b (b-t)|f'(t)|\,dt \\
&\leq (x-a)\int_a^x |f'(t)|\,dt + (b-x)\int_x^b |f'(t)|\,dt \\
&= (x-a)\|f'\|_{[a,x],1} + (b-x)\|f'\|_{[x,b],1}
\end{aligned} \tag{2.6}$$

and the first inequality in (2.2) is proved.

Now, let us observe that

$$\begin{aligned}&(x-a)\left\|f'\right\|_{[a,x],1}+(b-x)\left\|f'\right\|_{[x,b],1}\\&\leq \max\left\{\left\|f'\right\|_{[a,x],1},\left\|f'\right\|_{[x,b],1}\right\}(b-a)\\&=\frac{1}{2}\left[\left\|f'\right\|_{[a,x],1}+\left\|f'\right\|_{[x,b],1}+\left|\left\|f'\right\|_{[a,x],1}-\left\|f'\right\|_{[x,b],1}\right|\right](b-a)\\&=\frac{1}{2}\left[\left\|f'\right\|_{[a,b],1}+\left|\left\|f'\right\|_{[a,x],1}-\left\|f'\right\|_{[x,b],1}\right|\right](b-a)\end{aligned}$$

and the first part of the second inequality is proved.

For the second inequality, we employ the elementary inequality for real numbers which can be derived from Hölder's discrete inequality

$$0\leq ms+nt\leq\left(m^{\alpha}+n^{\alpha}\right)^{\frac{1}{\alpha}}\times\left(s^{\beta}+t^{\beta}\right)^{\frac{1}{\beta}}, \tag{2.7}$$

provided that $m,s,n,t\geq 0$, $\alpha>1$ and $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Using (2.7), we obtain

$$\begin{aligned}&(x-a)\left\|f'\right\|_{[a,x],1}+(b-x)\left\|f'\right\|_{[x,b],1}\\&\leq\left(\left\|f'\right\|_{[a,x],1}^{\alpha}+\left\|f'\right\|_{[x,b],1}^{\alpha}\right)^{\frac{1}{\alpha}}\left[(x-a)^{\beta}+(b-x)^{\beta}\right]^{\frac{1}{\beta}}\end{aligned}$$

and the second part of the second inequality in (2.2) is also obtained.

Finally, we observe that

$$\begin{aligned}&(x-a)\left\|f'\right\|_{[a,x],1}+(b-x)\left\|f'\right\|_{[x,b],1}\\&\leq\max\{x-a,b-x\}\left[\left\|f'\right\|_{[a,x],1}+\left\|f'\right\|_{[x,b],1}\right]\\&=\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\left\|f'\right\|_{[a,b],1}\end{aligned}$$

and the last part of the second inequality in (2.2) is proved. ∎

The following corollary is also natural.

COROLLARY 2.3. *Under the above assumptions, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{2}\left\|f'\right\|_{[a,b],1}. \tag{2.8}$$

Another interesting result is the following one.

COROLLARY 2.4. *Under the above assumptions, and if there is an* $x_0\in[a,b]$ *with*

$$\int_a^{x_0}\left|f'(t)\right|dt=\int_{x_0}^b\left|f'(t)\right|dt \tag{2.9}$$

then we have the inequality

$$\left|f(x_0)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{2}\left\|f'\right\|_{[a,b],1}. \tag{2.10}$$

3. OSTROWSKI FOR L_p-NORM

3.1. L_p-norm Inequality. In 1998, Dragomir and Wang proved the following Ostrowski type inequality [**6**].

THEOREM 3.1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, then we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_{[a,b],p}, \tag{3.1}$$

for all $x\in[a,b]$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p}:=\left(\int_a^b |g(t)|^p\,dt\right)^{1/p}.$$

Note that the inequality (3.1) can also be obtained from a more general result obtained by A. M. Fink in [**8**] choosing $n=1$ and doing some appropriate computation. However the inequality (3.1) was not stated explicitly in [**8**].

From (3.1) we get the following midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{2(q+1)^{1/q}}(b-a)^{1/q}\|f'\|_{[a,b],p}, \tag{3.2}$$

and $\frac{1}{2}$ is a best possible constant.

Indeed, if we take $f:[a,b]\to\mathbb{R}$ with $f(t)=\left|t-\frac{a+b}{2}\right|$, then f is absolutely continuous $\int_a^b f(t)\,dt=\frac{(b-a)^2}{4}$, $\|f'\|_{[a,b],p}=(b-a)^{1/p}$ and if we assume that (3.2) holds with a constant $C>0$ instead of $\frac{1}{2}$, then we get $\frac{1}{4}(b-a)\le\frac{C}{(q+1)^{1/q}}(b-a)$ for any $q>1$. Letting $q\to 1+$, we obtain $C\ge\frac{1}{2}$, which proves the sharpness of the constant.

In the following, we provide some refinements of (3.1) and (3.2).

3.2. A Refinement for L_p-norm. The following new result, which is an improvement on the inequality (3.1), holds.

THEOREM 3.2 (Dragomir, 2013 [**4**]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, then*

$$
\begin{aligned}
(3.3)\qquad & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
& \le \frac{1}{(q+1)^{1/q}} \left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}} \|f'\|_{[a,x],p} + \left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}} \|f'\|_{[x,b],p} \right] (b-a)^{1/q} \\
& \le \frac{1}{(q+1)^{1/q}} \\
& \times \begin{cases}
\frac{1}{2}\left[\|f'\|_{[a,x],p} + \|f'\|_{[x,b],p} + \left| \|f'\|_{[a,x],p} - \|f'\|_{[x,b],p} \right| \right] \\
\times \left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}} + \left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}} \right] (b-a)^{1/q} \\
\\
\left(\|f'\|^{\alpha}_{[a,x],p} + \|f'\|^{\alpha}_{[x,b],p} \right)^{\frac{1}{\alpha}} \left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}\beta} + \left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}\beta} \right]^{\frac{1}{\beta}} (b-a)^{1/q} \\
\textit{where } \alpha > 1 \textit{ and } \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\
\\
\left[\|f'\|_{[a,x],p} + \|f'\|_{[x,b],p} \right] \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right]^{\frac{q+1}{q}} (b-a)^{1/q}
\end{cases}
\end{aligned}
$$

for all $x \in [a,b]$, *where* $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$ *and* $\|\cdot\|_{[m,n],p}$ *denotes the usual* p*-norm on* $L_p[m,n]$ *with* $m < n$, *i.e., we recall that*

$$\|g\|_{[m,n],p} := \left(\int_m^n |g(t)|\,dt \right)^{1/p} < \infty.$$

PROOF. Using the integration by parts formula for absolutely continuous functions on $[a,b]$, we have

$$\int_a^x (t-a) f'(t)\,dt = (x-a) f(x) - \int_a^x f(t)\,dt \tag{3.4}$$

and

$$\int_x^b (t-b) f'(t)\,dt = (b-x) f(x) - \int_x^b f(t)\,dt \tag{3.5}$$

for all $x \in [a,b]$.

Adding the two inequalities, we obtain the Montgomery identity for absolutely continuous functions (see for example [**9**, p. 565])

$$(b-a) f(x) - \int_a^b f(t)\,dt = \int_a^x (t-a) f'(t)\,dt + \int_x^b (t-b) f'(t)\,dt \tag{3.6}$$

for all $x \in [a,b]$.

Taking the modulus, we deduce

$$
\begin{aligned}
(3.7)\qquad & \left| (b-a) f(x) - \int_a^b f(t)\,dt \right| \\
& \le \left| \int_a^x (t-a) f'(t)\,dt \right| + \left| \int_x^b (t-b) f'(t)\,dt \right| \\
& \le \int_a^x (t-a) |f'(t)|\,dt + \int_x^b (b-t) |f'(t)|\,dt.
\end{aligned}
$$

Utilizing Hölder's integral inequality we have

$$\begin{aligned}
&\int_a^x (t-a)\left|f'(t)\right| dt + \int_x^b (b-t)\left|f'(t)\right| dt \\
&\leq \left(\int_a^x (t-a)^q\, dt\right)^{1/q} \left(\int_a^x \left|f'(t)\right|^p dt\right)^{1/p} \\
&+ \left(\int_x^b (b-t)^q\, dt\right)^{1/q} \left(\int_x^b \left|f'(t)\right|^p dt\right)^{1/p} \\
&= \frac{1}{(b-a)(q+1)^{1/q}} \left[(x-a)^{\frac{q+1}{q}} \left\|f'\right\|_{[a,x],p} + (b-x)^{\frac{q+1}{q}} \left\|f'\right\|_{[x,b],p}\right]
\end{aligned}$$

for all $x \in [a,b]$, and the first inequality in (3.3) is proved.

Now, let us observe that

$$\begin{aligned}
&(x-a)^{\frac{q+1}{q}} \left\|f'\right\|_{[a,x],p} + (b-x)^{\frac{q+1}{q}} \left\|f'\right\|_{[x,b],p} \\
&\leq \max\left\{\left\|f'\right\|_{[a,x],p}, \left\|f'\right\|_{[x,b],p}\right\} \left[(x-a)^{\frac{q+1}{q}} + (b-x)^{\frac{q+1}{q}}\right] \\
&= \frac{1}{2}\left[\left\|f'\right\|_{[a,x],p} + \left\|f'\right\|_{[x,b],p} + \left|\left\|f'\right\|_{[a,x],p} - \left\|f'\right\|_{[x,b],p}\right|\right] \\
&\times \left[(x-a)^{\frac{q+1}{q}} + (b-x)^{\frac{q+1}{q}}\right]
\end{aligned}$$

and the first part of the second inequality is proved.

For the second inequality, we employ the elementary inequality for real numbers which can be derived from Hölder's discrete inequality

$$0 \leq ms + nt \leq (m^\alpha + n^\alpha)^{\frac{1}{\alpha}} \times \left(s^\beta + t^\beta\right)^{\frac{1}{\beta}}, \tag{3.8}$$

provided that $m, s, n, t \geq 0$, $\alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Using (3.8), we obtain

$$\begin{aligned}
&(x-a)^{\frac{q+1}{q}} \left\|f'\right\|_{[a,x],p} + (b-x)^{\frac{q+1}{q}} \left\|f'\right\|_{[x,b],p} \\
&\leq \left(\left\|f'\right\|_{[a,x],p}^\alpha + \left\|f'\right\|_{[x,b],p}^\alpha\right)^{\frac{1}{\alpha}} \left[(x-a)^{\frac{q+1}{q}\beta} + (b-x)^{\frac{q+1}{q}\beta}\right]^{\frac{1}{\beta}}
\end{aligned}$$

and the second part of the second inequality in (3.3) is also obtained.

Finally, we observe that

$$\begin{aligned}
&(x-a)^{\frac{q+1}{q}} \left\|f'\right\|_{[a,x],p} + (b-x)^{\frac{q+1}{q}} \left\|f'\right\|_{[x,b],p} \\
&\leq \max\left\{(x-a)^{\frac{q+1}{q}}, (b-x)^{\frac{q+1}{q}}\right\} \left[\left\|f'\right\|_{[a,x],p} + \left\|f'\right\|_{[x,b],p}\right] \\
&= \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^{\frac{q+1}{q}} \left[\left\|f'\right\|_{[a,x],p} + \left\|f'\right\|_{[x,b],p}\right]
\end{aligned}$$

and the last part of the second inequality in (3.3) is proved. ∎

The following corollary is also natural.

COROLLARY 3.3. *Under the above assumptions, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{2^{(q+1)/q}(q+1)^{1/q}}\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],p}+\|f'\|_{\left[\frac{a+b}{2},b\right],p}\right](b-a)^{1/q}. \tag{3.9}$$

Another interesting result is the following one.

COROLLARY 3.4. *Under the above assumptions, and if there is an $x_0 \in [a,b]$ with*

$$\int_a^{x_0}|f'(t)|^p\,dt=\int_{x_0}^b|f'(t)|^p\,dt \tag{3.10}$$

then we have the inequality

$$\left|f(x_0)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x_0-a}{b-a}\right)^{\frac{q+1}{q}}+\left(\frac{b-x_0}{b-a}\right)^{\frac{q+1}{q}}\right]\|f'\|_{[a,x_0],p}(b-a)^{1/q}. \tag{3.11}$$

REMARK 3.1. If we take in (3.3) $\alpha=p$ and $\beta=q$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then we get the following refinement of (3.1)

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| &\le \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\|f'\|_{[a,x],p}+\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}\|f'\|_{[x,b],p}\right](b-a)^{1/q}\\ &\le \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_{[a,b],p},\end{aligned} \tag{3.12}$$

for all $x\in[a,b]$.

This is true, since for $\alpha=p$ we have

$$\begin{aligned}\left(\|f'\|^{\alpha}_{[a,x],p}+\|f'\|^{\alpha}_{[x,b],p}\right)^{\frac{1}{\alpha}} &= \left(\|f'\|^{p}_{[a,x],p}+\|f'\|^{p}_{[x,b],p}\right)^{\frac{1}{p}}\\ &= \left(\int_a^x|f'(t)|^p\,dt+\int_x^b|f'(t)|^p\right)^{1/p}=\|f'\|_{[a,b],p}.\end{aligned}$$

4. OSTROWSKI FOR BOUNDED DERIVATIVES

We start with the following result.

THEOREM 4.1 (Dragomir, 2003 [**3**]). *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x\in[a,b]$. Suppose that there exist the functions $m_i, M_i:[a,b]\to\mathbb{R}$ $\left(i=\overline{1,2}\right)$ with the properties:*

$$m_1(x)\le f'(t)\le M_1(x) \quad \textit{for a.e. } t\in[a,x] \tag{4.1}$$

and

$$m_2(x)\le f'(t)\le M_2(x) \quad \textit{for a.e. } t\in(x,b]. \tag{4.2}$$

Then we have the inequalities:

$$\begin{aligned}(4.3)\qquad &\frac{1}{2(b-a)}\left[m_1(x)(x-a)^2 - M_2(x)(b-x)^2\right] \\ &\leq f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\leq \frac{1}{2(b-a)}\left[M_1(x)(x-a)^2 - m_2(x)(b-x)^2\right].\end{aligned}$$

The constant $\frac{1}{2}$ is sharp on both sides.

PROOF. As $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, it is differentiable a.e. on $[a,b]$ and, by applying the integration by parts formula, we may write the Montgomery identity

$$(4.4)\qquad f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^x (t-a) f'(t)\,dt + \frac{1}{b-a}\int_x^b (t-b) f'(t)\,dt$$

for any $x \in [a,b]$.

Using the assumption (4.1) and (4.2), we have:

$$(4.5)\qquad m_1(x)(t-a) \leq (t-a) f'(t) \leq M_1(x)(t-a) \quad \text{for a.e. } t \in [a,x]$$

and

$$(4.6)\qquad M_2(x)(t-b) \leq f'(t)(t-b) \leq m_2(x)(t-b) \quad \text{for a.e. } t \in (x,b].$$

Integrating (4.5) on $[a,x]$ and (4.6) on $[x,b]$ and summing the obtained inequalities, we have

$$\begin{aligned}&\frac{1}{2}m_1(x)(x-a)^2 - \frac{1}{2}M_2(x)(b-x)^2 \\ &\leq \int_a^x (t-a) f'(t)\,dt + \int_x^b (t-b) f'(t)\,dt \\ &\leq \frac{1}{2}M_1(x)(x-a)^2 - \frac{1}{2}m_2(x)(b-x)^2.\end{aligned}$$

Using the representation (4.4), we deduce (4.3).

Assume that the first inequality in (4.3) holds with a constant $c > 0$; that is,

$$(4.7)\qquad \frac{c}{b-a}\left[m_1(x)(x-a)^2 - M_2(x)(b-x)^2\right] \leq f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Consider the function $f : [a,b] \to \mathbb{R}$, $f(t) = M|t-x|$, $M > 0$. Then f is absolutely continuous and

$$f'(t) = \begin{cases} -M & \text{if } \ t \in [a,x] \\ M & \text{if } \ t \in (x,b]. \end{cases}$$

Thus, if we choose $m_1 = -M$, $m_2 = M$ in (4.7), we get

$$\begin{aligned}-M\frac{c}{b-a}\left[(x-a)^2 + (b-x)^2\right] &\leq -\frac{M}{b-a}\int_a^b |t-x|\,dt \\ &= -\frac{M}{b-a}\left[\frac{(b-x)^2 + (x-a)^2}{2}\right]\end{aligned}$$

for all $x \in [a,b]$, implying that $c \geq \frac{1}{2}$, that is, $\frac{1}{2}$ is the best constant in the first member of (4.3).

Using a similar process, we may prove that $\frac{1}{2}$ is the best constant in the third member of (4.3) and the theorem is completely proved. ∎

COROLLARY 4.2. *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and the derivative $f':[a,b]\to\mathbb{R}$ is bounded above and below, that is,*

$$-\infty<m\le f'(t)\le M<\infty \text{ for a.e. } t\in[a,b], \tag{4.8}$$

then we have the inequality

$$\begin{aligned}\frac{1}{2(b-a)}&\left[m(x-a)^2-M(b-x)^2\right]\\&\le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\\&\le\frac{1}{2(b-a)}\left[M(x-a)^2-m(b-x)^2\right]\end{aligned} \tag{4.9}$$

for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best in both inequalities.

Applying Taylor's formula

$$g(x)=g\left(\frac{a+b}{2}\right)+\left(x-\frac{a+b}{2}\right)g'\left(\frac{a+b}{2}\right)+\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2 g''\left(\frac{a+b}{2}\right)$$

for $g(x)=M(x-a)^2-m(b-x)^2$, we obtain

$$\begin{aligned}g(x)&=\frac{1}{4}(M-m)(b-a)^2+2\left(x-\frac{a+b}{2}\right)\left(\frac{M+m}{2}\right)(b-a)\\&\quad+(M-m)\left(x-\frac{a+b}{2}\right)^2.\end{aligned}$$

The same formula applied for $h(x)=m(x-a)^2-M(b-x)^2$, will reveal that

$$\begin{aligned}h(x)&=\frac{1}{4}(M-m)(b-a)^2+2\left(x-\frac{a+b}{2}\right)\left(\frac{M+m}{2}\right)(b-a)\\&\quad-(M-m)\left(x-\frac{a+b}{2}\right)^2.\end{aligned}$$

Consequently, we may rewrite Corollary 4.2 in the following equivalent manner:

COROLLARY 4.3. *With the assumptions on Corollary 4.2, we have:*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\left(x-\frac{a+b}{2}\right)\left(\frac{M+m}{2}\right)\right|\\&\le\frac{1}{2}(M-m)(b-a)\left[\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2+\frac{1}{4}\right]\end{aligned} \tag{4.10}$$

for all $x\in[a,b]$.

REMARK 4.1. If we assume that $\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)|$, then obviously we may choose in (4.9) $m = \|f'\|_\infty$ and $M = \|f'\|_\infty$, obtaining Ostrowski's inequality for absolutely continuous functions whose derivatives are essentially bounded:

$$\begin{aligned}\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| &\leq \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right] \\ &= \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty,\end{aligned}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ here is best.

REMARK 4.2. Ostrowski's inequality for absolutely continuous mappings in terms of $\|f'\|_\infty$ basically states that

$$\begin{aligned}-\frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right] &\leq f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\leq \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right]\end{aligned} \tag{4.11}$$

for all $x \in [a,b]$.

Now, if we assume that (4.1) and (4.2) hold, then $-\|f'\|_\infty \leq m_1(x)$, $m_2(x)$ and $M_1(x)$, $M_2(x) \leq \|f'\|_\infty$, which implies:

$$\begin{aligned}&-\frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right] \\ &\leq \frac{1}{2(b-a)}\left[m_1(x)(x-a)^2 - M_2(x)(b-x)^2\right] \leq f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\leq \frac{1}{2(b-a)}\left[M_1(x)(x-a)^2 - m_2(x)(b-x)^2\right] \\ &\leq \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right].\end{aligned} \tag{4.12}$$

Thus, the inequality (4.3) may also be regarded as a refinement of the classical Ostrowski result.

An important particular case is $x = \frac{a+b}{2}$ providing the following corollary.

COROLLARY 4.4. *Assume that the derivative* $f' : [a,b] \to \mathbb{R}$ *satisfy the conditions:*

$$-\infty < m_1 \leq f'(t) \leq M_1 < \infty \text{ for a.e. } t \in \left[a, \frac{a+b}{2}\right] \tag{4.13}$$

and

$$-\infty < m_2 \leq f'(t) \leq M_2 < \infty \text{ for a.e. } t \in \left(\frac{a+b}{2}, b\right]. \tag{4.14}$$

Then we have the inequalities

$$\begin{aligned}\frac{1}{8}(m_1 - M_2)(b-a) &\leq f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\leq \frac{1}{8}(M_1 - m_2)(b-a).\end{aligned} \tag{4.15}$$

The constant $\frac{1}{8}$ *is the best in both inequalities.*

Finally, if we know some global bounds for the derivative f' on $[a,b]$, then we may state the following corollary.

COROLLARY 4.5. *Under the assumptions of Corollary 4.2, we have the midpoint inequality:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{8}(M-m)(b-a). \tag{4.16}$$

The constant $\frac{1}{8}$ is best.

PROOF. The inequality is obvious by Corollary 4.2 putting $x = \frac{a+b}{2}$. We observe that the function $f : [a,b] \to \mathbb{R}$, $f(x) = k\left|x - \frac{a+b}{2}\right|$, $k > 0$ is absolutely continuous and $-k \leq f'(t) \leq k$ for all $t \in [a,b]$. Thus, we may choose $M = k$, $m = -k$ and as

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| &= \frac{k(b-a)}{4}, \\ \frac{1}{8}(M-m)(b-a) &= \frac{k(b-a)}{4} \end{aligned}$$

we conclude that the constant $\frac{1}{8}$ is best in (4.16). ∎

5. FUNCTIONAL OSTROWSKI INEQUALITY FOR CONVEX MAPPINGS

5.1. A Generalization of Ostrowski's Inequality.

The following result holds:

THEOREM 5.1 (Dragomir, 2013 [**4**]). *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. If $\Phi : \mathbb{R} \to \mathbb{R}$ is convex (concave) on $\mathbb{R}$ then we have the inequalities*

$$\begin{aligned} &\Phi\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right) \\ &\leq (\geq) \frac{1}{b-a}\left[\int_a^x \Phi[(t-a)f'(t)]\,dt + \int_x^b \Phi[(t-b)f'(t)]\,dt\right] \end{aligned} \tag{5.1}$$

for any $x \in [a,b]$.

PROOF. Utilising the Montgomery identity

$$\begin{aligned} &f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\left[\int_a^x (t-a)f'(t)\,dt + \int_x^b (t-b)f'(t)\,dt\right] \\ &= \frac{x-a}{b-a}\left(\frac{1}{x-a}\int_a^x (t-a)f'(t)\,dt\right) \\ &\quad + \frac{b-x}{b-a}\left(\frac{1}{b-x}\int_x^b (t-b)f'(t)\,dt\right), \end{aligned} \tag{5.2}$$

which holds for any $x \in (a,b)$ and the convexity of $\Phi : \mathbb{R} \to \mathbb{R}$, we have

$$\begin{aligned} &\Phi\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right) \\ &\leq \frac{x-a}{b-a}\Phi\left(\frac{1}{x-a}\int_a^x (t-a)f'(t)\,dt\right) \\ &\quad + \frac{b-x}{b-a}\Phi\left(\frac{1}{b-x}\int_x^b (t-b)f'(t)\,dt\right) \end{aligned} \tag{5.3}$$

for any $x \in (a, b)$, which is an inequality of interest in itself as well.

If we use Jensen's integral inequality

$$\Phi\left(\frac{1}{d-c}\int_c^d g(t)\,dt\right) \le \frac{1}{d-c}\int_c^d \Phi[g(t)]\,dt$$

we have

$$\Phi\left(\frac{1}{x-a}\int_a^x (t-a) f'(t)\,dt\right) \le \frac{1}{x-a}\int_a^x \Phi[(t-a) f'(t)]\,dt \tag{5.4}$$

and

$$\Phi\left(\frac{1}{b-x}\int_x^b (t-b) f'(t)\,dt\right) \le \frac{1}{b-x}\int_x^b \Phi[(t-b) f'(t)]\,dt \tag{5.5}$$

for any $x \in (a, b)$.

Making use of (5.3)-(5.5) we get the desired result (5.1) for the convex functions.

If $x = b$, then

$$f(b) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b (t-a) f'(t)\,dt$$

and by Jensen's inequality we get

$$\Phi\left(f(b) - \frac{1}{b-a}\int_a^b f(t)\,dt\right) \le \frac{1}{b-a}\int_a^b \Phi[(t-a) f'(t)]\,dt,$$

which proves the inequality (5.1) for $x = b$.

The same argument can be applied for $x = a$.

The case of concave functions goes likewise and the theorem is proved. ∎

COROLLARY 5.2. *With the assumptions of Theorem 5.1 we have*

$$\begin{aligned}\Phi(0) &\le (\ge) \frac{1}{b-a}\int_a^b \Phi\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right)dx \\ &\le (\ge) \frac{1}{(b-a)^2}\left[\int_a^b (b-x)\,\Phi[(x-a) f'(x)]\,dx \right.\\ &\left.\quad + \int_a^b (x-a)\,\Phi[(x-b) f'(x)]\,dx\right].\end{aligned} \tag{5.6}$$

PROOF. By Jensen's integral inequality we have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \Phi\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right)dx \\ &\ge (\le)\, \Phi\left[\frac{1}{b-a}\int_a^b \left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right)dx\right] \\ &= \Phi(0),\end{aligned}$$

which proves the first inequality in (5.6).

Integrating the inequality (5.1) over x we have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \Phi\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right)dx \\ &\le (\ge) \frac{1}{(b-a)^2}\int_a^b \left[\int_a^x \Phi[(t-a) f'(t)]\,dt + \int_x^b \Phi[(t-b) f'(t)]\,dt\right]dx.\end{aligned} \tag{5.7}$$

Integrating by parts we have

$$\begin{aligned}
&\int_a^b \left(\int_a^x \Phi\left[(t-a) f'(t)\right] dt\right) dx \\
&= x \int_a^x \Phi\left[(t-a) f'(t)\right] dt\Big|_a^b - \int_a^b x d\left(\int_a^x \Phi\left[(t-a) f'(t)\right] dt\right) \\
&= b \int_a^b \Phi\left[(t-a) f'(t)\right] dt - \int_a^b x\Phi\left[(x-a) f'(x)\right] dx \\
&= \int_a^b (b-x) \Phi\left[(x-a) f'(x)\right] dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \left(\int_x^b \Phi\left[(t-b) f'(t)\right] dt\right) dx \\
&= x \left(\int_x^b \Phi\left[(t-b) f'(t)\right] dt\right)\Big|_a^b - \int_a^b x d\left(\int_x^b \Phi\left[(t-b) f'(t)\right] dt\right) \\
&= -a \left(\int_a^b \Phi\left[(t-b) f'(t)\right] dt\right) + \int_a^b x\Phi\left[(x-b) f'(x)\right] dx \\
&= \int_a^b (x-a) \Phi\left[(x-b) f'(x)\right] dx.
\end{aligned}$$

Utilising the inequality (5.7) we deduce the desired inequality (5.6). ∎

REMARK 5.1. If we write the inequality (5.1) for the convex function $\Phi(x) = |x|^p, p \geq 1$ then we get the inequality

$$\begin{aligned}
&\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\, dt\right|^p \\
&\leq \frac{1}{b-a}\left[\int_a^x (t-a)^p \left|f'(t)\right|^p dt + \int_x^b (b-t)^p \left|f'(t)\right|^p dt\right]
\end{aligned} \tag{5.8}$$

for $x \in [a,b]$.

Utilising Hölder's inequality we have

$$B(x) := \int_a^x (t-a)^p |f'(t)|^p \, dt + \int_x^b (b-t)^p |f'(t)|^p \, dt \tag{5.9}$$
$$\leq \begin{cases} \frac{(x-a)^{p+1}}{p+1} \|f'\|_{[a,x],\infty}^p & \text{if } f' \in L_\infty[a,x]; \\ \\ \frac{(x-a)^{p+1/\alpha}}{(p\alpha+1)^{1/\alpha}} \|f'\|_{[a,x],p\beta}^p & \begin{array}{c} \text{if } f' \in L_{p\beta}[a,x], \\ \alpha > 1, 1/\alpha + 1/\beta = 1; \end{array} \\ \\ (x-a)^p \|f'\|_{[a,x],p}^p \end{cases}$$
$$+ \begin{cases} \frac{(b-x)^{p+1}}{p+1} \|f'\|_{[x,b],\infty}^p & \text{if } f' \in L_\infty[x,b]; \\ \\ \frac{(b-x)^{p+1/\alpha}}{(p\alpha+1)^{1/\alpha}} \|f'\|_{[x,b],p\beta}^p & \begin{array}{c} \text{if } f' \in L_{p\beta}[x,b], \\ \alpha > 1, 1/\alpha + 1/\beta = 1; \end{array} \\ \\ (b-x)^p \|f'\|_{[x,b],p}^p \end{cases}$$

for $x \in [a,b]$.

Utilising the inequalities (5.8) and (5.9) we have for $x \in [a,b]$ that

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right|^p \\ &\leq \frac{1}{(b-a)(p+1)} \left[(x-a)^{p+1} \|f'\|_{[a,x],\infty}^p + (b-x)^{p+1} \|f'\|_{[x,b],\infty}^p \right] \\ &\leq \frac{1}{(p+1)} \left[\left(\frac{x-a}{b-a} \right)^{p+1} + \left(\frac{b-x}{b-a} \right)^{p+1} \right] (b-a)^p \|f'\|_{[a,b],\infty}^p \end{aligned} \tag{5.10}$$

provided $f' \in L_\infty[a,b]$,

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right|^p \\ &\leq \frac{1}{(b-a)(p\alpha+1)^{1/\alpha}} \left[(x-a)^{p+1/\alpha} \|f'\|_{[a,x],p\beta}^p + (b-x)^{p+1/\alpha} \|f'\|_{[x,b],p\beta}^p \right] \\ &\leq \frac{1}{(p\alpha+1)^{1/\alpha}} \left[\left(\frac{x-a}{b-a} \right)^{p+1/\alpha} + \left(\frac{b-x}{b-a} \right)^{p+1/\alpha} \right] (b-a)^{p-1/\beta} \|f'\|_{[a,b],p\beta}^p \end{aligned} \tag{5.11}$$

provided $f' \in L_{p\beta}[a,b], \alpha > 1, 1/\alpha + 1/\beta = 1$ and

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right|^p \\ &\leq \frac{1}{b-a} \left[(x-a)^p \|f'\|_{[a,x],p}^p + (b-x)^p \|f'\|_{[x,b],p}^p \right] \\ &\leq \max \left\{ \left(\frac{x-a}{b-a} \right)^p, \left(\frac{b-x}{b-a} \right)^p \right\} (b-a)^{p-1} \|f'\|_{[a,b],p}^p \\ &= \left\{ \frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right\}^p (b-a)^{p-1} \|f'\|_{[a,b],p}^p \end{aligned} \tag{5.12}$$

provided $f' \in L_p[a,b]$.

REMARK 5.2. If we take $p = 1$ in the above inequalities (5.11)-(5.12), then we obtain

$$\begin{aligned}(5.13)\quad &\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{2(b-a)}\left[(x-a)^2\|f'\|_{[a,x],\infty} + (b-x)^2\|f'\|_{[x,b],\infty}\right] \\ &\leq \frac{1}{2}\left[\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{b-x}{b-a}\right)^2\right](b-a)\|f'\|_{[a,b],\infty} \\ &= \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_{[a,b],\infty}\end{aligned}$$

for $x \in [a,b]$, provided $f' \in L_\infty[a,b]$,

$$\begin{aligned}(5.14)\quad &\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{(b-a)(\alpha+1)^{1/\alpha}}\left[(x-a)^{1+1/\alpha}\|f'\|_{[a,x],\beta} + (b-x)^{1+1/\alpha}\|f'\|_{[x,b],\beta}\right] \\ &\leq \frac{1}{(\alpha+1)^{1/\alpha}}\left[\left(\frac{x-a}{b-a}\right)^{1+1/\alpha} + \left(\frac{b-x}{b-a}\right)^{1+1/\alpha}\right](b-a)^{1/\alpha}\|f'\|_{[a,b],\beta}\end{aligned}$$

for $x \in [a,b]$, provided $f' \in L_\beta[a,b]$, $\alpha > 1, 1/\alpha + 1/\beta = 1$ and

$$\begin{aligned}(5.15)\quad &\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{b-a}\left[(x-a)\|f'\|_{[a,x],1} + (b-x)\|f'\|_{[x,b],1}\right] \\ &= \left\{\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right\}\|f'\|_{[a,b],1}\end{aligned}$$

for $x \in [a,b]$.

5.2. Applications for p-Norms. We have the following inequalities for Lebesgue norms of the deviation of a function from its integral mean:

THEOREM 5.3 (Dragomir, 2013 [**4**]). *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$.*

(i) *If $f' \in L_\infty[a,b]$, then*

$$(5.16)\quad \left\|f - \frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p} \leq \left[\frac{2}{(p+1)(p+2)}\right]^{1/p}(b-a)^{1+\frac{1}{p}}\|f'\|_{[a,b],\infty}.$$

(ii) *If $f' \in L_{p\beta}[a,b]$, with $\alpha > 1, 1/\alpha + 1/\beta = 1$ then*

$$\begin{aligned}(5.17)\quad &\left\|f - \frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p} \\ &\leq \left[\frac{2}{(p\alpha+1)^{1/\alpha}(p+1/\alpha+1)}\right]^{1/p}\|f'\|_{[a,b],p\beta}(b-a)^{1+\frac{1}{\alpha p}}.\end{aligned}$$

(iii) *We have*

$$\left\| f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p}\le\frac{1}{2}\left(\frac{2^{p+1}-1}{p+1}\right)^{1/p}(b-a)\left\|f'\right\|_{[a,b],p}. \tag{5.18}$$

PROOF. Integrating on $[a,b]$ the inequality (5.10) we have

$$\begin{aligned} &\int_a^b\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|^p dx \\ &\le\frac{1}{(b-a)(p+1)}\left\|f'\right\|_{[a,b],\infty}^p\int_a^b\left[(x-a)^{p+1}+(b-x)^{p+1}\right]dx \\ &=\frac{1}{(b-a)(p+1)}\left\|f'\right\|_{[a,b],\infty}^p\left[\frac{2(b-a)^{p+2}}{p+2}\right] \\ &=\frac{2}{(p+1)(p+2)}\left\|f'\right\|_{[a,b],\infty}^p(b-a)^{p+1}\end{aligned} \tag{5.19}$$

which is equivalent with (5.16).

Integrating the inequality (5.11)

$$\begin{aligned} &\int_a^b\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|^p dx \\ &\le\frac{1}{(b-a)(p\alpha+1)^{1/\alpha}}\left\|f'\right\|_{[a,b],p\beta}^p\int_a^b\left[(x-a)^{p+1/\alpha}+(b-x)^{p+1/\alpha}\right]dx \\ &=\frac{1}{(b-a)(p\alpha+1)^{1/\alpha}}\left\|f'\right\|_{[a,b],p\beta}^p\left[\frac{2(b-a)^{p+1/\alpha+1}}{p+1/\alpha+1}\right] \\ &=\frac{2}{(p\alpha+1)^{1/\alpha}(p+1/\alpha+1)}\left\|f'\right\|_{[a,b],p\beta}^p(b-a)^{p+1/\alpha}\end{aligned} \tag{5.20}$$

which is equivalent with (5.17).

Integrating the inequality (5.12) we have

$$\begin{aligned} &\int_a^b\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|^p dx \\ &\le\frac{1}{b-a}\left\|f'\right\|_{[a,b],p}^p\int_a^b\max\left\{(x-a)^p,(b-x)^p\right\}dx.\end{aligned} \tag{5.21}$$

Since

$$\begin{aligned} &\int_a^b\max\left\{(x-a)^p,(b-x)^p\right\}dx \\ &=\int_a^{\frac{a+b}{2}}(b-x)^p\,dx+\int_{\frac{a+b}{2}}^b(x-a)^p\,dx \\ &=-\frac{\left(\frac{b-a}{2}\right)^{p+1}}{p+1}+\frac{(b-a)^{p+1}}{p+1}+\frac{(b-a)^{p+1}}{p+1}-\frac{\left(\frac{b-a}{2}\right)^{p+1}}{p+1} \\ &=\frac{1}{p+1}\left(\frac{2^{p+1}-1}{2^p}\right)(b-a)^{p+1}\end{aligned}$$

then from (5.21) we get

$$\int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right|^p dx \le \frac{1}{p+1} \left(\frac{2^{p+1}-1}{2^p} \right) (b-a)^p \left\| f' \right\|_{[a,b],p}^p ,$$

which is equivalent with (5.18). ∎

5.3. Applications for the Exponential. If we write the inequality (5.1) for the convex function $\Phi(x) = \exp(x)$ then we get the inequality

$$\exp\left[f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right] \le \frac{1}{b-a} \left[\int_a^x \exp\left[(t-a) f'(t) \right] dt + \int_x^b \exp\left[(t-b) f'(t) \right] dt \right] \tag{5.22}$$

for $x \in [a,b]$.

If we write the inequality (5.1) for the convex function $\Phi(x) = \cosh(x) := \frac{e^x + e^{-x}}{2}$ then we get the inequality

$$\cosh\left[f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right] \le \frac{1}{b-a} \left[\int_a^x \cosh\left[(t-a) f'(t) \right] dt + \int_x^b \cosh\left[(t-b) f'(t) \right] dt \right] \tag{5.23}$$

for $x \in [a,b]$.

Utilising the inequality (5.22) we have the following multiplicative version of Ostrowski's inequality:

THEOREM 5.4 (Dragomir, 2013 [**4**]). *Let $f : [a,b] \to (0,\infty)$ be absolutely continuous on $[a,b]$. Then we have the inequalities*

$$\frac{f(x)}{\exp\left[\frac{1}{b-a} \int_a^b \ln f(t)\, dt \right]} \le \frac{1}{b-a} \left[\int_a^x \exp\left[(t-a) \frac{f'(t)}{f(t)} \right] dt + \int_x^b \exp\left[(t-b) \frac{f'(t)}{f(t)} \right] dt \right] \tag{5.24}$$

for any $x \in [a,b]$ and

$$\begin{aligned} \frac{\int_a^b f(x)\, dx}{\exp\left[\frac{1}{b-a} \int_a^b \ln f(t)\, dt \right]} &\le \frac{1}{b-a} \left[\int_a^b (b-x) \exp\left[(x-a) \frac{f'(x)}{f(x)} \right] dx \right. \\ &\quad \left. + \int_a^b (x-a) \exp\left[(x-b) \frac{f'(x)}{f(x)} \right] dx \right]. \end{aligned} \tag{5.25}$$

PROOF. If we replace f by $\ln f$ in (5.22) we get

$$\exp\left[\ln f(x) - \frac{1}{b-a} \int_a^b \ln f(t)\, dt \right] \le \frac{1}{b-a} \left[\int_a^x \exp\left[(t-a) \frac{f'(t)}{f(t)} \right] dt + \int_x^b \exp\left[(t-b) \frac{f'(t)}{f(t)} \right] dt \right] \tag{5.26}$$

for any $x \in [a, b]$.

Since

$$\begin{aligned}
&\exp\left[\ln f(x) - \frac{1}{b-a}\int_a^b \ln f(t)\,dt\right] \\
&= \exp\left[\ln f(x) - \ln\left\{\exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)\right\}\right] \\
&= \exp\left[\ln\left(\frac{f(x)}{\exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)}\right)\right] \\
&= \frac{f(x)}{\exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)}
\end{aligned}$$

for any $x \in [a, b]$, then we get from (5.26) the desired inequality (5.24).

If we integrate the inequality (5.24) we get

$$\begin{aligned}
&\frac{\int_a^b f(x)\,dx}{\exp\left[\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right]} \\
&\leq \frac{1}{b-a}\int_a^b \left[\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt + \int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt\right]dx.
\end{aligned} \tag{5.27}$$

Integrating by parts we have

$$\begin{aligned}
&\int_a^b \left(\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt\right)dx \\
&= x\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt\bigg|_a^b - \int_a^b x\,d\left(\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt\right) \\
&= b\int_a^b \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt - \int_a^b x\exp\left[(x-a)\frac{f'(x)}{f(x)}\right]dx \\
&= \int_a^b (b-x)\exp\left[(x-a)\frac{f'(x)}{f(x)}\right]dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \left(\int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt\right)dx \\
&= x\int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt\bigg|_a^b - \int_a^b x\,d\left(\int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt\right) \\
&= -a\int_a^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt + \int_a^b x\exp\left[(x-b)\frac{f'(x)}{f(x)}\right]dx \\
&= \int_a^b (x-a)\exp\left[(x-b)\frac{f'(x)}{f(x)}\right]dx,
\end{aligned}$$

then by (5.27) we deduce the desired inequality (5.25). ∎

5.4. Applications for Midpoint-Inequalities. We have from the inequality (5.1) written for $-f$ the following result:

PROPOSITION 5.5. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. If $\Phi:\mathbb{R}\to\mathbb{R}$ is convex (concave) on $\mathbb{R}$ then from (5.1) we have the inequalities*

$$\Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right) \leq (\geq)\, \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b \Phi\left[(b-t)\,f'(t)\right]dt + \int_a^{\frac{a+b}{2}} \Phi\left[(a-t)\,f'(t)\right]dt\right]. \tag{5.28}$$

If $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$, then by Hermite-Hadamard inequality we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq f\left(\frac{a+b}{2}\right).$$

We can state the following result in which the function Φ is assumed be convex only on $[0,\infty)$ or $(0,\infty)$.

PROPOSITION 5.6. *If $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$, monotonic nondecreasing on $\left[a,\frac{a+b}{2}\right]$ and monotonic nonincreasing $\left[a,\frac{a+b}{2}\right]$ .If $\Phi:[0,\infty),(0,\infty)\to\mathbb{R}$ is convex (concave) on $[0,\infty)$ or $(0,\infty)$, then (5.28) holds true.*

If $f:[a,b]\to\mathbb{R}$ is strictly convex on $[a,b]$, monotonic nondecreasing on $\left[a,\frac{a+b}{2}\right]$ and monotonic nonincreasing $\left[a,\frac{a+b}{2}\right]$, then by taking $\Phi(x)=\ln x$, which is strictly concave on $(0,\infty)$, we get the logarithmic inequality

$$\ln\left(\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right) \geq \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b \ln\left[(b-t)\,f'(t)\right]dt + \int_a^{\frac{a+b}{2}} \ln\left[(a-t)\,f'(t)\right]dt\right]. \tag{5.29}$$

If $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$, monotonic nondecreasing on $\left[a,\frac{a+b}{2}\right]$ and monotonic nonincreasing $\left[a,\frac{a+b}{2}\right]$, then by taking $\Phi(x)=x^q$, with $q\in(0,1)$ we also have

$$\left(\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right)^q \geq \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b \left[(b-t)\,f'(t)\right]^q dt + \int_a^{\frac{a+b}{2}} \left[(a-t)\,f'(t)\right]^q dt\right]. \tag{5.30}$$

If $\Phi:[0,\infty),(0,\infty)\to\mathbb{R}$ is convex (concave) on $[0,\infty)$ or $(0,\infty)$, and if we take $f(t):=\left|t-\frac{a+b}{2}\right|^p$, $p\geq 1$, then we get from (5.28)

$$\Phi\left(\frac{(b-a)^p}{2^p(p+1)}\right) \leq (\geq)\, \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b \Phi\left[p(b-t)\left(t-\frac{a+b}{2}\right)^{p-1}\right]dt + \int_a^{\frac{a+b}{2}} \Phi\left[(t-a)\left(\frac{a+b}{2}-t\right)^{p-1}\right]dt\right]. \tag{5.31}$$

Assume that $\Phi:\mathbb{R}\to\mathbb{R}$ is convex (concave) on $\mathbb{R}$.

Now, if we take $f(t) = \frac{1}{t}$ in (5.28), where $t \in [a,b] \subset (0,\infty)$, then we have

$$\Phi\left(\frac{A(a,b) - L(a,b)}{A(a,b)L(a,b)}\right) \leq (\geq) \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^{b} \Phi\left(\frac{t-b}{t^2}\right)dt + \int_{a}^{\frac{a+b}{2}} \Phi\left(\frac{t-a}{t^2}\right)dt\right]. \tag{5.32}$$

If we take $f(t) = -\ln t$ in (5.28), where $t \in [a,b] \subset (0,\infty)$, then we have

$$\Phi\left(\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right) \leq (\geq) \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^{b} \Phi\left(\frac{t-b}{t}\right)dt + \int_{a}^{\frac{a+b}{2}} \Phi\left(\frac{t-a}{t}\right)dt\right]. \tag{5.33}$$

If we take $f(t) = t^p$, $p \in \mathbb{R}\setminus\{0,-1\}$ in (5.28), where $t \in [a,b] \subset (0,\infty)$, then we have

$$\Phi\left(L_p^p(a,b) - A^p(a,b)\right) \leq (\geq) \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^{b} \Phi\left[p(b-t)t^{p-1}\right]dt + \int_{a}^{\frac{a+b}{2}} \Phi\left[p(a-t)t^{p-1}\right]dt\right]. \tag{5.34}$$

REFERENCES

[1] S. S. Dragomir, A refinement of Ostrowski's inequality for absolutely continuous functions and applications. *Acta Math. Vietnam.* **27** (2002), no. 2, 203–217.

[2] S. S. Dragomir, A refinement of Ostrowski's inequality for absolutely continuous functions whose derivatives belong to L_∞ and applications. *Libertas Math.* **22** (2002), 49–63.

[3] S. S. Dragomir, Improvements of Ostrowski and generalised trapezoid inequality in terms of the upper and lower bounds of the first derivative. *Tamkang J. Math.* **34** (2003), No. 3, 213–222.

[4] S. S. Dragomir, A functional generalization of Ostrowski inequality via Montgomery identity, *Acta Math. Univ. Comenian.* (N.S.) **84** (2015), no. 1, 63–78. Preprint *RGMIA Res. Rep. Coll.* **16** (2013), Art. 65, pp. 15 [Online http://rgmia.org/papers/v16/v16a65.pdf].

[5] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1 norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.

[6] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_p norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.*, **40** (1998), No. 3, 299-304.

[7] S. S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rules, *Appl. Math. Lett.,* **11** (1) (1998), 105-109.

[8] A. M. Fink, Bounds on the derivative of a function from its averages, *Czechoslovak Math. J.,* **42** (117) (1992), 289-310.

[9] D. S. Mitrinović, J. E. Pečarić and A .M. Fink, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.

[10] A. Ostrowski, Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert, *Comment. Math. Helv.,* **10** (1938), 226-227.

[11] T. C. Peachey, A. McAndrew and S. S. Dragomir, The best constant in an inequality of Ostrowski type, *Tamkang J. of Math.*, **30** (1999), No. 3, 219-222.

CHAPTER 3

Inequalities for Differentiable Functions

1. Ostrowski Via Cauchy Mean Value Theorem

1.1. A Local Result. The following theorem holds.

THEOREM 1.1 (Dragomir, 2003 [**1**]). *Let* $f:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b)*. Let* $p\in(0,\infty)$ *and assume, for a given* $x\in(a,b)$*, we have that*

$$M_p(x):=\sup_{u\in(a,b)}\left\{|x-u|^{1-p}\,|f'(u)|\right\}<\infty. \tag{1.1}$$

Then we have the inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{M_p(x)}{p(p+1)(b-a)}\left[(x-a)^{p+1}+(b-x)^{p+1}\right]. \tag{1.2}$$

PROOF. Let $x\in(a,b)$ and define the mapping $g_{1,x}:(a,x)\to\mathbb{R}$, $g_{1,x}(t)=(x-t)^p$.

Applying the Cauchy mean value theorem, for any $t\in(a,x)$ there exists a $\eta\in(t,x)$ such that

$$[f(t)-f(x)]\,g_{1,x}'(\eta)=[g_{1,x}(t)-g_{1,x}(x)]\,f'(\eta)$$

i.e.,

$$(-p)(f(t)-f(x))(x-\eta)^{p-1}=(x-t)^p f'(\eta)$$

from where we obtain

$$|f(t)-f(x)|=\frac{(x-t)^p\,|f'(\eta)|}{p(x-\eta)^{p-1}}\le\frac{(x-t)^p}{p}M_p(x)\,,\quad t\in(a,x)\,. \tag{1.3}$$

We define the mapping $g_{2,x}:(x,b)\to\mathbb{R}$, $g_{2,x}(t)=(t-x)^p$. Applying the Cauchy mean value theorem, we can find a $\xi\in(x,t)$ such that

$$[f(t)-f(x)]\,p(\xi-x)^{p-1}=(t-x)^p f'(\xi)$$

from where we get

$$|f(t)-f(x)|=\frac{(t-x)^p\,|f'(\xi)|}{p(\xi-x)^{p-1}}\le\frac{(t-x)^p}{p}M_p(x)\,,\quad t\in(x,b)\,. \tag{1.4}$$

In conclusion, by (1.3) and (1.4) we may write

$$|f(t)-f(x)|\le\frac{1}{p}M_p(x)\,|t-x|^p\quad\text{for all }\ t\in(a,b)\,. \tag{1.5}$$

Integrating (1.5) over t on $[a,b]$, we get

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{b-a}\int_a^b |f(t)-f(x)|\,dt \leq \frac{1}{p}M_p(x)\frac{1}{b-a}\int_a^b |t-x|^p\,dt \\
&= \frac{1}{p}M_p(x)\frac{1}{b-a}\left[\int_a^x (x-t)^p\,dt+\int_x^b (t-x)^p\,dt\right] \\
&= \frac{1}{p}M_p(x)\frac{(x-a)^{p+1}+(b-x)^{p+1}}{(p+1)(b-a)},
\end{aligned}$$

and the inequality (1.2) is proved. ∎

REMARK 1.1. For $p=1$, we obtain

$$\begin{aligned}
\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| &\leq \frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f'\|_\infty \\
&= \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty (b-a), \ x\in[a,b],
\end{aligned}$$

where $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)| < \infty$, which is Ostrowski's inequality [**5**]. It is obvious that for $p>1$, the accuracy order provided by (1.2) is higher than 1, as provided by the classical Ostrowski's inequality.

REMARK 1.2. If $p\in(0,1)$ and $f'\in L_\infty[a,b]$, then obviously

$$M_p(x)\leq[\max\{x-a,b-x\}]^{1-p}\|f'\|_\infty = \left[\frac{a+b}{2}+\left|x-\frac{a+b}{2}\right|\right]^{1-p}\|f'\|_\infty$$

for all $x\in[a,b]$.

The following midpoint formula holds.

COROLLARY 1.2. *Let f and p be as in Theorem 1.1. Assume that*

$$M_p\left(\frac{a+b}{2}\right) := \sup_{u\in(a,b)}\left\{\left|\frac{a+b}{2}-u\right|^{1-p}|f'(u)|\right\} < \infty.$$

Then we have the midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^p}{p(p+1)2^p}M_p\left(\frac{a+b}{2}\right). \tag{1.6}$$

1.2. Some Global Results. Before we continue our presentation, we recall the following special means:

(a) The *arithmetic mean*

$$A=A(a,b):=\frac{a+b}{2}, \quad a,b\geq 0;$$

(b) The *geometric mean*

$$G=G(a,b):=\sqrt{ab}, \quad a,b\geq 0;$$

(c) The *harmonic mean*

$$H = H(a,b) := \frac{2ab}{a+b}, \quad a,b>0;$$

(d) The *logarithmic mean*

$$L = L(a,b) := \begin{cases} a & \text{if} \quad a=b, \\ \\ \frac{b-a}{\ln b - \ln a} & \text{if} \quad a \neq b, \end{cases} \quad a,b>0;$$

(e) The *identric mean*

$$I = I(a,b) := \begin{cases} a & \text{if} \quad a=b, \\ \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a \neq b, \end{cases} \quad a,b>0;$$

(f) The p-*logarithmic mean*

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a \neq b, \\ \\ a & \text{if} \quad a=b, \end{cases} \quad a,b>0;$$

where $p \in \mathbb{R}\backslash\{-1,0\}$.

The following result also holds.

THEOREM 1.3 (Dragomir, 2003 [**1**]). *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ with $a>0$, and differentiable on (a,b). Let $p\in\mathbb{R}\diagdown\{0\}$ and assume that*

$$K_p(f') := \sup_{u\in(a,b)}\left\{u^{1-p}\left|f'(u)\right|\right\} < \infty. \tag{1.7}$$

Then we have the inequality:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{K_p(f')}{|p|(b-a)} \tag{1.8}$$
$$\times \begin{cases} 2x^p(x-A) + (b-x)L_p^p(b,x) - (x-a)L_p^p(x,a) & \textit{if} \quad p\in(0,\infty); \\ \\ (x-a)L_p^p(x,a) - (b-x)L_p^p(b,x) - 2x^p(x-A) & \textit{if} \quad \begin{array}{l} p\in(-\infty,-1) \\ \cup(-1,0); \end{array} \\ \\ (x-a)L^{-1}(a,x) - (b-x)L^{-1}(b,x) - \frac{2}{x}(x-A) & \textit{if} \quad p=-1, \end{cases}$$

for all $x\in[a,b]$.

PROOF. Consider the mapping $g:[a,b]\to\mathbb{R}$, $g(x)=x^p$. Applying the Cauchy mean value theorem, then for any x and $t\in[a,b]$, there exists a η between x and t such that

$$[f(t)-f(x)]\,g'(\eta) = [g(t)-g(x)]\,f'(\eta)$$

i.e.,

$$(f(t)-f(x))\,p\eta^{p-1} = (t^p - x^p)\,f'(\eta)$$

from where we obtain:

$$|f(t)-f(x)| = \frac{|f'(\eta)|\,|t^p-x^p|}{|p|\,\eta^{p-1}} \leq \frac{K_p(f')}{|p|}\,|t^p-x^p|\,.$$

In conclusion, for any $t, x \in [a,b]$, we have the inequality

$$|f(t) - f(x)| \leq \frac{K_p(f')}{|p|} |t^p - x^p|. \tag{1.9}$$

Integrating (1.9) over t on $[a,b]$, we get

$$\begin{aligned}\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| &\leq \frac{1}{b-a}\int_a^b |f(t) - f(x)|\,dt \\ &\leq \frac{K_p(f')}{p}\frac{1}{b-a}\int_a^b |t^p - x^p|\,dt.\end{aligned}$$

For $p > 0$, we have

$$\begin{aligned}\int_a^b |t^p - x^p|\,dt &= \int_a^x (x^p - t^p)\,dt + \int_x^b (t^p - x^p)\,dt \\ &= 2x^p(x - A) + (b - x) L_p^p(b,x) - (x - a) L_p^p(x,a).\end{aligned}$$

For $p \in (-\infty, -1) \cup (-1, 0)$, we have

$$\begin{aligned}\int_a^b |x^p - t^p|\,dt &= \int_a^x (t^p - x^p)\,dt + \int_x^b (x^p - t^p)\,dt \\ &= (x - a) L_p^p(x,a) - (b - x) L_p^p(b,x) - 2x^p(x - A)\end{aligned}$$

and, finally, for $p = -1$, we have

$$\begin{aligned}\int_a^b \left|\frac{1}{x} - \frac{1}{t}\right| dt &= \int_a^x \left(\frac{1}{t} - \frac{1}{x}\right) dt + \int_x^b \left(\frac{1}{x} - \frac{1}{t}\right) dt \\ &= (x - a) L^{-1}(a,x) - (b - x) L^{-1}(b,x) - \frac{2}{x}(x - A)\end{aligned}$$

and the theorem is proved. ∎

The following corollary is natural.

COROLLARY 1.4. *With the assumptions in Theorem 1.3, we have the midpoint inequality*

$$\begin{aligned}\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| &\leq \frac{K_p(f')}{|p|} \\ &\times \begin{cases} \frac{1}{2}\left(L_p^p(b,A) - L_p^p(A,a)\right) & \text{if } \ p > 0; \\ \frac{1}{2}\left(L_p^p(A,a) - L_p^p(A,b)\right) & \text{if } \ p \in (-\infty,-1) \cup (-1,0); \\ \frac{1}{2}\left(L^{-1}(a,A) - L^{-1}(A,b)\right) & \text{if } \ p = -1. \end{cases}\end{aligned} \tag{1.10}$$

The following theorem also holds.

THEOREM 1.5 (Dragomir, 2003 [**1**]). *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ (with $a > 0$) and differentiable on (a,b). If*

$$P(f') := \sup_{u \in (a,b)} |u f'(u)| < \infty \tag{1.11}$$

then we have the inequality

$$\begin{aligned}(1.12)\qquad & \left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ & \le \frac{P(f')}{b-a}\left[\ln\left[\frac{[I(x,b)]^{b-x}}{[I(a,x)]^{x-a}}\right]+2(x-A)\ln x\right]\end{aligned}$$

for all $x\in[a,b]$.

PROOF. Consider the mapping $g:[a,b]\to\mathbb{R}$, $g(t)=\ln t$. Applying the Cauchy mean value theorem for any x and $t\in[a,b]$, then there exists a η between x and t such that

$$(f(t)-f(x))\,g'(\eta)=(g(t)-g(x))\,f'(\eta)$$

i.e.,

$$(f(t)-f(x))\frac{1}{\eta}=(\ln t-\ln x)\,f'(\eta)$$

from where we get

$$|f(t)-f(x)|=|\eta f'(\eta)|\,|\ln t-\ln x|\le P(f')\,|\ln t-\ln x|\,.$$

In conclusion, for any $t,x\in[a,b]$, we have the inequality

$$|f(t)-f(x)|\le P(f')\,|\ln t-\ln x|\,. \tag{1.13}$$

Integrating (1.13) over t on $[a,b]$, we get

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ &\le \frac{1}{b-a}\int_a^b |f(t)-f(x)|\,dt\le P(f')\frac{1}{b-a}\int_a^b|\ln t-\ln x|\,dt\\ &=\frac{P(f')}{b-a}\left[\int_a^x(\ln x-\ln t)\,dt+\int_x^b(\ln t-\ln x)\,dt\right]\\ &=\frac{P(f')}{b-a}\left[(x-a)\ln x-(x-a)\ln I(a,x)+(b-x)\ln I(b,x)-(b-x)\ln x\right]\\ &=\frac{P(f')}{b-a}\left[2(x-A)\ln x+(b-x)\ln I(x,b)-(x-a)\ln I(a,x)\right]\end{aligned}$$

and the theorem is proved. ∎

The following corollary is natural.

COROLLARY 1.6. *With the assumptions of Theorem 1.5, we have the inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{2}P(f')\ln\left[\frac{I(A,b)}{I(a,A)}\right], \tag{1.14}$$

where $A=A(a,b)=\frac{a+b}{2}$.

2. Ostrowski Via General Cauchy Mean Value Theorem

2.1. A General Result.

We may state the following theorem.

THEOREM 2.1 (Dragomir, 2005 [**2**]). *Let* $f, g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b). *If* $g'(t) \neq 0$ *for each* $t \in (a,b)$ *and*

$$\left\| \frac{f'}{g'} \right\|_{\infty} := \sup_{t \in (a,b)} \left| \frac{f'(t)}{g'(t)} \right| < \infty, \tag{2.1}$$

then for any $x \in [a,b]$ *one has the inequality*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \\ &\leq \left| 2\left(\frac{x - \frac{a+b}{2}}{b-a} \right) g(x) + \frac{\int_x^b g(t)\,dt - \int_a^x g(t)\,dt}{b-a} \right| \cdot \left\| \frac{f'}{g'} \right\|_{\infty}. \end{aligned} \tag{2.2}$$

PROOF. Let $x, t \in [a,b]$ with $t \neq x$. Applying Cauchy's mean value theorem, there exists a η between t and x such that

$$(f(x) - f(t)) = \frac{f'(\eta)}{g'(\eta)} (g(x) - g(t))$$

from where we get

$$|f(x) - f(t)| = \left| \frac{f'(\eta)}{g'(\eta)} \right| |g(x) - g(t)| \leq \left\| \frac{f'}{g'} \right\|_{\infty} |g(x) - g(t)|, \tag{2.3}$$

for any $t, x \in [a,b]$.

Using the properties of the integral, we deduce by (2.3), that

$$\begin{aligned} \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| &\leq \frac{1}{b-a} \int_a^b |f(x) - f(t)|\,dt \\ &\leq \left\| \frac{f'}{g'} \right\|_{\infty} \frac{1}{b-a} \int_a^b |g(x) - g(t)|\,dt. \end{aligned} \tag{2.4}$$

Since $g'(t) \neq 0$ on (a,b), it follows that either $g'(t) > 0$ or $g'(t) < 0$ for any $t \in (a,b)$.

If $g'(t) > 0$ for all $t \in (a,b)$, then g is strictly monotonic increasing on (a,b) and

$$\begin{aligned} \int_a^b |g(x) - g(t)|\,dt &= \int_a^x (g(x) - g(t))\,dt + \int_x^b (g(t) - g(x))\,dt \\ &= 2\left(x - \frac{a+b}{2} \right) g(x) + \int_x^b g(t)\,dt - \int_a^x g(t)\,dt. \end{aligned}$$

If $g'(t) < 0$ for all $t \in (a,b)$, then

$$\int_a^b |g(x) - g(t)|\,dt = -\left[2\left(x - \frac{a+b}{2} \right) g(x) + \int_x^b g(t)\,dt - \int_a^x g(t)\,dt \right]$$

and the inequality (2.2) is proved. ∎

The following midpoint inequality is a natural consequence of the above result.

COROLLARY 2.2. *With the above assumptions for f and g, one has the inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{b-a}\left|\int_{\frac{a+b}{2}}^b g(t)\,dt-\int_a^{\frac{a+b}{2}} g(t)\,dt\right|\left\|\frac{f'}{g'}\right\|_\infty . \tag{2.5}$$

REMARK 2.1. (1)
(2) If in the above theorem, we choose $g(t)=t$, then from (2.2) we recapture Ostrowski's inequality [**5**].
(3) If in Theorem 2.1 we choose $g(t)=t^p$, $p\in\mathbb{R}\backslash\{0\}$, or $g(t)=\ln t$ with $t\in(a,b)\subset(0,\infty)$, then we obtain the results from [**1**].

One may obtain many inequalities from Theorem 2.1 on choosing different instances of functions g.

PROPOSITION 2.3 (Dragomir, 2005 [**2**]). *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If there exists a constant $\Gamma<\infty$ such that*

$$|f'(t)|\le \Gamma e^{-t} \text{ for any } t\in(a,b), \tag{2.6}$$

then one has the inequality:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \Gamma\left[2\left(\frac{x-A}{b-a}\right)e^x+\frac{(b-x)E(x,b)-(x-a)E(a,x)}{b-a}\right] \tag{2.7}$$

for any $x\in(a,b)$, where $A=A(a,b)=\frac{a+b}{2}$ and E is the exponential men, i.e.,

$$E(x,y):=\begin{cases}\dfrac{e^x-e^y}{x-y} & \text{if } x\neq y\\ e^y & \text{if } x=y\end{cases},\ x,y\in\mathbb{R}.$$

In particular, we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{2}\left[E(A,b)-E(a,A)\right]\Gamma. \tag{2.8}$$

The proof is obvious by Theorem 2.1 on choosing $g(t)=e^t$ and we omit the details.
Another example is considered in the following proposition.

PROPOSITION 2.4 (Dragomir, 2005 [**2**]). *Let $f:[a,b]\subset\left(0,\frac{\pi}{2}\right)\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b).*

(i) *If there exists a constant $\Gamma_1<\infty$ such that*

$$|f'(t)|\le\Gamma_1\cos t,\ t\in(a,b), \tag{2.9}$$

then one has the inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \Gamma_1\left[2\left(\frac{x-A}{b-a}\right)\sin x+\frac{(x-a)C(a,x)-(b-x)C(x,b)}{b-a}\right] \tag{2.10}$$

for any $x \in (a,b)$, *where* C *is the cos-mean value, i.e.,*

$$C(x,y) := \begin{cases} \dfrac{\cos x - \cos y}{x-y} & \text{if } x \neq y \\ \\ -\sin y & \text{if } x = y \end{cases}.$$

In particular we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{2}\left[C(a,A) - C(A,b)\right]\Gamma_1. \tag{2.11}$$

(ii) *If there exists a constant* $\Gamma_2 < \infty$ *such that*

$$|f'(t)| \leq \Gamma_1 \sin t,\ t \in (a,b), \tag{2.12}$$

then one has the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \Gamma_2\left[2\left(\frac{x-A}{b-a}\right)\cos x + \frac{(b-x)\,S(x,b) - (x-a)\,S(a,x)}{b-a}\right],$$

for any $x \in (a,b)$, *where* S *is the sin-mean value, i.e.,*

$$S(x,y) := \begin{cases} \dfrac{\sin x - \sin y}{x-y} & \text{if } x \neq y \\ \\ \cos y & \text{if } x = y \end{cases}.$$

In particular, we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{2}\left[S(A,b) - S(a,A)\right]\Gamma_2. \tag{2.13}$$

2.2. A Result on Sub-intervals.

The following result also holds.

THEOREM 2.5 (Dragomir, 2005 [**2**]). *Let* $f, g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* $(a,b) \setminus \{x\}$, $x \in (a,b)$. *If* $g'(t) \neq 0$ *for* $t \in (a,x) \cup (x,b)$, *then we have the inequality*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\leq \frac{1}{b-a}\left| g(x)(x-a) - \int_a^x g(t)\,dt \right| \cdot \left\| \frac{f'}{g'} \right\|_{(a,x),\infty} \\ &+ \frac{1}{b-a}\left| g(x)(b-x) - \int_x^b g(t)\,dt \right| \cdot \left\| \frac{f'}{g'} \right\|_{(x,b),\infty}. \end{aligned} \tag{2.14}$$

PROOF. We obviously have:

$$\begin{aligned}
(2.15) \qquad & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&= \left| \frac{1}{b-a}\int_a^b (f(x) - f(t))\,dt \right| \\
&\le \frac{1}{b-a}\int_a^b |f(x) - f(t)|\,dt \\
&= \frac{1}{b-a}\left[\int_a^x |f(x) - f(t)|\,dt + \int_x^b |f(x) - f(t)|\,dt\right].
\end{aligned}$$

Applying Cauchy's mean value theorem on the interval (a, x), we deduce (see the proof of Theorem 2.1) that

$$(2.16) \qquad |f(x) - f(t)| \le \left\| \frac{f'}{g'} \right\|_{(a,x),\infty} |g(x) - g(t)|$$

for any $t \in (a, x)$, and, similarly

$$(2.17) \qquad |f(x) - f(t)| \le \left\| \frac{f'}{g'} \right\|_{(x,b),\infty} |g(x) - g(t)|$$

for any $t \in (x, b)$.

Consequently

$$\int_a^x |f(x) - f(t)|\,dt \le \left\| \frac{f'}{g'} \right\|_{(a,x),\infty} \int_a^x |g(x) - g(t)|\,dt$$

and

$$\int_x^b |f(x) - f(t)|\,dt \le \left\| \frac{f'}{g'} \right\|_{(x,b),\infty} \int_x^b |g(x) - g(t)|\,dt.$$

Since g' has a constant sign in either (a, x) or (x, b), it follows that g is strictly increasing or strictly decreasing in (a, x) and (x, b).

Thus

$$\begin{aligned}
&\int_a^x |g(x) - g(t)|\,dt \\
&= \begin{cases} g(x)(x-a) - \int_a^x g(t)\,dt & \text{if } g \text{ is increasing on } [a,x] \\ \int_a^x g(t)\,dt - g(x)(x-a) & \text{if } g \text{ is decreasing} \end{cases} \\
&= \left| g(x)(x-a) - \int_a^x g(t)\,dt \right|
\end{aligned}$$

and, in a similar way

$$\int_x^b |g(x) - g(t)|\,dt = \left| g(x)(b-x) - \int_x^b g(t)\,dt \right|.$$

Consequently, by the use of (2.15), we deduce the desired inequality (2.14). ∎

The following particular case may be of interest.

COROLLARY 2.6. *Let* $f, g : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* $(a, b) \setminus \left\{\frac{a+b}{2}\right\}$. *If* $g'(t) \neq 0$ *on* $\left(a, \frac{a+b}{2}\right) \cup \left(\frac{a+b}{2}, b\right)$, *then we have the inequality*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\quad \leq \frac{1}{2} \left\{ \left| g\left(\frac{a+b}{2}\right) - \frac{2}{b-a} \int_a^{\frac{a+b}{2}} g(t)\, dt \right| \cdot \left\| \frac{f'}{g'} \right\|_{\left(a, \frac{a+b}{2}\right), \infty} \right. \\ &\qquad \left. + \left| g\left(\frac{a+b}{2}\right) - \frac{2}{b-a} \int_{\frac{a+b}{2}}^b g(t)\, dt \right| \cdot \left\| \frac{f'}{g'} \right\|_{\left(\frac{a+b}{2}, b\right), \infty} \right\}. \end{aligned} \tag{2.18}$$

The following result also holds.

PROPOSITION 2.7 (Dragomir, 2005 [**2**]). *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* $(a, b) \setminus \{x\}$, $x \in (a, b)$. *Assume that, for* $p > 0$, *we have*

$$|f'(t)| \leq \begin{cases} M_{1,p}(x)(x-t)^{1-p} & \text{for any } t \in (a, x), \\ \\ M_{2,p}(x)(t-x)^{1-p} & \text{for any } t \in (x, b). \end{cases} \tag{2.19}$$

where $M_{1,p}(x)$ *and* $M_{2,p}(x)$ *are positive constants depending on* x. *Then we have the inequality*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\leq \frac{1}{p(p+1)} (b-a) \left[M_{1,p}(x)(x-a)^{p+1} + M_{2,p}(x)(b-x)^{p+1} \right]. \end{aligned} \tag{2.20}$$

The proof follows by Theorem 2.5 applied for $g(x) = |x-t|^p$, $p > 0$. We omit the details.

REMARK 2.2. If f is as in Proposition 2.7 and

$$|f'(t)| \leq \begin{cases} M_1\left(\frac{a+b}{2}\right)\left(\frac{a+b}{2} - t\right)^{1-p} & \text{for any } t \in \left(a, \frac{a+b}{2}\right), \\ \\ M_2\left(\frac{a+b}{2}\right)\left(t - \frac{a+b}{2}\right)^{1-p} & \text{for any } t \in \left(\frac{a+b}{2}, b\right), \end{cases} \tag{2.21}$$

then, by (2.20), we get

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\leq \frac{(b-a)^{p+1}}{2^{p+1} p(p+1)} \left[M_1\left(\frac{a+b}{2}\right) + M_2\left(\frac{a+b}{2}\right) \right]. \end{aligned} \tag{2.22}$$

REMARK 2.3. If f is as in Proposition 2.7 and

$$|f'(t)| \leq M_p(x)|x-t|^{1-p} \ \ t \in (a, b),$$

then, by (2.20), we get

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\leq \frac{1}{p(p+1)(b-a)} \left[(x-a)^{p+1} + (b-x)^{p+1} \right] M_p(x), \end{aligned} \tag{2.23}$$

which is the result obtained in [**1**].

2.3. Some Inequalities of Midpoint Type.

(1) Let $0 < a < b$. Consider the function $g : [a,b] \to \mathbb{R}$, $g(t) = t^p$, $t \in \mathbb{R} \backslash \{0,-1\}$. Then $g'(t) = pt^{p-1}$, $g\left(\frac{a+b}{2}\right) = A^p(a,b)$,

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}} g(t)\,dt = L_p^p(a,A),$$
$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} g(t)\,dt = L_p^p(A,b),$$

and by Corollary 2.6, we may state the following proposition.

PROPOSITION 2.8. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b) \setminus \left\{\frac{a+b}{2}\right\}$. If*

$$|f'(t)| \le \begin{cases} M_1\left(\frac{a+b}{2}\right) t^p, & t \in \left(a, \frac{a+b}{2}\right), \\ M_2\left(\frac{a+b}{2}\right) t^p, & t \in \left(\frac{a+b}{2}, b\right), \end{cases} \tag{2.24}$$

then we have the inequality

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\qquad \le \frac{1}{2p}\left\{ M_1\left(\frac{a+b}{2}\right)\left|A^p(a,b) - L_p^p(a,A)\right| \right. \\ &\qquad\qquad \left. + M_2\left(\frac{a+b}{2}\right)\left|L_p^p(A,b) - A^p(a,b)\right| \right\}. \end{aligned} \tag{2.25}$$

The particular case $p = 1$ is of interest and so we may state the following corollary.

COROLLARY 2.9. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b) \setminus \left\{\frac{a+b}{2}\right\}$. If*

$$|f'(t)| \le \begin{cases} N_1\left(\frac{a+b}{2}\right) t, & t \in \left(a, \frac{a+b}{2}\right), \\ N_2\left(\frac{a+b}{2}\right) t, & t \in \left(\frac{a+b}{2}, b\right), \end{cases} \tag{2.26}$$

then we have the inequality:

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\le \frac{1}{8}\left[N_1\left(\frac{a+b}{2}\right) + N_2\left(\frac{a+b}{2}\right)\right](b-a). \end{aligned} \tag{2.27}$$

(2) Let $0 < a < b$. Consider the function $g : [a,b] \to \mathbb{R}$, $g(t) = \frac{1}{t}$. Then $g'(t) = -\frac{1}{t^2}$, $g\left(\frac{a+b}{2}\right) = A^{-1}(a,b)$,

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}} g(t)\,dt = L^{-1}(a,A),$$
$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} g(t)\,dt = L^{-1}(A,b),$$

and by Corollary 2.6 we may state the following Proposition.

PROPOSITION 2.10. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b) \setminus \left\{\frac{a+b}{2}\right\}$. If*

$$|f'(t)| \le \begin{cases} M_1\left(\frac{a+b}{2}\right)t^{-2}, & t \in \left(a, \frac{a+b}{2}\right), \\ M_2\left(\frac{a+b}{2}\right)t^{-2}, & t \in \left(\frac{a+b}{2}, b\right), \end{cases} \tag{2.28}$$

then we have the inequality:

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| &\le \frac{1}{2}\left[M_1\left(\frac{a+b}{2}\right)\cdot \frac{[A - L(a,A)]}{L(a,A)\,A} \right. \\ &\qquad \left. + M_2\left(\frac{a+b}{2}\right)\cdot\frac{[L(A,b)-A]}{L(A,b)\,A}\right]. \end{aligned} \tag{2.29}$$

(3) Let $0 < a < b$. Consider the function $g : [a,b] \to \mathbb{R}$, $g(t) = \ln t$. Then $g'(t) = \frac{1}{t}$, $g\left(\frac{a+b}{2}\right) = \ln A$,

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}} g(t)\,dt = \ln I(a,A),$$
$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} g(t)\,dt = \ln I(A,b),$$

and by Corollary 2.6 we may state the following proposition.

PROPOSITION 2.11. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b) \setminus \left\{\frac{a+b}{2}\right\}$. If*

$$|f'(t)| \le \begin{cases} M_1\left(\frac{a+b}{2}\right)t^{-1}, & t \in \left(a, \frac{a+b}{2}\right), \\ M_2\left(\frac{a+b}{2}\right)t^{-1}, & t \in \left(\frac{a+b}{2}, b\right), \end{cases} \tag{2.30}$$

then we have the inequality:

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\le \ln\left\{ G\left(\left[\frac{A}{I(a,A)}\right]^{M_1\left(\frac{a+b}{2}\right)}, \left[\frac{I(A,b)}{A}\right]^{M_2\left(\frac{a+b}{2}\right)} \right) \right\}. \end{aligned} \tag{2.31}$$

3. OSTROWSKI VIA POMPEIU MEAN VALUE THEOREM

3.1. Pompeiu Mean Value Theorem. In 1946, Pompeiu [**6**] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem* (see also [**7**, p. 83]).

LEMMA 3.1 (Pompeiu, 1946 [**6**]). *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{3.1}$$

PROOF. Define a real valued function F on the interval $\left[\frac{1}{b},\frac{1}{a}\right]$ by

$$F(t)=tf\left(\frac{1}{t}\right). \tag{3.2}$$

Since f is differentiable on $\left(\frac{1}{b},\frac{1}{a}\right)$ and

$$F'(t)=f\left(\frac{1}{t}\right)-\frac{1}{t}f'\left(\frac{1}{t}\right), \tag{3.3}$$

then applying the mean value theorem to F on the interval $[x,y]\subset\left[\frac{1}{b},\frac{1}{a}\right]$ we get

$$\frac{F(x)-F(y)}{x-y}=F'(\eta) \tag{3.4}$$

for some $\eta\in(x,y)$.

Let $x_2=\frac{1}{x}$, $x_1=\frac{1}{y}$ and $\xi=\frac{1}{\eta}$. Then, since $\eta\in(x,y)$, we have

$$x_1<\xi<x_2.$$

Now, using (3.2) and (3.3) on (3.4), we have

$$\frac{xf\left(\frac{1}{x}\right)-yf\left(\frac{1}{y}\right)}{x-y}=f\left(\frac{1}{\eta}\right)-\frac{1}{\eta}f'\left(\frac{1}{\eta}\right),$$

that is

$$\frac{x_1f(x_2)-x_2f(x_1)}{x_1-x_2}=f(\xi)-\xi f'(\xi).$$

This completes the proof of the theorem. ∎

REMARK 3.1. Following [**7**, p. 84 – 85], we will mention here a geometrical interpretation of Pompeiu's theorem. The equation of the secant line joining the points $(x_1,f(x_1))$ and $(x_2,f(x_2))$ is given by

$$y=f(x_1)+\frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1).$$

This line intersects the $y-$axis at the point $(0,y)$, where y is

$$\begin{aligned}y&=f(x_1)+\frac{f(x_2)-f(x_1)}{x_2-x_1}(0-x_1)\\&=\frac{x_1f(x_2)-x_2f(x_1)}{x_1-x_2}.\end{aligned}$$

The equation of the tangent line at the point $(\xi,f(\xi))$ is

$$y=(x-\xi)f'(\xi)+f(\xi).$$

The tangent line intersects the $y-$axis at the point $(0,y)$, where

$$y=-\xi f'(\xi)+f(\xi).$$

Hence, the geometric meaning of Pompeiu's mean value theorem is that the tangent of the point $(\xi,f(\xi))$ intersects on the $y-$axis at the same point as the secant line connecting the points $(x_1,f(x_1))$ and $(x_2,f(x_2))$.

3.2. Evaluating the Integral Mean. The following result holds.

THEOREM 3.2 (Dragomir, 2005 [**3**]). *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $[a,b]$ not containing 0. Then for any $x\in[a,b]$, we have the inequality*

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{|x|}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f-\ell f'\|_\infty, \tag{3.5}$$

where $\ell(t)=t$, $t\in[a,b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

PROOF. Applying Pompeiu's mean value theorem, for any $x,t\in[a,b]$, there is a ξ between x and t such that

$$tf(x)-xf(t)=[f(\xi)-\xi f'(\xi)](t-x)$$

giving

$$|tf(x)-xf(t)| \leq \sup_{\xi\in[a,b]}|f(\xi)-\xi f'(\xi)|\,|x-t| = \|f-\ell f'\|_\infty|x-t| \tag{3.6}$$

for any $t,x\in[a,b]$.

Integrating over $t\in[a,b]$, we get

$$\begin{aligned}\left|f(x)\int_a^b t\,dt-x\int_a^b f(t)\,dt\right| &\leq \int_a^b|tf(x)-xf(t)|\,dt \\ &\leq \|f-\ell f'\|_\infty\int_a^b|x-t|\,dt \\ &= \|f-\ell f'\|_\infty\left[\frac{(x-a)^2+(b-x)^2}{2}\right] \\ &= \|f-\ell f'\|_\infty\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\end{aligned} \tag{3.7}$$

and since $\int_a^b t\,dt=\frac{b^2-a^2}{2}$, we deduce from (3.7) the desired result (3.5).

Now, assume that (3.6) holds with a constant $k>0$, i.e.,

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{|x|}\left[k+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f-\ell f'\|_\infty, \tag{3.8}$$

for any $x\in[a,b]$.

Consider $f:[a,b]\to\mathbb{R}$, $f(t)=\alpha t+\beta;\ \alpha,\beta\neq 0$. Then

$$\begin{aligned}\left\|f-\ell f'\right\|_{\infty} &= |\beta|,\\ \frac{1}{b-a}\int_a^b f(t)\,dt &= \frac{a+b}{2}\cdot\alpha+\beta,\end{aligned}$$

and by (3.8) we deduce

$$\left|\frac{a+b}{2}\left(\alpha+\frac{\beta}{x}\right)-\left(\frac{a+b}{2}\alpha+\beta\right)\right|\leq\frac{b-a}{|x|}\left[k+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]|\beta|$$

giving

$$\left|\frac{a+b}{2}-x\right|\leq(b-a)\,k+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2 \tag{3.9}$$

for any $x\in[a,b]$.

If in (3.9) we let $x=a$ or $x=b$, we deduce $k\geq\frac{1}{4}$, and the sharpness of the constant is proved. ∎

The following interesting particular case holds.

COROLLARY 3.3. *With the assumptions in Theorem 3.2, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{b-a}{2|a+b|}\left\|f-\ell f'\right\|_{\infty}. \tag{3.10}$$

4. ANOTHER OSTROWSKI TYPE INEQUALITY VIA POMPEIU'S RESULT

4.1. Evaluating the Integral Mean.

The following new result holds.

THEOREM 4.1 (Dragomir, 2013 [**4**]). *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $b>a>0$. Then for any $x\in[a,b]$, we have the inequality*

$$\begin{aligned}&\left|\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt\right|\\ &\leq\frac{2}{b-a}\left\|f-\ell f'\right\|_{\infty}\left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right),\end{aligned} \tag{4.1}$$

where $\ell(t)=t$, $t\in[a,b]$.

The constant 2 is best possible in (4.1).

PROOF. Applying Pompeiu's mean value theorem [**6**] (see also [**7**, p. 83]), for any $x,t\in[a,b]$, there is a ξ between x and t such that

$$tf(x)-xf(t)=\left[f(\xi)-\xi f'(\xi)\right](t-x)$$

giving

$$|tf(x)-xf(t)|\leq\sup_{\xi\in[a,b]}\left|f(\xi)-\xi f'(\xi)\right||x-t|=\left\|f-\ell f'\right\|_{\infty}|x-t|$$

for any $t,x\in[a,b]$, or, by dividing with $x,t>0$, equivalently to

$$\left|\frac{f(x)}{x}-\frac{f(t)}{t}\right|\leq\left\|f-\ell f'\right\|_{\infty}\left|\frac{1}{x}-\frac{1}{t}\right| \tag{4.2}$$

for any $t,x\in[a,b]$.

Integrating over $t \in [a,b]$, we get

$$\left| \frac{f(x)}{x}(b-a) - \int_a^b \frac{f(t)}{t} dt \right| \leq \int_a^b \left| \frac{f(x)}{x} - \frac{f(t)}{t} \right| dt \tag{4.3}$$
$$\leq \|f - \ell f'\|_\infty \int_a^b \left| \frac{1}{x} - \frac{1}{t} \right| dt$$

and since

$$\int_a^b \left| \frac{1}{x} - \frac{1}{t} \right| dt = \left[\int_a^x \left(\frac{1}{t} - \frac{1}{x} \right) dt + \int_x^b \left(\frac{1}{x} - \frac{1}{t} \right) dt \right]$$
$$= \left(\ln \frac{x}{a} - \frac{x-a}{x} + \frac{b-x}{x} - \ln \frac{b}{x} \right)$$
$$= \left(\ln \frac{x^2}{ab} + \frac{a+b-2x}{x} \right)$$

for any $x \in [a,b]$, then we deduce from (4.3) the desired result (4.1).

Now, assume that (4.1) holds with a constant $k > 0$, i.e.,

$$\left| \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b \frac{f(t)}{t} dt \right| \tag{4.4}$$
$$\leq \frac{k}{b-a} \|f - \ell f'\|_\infty \left(\ln \frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2} - x}{x} \right),$$

for any $x \in [a,b]$.

Consider $f : [a,b] \to \mathbb{R}$, $f(t) = 1$. Then

$$\|f - \ell f'\|_\infty = 1, \frac{1}{b-a} \int_a^b \frac{f(t)}{t} dt = \frac{1}{b-a} \ln \frac{b}{a},$$

and by (4.4) we deduce

$$\left| \frac{1}{x} - \frac{1}{b-a} \ln \frac{b}{a} \right| \leq \frac{k}{b-a} \left(\ln \frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2} - x}{x} \right)$$

for any $x \in [a,b]$.

If we take in this inequality $x = a$, we get

$$\left| \frac{1}{a} - \frac{1}{b-a} \ln \frac{b}{a} \right| \leq \frac{k}{b-a} \left(\ln \frac{a}{\sqrt{ab}} + \frac{b-a}{2a} \right) \tag{4.5}$$
$$= \frac{k}{2(b-a)} \left(\ln \frac{a^2}{ab} + \frac{b-a}{a} \right)$$
$$= \frac{k}{2(b-a)} \left(\ln \frac{a}{b} + \frac{b-a}{a} \right).$$

In we multiply (4.5) with $2(b-a)$ we get

$$2 \left| \frac{b-a}{a} - \ln \frac{b}{a} \right| \leq k \left(\frac{b-a}{a} - \ln \frac{b}{a} \right)$$

which implies that $k \geq 2$. ∎

The following interesting particular case holds.

COROLLARY 4.2. *With the assumptions in Theorem 4.1, we have*

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}}-\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt\right|\le\frac{2}{b-a}\left\|f-\ell f'\right\|_\infty\ln\left(\frac{\frac{a+b}{2}}{\sqrt{ab}}\right). \tag{4.6}$$

REMARK 4.1. If we consider the function $\psi:[a,b]\to\mathbb{R}$ given by

$$\psi(x):=\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x},$$

then we observe that

$$\psi'(x)=\frac{x-\frac{a+b}{2}}{x^2},$$

which shows that

$$\inf_{x\in[a,b]}\psi(x)=\psi\left(\frac{a+b}{2}\right)=\ln\left(\frac{\frac{a+b}{2}}{\sqrt{ab}}\right),$$

meaning that the inequality (4.6) is the best possible one can get from (4.1).

REMARK 4.2. We can state from (4.1) the following inequality as well:

$$\left|\frac{f\left(\sqrt{ab}\right)}{\sqrt{ab}}-\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt\right|\le\frac{2}{b-a}\left\|f-\ell f'\right\|_\infty\left(\frac{\frac{a+b}{2}-\sqrt{ab}}{\sqrt{ab}}\right). \tag{4.7}$$

REFERENCES

[1] S. S. Dragomir, Some new inequalities of Ostrowski type, *Austral. Math. Soc. Gaz.* **30** (2003), No. 1, 25–30. Preprint: *RGMIA Res. Rep. Coll.*, **5** (2002), Supplement, Article 11, [ONLINE:http://rgmia.org/papers/v5e/IOT.pdf].

[2] S. S. Dragomir, Some Ostrowski type inequalities via Cauchy's mean value theorem. *New Zealand J. Math.* **34** (2005), No. 1, 31–42.

[3] S. S. Dragomir, An inequality of Ostrowski type via Pompeiu's mean value theorem. *J. Inequal. Pure Appl. Math.* **6** (2005), No. 3, Article 83, 9 pp. [Online http://www.emis.de/journals/JIPAM/article556.html?sid=556].

[4] S. S. Dragomir, Another Ostrowski type inequality via Pompeiu's mean value theorem, *General Mathematics* (Sibiu, Romania), **21** (2013), No. 2, 3-16. Preprint *RGMIA Res. Rep. Coll.* **16** (2013), Art. 64, pp. 9 [Online http://rgmia.org/papers/v16/v16a64.pdf].

[5] A. Ostrowski, Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel*, **10** (1938), 226-227.

[6] D. Pompeiu, Sur une proposition analogue au théorème des accroissements finis, *Mathematica* (Cluj, Romania), **22** (1946), 143-146.

[7] P. K. Sahoo and T. Riedel, *Mean Value Theorems and Functional Equations*, World Scientific, Singapore, New Jersey, London, Hong Kong, 2000.

CHAPTER 4

Inequalities of Pompeiu Type

1. Ostrowski Via a Generalized Pompeiu's Inequality

1.1. Pompeiu's Inequality for p-Norms. In 1946, Pompeiu [**6**] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem* (see also [**8**, p. 83]).

LEMMA 1.1 (Pompeiu, 1946 [**6**]). *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{1.1}$$

The following inequality is useful to derive some Ostrowski type inequalities.

COROLLARY 1.2 (Pompeiu's Inequality). *With the assumptions of Lemma 1.1 and if $\|f - \ell f'\|_\infty = \sup_{t\in(a,b)} |f(t) - tf'(t)| < \infty$ where $\ell(t) = t$, $t \in [a,b]$, then*

$$|tf(x) - xf(t)| \le \|f - \ell f'\|_\infty |x - t| \tag{1.2}$$

for any $t, x \in [a,b]$.

The inequality (1.2) was obtained by the author in [**1**].

We can generalize the above inequality for the larger class of functions that are absolutely continuous and complex valued as well as for other norms of the difference $f - \ell f'$.

LEMMA 1.3 (Dragomir, 2013 [**3**]). *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b > a > 0$. Then for any $t, x \in [a,b]$ we have*

$$\begin{aligned} &|tf(x) - xf(t)| \\ &\le \begin{cases} \|f - \ell f'\|_\infty |x - t| & \text{if } f - \ell f' \in L_\infty[a,b] \\[2ex] \frac{1}{2q-1} \|f - \ell f'\|_p \left|\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}}\right|^{1/q} & \begin{array}{c}\text{if } f - \ell f' \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1\end{array} \\[2ex] \|f - \ell f'\|_1 \frac{\max\{t,x\}}{\min\{t,x\}} \end{cases} \end{aligned} \tag{1.3}$$

or, equivalently

$$\left|\frac{f(x)}{x}-\frac{f(t)}{t}\right| \tag{1.4}$$
$$\leq \begin{cases} \|f-\ell f'\|_{\infty}\left|\frac{1}{t}-\frac{1}{x}\right| & \text{if } f-\ell f'\in L_{\infty}[a,b] \\ \\ \frac{1}{2q-1}\|f-\ell f'\|_{p}\left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right|^{1/q} & \begin{array}{c}\text{if } f-\ell f'\in L_{p}[a,b] \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1\end{array} \\ \\ \|f-\ell f'\|_{1}\frac{1}{\min\{t^{2},x^{2}\}} \end{cases}$$

PROOF. If f is absolutely continuous, then f/ℓ is absolutely continuous on the interval $[a,b]$ that does not containing 0 and

$$\int_t^x\left(\frac{f(s)}{s}\right)'ds=\frac{f(x)}{x}-\frac{f(t)}{t}$$

for any $t,x\in[a,b]$ with $x\neq t.$

Since

$$\int_t^x\left(\frac{f(s)}{s}\right)'ds=\int_t^x\frac{f'(s)s-f(s)}{s^2}ds$$

then we get the following identity

$$tf(x)-xf(t)=xt\int_t^x\frac{f'(s)s-f(s)}{s^2}ds \tag{1.5}$$

for any $t,x\in[a,b]$.

We notice that the equality (1.5) was proved for the smaller class of differentiable function and in a different manner in [**7**].

Taking the modulus in (1.5) we have

$$\begin{aligned} |tf(x)-xf(t)| &= \left|xt\int_t^x\frac{f'(s)s-f(s)}{s^2}ds\right| \\ &\leq xt\left|\int_t^x\left|\frac{f'(s)s-f(s)}{s^2}\right|ds\right|:=I \end{aligned} \tag{1.6}$$

and utilizing Hölder's integral inequality we deduce

$$I \leq xt\begin{cases} \sup_{s\in[t,x]([x,t])}|f'(s)s-f(s)|\left|\int_t^x\frac{1}{s^2}ds\right| \\ \\ \left|\int_t^x|f'(s)s-f(s)|^p ds\right|^{1/p}\left|\int_t^x\frac{1}{s^{2q}}ds\right|^{1/q} & \begin{array}{c}p>1, \\ \frac{1}{p}+\frac{1}{q}=1\end{array} \\ \\ \left|\int_t^x|f'(s)s-f(s)|ds\right|\sup_{s\in[t,x]([x,t])}\left\{\frac{1}{s^2}\right\} \end{cases} \tag{1.7}$$
$$\leq \begin{cases} \|f-\ell f'\|_{\infty}|x-t| \\ \\ \frac{1}{2q-1}\|f-\ell f'\|_{p}\left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|^{1/q} & \begin{array}{c}p>1, \\ \frac{1}{p}+\frac{1}{q}=1\end{array} \\ \\ \|f-\ell f'\|_{1}\frac{\max\{t,x\}}{\min\{t,x\}} \end{cases}$$

and the inequality (1.4) is proved. ∎

REMARK 1.1. The first inequality in (1.3) also holds in the same form for $0 > b > a$.

REMARK 1.2. If we take in (1.3) $x = A = A(a,b) := \frac{a+b}{2}$ (the arithmetic mean) and $t = G = G(a,b) := \sqrt{ab}$ (the geometric mean) then we get the simple inequality for functions of means:

$$|Gf(A) - Af(G)| \leq \begin{cases} \|f - \ell f'\|_\infty (A - G) & \text{if } f - \ell f' \in L_\infty[a,b] \\ \frac{1}{2q-1}\|f - \ell f'\|_p \frac{\left(A^{2q-1} - G^{2q-1}\right)^{1/q}}{A^{1/p}G^{1/p}} & \begin{array}{l}\text{if } f - \ell f' \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1\end{array} \\ \|f - \ell f'\|_1 \frac{A}{G} \end{cases} \tag{1.8}$$

1.2. Evaluating the Integral Mean. The following new result holds.

THEOREM 1.4 (Dragomir, 2013 [**3**]). *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b > a > 0$. Then for any $x \in [a,b]$ we have*

$$\left|\frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \begin{cases} \frac{b-a}{x}\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right]\|f - \ell f'\|_\infty & \textit{if } f - \ell f' \in L_\infty[a,b] \\ \frac{1}{(2q-1)x(b-a)^{1/q}}\|f - \ell f'\|_p \left[B_q(a,b;x)\right]^{1/q} & \begin{array}{l}\textit{if } f - \ell f' \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1\end{array} \\ \frac{1}{b-a}\|f - \ell f'\|_1 \left(\ln\frac{x}{a} + \frac{b^2 - x^2}{2x^2}\right), \end{cases}, \tag{1.9}$$

where

$$B_q(a,b;x) = \begin{cases} \frac{x^q}{2-q}\left(2x^{q-2} - a^{q-2} - b^{q-2}\right) \\ \quad + \frac{1}{x^{q-1}(q+1)}\left(b^{q+1} + a^{q+1} - 2x^{q+1}\right), & q \neq 2 \\ x^2 \ln\frac{x^2}{ab} + \frac{b^3 + a^3 - 2x^3}{3x}, & q = 2 \end{cases} \tag{1.10}$$

PROOF. The first inequality can be proved in an identical way to the case of differentiable functions from [**1**] by utilizing the first inequality in (1.3).

Utilising the second inequality in (1.3) we have

$$\begin{aligned} &\left|\frac{a+b}{2} \cdot f(x) - \frac{x}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{b-a}\int_a^b |tf(x) - xf(t)|\,dt \\ &\leq \frac{1}{(2q-1)(b-a)}\|f - \ell f'\|_p \int_a^b \left|\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}}\right|^{1/q} dt \end{aligned} \tag{1.11}$$

Utilising Hölder's integral inequality we have

$$\begin{aligned}
(1.12)\qquad & \int_a^b \left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|^{1/q} dt \\
&\le \left(\int_a^b dt\right)^{1/p}\left(\int_a^b \left[\left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|^{1/q}\right]^q dt\right)^{1/q} \\
&= (b-a)^{1/p}\left(\int_a^b \left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right| dt\right)^{1/q}.
\end{aligned}$$

For $q \neq 2$ we have

$$\begin{aligned}
&\int_a^b \left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right| dt \\
&= \int_a^x \left(\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right) dt + \int_x^b \left(\frac{t^q}{x^{q-1}}-\frac{x^q}{t^{q-1}}\right) dt \\
&= x^q \int_a^x \frac{dt}{t^{q-1}} - \frac{1}{x^{q-1}}\int_a^x t^q dt + \frac{1}{x^{q-1}}\int_x^b t^q dt - x^q \int_x^b \frac{1}{t^{q-1}} dt \\
&= \frac{x^q}{2-q}\left(\frac{1}{x^{2-q}}-\frac{1}{a^{2-q}}\right) - \frac{1}{x^{q-1}(q+1)}\left(x^{q+1}-a^{q+1}\right) \\
&+ \frac{1}{x^{q-1}(q+1)}\left(b^{q+1}-x^{q+1}\right) - \frac{x^q}{2-q}\left(\frac{1}{b^{2-q}}-\frac{1}{x^{2-q}}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{x^q}{2-q}\left(\frac{1}{x^{2-q}}-\frac{1}{a^{2-q}}-\frac{1}{b^{2-q}}+\frac{1}{x^{2-q}}\right) \\
&+ \frac{1}{x^{q-1}(q+1)}\left(b^{q+1}-x^{q+1}-x^{q+1}+a^{q+1}\right) \\
&= \frac{x^q}{2-q}\left(2x^{q-2}-a^{q-2}-b^{q-2}\right) + \frac{1}{x^{q-1}(q+1)}\left(b^{q+1}+a^{q+1}-2x^{q+1}\right) \\
&= B_q(a,b;x).
\end{aligned}$$

For $q = 2$ we have

$$\begin{aligned}
& \int_a^b \left|\frac{x^2}{t}-\frac{t^2}{x}\right| dt \\
&= \int_a^x \left(\frac{x^2}{t}-\frac{t^2}{x}\right) dt + \int_x^b \left(\frac{t^2}{x}-\frac{x^2}{t}\right) dt \\
&= x^2\int_a^x \frac{dt}{t} - \frac{1}{x}\int_a^x t^2 dt + \frac{1}{x}\int_x^b t^2 dt - x^2\int_x^b \frac{1}{t} dt \\
&= x^2 \ln\frac{x}{a} - \frac{1}{x}\frac{x^3-a^3}{3} + \frac{1}{x}\frac{b^3-x^3}{3} - x^2\ln\frac{b}{x} \\
&= x^2 \ln\frac{x^2}{ab} + \frac{1}{x}\frac{b^3+a^3-2x^3}{3} = B_2(a,b;x).
\end{aligned}$$

Utilizing (1.11) and (1.12) we get the second inequality in (1.9).

Utilising the third inequality in (1.3) we have

$$\left|\frac{a+b}{2}\cdot f(x)-\frac{x}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{b-a}\int_a^b |tf(x)-xf(t)|\,dt \tag{1.13}$$
$$\le \frac{1}{b-a}\|f-\ell f'\|_1\int_a^b \frac{\max\{t,x\}}{\min\{t,x\}}dt.$$

Since

$$\int_a^b \frac{\max\{t,x\}}{\min\{t,x\}}dt=\int_a^x \frac{x}{t}dt+\int_x^b \frac{t}{x}dt = x\ln\frac{x}{a}+\frac{1}{x}\frac{b^2-x^2}{2},$$

then by (1.13) we have

$$\left|\frac{a+b}{2}\cdot f(x)-\frac{x}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{b-a}\int_a^b |tf(x)-xf(t)|\,dt$$
$$\le \frac{1}{b-a}\|f-\ell f'\|_1\left[x\ln\frac{x}{a}+\frac{1}{x}\frac{b^2-x^2}{2}\right],$$

and the last part of (1.9) is thus proved. ∎

REMARK 1.3. If we take in (1.9) $x=A=A(a,b):=\frac{a+b}{2}$, then we get

$$\left|f(A)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.14}$$
$$\le \begin{cases} \frac{b-a}{4A}\|f-\ell f'\|_\infty & \text{if } f-\ell f'\in L_\infty[a,b] \\ \frac{1}{(2q-1)A(b-a)^{1/q}}\|f-\ell f'\|_p\left[B_q(a,b;A)\right]^{1/q} & \begin{array}{c}\text{if } f-\ell f'\in L_p[a,b]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1\end{array} \\ \frac{1}{b-a}\|f-\ell f'\|_1\left(\ln\frac{A}{a}+\frac{1}{2}(b-a)\frac{a+3b}{4}A\right), \end{cases},$$

where

$$B_q(a,b;A)=\begin{cases} \frac{2A^q}{2-q}\left(A^{q-2}-A\left(a^{q-2},b^{q-2}\right)\right) & \\ +\frac{2}{(q+1)A^{q-1}}\left(A\left(b^{q+1},a^{q+1}\right)-A^{q+1}\right), & q\neq 2 \\ 2A^2\ln\frac{A}{G}+\frac{1}{2}(b-a)^2, & q=2 \end{cases}$$

1.3. A Related Result.
The following new result also holds.

THEOREM 1.5 (Dragomir, 2013 [**3**]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b>a>0$. Then for any $x\in[a,b]$ we have*

$$\left|\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt\right| \tag{1.15}$$
$$\le \begin{cases} \frac{2}{b-a}\|f-\ell f'\|_\infty\left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right) & \textit{if } f-\ell f'\in L_\infty[a,b] \\ \frac{1}{(2q-1)(b-a)^{1/q}}\|f-\ell f'\|_p\left(C_q(a,b;x)\right)^{1/q} & \begin{array}{c}\textit{if } f-\ell f'\in L_p[a,b]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1\end{array} \\ \frac{1}{b-a}\|f-\ell f'\|_1\frac{x^2+ab-2ax}{x^2a}, \end{cases},$$

where

$$C_q(a,b;x) = \frac{1}{x^{2q-1}}(b+a-2x) + \frac{a^{2-2q}+b^{2-2q}-2x^{2-2q}}{2(q-1)}, q > 1. \tag{1.16}$$

PROOF. From the first inequality in (1.10) we have

$$\begin{aligned} \left|\frac{f(x)}{x} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt\right| &\le \frac{1}{b-a}\int_a^b \left|\frac{f(x)}{x} - \frac{f(t)}{t}\right| dt \\ &\le \|f-\ell f'\|_\infty \frac{1}{b-a}\int_a^b \left|\frac{1}{t}-\frac{1}{x}\right| dt. \end{aligned} \tag{1.17}$$

Since

$$\begin{aligned} \int_a^b \left|\frac{1}{x}-\frac{1}{t}\right| dt &= \left[\int_a^x \left(\frac{1}{t}-\frac{1}{x}\right)dt + \int_x^b \left(\frac{1}{x}-\frac{1}{t}\right)dt\right] \\ &= \left(\ln\frac{x}{a} - \frac{x-a}{x} + \frac{b-x}{x} - \ln\frac{b}{x}\right) \\ &= \left(\ln\frac{x^2}{ab} + \frac{a+b-2x}{x}\right) \end{aligned}$$

for any $x \in [a,b]$, then we deduce from (1.17) the first inequality in (1.15).
From the second inequality in (1.10) we have

$$\begin{aligned} &\left|\frac{f(x)}{x} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt\right| \\ &\le \frac{1}{b-a}\int_a^b \left|\frac{f(x)}{x}-\frac{f(t)}{t}\right| dt \\ &\le \frac{1}{(2q-1)(b-a)}\|f-\ell f'\|_p \int_a^b \left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right|^{1/q} dt. \end{aligned} \tag{1.18}$$

Utilising Hölder's integral inequality we have

$$\begin{aligned} &\int_a^b \left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right|^{1/q} dt \\ &\le \left(\int_a^b dt\right)^{1/p} \left(\int_a^b \left[\left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right|^{1/q}\right]^q dt\right)^{1/q} \\ &= (b-a)^{1/p}\left(\int_a^b \left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right| dt\right)^{1/q}. \end{aligned} \tag{1.19}$$

Since

$$\begin{aligned}
&\int_a^b \left|\frac{1}{t^{2q-1}} - \frac{1}{x^{2q-1}}\right| dt \\
&= \int_a^x \left(\frac{1}{t^{2q-1}} - \frac{1}{x^{2q-1}}\right) dt + \int_x^b \left(\frac{1}{x^{2q-1}} - \frac{1}{t^{2q-1}}\right) dt \\
&= \frac{x^{2-2q} - a^{2-2q}}{2-2q} - \frac{1}{x^{2q-1}}(x-a) + \frac{1}{x^{2q-1}}(b-x) - \frac{b^{2-2q} - x^{2-2q}}{2-2q} \\
&= \frac{1}{x^{2q-1}}(b+a-2x) + \frac{2x^{2-2q} - a^{2-2q} - b^{2-2q}}{2-2q} \\
&= \frac{1}{x^{2q-1}}(b+a-2x) + \frac{a^{2-2q} + b^{2-2q} - 2x^{2-2q}}{2(q-1)} = C_q(a,b;x)
\end{aligned}$$

then by (1.18) and (1.19) we get

$$\begin{aligned}
&\left|\frac{f(x)}{x} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t} dt\right| \\
&\le \frac{1}{(2q-1)(b-a)} \|f - \ell f'\|_p (b-a)^{1/p} (C_q(a,b;x))^{1/q}
\end{aligned}$$

and the second inequality in (1.15) is proved.

From the third inequality in (1.10) we have

$$\begin{aligned}
\left|\frac{f(x)}{x} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t} dt\right| &\le \frac{1}{b-a}\int_a^b \left|\frac{f(x)}{x} - \frac{f(t)}{t}\right| dt \\
&\le \frac{1}{b-a}\|f - \ell f'\|_1 \int_a^b \frac{1}{\min\{t^2, x^2\}} dt.
\end{aligned} \tag{1.20}$$

Since

$$\begin{aligned}
\int_a^b \frac{1}{\min\{t^2, x^2\}} dt &= \int_a^x \frac{dt}{t^2} + \int_x^b \frac{dt}{x^2} = \frac{x-a}{xa} + \frac{b-x}{x^2} \\
&= \frac{x^2 + ab - 2ax}{x^2 a},
\end{aligned}$$

then by (1.20) we deduce the last part of (1.15). ∎

REMARK 1.4. If we take in (1.15) $x = A = A(a,b) := \frac{a+b}{2}$, then we get

$$\begin{aligned}
&\left|\frac{f(A)}{A} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t} dt\right| \\
&\le \begin{cases}
\frac{2}{b-a}\|f - \ell f'\|_\infty \ln\left(\frac{A}{G}\right) & \text{if } f - \ell f' \in L_\infty[a,b] \\[2ex]
\frac{1}{(2q-1)(b-a)^{1/q}}\|f - \ell f'\|_p (C_q(a,b;A))^{1/q} & \begin{array}{c}\text{if } f - \ell f' \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1\end{array} \\[2ex]
\frac{1}{2}\|f - \ell f'\|_1 \frac{A+a}{A^2 a},
\end{cases}
\end{aligned} \tag{1.21}$$

where

$$C_q(a,b;A) = \frac{A(a^{2-2q}, b^{2-2q}) - A^{2-2q}}{q-1}, q > 1.$$

2. OSTROWSKI VIA POWER POMPEIU'S INEQUALITY

2.1. Power Pompeiu's Inequality. In 1946, Pompeiu [**6**] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem* (see also [**8**, p. 83]).

We can generalize the above Pompeiu's inequality for the power function as follows.

LEMMA 2.1 (Dragomir, 2013 [**2**]). *Let* $f:[a,b]\to\mathbb{C}$ *be an absolutely continuous function on the interval* $[a,b]$ *with* $b>a>0$. *If* $r\in\mathbb{R}$, $r\neq 0$, *then for any* $t,x\in[a,b]$ *we have*

$$|t^r f(x)-x^r f(t)| \tag{2.1}$$
$$\leq \begin{cases} \frac{1}{|r|}\|f'\ell-rf\|_\infty |t^r-x^r|, \text{ if } f'\ell-rf\in L_\infty[a,b] \\ \\ \|f'\ell-rf\|_p \begin{cases} \frac{1}{|1-q(r+1)|}\left|\frac{t^r}{x^{1-q(r+1)-r}}-\frac{x^r}{t^{1-q(r+1)-r}}\right|, \text{ for } r\neq -\frac{1}{p} \\ t^r x^r |\ln x-\ln t|, \text{ for } r=-\frac{1}{p} \end{cases} \\ \text{if } f'\ell-rf\in L_p[a,b] \\ \\ \|f'\ell-rf\|_1 \frac{t^r x^r}{\min\{x^{r+1},t^{r+1}\}} \end{cases}$$

or, equivalently

$$\left|\frac{f(x)}{x^r}-\frac{f(t)}{t^r}\right| \tag{2.2}$$
$$\leq \begin{cases} \frac{1}{|r|}\|f'\ell-rf\|_\infty \left|\frac{1}{x^r}-\frac{1}{t^r}\right|, \text{ if } f'\ell-rf\in L_\infty[a,b] \\ \\ \|f'\ell-rf\|_p \begin{cases} \frac{1}{|1-q(r+1)|}\left|\frac{1}{x^{1-q(r+1)}}-\frac{1}{t^{1-q(r+1)}}\right|, \text{ for } r\neq -\frac{1}{p} \\ |\ln x-\ln t|, \text{ for } r=-\frac{1}{p} \end{cases} \\ \text{if } f'\ell-rf\in L_p[a,b] \\ \\ \|f'\ell-rf\|_1 \frac{1}{\min\{x^{r+1},t^{r+1}\}} \end{cases}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

PROOF. If f is absolutely continuous, then $f/(\cdot)^r$ is absolutely continuous on the interval $[a,b]$ and

$$\int_t^x \left(\frac{f(s)}{s^r}\right)' ds=\frac{f(x)}{x^r}-\frac{f(t)}{t^r}$$

for any $t,x\in[a,b]$ with $x\neq t$.

Since

$$\int_t^x \left(\frac{f(s)}{s^r}\right)' ds=\int_t^x \frac{f'(s)s^r-rs^{r-1}f(s)}{s^{2r}}ds=\int_t^x \frac{f'(s)s-rf(s)}{s^{r+1}}ds$$

then we get the following identity

$$t^r f(x)-x^r f(t)=x^r t^r\int_t^x \frac{f'(s)s-rf(s)}{s^{r+1}}ds \tag{2.3}$$

for any $t,x\in[a,b]$.

Taking the modulus in (2.3) we have

$$
\begin{aligned}
|t^r f(x) - x^r f(t)| &= x^r t^r \left| \int_t^x \frac{f'(s)s - rf(s)}{s^{r+1}} ds \right| \\
&\le x^r t^r \left| \int_t^x \frac{|f'(s)s - rf(s)|}{s^{r+1}} ds \right| := I
\end{aligned} \tag{2.4}
$$

and utilizing Hölder's integral inequality we deduce

$$
\begin{aligned}
I &\le x^r t^r \begin{cases} \sup_{s\in[t,x]([x,t])} |f'(s)s - rf(s)| \left| \int_t^x \frac{1}{s^{r+1}} ds \right| \\ \\ \left| \int_t^x |f'(s)s - rf(s)|^p ds \right|^{1/p} \left| \int_t^x \frac{1}{s^{q(r+1)}} ds \right|^{1/q} \\ \\ \left| \int_t^x |f'(s)s - rf(s)| ds \right| \sup_{s\in[t,x]([x,t])} \left\{ \frac{1}{s^{r+1}} \right\} \end{cases} \\
&\le x^r t^r \begin{cases} \frac{1}{|r|} \|f'\ell - rf\|_\infty \left| \frac{1}{x^r} - \frac{1}{t^r} \right| \\ \\ \|f'\ell - rf\|_p \begin{cases} \frac{1}{|1-q(r+1)|} \left| \frac{1}{x^{1-q(r+1)}} - \frac{1}{t^{1-q(r+1)}} \right|, r \ne -\frac{1}{p} \\ |\ln x - \ln t|, r = -\frac{1}{p} \end{cases} \\ \\ \|f'\ell - rf\|_1 \frac{1}{\min\{x^{r+1}, t^{r+1}\}}. \end{cases}
\end{aligned} \tag{2.5}
$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and the inequality (2.1) is proved. ∎

2.2. Some Ostrowski Type Results. The following new result also holds.

THEOREM 2.2 (Dragomir, 2013 [**2**]). *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b > a > 0$. If $r \in \mathbb{R}$, $r \ne 0$, and $f'\ell - rf \in L_\infty[a,b]$, then for any $x \in [a,b]$ we have*

$$
\begin{aligned}
&\left| \frac{b^{r+1} - a^{r+1}}{r+1} f(x) - x^r \int_a^b f(t) dt \right| \\
&\le \frac{1}{|r|} \|f'\ell - rf\|_\infty \\
&\times \begin{cases} \frac{2rx^{r+1} - x^r(a+b)(r+1) + b^{r+1} + a^{r+1}}{r+1}, & \text{if } r > 0 \\ \\ \frac{x^r(a+b)(r+1) - 2rx^{r+1} - b^{r+1} - a^{r+1}}{r+1}, & \text{if } r \in (-\infty, 0) \setminus \{-1\}. \end{cases}
\end{aligned} \tag{2.6}
$$

Also, for $r = -1$, we have

$$
\left| f(x) \ln \frac{b}{a} - \frac{1}{x} \int_a^b f(t) dt \right| \le 2 \|f'\ell + f\|_\infty \left(\ln \frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2} - x}{x} \right) \tag{2.7}
$$

for any $x \in [a,b]$, provided $f'\ell + f \in L_\infty[a,b]$

The constant 2 in (2.7) is best possible.

PROOF. Utilising the first inequality in (2.1) for $r \ne -1$ we have

$$
\begin{aligned}
\left| \frac{b^{r+1} - a^{r+1}}{r+1} f(x) - x^r \int_a^b f(t) dt \right| &\le \int_a^b |t^r f(x) - x^r f(t)| dt \\
&\le \frac{1}{|r|} \|f'\ell - rf\|_\infty \int_a^b |t^r - x^r| dt.
\end{aligned} \tag{2.8}
$$

Observe that

$$\int_a^b |t^r - x^r|\,dt$$
$$= \begin{cases} \int_a^x (x^r - t^r)\,dt + \int_x^b (t^r - x^r)\,dt, \text{ if } r > 0 \\ \\ \int_a^x (t^r - x^r)\,dt + \int_x^b (x^r - t^r)\,dt, \text{ if } r \in (-\infty, 0) \diagdown \{-1\}. \end{cases}$$

Then for $r > 0$ we have

$$\int_a^x (x^r - t^r)\,dt + \int_x^b (t^r - x^r)\,dt$$
$$= x^r (x-a) - \frac{x^{r+1} - a^{r+1}}{r+1} + \frac{b^{r+1} - x^{r+1}}{r+1} - x^r (b-x)$$
$$= 2x^{r+1} - x^r (a+b) + \frac{b^{r+1} + a^{r+1} - 2x^{r+1}}{r+1}$$
$$= \frac{2rx^{r+1} + 2x^{r+1} - x^r (a+b)(r+1) + b^{r+1} + a^{r+1} - 2x^{r+1}}{r+1}$$
$$= \frac{2rx^{r+1} - x^r (a+b)(r+1) + b^{r+1} + a^{r+1}}{r+1}$$

and for $r \in (-\infty, 0) \diagdown \{-1\}$ we have

$$\int_a^x (t^r - x^r)\,dt + \int_x^b (x^r - t^r)\,dt$$
$$= -\frac{2rx^{r+1} - x^r (a+b)(r+1) + b^{r+1} + a^{r+1}}{r+1}.$$

Making use of (2.8) we get (2.6).

Utilizing the inequality (2.1) for $r = -1$ we have

$$\left|t^{-1} f(x) - x^{-1} f(t)\right| \le \|f'\ell + f\|_\infty \left|t^{-1} - x^{-1}\right|$$

if $f'\ell + f \in L_\infty [a,b]$.

Integrating this inequality, we have

$$\left|f(x) \ln \frac{b}{a} - x^{-1} \int_a^b f(t)\,dt\right| \le \int_a^b \left|t^{-1} f(x) - x^{-1} f(t)\right| dt \tag{2.9}$$
$$\le \|f'\ell + f\|_\infty \int_a^b \left|t^{-1} - x^{-1}\right| dt.$$

Since

$$\int_a^b \left|\frac{1}{x} - \frac{1}{t}\right| dt = \left[\int_a^x \left(\frac{1}{t} - \frac{1}{x}\right) dt + \int_x^b \left(\frac{1}{x} - \frac{1}{t}\right) dt\right]$$
$$= \left(\ln \frac{x}{a} - \frac{x-a}{x} + \frac{b-x}{x} - \ln \frac{b}{x}\right)$$
$$= \ln \frac{x^2}{ab} + \frac{a+b-2x}{x}$$
$$= 2\left(\ln \frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2} - x}{x}\right),$$

then by (2.9) we get the desired inequality (2.7).

Now, assume that (2.7) holds with a constant $C > 0$, i.e.

$$\left|f(x)\ln\frac{b}{a} - x^{-1}\int_a^b f(t)\,dt\right| \le C\left\|f'\ell + f\right\|_\infty \left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2} - x}{x}\right) \tag{2.10}$$

for any $x \in [a,b]$.

If we take in (2.10) $f(t) = 1, t \in [a,b]$, then we get

$$\left|\ln\frac{b}{a} - \frac{b-a}{x}\right| \le C\left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2} - x}{x}\right) \tag{2.11}$$

for any for any $x \in [a,b]$.

Making $x = a$ in (2.10) produces the inequality

$$\left|\ln\frac{b}{a} - \frac{b-a}{a}\right| \le C\left(\frac{b-a}{2a} - \frac{1}{2}\ln\frac{b}{a}\right)$$

which implies that $C \ge 2$.

This proves the sharpness of the constant 2 in (2.7). ∎

REMARK 2.1. Consider the r-Logarithmic mean

$$L_r = L_r(a,b) := \left[\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}\right]^{1/r}$$

defined for $r \in \mathbb{R}\setminus\{0,-1\}$ and the Logarithmic mean, defined as

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

If $A = A(a,b) := \frac{a+b}{2}$, then from (2.6) we get for $x = A$ the inequality

$$\left|L_r^r(b-a)f(A) - A^r\int_a^b f(t)\,dt\right| \tag{2.12}$$
$$\le \frac{2}{|r|}\left\|f'\ell - rf\right\|_\infty \begin{cases} \frac{A\left(b^{r+1},a^{r+1}\right) - A^{r+1}}{r+1}, & \text{if } r > 0 \\ \\ \frac{A^{r+1} - A\left(b^{r+1},a^{r+1}\right)}{r+1}, & \text{if } r \in (-\infty,0)\setminus\{-1\}, \end{cases}$$

while from (2.7) we get

$$\left|L^{-1}(b-a)f(A) - A^{-1}\int_a^b f(t)\,dt\right| \le 2\left\|f'\ell + f\right\|_\infty \ln\frac{A}{G} \tag{2.13}$$

The following related result holds.

THEOREM 2.3 (Dragomir, 2013 [**2**]). *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b > a > 0$. If $r \in \mathbb{R}$, $r \ne 0$, then for any $x \in [a,b]$ we have*

$$\left|\frac{f(x)}{x^r}(b-a) - \int_a^b \frac{f(t)}{t^r}dt\right| \tag{2.14}$$
$$\le \frac{1}{|r|}\left\|f'\ell - rf\right\|_\infty$$
$$\times \begin{cases} \frac{2x^{1-r} - a^{1-r} - b^{1-r}}{1-r} + \frac{1}{x^r}(b + a - 2x), & r \in (0,\infty)\setminus\{1\} \\ \\ \frac{a^{1-r} + b^{1-r} - 2x^{1-r}}{1-r} + \frac{1}{x^r}(2x - a - b), & \textit{if } r < 0. \end{cases},$$

Also, for $r=1$, *we have*

$$\left|\frac{f(x)}{x}(b-a)-\int_a^b \frac{f(t)}{t}dt\right| \le 2\left\|f'\ell-f\right\|_\infty \left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right) \tag{2.15}$$

for any $x\in[a,b]$, *provided* $f'\ell-f\in L_\infty[a,b]$.

The constant 2 *is best possible in (2.15).*

PROOF. From the first inequality in (2.2) we have

$$\left|\frac{f(x)}{x^r}-\frac{f(t)}{t^r}\right| \le \frac{1}{|r|}\left\|f'\ell-rf\right\|_\infty \left|\frac{1}{x^r}-\frac{1}{t^r}\right|, \tag{2.16}$$

for any $t,x\in[a,b]$, provided $f'\ell-rf\in L_\infty[a,b]$.

Integrating over $t\in[a,b]$ we get

$$\begin{aligned} \left|\frac{f(x)}{x^r}(b-a)-\int_a^b \frac{f(t)}{t^r}dt\right| &\le \int_a^b \left|\frac{f(x)}{x^r}-\frac{f(t)}{t^r}\right| dt \\ &\le \frac{1}{|r|}\left\|f'\ell-rf\right\|_\infty \int_a^b \left|\frac{1}{x^r}-\frac{1}{t^r}\right| dt, \end{aligned} \tag{2.17}$$

for $r\in\mathbb{R}$, $r\neq 0$.

For $r\in(0,\infty)\setminus\{1\}$ we have

$$\begin{aligned} &\int_a^b \left|\frac{1}{x^r}-\frac{1}{t^r}\right| dt \\ &= \int_a^x \left(\frac{1}{t^r}-\frac{1}{x^r}\right) dt+\int_x^b \left(\frac{1}{x^r}-\frac{1}{t^r}\right) dt \\ &= \frac{x^{1-r}-a^{1-r}}{1-r}-\frac{1}{x^r}(x-a)+\frac{1}{x^r}(b-x)-\frac{b^{1-r}-x^{1-r}}{1-r} \\ &= \frac{2x^{1-r}-a^{1-r}-b^{1-r}}{1-r}+\frac{1}{x^r}(b+a-2x) \end{aligned}$$

for any $x\in[a,b]$.

For $r<0$, we also have

$$\int_a^b \left|\frac{1}{x^r}-\frac{1}{t^r}\right| dt = \frac{a^{1-r}+b^{1-r}-2x^{1-r}}{1-r}+\frac{1}{x^r}(2x-a-b)$$

for any $x\in[a,b]$.

For $r=1$ we have

$$\int_a^b \left|\frac{1}{x}-\frac{1}{t}\right| dt = 2\left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right)$$

for any $x\in[a,b]$, and the inequality (2.15) is obtained.

The sharpness of the constant 2 follows as in the proof of Theorem 2.2 and the details are omitted. ∎

REMARK 2.2. If we take $x = A$ in Theorem 2.3, then we we have

$$\left|\frac{f(A)}{A^r}(b-a) - \int_a^b \frac{f(t)}{t^r}dt\right| \leq \frac{2}{|r|}\|f'\ell - rf\|_\infty \begin{cases} \frac{A^{1-r}-A\left(a^{1-r},b^{1-r}\right)}{1-r}, & r \in (0,\infty)\setminus\{1\} \\ \frac{A\left(a^{1-r},b^{1-r}\right)-A^{1-r}}{1-r}, & \text{if } r < 0. \end{cases}, \tag{2.18}$$

Also, for $r = 1$, we have

$$\left|\frac{f(A)}{A}(b-a) - \int_a^b \frac{f(t)}{t}dt\right| \leq 2\|f'\ell - f\|_\infty \ln\frac{A}{G}. \tag{2.19}$$

REMARK 2.3. The interested reader may obtain other similar results in terms of the p-norms $\|f'\ell - rf\|_p$ with $p \geq 1$. However, since some calculations are too complicated, the details are not presented here.

3. OSTROWSKI VIA AN EXPONENTIAL POMPEIU'S INEQUALITY

3.1. An Exponential Pompeiu's Inequality. In 1946, Pompeiu [**6**] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem* (see also [**8**, p. 83]).

We can provide some similar results for complex-valued functions and the exponential as follows.

LEMMA 3.1 (Dragomir, 2013 [**4**]). *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) \neq 0$. Then for any $t, x \in [a,b]$ we have*

$$\left|\frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)}\right| \tag{3.1}$$

$$\leq \begin{cases} |\operatorname{Re}(\alpha)|\,\|f' - \alpha f\|_\infty \\ \times\left|\frac{1}{\exp(t\operatorname{Re}(\alpha))} - \frac{1}{\exp(x\operatorname{Re}(\alpha))}\right| & \text{if } f'-\alpha f \in L_\infty[a,b], \\ \\ q^{1/q}|\operatorname{Re}(\alpha)|^{1/q}\|f' - \alpha f\|_p \\ \times\left|\frac{1}{\exp(tq\operatorname{Re}(\alpha))} - \frac{1}{\exp(xq\operatorname{Re}(\alpha))}\right|^{1/q} & \text{if } f'-\alpha f \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \|f' - \alpha f\|_1 \frac{1}{\min\{\exp(t\operatorname{Re}(\alpha)),\exp(x\operatorname{Re}(\alpha))\}}, \end{cases}$$

or, equivalently

$$(3.2)\quad \left|\exp(\alpha t) f(x) - f(t)\exp(\alpha x)\right|$$

$$\leq \begin{cases} \left|\operatorname{Re}(\alpha)\right| \left\|f' - \alpha f\right\|_\infty & \text{if } f' - \alpha f \\ \times \left|\exp(x\operatorname{Re}(\alpha)) - \exp(t\operatorname{Re}(\alpha))\right| & \in L_\infty[a,b], \\ & \\ & \text{if } f' - \alpha f \\ q^{1/q}\left|\operatorname{Re}(\alpha)\right|^{1/q}\left\|f' - \alpha f\right\|_p & \in L_p[a,b] \\ \times \left|\exp(xq\operatorname{Re}(\alpha)) - \exp(tq\operatorname{Re}(\alpha))\right|^{1/q} & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ & \\ \left\|f' - \alpha f\right\|_1 \max\left\{\exp(t\operatorname{Re}(\alpha)), \exp(x\operatorname{Re}(\alpha))\right\}. & \end{cases}$$

PROOF. If f is absolutely continuous, then $f/\exp(\alpha\cdot)$ is absolutely continuous on the interval $[a,b]$ and

$$\int_t^x \left(\frac{f(s)}{\exp(\alpha s)}\right)' ds = \frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)}$$

for any $t, x \in [a,b]$ with $x \neq t$.

Since

$$\begin{aligned} \int_t^x \left(\frac{f(s)}{\exp(\alpha s)}\right)' ds &= \int_t^x \frac{f'(s)\exp(\alpha s) - \alpha f(s)\exp(\alpha s)}{\exp(2\alpha s)} ds \\ &= \int_t^x \frac{f'(s) - \alpha f(s)}{\exp(\alpha s)} ds, \end{aligned}$$

then we get the following identity

$$(3.3)\qquad \frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)} = \int_t^x \frac{f'(s) - \alpha f(s)}{\exp(\alpha s)} ds$$

for any $t, x \in [a,b]$ with $x \neq t$.

Taking the modulus in (3.3) we have

$$(3.4)\qquad \begin{aligned} \left|\frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)}\right| &= \left|\int_t^x \frac{f'(s) - \alpha f(s)}{\exp(\alpha s)} ds\right| \\ &\leq \left|\int_t^x \frac{\left|f'(s) - \alpha f(s)\right|}{\left|\exp(\alpha s)\right|} ds\right| := I \end{aligned}$$

and utilizing Hölder's integral inequality we deduce

$$(3.5)\qquad I \leq \begin{cases} \sup_{s\in[t,x]([x,t])} \left|f'(s) - \alpha f(s)\right| \left|\int_t^x \frac{1}{|\exp(\alpha s)|} ds\right|, \\ \\ \left|\int_t^x \left|f'(s) - \alpha f(s)\right|^p ds\right|^{1/p} \left|\int_t^x \frac{1}{|\exp(\alpha s)|^q} ds\right|^{1/q}, \\ \\ \left|\int_t^x \left|f'(s) - \alpha f(s)\right| ds\right| \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|\exp(\alpha s)|}\right\}, \end{cases}$$

$$\leq \begin{cases} \left\|f' - \alpha f\right\|_\infty \left|\int_t^x \frac{1}{|\exp(\alpha s)|} ds\right|, \\ \\ \left\|f' - \alpha f\right\|_p \left|\int_t^x \frac{1}{|\exp(\alpha s)|^q} ds\right|^{1/q}, \\ \\ \left\|f' - \alpha f\right\|_1 \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|\exp(\alpha s)|}\right\}. \end{cases}$$

Now, since $\alpha = \operatorname{Re}(\alpha) + i \operatorname{Im}(\alpha)$ and $s \in [a, b]$, then

$$\begin{aligned} |\exp(\alpha s)| &= \exp(s \operatorname{Re}(\alpha) + is \operatorname{Im}(\alpha)) = |\exp(s \operatorname{Re}(\alpha)) \exp(is \operatorname{Im}(\alpha))| \\ &= |\exp(s \operatorname{Re}(\alpha))| \, |\exp(is \operatorname{Im}(\alpha))| = \exp(s \operatorname{Re}(\alpha)). \end{aligned}$$

We have

$$\begin{aligned} \int_t^x \frac{1}{|\exp(\alpha s)|} ds = \int_t^x \frac{1}{\exp(s \operatorname{Re}(\alpha))} ds &= \int_t^x \exp(-s \operatorname{Re}(\alpha)) \, ds \\ &= -\operatorname{Re}(\alpha) \exp(-s \operatorname{Re}(\alpha))\big|_t^x \\ &= -\operatorname{Re}(\alpha) \exp(-x \operatorname{Re}(\alpha)) + \operatorname{Re}(\alpha) \exp(-t \operatorname{Re}(\alpha)) \\ &= \operatorname{Re}(\alpha) \left[\exp(-t \operatorname{Re}(\alpha)) - \exp(-x \operatorname{Re}(\alpha))\right] \\ &= \operatorname{Re}(\alpha) \left[\frac{1}{\exp(t \operatorname{Re}(\alpha))} - \frac{1}{\exp(x \operatorname{Re}(\alpha))}\right] \end{aligned}$$

and by (3.4) and (3.5) we get

$$\begin{aligned} &\left|\frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)}\right| \\ &\leq \left\|f' - \alpha f\right\|_\infty \left|\operatorname{Re}(\alpha)\right| \left|\frac{1}{\exp(t \operatorname{Re}(\alpha))} - \frac{1}{\exp(x \operatorname{Re}(\alpha))}\right| \end{aligned}$$

and the first part of (3.1) is proved.

We have

$$\begin{aligned} \int_t^x \frac{1}{|\exp(\alpha s)|^q} ds &= \int_t^x \frac{1}{\exp(sq \operatorname{Re}(\alpha))} ds \\ &= q \operatorname{Re}(\alpha) \left[\frac{1}{\exp(tq \operatorname{Re}(\alpha))} - \frac{1}{\exp(xq \operatorname{Re}(\alpha))}\right] \end{aligned}$$

and by (3.4) and (3.5) we get the second part of (3.1).

We have

$$\sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|\exp(\alpha s)|}\right\} = \frac{1}{\min\left\{\exp(t \operatorname{Re}(\alpha)), \exp(x \operatorname{Re}(\alpha))\right\}}$$

and by (3.4) and (3.5) we get the last part of (3.1).

The inequality (3.2) follows by (3.1) on multiplying with $|\exp(\alpha x)\exp(\alpha t)|$ and performing the required calculation. ∎

The following particular case is of interest.

COROLLARY 3.2. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$. Then for any $t,x\in[a,b]$ we have*

$$\left|\frac{f(x)}{\exp(x)}-\frac{f(t)}{\exp(t)}\right| \tag{3.6}$$

$$\leq\begin{cases}\|f'-f\|_\infty\left|\frac{1}{\exp(t)}-\frac{1}{\exp(x)}\right| & \text{if } f'-f \in L_\infty[a,b],\\[2ex] q^{1/q}\|f'-f\|_p\left|\frac{1}{\exp(tq)}-\frac{1}{\exp(xq)}\right|^{1/q} & \text{if } f'-f\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \|f'-f\|_1\frac{1}{\min\{\exp(t),\exp(x)\}},\end{cases}$$

or, equivalently

$$|\exp(t)f(x)-f(t)\exp(x)| \tag{3.7}$$

$$\leq\begin{cases}\|f'-f\|_\infty|\exp(x)-\exp(t)| & \text{if } f'-f \in L_\infty[a,b],\\[2ex] q^{1/q}\|f'-f\|_p|\exp(xq)-\exp(tq)|^{1/q} & \text{if } f'-f\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \|f'-f\|_1\max\{\exp(t),\exp(x)\}.\end{cases}$$

REMARK 3.1. If $\operatorname{Re}(\alpha) = 0$ then the inequality (3.5) becomes

$$I \leq \begin{cases} \sup_{s\in[t,x]([x,t])} |f'(s) - i\operatorname{Im}(\alpha) f(s)| \left|\int_t^x \frac{1}{|\exp(i\operatorname{Im}(\alpha)s)|} ds\right|, \\ \left|\int_t^x |f'(s) - i\operatorname{Im}(\alpha) f(s)|^p ds\right|^{1/p} \left|\int_t^x \frac{1}{|\exp(i\operatorname{Im}(\alpha)s)|^q} ds\right|^{1/q}, \\ \left|\int_t^x |f'(s) - i\operatorname{Im}(\alpha) f(s)| ds\right| \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|\exp(i\operatorname{Im}(\alpha)s)|}\right\}, \end{cases}$$

$$\leq \begin{cases} \|f' - i\operatorname{Im}(\alpha) f\|_\infty \left|\int_t^x ds\right|, \\ \|f' - i\operatorname{Im}(\alpha) f\|_p \left|\int_t^x ds\right|^{1/q}, \\ \|f' - i\operatorname{Im}(\alpha) f\|_1, \end{cases}$$

$$= \begin{cases} \|f' - i\operatorname{Im}(\alpha) f\|_\infty |x-t|, \\ \|f' - i\operatorname{Im}(\alpha) f\|_p |x-t|^{1/q}, \\ \|f' - i\operatorname{Im}(\alpha) f\|_1. \end{cases}$$

Therefore we have

$$\left|\frac{f(x)}{\exp(i\operatorname{Im}(\alpha)x)} - \frac{f(t)}{\exp(i\operatorname{Im}(\alpha)t)}\right| \leq \begin{cases} \|f' - i\operatorname{Im}(\alpha) f\|_\infty |x-t|, \\ \|f' - i\operatorname{Im}(\alpha) f\|_p |x-t|^{1/q}, \\ \|f' - i\operatorname{Im}(\alpha) f\|_1, \end{cases} \tag{3.8}$$

or, equivalently

$$|\exp(i\operatorname{Im}(\alpha)t) f(x) - f(t)\exp(i\operatorname{Im}(\alpha)x)| \leq \begin{cases} \|f' - i\operatorname{Im}(\alpha) f\|_\infty |x-t|, \\ \|f' - i\operatorname{Im}(\alpha) f\|_p |x-t|^{1/q}, \\ \|f' - i\operatorname{Im}(\alpha) f\|_1 \end{cases} \tag{3.9}$$

for any $t, x \in [a,b]$.

In particular, we have

$$\left|\frac{f(x)}{\exp(ix)} - \frac{f(t)}{\exp(it)}\right| \leq \begin{cases} \|f' - if\|_\infty |x-t|, \\ \|f' - if\|_p |x-t|^{1/q}, \\ \|f' - if\|_1, \end{cases} \tag{3.10}$$

or, equivalently

$$\left|\exp(it) f(x) - f(t)\exp(ix)\right| \leq \begin{cases} \|f' - if\|_\infty |x-t|, \\ \\ \|f' - if\|_p |x-t|^{1/q}, \\ \\ \|f' - if\|_1, \end{cases} \tag{3.11}$$

for any $t, x \in [a,b]$.

3.2. Inequalities of Ostrowski Type.

The following result holds:

THEOREM 3.3 (Dragomir, 2013 [**4**]). *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ and $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$. Then for any $x \in [a,b]$ we have*

$$\left| f(x)\frac{\exp(\alpha b) - \exp(\alpha a)}{\alpha} - \exp(\alpha x)\int_a^b f(t)\,dt \right| \tag{3.12}$$

$$\leq \begin{cases} |\mathrm{Re}(\alpha)|\,\|f' - \alpha f\|_\infty B_1(a,b,x,\alpha) & \textit{if } f'-\alpha f \in L_\infty[a,b], \\ \\ q^{1/q}|\mathrm{Re}(\alpha)|^{1/q}(b-a)^{1/p} & \textit{if } f'-\alpha f \in L_p[a,b] \\ \times \|f' - \alpha f\|_p |B_q(a,b,x,\alpha)|^{1/q} & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \|f' - \alpha f\|_1 B_\infty(a,b,x,\alpha) \end{cases}$$

where

$$B_q(a,b,x,\alpha) := 2\left[\exp(xq\,\mathrm{Re}(\alpha))\left(x - \frac{a+b}{2}\right) + \frac{1}{q\,\mathrm{Re}(\alpha)}\left(\frac{\exp(bq\,\mathrm{Re}(\alpha)) + \exp(aq\,\mathrm{Re}(\alpha))}{2} - \exp(xq\,\mathrm{Re}(\alpha))\right)\right]$$

for $q \geq 1$ and

$$B_\infty(a,b,x,\alpha) := \exp(x\,\mathrm{Re}(\alpha))(x-a) + \frac{\exp(b\,\mathrm{Re}(\alpha)) - \exp(x\,\mathrm{Re}(\alpha))}{\mathrm{Re}(\alpha)}.$$

PROOF. Utilising the first inequality in (3.2) we have

$$\begin{aligned} &\left| f(x)\int_a^b \exp(\alpha t)\,dt - \exp(\alpha x)\int_a^b f(t)\,dt \right| \\ &\leq \int_a^b \left|\exp(\alpha t) f(x) - f(t)\exp(\alpha x)\right| dt \\ &\leq |\mathrm{Re}(\alpha)|\,\|f' - \alpha f\|_\infty \int_a^b \left|\exp(x\,\mathrm{Re}(\alpha)) - \exp(t\,\mathrm{Re}(\alpha))\right| dt \end{aligned} \tag{3.13}$$

for any $x \in [a,b]$.

Observe that, since $\operatorname{Re}(\alpha) > 0$, then

$$\begin{aligned}
&\int_a^b |\exp(x\operatorname{Re}(\alpha)) - \exp(t\operatorname{Re}(\alpha))|\,dt \\
&= \int_a^x (\exp(x\operatorname{Re}(\alpha)) - \exp(t\operatorname{Re}(\alpha)))\,dt \\
&+ \int_x^b (\exp(t\operatorname{Re}(\alpha)) - \exp(x\operatorname{Re}(\alpha)))\,dt \\
&= \exp(x\operatorname{Re}(\alpha))(x-a) - \left.\frac{\exp(t\operatorname{Re}(\alpha))}{\operatorname{Re}(\alpha)}\right|_a^x \\
&+ \left.\frac{\exp(t\operatorname{Re}(\alpha))}{\operatorname{Re}(\alpha)}\right|_x^b - (b-x)\exp(x\operatorname{Re}(\alpha)) \\
&= \exp(x\operatorname{Re}(\alpha))(2x-a-b) - \frac{1}{\operatorname{Re}(\alpha)}(\exp(x\operatorname{Re}(\alpha)) - \exp(a\operatorname{Re}(\alpha))) \\
&+ \frac{1}{\operatorname{Re}(\alpha)}(\exp(b\operatorname{Re}(\alpha)) - \exp(x\operatorname{Re}(\alpha))) \\
&= \exp(x\operatorname{Re}(\alpha))(2x-a-b) \\
&+ \frac{1}{\operatorname{Re}(\alpha)}(\exp(b\operatorname{Re}(\alpha)) + \exp(a\operatorname{Re}(\alpha)) - 2\exp(x\operatorname{Re}(\alpha))) \\
&= 2\left[\exp(x\operatorname{Re}(\alpha))\left(x - \frac{a+b}{2}\right)\right. \\
&\left.+ \frac{1}{\operatorname{Re}(\alpha)}\left(\frac{\exp(b\operatorname{Re}(\alpha)) + \exp(a\operatorname{Re}(\alpha))}{2} - \exp(x\operatorname{Re}(\alpha))\right)\right]
\end{aligned}$$

for any $x \in [a,b]$.

Also

$$\begin{aligned}
&f(x)\int_a^b \exp(\alpha t)\,dt - \exp(\alpha x)\int_a^b f(t)\,dt \\
&= f(x)\frac{\exp(\alpha b) - \exp(\alpha a)}{\alpha} - \exp(\alpha x)\int_a^b f(t)\,dt
\end{aligned}$$

for any $x \in [a,b]$ and by (3.13) we get the first inequality in (3.12).

Using the second inequality in (3.2) we have

$$\begin{aligned}
&\left|f(x)\int_a^b \exp(\alpha t)\,dt - \exp(\alpha x)\int_a^b f(t)\,dt\right| \\
&\le \int_a^b |\exp(\alpha t) f(x) - f(t)\exp(\alpha x)|\,dt \\
&\le q^{1/q}|\operatorname{Re}(\alpha)|^{1/q}\|f' - \alpha f\|_p \int_a^b |\exp(xq\operatorname{Re}(\alpha)) - \exp(tq\operatorname{Re}(\alpha))|^{1/q}\,dt
\end{aligned} \tag{3.14}$$

for any $x \in [a,b]$.

By Hölder's integral inequality we also have

$$\int_a^b \left|\exp\left(xq\operatorname{Re}(\alpha)\right)-\exp\left(tq\operatorname{Re}(\alpha)\right)\right|^{1/q} dt$$
$$\leq \left(\int_a^b dt\right)^{1/p}\left[\int_a^b \left(\left|\exp\left(xq\operatorname{Re}(\alpha)\right)-\exp\left(tq\operatorname{Re}(\alpha)\right)\right|^{1/q}\right)^q dt\right]^{1/q}$$
$$=(b-a)^{1/p}\left[\int_a^b \left|\exp\left(xq\operatorname{Re}(\alpha)\right)-\exp\left(tq\operatorname{Re}(\alpha)\right)\right| dt\right]^{1/q},$$

for any $x\in[a,b]$.

Observe that, as above, we have

$$\int_a^b \left|\exp\left(xq\operatorname{Re}(\alpha)\right)-\exp\left(tq\operatorname{Re}(\alpha)\right)\right| dt$$
$$=2\left[\exp\left(xq\operatorname{Re}(\alpha)\right)\left(x-\frac{a+b}{2}\right)\right.$$
$$\left.+\frac{1}{q\operatorname{Re}(\alpha)}\left(\frac{\exp\left(bq\operatorname{Re}(\alpha)\right)+\exp\left(aq\operatorname{Re}(\alpha)\right)}{2}-\exp\left(xq\operatorname{Re}(\alpha)\right)\right)\right]$$
$$=B_q(a,b,x,\alpha)$$

for any $x\in[a,b]$ and by (3.14) we get the second part of (3.12).

Using the third inequality in (3.2) we have

$$\left|f(x)\int_a^b \exp(\alpha t)\,dt-\exp(\alpha x)\int_a^b f(t)\,dt\right| \tag{3.15}$$
$$\leq \int_a^b \left|\exp(\alpha t)\,f(x)-f(t)\exp(\alpha x)\right| dt$$
$$\leq \|f'-\alpha f\|_1\int_a^b \max\left\{\exp\left(t\operatorname{Re}(\alpha)\right),\exp\left(x\operatorname{Re}(\alpha)\right)\right\} dt$$

for any $x\in[a,b]$.

Observe that,

$$\int_a^b \max\left\{\exp\left(t\operatorname{Re}(\alpha)\right),\exp\left(x\operatorname{Re}(\alpha)\right)\right\} dt$$
$$=\int_a^x \max\left\{\exp\left(t\operatorname{Re}(\alpha)\right),\exp\left(x\operatorname{Re}(\alpha)\right)\right\} dt$$
$$+\int_x^b \max\left\{\exp\left(t\operatorname{Re}(\alpha)\right),\exp\left(x\operatorname{Re}(\alpha)\right)\right\} dt$$
$$=\int_a^x \exp\left(x\operatorname{Re}(\alpha)\right) dt+\int_x^b \exp\left(t\operatorname{Re}(\alpha)\right) dt=$$
$$=\exp\left(x\operatorname{Re}(\alpha)\right)(x-a)+\frac{\exp\left(b\operatorname{Re}(\alpha)\right)-\exp\left(x\operatorname{Re}(\alpha)\right)}{\operatorname{Re}(\alpha)}$$

and by (3.15) we get the third part of (3.12). ∎

REMARK 3.2. If $\operatorname{Re}(\alpha)<0$, then a similar result may be stated. However the details are left to the interested reader.

COROLLARY 3.4. *Let* $f : [a,b] \to \mathbb{C}$ *be an absolutely continuous function on the interval* $[a,b]$*. Then for any* $x \in [a,b]$ *we have*

$$\left| f(x)[\exp(b) - \exp(a)] - \exp(x) \int_a^b f(t)\,dt \right| \tag{3.16}$$

$$\leq \begin{cases} \|f' - f\|_\infty B_1(a,b,x) & \text{if } f'-f \in L_\infty[a,b], \\ \\ q^{1/q}(b-a)^{1/p} \|f' - f\|_p & \text{if } f'-f \in L_p[a,b] \\ \times |B_q(a,b,x)|^{1/q} & p>1, \\ & \frac{1}{p}+\frac{1}{q}=1, \\ \\ \|f' - f\|_1 B_\infty(a,b,x) \end{cases}$$

where

$$\begin{aligned} &B_q(a,b,x) \\ &:= 2\left[\left(x - \frac{a+b}{2}\right)\exp(xq) + \frac{1}{q}\left(\frac{\exp(bq)+\exp(aq)}{2} - \exp(xq)\right)\right] \end{aligned}$$

for $q \geq 1$ *and*

$$B_\infty(a,b,x) := (x-a)\exp(x) + \exp(b) - \exp(x).$$

REMARK 3.3. The midpoint case is as follows:

$$\left| f\left(\frac{a+b}{2}\right)[\exp(b) - \exp(a)] - \exp\left(\frac{a+b}{2}\right) \int_a^b f(t)\,dt \right| \tag{3.17}$$

$$\leq \begin{cases} \|f' - f\|_\infty B_1(a,b) & \text{if } f'-f \in L_\infty[a,b], \\ \\ q^{1/q}(b-a)^{1/p} \|f' - f\|_p & \text{if } f'-f \in L_p[a,b] \\ \times |B_q(a,b)|^{1/q} & p>1, \\ & \frac{1}{p}+\frac{1}{q}=1, \\ \\ \|f' - f\|_1 B_\infty(a,b) \end{cases}$$

where

$$B_q(a,b,x) := \frac{2}{q}\left(\frac{\exp(bq)+\exp(aq)}{2} - \exp\left(\frac{a+b}{2}q\right)\right)$$

for $q \geq 1$ and

$$B_\infty(a,b) := \frac{b-a}{2}\exp\left(\frac{a+b}{2}\right) + \exp(b) - \exp\left(\frac{a+b}{2}\right).$$

The case $\operatorname{Re}(\alpha) = 0$ is different and may be stated as follows.

THEOREM 3.5 (Dragomir, 2013 [**4**]). *Let* $f : [a,b] \to \mathbb{C}$ *be an absolutely continuous function on the interval* $[a,b]$ *and* $\alpha \in \mathbb{C}$ *with* $\operatorname{Re}(\alpha) = 0$ *and* $\operatorname{Im}(\alpha) \neq 0$*. Then for any* $x \in [a,b]$

we have

$$\left|f(x)\frac{\exp(i\operatorname{Im}(\alpha)b)-\exp(i\operatorname{Im}(\alpha)a)}{i\operatorname{Im}(\alpha)}-\exp(i\operatorname{Im}(\alpha)x)\int_a^b f(t)\,dt\right| \tag{3.18}$$

$$\leq\begin{cases}\|f'-i\operatorname{Im}(\alpha)f\|_\infty\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2 & \text{if } f'-i\operatorname{Im}(\alpha)f\in L_\infty[a,b],\\[2ex] \frac{q}{q+1}\|f'-i\operatorname{Im}(\alpha)f\|_p\\ \times\left[\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}+\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\right](b-a)^{\frac{q+1}{q}} & \text{if } f'-i\operatorname{Im}(\alpha)f\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \|f'-i\operatorname{Im}(\alpha)f\|_1(b-a).\end{cases}$$

PROOF. Utilizing the inequality (3.9) we have

$$\left|f(x)\frac{\exp(i\operatorname{Im}(\alpha)b)-\exp(i\operatorname{Im}(\alpha)a)}{i\operatorname{Im}(\alpha)}-\exp(i\operatorname{Im}(\alpha)x)\int_a^b f(t)\,dt\right| \tag{3.19}$$

$$\leq\int_a^b|\exp(i\operatorname{Im}(\alpha)t)f(x)-f(t)\exp(i\operatorname{Im}(\alpha)x)|\,dt$$

$$\leq\begin{cases}\|f'-i\operatorname{Im}(\alpha)f\|_\infty\int_a^b|x-t|\,dt,\\[1ex] \|f'-i\operatorname{Im}(\alpha)f\|_p\int_a^b|x-t|^{1/q}\,dt,\\[1ex] \|f'-i\operatorname{Im}(\alpha)f\|_1\int_a^b dt.\end{cases}$$

Since

$$\int_a^b|x-t|\,dt=\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2$$

and

$$\int_a^b|x-t|^{1/q}\,dt=\frac{(b-x)^{1+1/q}+(x-a)^{1+1/q}}{1+1/q}$$
$$=\frac{q}{q+1}\left[\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}+\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\right](b-a)^{\frac{q+1}{q}},$$

then we get from (3.19) the desired result (3.18). ∎

COROLLARY 3.6. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$. Then for any $x \in [a,b]$ we have*

$$\left| f(x) \frac{\exp(ib) - \exp(ia)}{i} - \exp(ix) \int_a^b f(t)\,dt \right| \tag{3.20}$$

$$\leq \begin{cases} \|f' - if\|_\infty \times \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)^2 & \text{if } f' - if \in L_\infty[a,b], \\ \\ \frac{q}{q+1} \|f' - if\|_p \times \left[\left(\frac{b-x}{b-a} \right)^{\frac{q+1}{q}} + \left(\frac{x-a}{b-a} \right)^{\frac{q+1}{q}} \right] (b-a)^{\frac{q+1}{q}} & \text{if } f' - if \in L_p[a,b] \; p > 1, \; \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \|f' - if\|_1 (b-a). \end{cases}$$

REMARK 3.4. The midpoint case is as follows

$$\left| f\left(\frac{a+b}{2}\right) \frac{\exp(ib) - \exp(ia)}{i} - \exp\left(i\frac{a+b}{2}\right) \int_a^b f(t)\,dt \right| \tag{3.21}$$

$$\leq \begin{cases} \frac{1}{4} \|f' - if\|_\infty (b-a)^2, \text{ if } f' - if \in L_\infty[a,b], \\ \\ \frac{q}{(q+1)2^{1/q}} \|f' - if\|_p (b-a)^{\frac{q+1}{q}}, \text{ if } f' - if \in L_p[a,b]. \end{cases}$$

Similar inequalities may be stated if one uses (3.1) and integrates over t on $[a,b]$. The details are left to the interested reader.

4. OSTROWSKI VIA A TWO FUNCTIONS POMPEIU'S INEQUALITY

4.1. A General Pompeiu's Inequality. We start with the following generalization of Pompeiu's inequality:

Assume that $f : [a,b] \to \mathbb{R}$ is differentiable on the interval (a,b) which is not containing 0 and if $\|f - \ell f'\|_\infty = \sup_{t \in (a,b)} |f(t) - tf'(t)| < \infty$ where $\ell(t) = t$, $t \in [a,b]$, then

$$|tf(x) - xf(t)| \leq \|f - \ell f'\|_\infty |x - t| \tag{4.1}$$

for any $t, x \in [a,b]$.

The inequality (4.1) was stated by the author in [**1**].

THEOREM 4.1 (Dragomir, 2013 [**5**]). *Let $f,g:[a,b]\to\mathbb{C}$ be absolutely continuous functions on the interval $[a,b]$ with $g(t)\neq 0$ for all $t\in[a,b]$. Then for any $t,x\in[a,b]$ we have*

$$\left|\frac{f(x)}{g(x)}-\frac{f(t)}{g(t)}\right| \tag{4.2}$$

$$\leq\begin{cases}\left\|f'g-fg'\right\|_{\infty}\left|\int_t^x\frac{1}{|g(s)|^2}ds\right| & \text{if } f'g-fg'\in L_{\infty}[a,b],\\[2ex] \left\|f'g-fg'\right\|_{p}\left|\int_t^x\frac{1}{|g(s)|^{2q}}ds\right|^{1/q} & \begin{array}{c}\text{if } f'g-fg'\in L_{p}[a,b]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1,\end{array}\\[2ex] \left\|f'g-fg'\right\|_{1}\sup_{s\in[t,x]([x,t])}\left\{\frac{1}{|g(s)|^2}\right\} & \end{cases}$$

or, equivalently

$$|g(t)f(x)-f(t)g(x)| \tag{4.3}$$

$$\leq\begin{cases}\left\|f'g-fg'\right\|_{\infty}|g(t)g(x)|\left|\int_t^x\frac{1}{|g(s)|^2}ds\right| & \text{if } f'g-fg'\in L_{\infty}[a,b],\\[2ex] \left\|f'g-fg'\right\|_{p}|g(t)g(x)|\left|\int_t^x\frac{1}{|g(s)|^{2q}}ds\right|^{1/q} & \begin{array}{c}\text{if } f'g-fg'\in L_{p}[a,b]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1,\end{array}\\[2ex] \left\|f'g-fg'\right\|_{1}|g(t)g(x)|\sup_{s\in[t,x]([x,t])}\left\{\frac{1}{|g(s)|^2}\right\}. & \end{cases}$$

PROOF. If f and g are absolutely continuous and $g(t)\neq 0$ for all $t\in[a,b]$, then f/g is absolutely continuous on the interval $[a,b]$ and

$$\int_t^x\left(\frac{f(s)}{g(s)}\right)'ds=\frac{f(x)}{g(x)}-\frac{f(t)}{g(t)}$$

for any $t,x\in[a,b]$ with $x\neq t$.

Since

$$\int_t^x\left(\frac{f(s)}{g(s)}\right)'ds=\int_t^x\frac{f'(s)g(s)-f(s)g'(s)}{g^2(s)}ds,$$

then we get the following identity

$$\frac{f(x)}{g(x)}-\frac{f(t)}{g(t)}=\int_t^x\frac{f'(s)g(s)-f(s)g'(s)}{g^2(s)}ds \tag{4.4}$$

for any $t,x\in[a,b]$.

Taking the modulus in (4.4) we have

$$\begin{aligned}\left|\frac{f(x)}{g(x)}-\frac{f(t)}{g(t)}\right|&=\left|\int_t^x\frac{f'(s)g(s)-f(s)g'(s)}{g^2(s)}ds\right|\\&\leq\left|\int_t^x\frac{|f'(s)g(s)-f(s)g'(s)|}{|g(s)|^2}ds\right|:=I\end{aligned} \tag{4.5}$$

and utilizing Hölder's integral inequality we deduce

$$I \leq \begin{cases} \sup_{s\in[t,x]([x,t])} \left|f'(s)g(s)-f(s)g'(s)\right| \left|\int_t^x \frac{1}{|g(s)|^2}ds\right|, \\ \\ \left|\int_t^x \left|f'(s)g(s)-f(s)g'(s)\right|^p ds\right|^{1/p} \left|\int_t^x \frac{1}{|g(s)|^{2q}}ds\right|^{1/q} & \begin{array}{c} p>1, \\ \frac{1}{p}+\frac{1}{q}=1, \end{array} \\ \\ \left|\int_t^x \left|f'(s)g(s)-f(s)g'(s)\right| ds\right| \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|g(s)|^2}\right\}, \end{cases}$$

$$\leq \begin{cases} \left\|f'g-fg'\right\|_\infty \left|\int_t^x \frac{1}{|g(s)|^2}ds\right|, \\ \\ \left\|f'g-fg'\right\|_p \left|\int_t^x \frac{1}{|g(s)|^{2q}}ds\right|^{1/q} & \begin{array}{c} p>1, \\ \frac{1}{p}+\frac{1}{q}=1, \end{array} \\ \\ \left\|f'g-fg'\right\|_1 \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|g(s)|^2}\right\} \end{cases}$$

and the inequality (4.2) is proved. ∎

The following particular case extends Pompeiu's inequality to other p-norms than $p=\infty$ obtained in (4.2).

COROLLARY 4.2. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b>a>0$. Then for any $t,x\in[a,b]$ we have*

$$(4.6)\quad \left|\frac{f(x)}{x}-\frac{f(t)}{t}\right|$$

$$\leq \begin{cases} \left\|f-\ell f'\right\|_\infty \left|\frac{1}{t}-\frac{1}{x}\right| & \text{if } f-\ell f'\in L_\infty[a,b], \\ \\ \frac{1}{2q-1}\left\|f-\ell f'\right\|_p \left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right|^{1/q} & \begin{array}{c} \text{if } f-\ell f'\in L_p[a,b] \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1, \end{array} \\ \\ \left\|f-\ell f'\right\|_1 \frac{1}{\min\{t^2,x^2\}} \end{cases}$$

or, equivalently

$$(4.7)\quad \left|tf(x)-xf(t)\right|$$

$$\leq \begin{cases} \left\|f-\ell f'\right\|_\infty |x-t| & \text{if } f-\ell f'\in L_\infty[a,b], \\ \\ \frac{1}{2q-1}\left\|f-\ell f'\right\|_p \left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|^{1/q} & \begin{array}{l} \text{if } f-\ell f'\in L_p[a,b] \\ p>1, \frac{1}{p}+\frac{1}{q}=1, \end{array} \\ \\ \left\|f-\ell f'\right\|_1 \frac{\max\{t,x\}}{\min\{t,x\}}, \end{cases}$$

where $\ell(t)=t, t\in[a,b]$.

The proof follows by (4.2) for $g(t)=\ell(t)=t$, $t\in[a,b]$.
The general case for power functions is as follows.

COROLLARY 4.3. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b>a>0$. If $r\in\mathbb{R}$, $r\neq 0$, then for any $t,x\in[a,b]$ we have*

$$\left|\frac{f(x)}{x^r}-\frac{f(t)}{t^r}\right| \tag{4.8}$$

$$\leq\begin{cases}\frac{1}{|r|}\left\|f'\ell-rf\right\|_\infty\left|\frac{1}{x^r}-\frac{1}{t^r}\right|, \text{ if } f'\ell-rf\in L_\infty[a,b],\\ \\ \left\|f'\ell-rf\right\|_p\\ \quad\times\begin{cases}\frac{1}{|1-q(r+1)|}\left|\frac{1}{x^{1-q(r+1)}}-\frac{1}{t^{1-q(r+1)}}\right|, \text{ for } r\neq-\frac{1}{p}\\ \\ \left|\ln x-\ln t\right|, \text{ for } r=-\frac{1}{p}\end{cases}\\ \text{if } f'\ell-rf\in L_p[a,b],\\ \\ \left\|f'\ell-rf\right\|_1\frac{1}{\min\{x^{r+1},t^{r+1}\}},\end{cases}$$

or, equivalently

$$\left|t^rf(x)-x^rf(t)\right| \tag{4.9}$$

$$\leq\begin{cases}\frac{1}{|r|}\left\|f'\ell-rf\right\|_\infty\left|t^r-x^r\right|, \text{ if } f'\ell-rf\in L_\infty[a,b],\\ \\ \left\|f'\ell-rf\right\|_p\\ \quad\times\begin{cases}\frac{1}{|1-q(r+1)|}\left|\frac{t^r}{x^{1-q(r+1)-r}}-\frac{x^r}{t^{1-q(r+1)-r}}\right|, \text{ for } r\neq-\frac{1}{p}\\ \\ t^rx^r\left|\ln x-\ln t\right|, \text{ for } r=-\frac{1}{p}\end{cases}\\ \text{if } f'\ell-rf\in L_p[a,b],\\ \\ \left\|f'\ell-rf\right\|_1\frac{t^rx^r}{\min\{x^{r+1},t^{r+1}\}},\end{cases}$$

where $p>1,\frac{1}{p}+\frac{1}{q}=1.$

The proof follows by (4.2) for $g(t)=t^r$, $t\in[a,b]$. The details for calculations are omitted. We have the following result for exponential.

COROLLARY 4.4. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ and $\alpha\in\mathbb{R}$, $\alpha\neq 0$. Then for any $t,x\in[a,b]$ we have*

$$\left|\frac{f(x)}{\exp(i\alpha x)}-\frac{f(t)}{\exp(i\alpha t)}\right| \tag{4.10}$$

$$\leq\begin{cases}\left\|f'-i\alpha f\right\|_\infty|x-t| & \text{if } f'-i\alpha f\in L_\infty[a,b],\\ \\ \left\|f'-i\alpha f\right\|_p|x-t|^{1/q} & \begin{array}{c}\text{if } f'-i\alpha f\in L_p[a,b]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1,\end{array}\\ \\ \left\|f'-i\alpha f\right\|_1\end{cases}$$

or, equivalently

$$|\exp(i\alpha t) f(x) - f(t)\exp(i\alpha x)| \tag{4.11}$$

$$\leq \begin{cases} \|f' - i\alpha f\|_\infty |x-t| & \text{if } f' - i\alpha f \in L_\infty[a,b], \\ \|f' - i\alpha f\|_p |x-t|^{1/q} & \begin{array}{c}\text{if } f' - i\alpha f \in L_p[a,b] \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1, \end{array} \\ \|f' - i\alpha f\|_1 . \end{cases}$$

4.2. An Inequality Generalizing Ostrowski's. The following result holds:

THEOREM 4.5 (Dragomir, 2013 [**5**]). *Let $f, g : [a,b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a,b]$. If $0 < m \leq |g(t)| \leq M < \infty$ for any $t \in [a,b]$, then*

$$\left| f(x) \int_a^b g(t)\,dt - g(x) \int_a^b f(t)\,dt \right| \tag{4.12}$$

$$\leq \left(\frac{M}{m}\right)^2 \begin{cases} \|f'g - fg'\|_\infty (b-a)^2 \left[\frac{1}{4} + \left(\frac{t - \frac{a+b}{2}}{b-a}\right)^2\right] & \text{if } f'g - fg' \in L_\infty[a,b], \\ \|f'g - fg'\|_p \left[\frac{(b-x)^{1+1/q} + (x-a)^{1+1/q}}{1+1/q}\right] & \begin{array}{c}\text{if } f'g - fg' \in L_p[a,b] \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1, \end{array} \\ \|f'g - fg'\|_1 (b-a) \end{cases}$$

for any $x \in [a,b]$.

PROOF. Utilizing (4.3) we have

$$\left| f(x) \int_a^b g(t)\,dt - g(x) \int_a^b f(t)\,dt \right| \tag{4.13}$$

$$\leq \int_a^b |g(t) f(x) - f(t) g(x)|\,dt$$

$$\leq \begin{cases} \|f'g - fg'\|_\infty |g(x)| \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \right) dt, \\ \|f'g - fg'\|_p |g(x)| \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \right) dt, \\ \|f'g - fg'\|_1 |g(x)| \int_a^b \left(|g(t)| \sup_{s\in[t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} \right) dt \end{cases}$$

for any $x \in [a,b]$, which is of interest in itself.

Since $0 < m \le |g(t)| \le M < \infty$ for any $t \in [a,b]$, then

$$|g(x)| \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \right) dt \le \left(\frac{M}{m} \right)^2 \int_a^b |x-t| \, dt$$
$$= \left(\frac{M}{m} \right)^2 \left[\frac{1}{4} + \left(\frac{t - \frac{a+b}{2}}{b-a} \right)^2 \right],$$

$$|g(x)| \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \right) dt$$
$$\le \left(\frac{M}{m} \right)^2 \int_a^b |x-t|^{1/q} \, dt = \left(\frac{M}{m} \right)^2 \frac{(b-x)^{1+1/q} + (x-a)^{1+1/q}}{1+1/q}$$

and

$$|g(x)| \int_a^b \left(|g(t)| \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} \right) dt \le \left(\frac{M}{m} \right)^2 \int_a^b dt = \left(\frac{M}{m} \right)^2 (b-a)$$

for any $x \in [a,b]$ and by (4.13) we get the desired result (4.12). ∎

REMARK 4.1. If we take $g(t) = 1, t \in [a,b]$ in the first inequality (4.12) we recapture Ostrowski's inequality.

COROLLARY 4.6. *With the assumptions in Theorem 4.5 we have the midpoint inequalities*

$$\left| f\left(\frac{a+b}{2} \right) \int_a^b g(t) \, dt - g\left(\frac{a+b}{2} \right) \int_a^b f(t) \, dt \right| \tag{4.14}$$
$$\le \left(\frac{M}{m} \right)^2 \begin{cases} \frac{1}{4}(b-a)^2 \|f'g - fg'\|_\infty & \text{if } f'g - fg' \in L_\infty[a,b], \\ \\ \frac{1}{2^{1/q}(1+1/q)} (b-a)^{1+1/q} \|f'g - fg'\|_p & \begin{array}{c} \text{if } f'g - fg' \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1. \end{array} \end{cases}$$

The following result also holds:

THEOREM 4.7 (Dragomir, 2013 [**5**]). *Let $f, g : [a,b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a,b]$, $g(x) \neq 0$ for $x \in [a,b]$ and $g^{-2} \in L_\infty[a,b]$. Then*

$$\left| \frac{f(x)}{g(x)} \int_a^b g(t) \, dt - \int_a^b f(t) \, dt \right| \tag{4.15}$$
$$\le \left\| g^{-2} \right\|_\infty \times \begin{cases} \|f'g - fg'\|_\infty \int_a^b |g(t)| \, |x-t| \, dt, & \text{if } f'g - fg' \in L_\infty[a,b], \\ \\ \|f'g - fg'\|_p \int_a^b |g(t)| \, |x-t|^{1/q} \, dt & \begin{array}{c} \text{if } f'g - fg' \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\ \\ \|f'g - fg'\|_1 \int_a^b |g(t)| \, dt \end{cases}$$

for any $x \in [a, b]$.

PROOF. Utilizing (4.3) we have

$$\left|\frac{f(x)}{g(x)}\int_a^b g(t)\,dt - \int_a^b f(t)\,dt\right| \tag{4.16}$$

$$\leq \begin{cases} \|f'g - fg'\|_\infty \int_a^b \left(|g(t)| \left|\int_t^x \frac{1}{|g(s)|^2} ds\right|\right) dt, \\ \\ \|f'g - fg'\|_p \int_a^b \left(|g(t)| \left|\int_t^x \frac{1}{|g(s)|^{2q}} ds\right|^{1/q}\right) dt, \\ \\ \|f'g - fg'\|_1 \int_a^b \left(|g(t)| \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|g(s)|^2}\right\}\right) dt \end{cases}$$

for any $x \in [a, b]$.

Since

$$\left|\int_t^x \frac{1}{|g(s)|^2} ds\right| \leq \left\|g^{-2}\right\|_\infty |x - t|,$$

$$\left|\int_t^x \frac{1}{|g(s)|^{2q}} ds\right|^{1/q} \leq \left\|g^{-2}\right\|_\infty |x - t|$$

and

$$\sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|g(s)|^2}\right\} \leq \left\|g^{-2}\right\|_\infty$$

for any $x, t \in [a, b]$, then on making use of (4.16) we get the desired result (4.15). ∎

We have the midpoint inequalities:

COROLLARY 4.8. *With the assumptions of Theorem 4.7 we have*

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\int_a^b g(t)\,dt - \int_a^b f(t)\,dt\right| \tag{4.17}$$

$$\leq \left\|g^{-2}\right\|_\infty \times \begin{cases} \|f'g - fg'\|_\infty \int_a^b |g(t)| \left|\frac{a+b}{2} - t\right| dt, & \textit{if } f'g - fg' \in L_\infty[a,b], \\ \\ \|f'g - fg'\|_p \int_a^b |g(t)| \left|\frac{a+b}{2} - t\right|^{1/q} dt & \begin{array}{c}\textit{if } f'g - fg' \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1.\end{array} \end{cases}$$

We have the following exponential version of Ostrowski's inequality as well:

THEOREM 4.9 (Dragomir, 2013 [**5**]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ and $\alpha\in\mathbb{R}$, $\alpha\neq 0$. Then for any $x\in[a,b]$ we have*

$$\left|\frac{\exp(i\alpha(b-x))-\exp(-i\alpha(x-a))}{i\alpha}f(x)-\int_a^b f(t)\,dt\right| \tag{4.18}$$

$$\leq\begin{cases}\|f'-i\alpha f\|_\infty(b-a)^2\left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right], & \text{if } f'-i\alpha f\in L_\infty[a,b],\\[2ex] \|f'-i\alpha f\|_p\frac{(b-x)^{1+1/q}+(x-a)^{1+1/q}}{1+1/q}, & \begin{array}{c}\text{if } f'-i\alpha f\\ \in L_p[a,b]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1,\end{array}\\[2ex] \|f'-i\alpha f\|_1.\end{cases}$$

PROOF. If we write the inequality (4.13) for $g(t)=\exp(i\alpha t)$, $t\in[a,b]$, then we get

$$\left|f(x)\int_a^b\exp(i\alpha t)\,dt-\exp(i\alpha x)\int_a^b f(t)\,dt\right|$$

$$\leq\begin{cases}\|f'-i\alpha f\|_\infty\int_a^b|x-t|\,dt, & \text{if } f'-i\alpha f\in L_\infty[a,b]\\[2ex] \|f'-i\alpha f\|_p|g(x)|\int_a^b|x-t|^{1/q}\,dt, & \begin{array}{c}\text{if } f'-i\alpha f\\ \in L_p[a,b]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1\end{array}\\[2ex] \|f'-i\alpha f\|_1,\end{cases}$$

which, after simple calculation, is equivalent with (4.18).

The details are omitted. ∎

COROLLARY 4.10. *With the assumptions of Theorem 4.9 we have the midpoint inequalities*

$$\left|\frac{\exp\left(i\alpha\left(\frac{b-a}{2}\right)\right)-\exp\left(-i\alpha\left(\frac{b-a}{2}\right)\right)}{i\alpha}f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right| \tag{4.19}$$

$$\leq\begin{cases}\frac{1}{4}\|f'-i\alpha f\|_\infty(b-a)^2, & \text{if } f'-i\alpha f\in L_\infty[a,b],\\[2ex] \frac{1}{2^{1/q}(1+1/q)}(b-a)^{1+1/q}\|f'-i\alpha f\|_p, & \begin{array}{l}\text{if } f'-i\alpha f\in L_p[a,b]\\ p>1,\frac{1}{p}+\frac{1}{q}=1,\end{array}\end{cases}$$

or, equivalently

$$\left|\frac{2\sin\left(\alpha\left(\frac{b-a}{2}\right)\right)}{\alpha}f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right| \tag{4.20}$$

$$\leq \begin{cases} \frac{1}{4}\left\|f'-i\alpha f\right\|_\infty (b-a)^2\,, & \textit{if } f'-i\alpha f\in L_\infty[a,b]\,, \\ \\ \frac{1}{2^{1/q}(1+1/q)}(b-a)^{1+1/q}\left\|f'-i\alpha f\right\|_p\,, & \begin{array}{l}\textit{if } f'-i\alpha f\in L_p[a,b] \\ p>1,\frac{1}{p}+\frac{1}{q}=1.\end{array} \end{cases}$$

4.3. An Application for CBS-Inequality. The following inequality is well known in the literature as the Cauchy-Bunyakovsky-Schwarz inequality, or the CBS-inequality, for short:

$$\left|\int_a^b f(t)\,g(t)\,dt\right|^2 \leq \int_a^b |f(t)|^2\,dt\int_a^b |g(t)|^2\,dt, \tag{4.21}$$

provided that $f,g\in L_2[a,b]$.

We have the following result concerning some reverses of the CBS-inequality:

THEOREM 4.11 (Dragomir, 2013 [**5**]). *Let $f,g:[a,b]\to\mathbb{C}$ be absolutely continuous functions on the interval $[a,b]$ with $g(t)\neq 0$ for all $t\in[a,b]$. Then*

$$0\leq \int_a^b |g(t)|^2\,dt\int_a^b |f(t)|^2\,dt-\left|\int_a^b f(t)\,g(t)\,dt\right|^2 \tag{4.22}$$

$$\leq \frac{1}{2}\times\begin{cases} \left\|f'\overline{g}-f\overline{g}'\right\|_\infty^2\left(\int_a^b|g(t)|^2dt\right)^2\left(\int_a^b\frac{1}{|g(t)|^2}dt\right)^2, & \textit{if } \begin{array}{c} f'\overline{g}-f\overline{g}'\in L_\infty[a,b]\,, \\ \frac{1}{|g|^2}\in L[a,b]\end{array} \\ \\ \left\|f'\overline{g}-f\overline{g}'\right\|_p^2\left(\int_a^b|g(t)|^2dt\right)^2\left(\int_a^b\frac{1}{|g(t)|^{2q}}dt\right)^{2/q}, & \textit{if } \begin{array}{c} f'\overline{g}-f\overline{g}'\in L_p[a,b]\,, \\ \frac{1}{|g|^{2q}}\in L[a,b] \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1,\end{array} \\ \\ \left\|f'\overline{g}-f\overline{g}'\right\|_1^2\left(\int_a^b|g(t)|^2dt\right)^2 ess\sup_{t\in[a,b]}\left\{\frac{1}{|g(t)|^4}\right\}, & \textit{if } \frac{1}{|g|}\in L_\infty[a,b]\,. \end{cases}$$

PROOF. Utilising the inequality (4.3) we have

$$\left|\overline{g(t)}f(x)-f(t)\overline{g(x)}\right| \tag{4.23}$$

$$\leq \begin{cases} \|f'\overline{g}-f\overline{g}'\|_\infty |g(t)g(x)| \left|\int_t^x \frac{1}{|g(s)|^2}ds\right| & \text{if } f'\overline{g}-f\overline{g}' \in L_\infty[a,b], \\ \\ \|f'\overline{g}-f\overline{g}'\|_p |g(t)g(x)| \left|\int_t^x \frac{1}{|g(s)|^{2q}}ds\right|^{1/q} & \text{if } f'\overline{g}-f\overline{g}' \in L_p[a,b] \\ & p>1, \\ & \frac{1}{p}+\frac{1}{q}=1, \\ \\ \|f'\overline{g}-f\overline{g}'\|_1 |g(t)g(x)| \sup_{s\in[t,x]([x,t])}\left\{\frac{1}{|g(s)|^2}\right\}. \end{cases}$$

for any $t,x\in[a,b]$.

Taking the square in (4.23) and integrating over $(t,x)\in[a,b]^2$ we have

$$\int_a^b\int_a^b \left|\overline{g(t)}f(x)-f(t)\overline{g(x)}\right|^2 dtdx \tag{4.24}$$

$$\leq \begin{cases} \|f'\overline{g}-f\overline{g}'\|_\infty^2 \int_a^b\int_a^b |g(t)g(x)|^2 \left|\int_t^x \frac{1}{|g(s)|^2}ds\right|^2 dtdx, \\ \\ \|f'\overline{g}-f\overline{g}'\|_p^2 \int_a^b\int_a^b |g(t)g(x)|^2 \left|\int_t^x \frac{1}{|g(s)|^{2q}}ds\right|^{2/q} dtdx, \\ \\ \|f'\overline{g}-f\overline{g}'\|_1^2 \int_a^b\int_a^b |g(t)g(x)|^2 \sup_{s\in[t,x]([x,t])}\left\{\frac{1}{|g(s)|^4}\right\} dtdx. \end{cases}$$

Observe that

$$\begin{aligned} &\int_a^b\int_a^b \left|\overline{g(t)}f(x)-f(t)\overline{g(x)}\right|^2 dtdx \\ &= \int_a^b\int_a^b \left(|g(t)|^2|f(x)|^2 - 2\operatorname{Re}\left[\overline{g(t)}f(x)\overline{f(t)\overline{g(x)}}\right] + |g(x)|^2|f(t)|^2\right)dtdx \\ &= \int_a^b |g(t)|^2dt\int_a^b|f(x)|^2dx - 2\operatorname{Re}\left[\int_a^b\overline{f(t)g(t)}dt\int_a^b f(x)g(x)dx\right] \\ &+ \int_a^b|g(x)|^2dx\int_a^b|f(t)|^2dt \\ &= 2\left[\int_a^b|g(t)|^2dt\int_a^b|f(t)|^2dt - \left|\int_a^b f(t)g(t)dt\right|^2\right], \end{aligned}$$

$$\begin{aligned} &\int_a^b\int_a^b\left[|g(t)g(x)|^2\left|\int_t^x\frac{1}{|g(s)|^2}ds\right|^2\right]dtdx \\ &\leq \left(\int_a^b|g(t)|^2dt\right)^2\left(\int_a^b\frac{1}{|g(t)|^2}dt\right)^2, \end{aligned}$$

$$\begin{aligned}&\int_a^b \int_a^b \left[|g(t)\, g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{2/q} \right] dtdx \\ &\leq \left(\int_a^b |g(t)|^2\, dt \right)^2 \left(\int_a^b \frac{1}{|g(t)|^{2q}} dt \right)^{2/q}\end{aligned}$$

and

$$\begin{aligned}&\int_a^b \int_a^b \left[|g(t)\, g(x)|^2 \sup_{s\in[t,x]([x,t])} \left\{ \frac{1}{|g(s)|^4} \right\} \right] dtdx \\ &\leq \left(\int_a^b |g(t)|^2\, dt \right)^2 ess \sup_{t\in[a,b]} \left\{ \frac{1}{|g(t)|^4} \right\},\end{aligned}$$

then by (4.24) we get the desired result (4.22). ∎

REFERENCES

[1] S. S. Dragomir, An inequality of Ostrowski type via Pompeiu's mean value theorem. *J. Inequal. Pure Appl. Math.* **6** (2005), no. 3, Article 83, 9 pp.

[2] S. S. Dragomir, Power Pompeiu's type inequalities for absolutely continuous functions with applications to Ostrowski's inequality, *Stud. Univ. Babeş-Bolyai Math.* **59** (2014), no. 2, 131–140. Preprint *RGMIA Res. Rep. Coll.* **16** (2013), Art. 68, pp. 8 [Online http://rgmia.org/papers/v16/v16a68.pdf].

[3] S. S. Dragomir, Inequalities of Pompeiu's type for absolutely continuous functions with applications to Ostrowski's inequality, *Acta Math. Acad. Paedagog. Nyházi.* (N.S.) **30** (2014), no. 1, 43–52. Preprint *RGMIA Res. Rep. Coll.* **16** (2013), Art. 66, pp. 10 [Online http://rgmia.org/papers/v16/v16a66].

[4] S. S. Dragomir, Exponential Pompeiu's type inequalities with applications to Ostrowski's inequality, *Acta Math. Univ. Comenian.* (N.S.) **84** (2015), No. 1, 39–50. Preprint *RGMIA Res. Rep. Coll.* **16** (2013), Art. 70, pp. 12 [Online http://rgmia.org/papers/v16/v16a70.pdf].

[5] S. S. Dragomir, Ostrowski via a two functions Pompeiu's inequality, *An. St. Univ. Ovidius Constanta*, Vol. **24** (3), 2016, 123-139. Preprint *RGMIA Res. Rep. Coll.* **16** (2013), Art. 90, 13pp. [Online http://rgmia.org/papers/v16/v16a90.pdf].

[6] D. Pompeiu, Sur une proposition analogue au théorème des accroissements finis, *Mathematica* (Cluj, Romania), **22**(1946), 143-146.

[7] J. Pečarić and Š. Ungar, On an inequality of Ostrowski type, *J. Ineq. Pure & Appl. Math.* **7** (2006). No. 4. Art. 151.

[8] P. K. Sahoo and T. Riedel, *Mean Value Theorems and Functional Equations*, World Scientific, Singapore, New Jersey, London, Hong Kong, 2000.

CHAPTER 5

Inequalities for Derivatives with Special Properties

1. Ostrowski for Convex Derivatives

1.1. A Representation Result. In [**8**], the author pointed out the following identity in representing an absolutely continuous function.

LEMMA 1.1 (Dragomir, 2002 [**8**]). *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then for any $x\in[a,b]$, one has the equality:*

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{b-a}\int_a^b (x-t)\left(\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right)dt. \tag{1.1}$$

PROOF. For any $t,x\in[a,b]$, $x\neq t$, one has

$$\frac{f(x)-f(t)}{x-t}=\frac{1}{x-t}\int_t^x f'(u)\,du=\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda,$$

showing that

$$f(x)=f(t)+(x-t)\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda \tag{1.2}$$

for any $t,x\in[a,b]$.

If we integrate (1.2) over t on $[a,b]$ and divide by $(b-a)$, we deduce the desired identity (1.1). ∎

1.2. Ostrowski for Convex Derivatives. The following result holds.

THEOREM 1.2 (Dragomir & Sofo, 2002 [**17**]). *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that its derivative f' is convex on (a,b). Then for any $x\in(a,b)$ one has the inequality*

$$\begin{aligned}\frac{1}{2}\cdot\frac{b-x}{b-a}&\left[\frac{1}{b-a}\int_x^b f(u)\,du-f(b)-\frac{1}{2}(b-x)f'(x)\right]\\&\qquad+2\frac{(x-a)}{b-a}\left[\frac{2}{x-a}\int_{\frac{a+x}{2}}^x f(u)\,du-f\left(\frac{a+x}{2}\right)\right]\\&\leq f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\\&\leq\frac{1}{2}\cdot\frac{x-a}{b-a}\left[\frac{1}{x-a}\int_a^x f(u)\,du-f(a)-\frac{1}{2}(x-a)f'(x)\right]\\&\qquad+2\frac{(b-x)}{b-a}\left[\frac{2}{b-x}\int_x^{\frac{x+b}{2}} f(u)\,du-f\left(\frac{x+b}{2}\right)\right].\end{aligned} \tag{1.3}$$

PROOF. For any $x \in (a, b)$, one has the identity (see Lemma 1.1)

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\left[\int_a^x (x-t)\left(\int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda\right)dt + \int_x^b (x-t)\left(\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right)dt\right]. \tag{1.4}$$

Since f' is convex, then by the Hermite-Hadamard inequality for f' we have

$$f'\left(\frac{x+t}{2}\right) \le \int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda \le \frac{f'(x)+f'(t)}{2} \tag{1.5}$$

for any $t, x \in (a, b)$.

Assume $a \le t \le x$. Then by (1.5) we get

$$\int_a^x (x-t) f'\left(\frac{x+t}{2}\right)dt \le \int_a^x (x-t)\left(\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right)dt \le \int_a^x \left[\frac{f'(x)+f'(t)}{2}\right](x-t)\,dt. \tag{1.6}$$

Assume $x \le t \le b$. Then by (1.5) we also get

$$\int_x^b (x-t)\left[\frac{f'(x)+f'(t)}{2}\right]dt \le \int_x^b (x-t)\left(\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right)dt \le \int_x^b (x-t) f'\left(\frac{x+t}{2}\right)dt. \tag{1.7}$$

Summing (1.6) with (1.7), dividing with $b-a$ and using the identity (1.4) we deduce

$$\begin{aligned}\frac{1}{b-a}&\left[\int_a^x (x-t) f'\left(\frac{x+t}{2}\right)dt + \int_x^b (x-t)\left[\frac{f'(x)+f'(t)}{2}\right]dt\right] \\ &\le f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le \frac{1}{b-a}\left[\int_a^x \left[\frac{f'(x)+f'(t)}{2}\right](x-t)\,dt + \int_x^b (x-t) f'\left(\frac{x+t}{2}\right)dt\right].\end{aligned} \tag{1.8}$$

Since

$$\begin{aligned}\int_a^x (x-t) f'\left(\frac{x+t}{2}\right)dt &= 2\int_a^x f\left(\frac{x+t}{2}\right)dt - 2f\left(\frac{x+a}{2}\right)(x-a) \\ &= 4\int_{\frac{a+x}{2}}^x f(u)\,du - 2f\left(\frac{a+x}{2}\right)(x-a),\end{aligned}$$

$$\int_x^b (x-t)\left[\frac{f'(x)+f'(t)}{2}\right]dt = \frac{1}{2}\int_x^b f(t)\,dt - \frac{1}{2}(b-x) f(b) - \frac{1}{4}(b-x)^2 f'(x),$$

$$\int_a^x \left[\frac{f'(x)+f'(t)}{2}\right](x-t)\,dt = \frac{1}{2}\int_a^x f(t)\,dt - \frac{1}{2}(x-a) f(a) + \frac{1}{4}(x-a)^2 f'(x),$$

and

$$\begin{aligned}\int_x^b (x-t) f'\left(\frac{x+t}{2}\right) dt &= 2\int_x^b f\left(\frac{x+t}{2}\right) dt - 2f\left(\frac{x+b}{2}\right)(b-x)\\ &= 4\int_x^{\frac{x+b}{2}} f(u)\,du - 2f\left(\frac{x+b}{2}\right)(b-x),\end{aligned}$$

then by (1.8) we deduce the desired result. ∎

The following corollary is natural to be considered.

COROLLARY 1.3. *Assume that $f:[a,b]\to\mathbb{R}$ is as in Theorem 1.2. Then*

$$\begin{aligned}&\frac{1}{4}\left[\frac{2}{b-a}\int_{\frac{a+b}{2}}^b f(u)\,du - f(b) - \frac{1}{4}(b-a) f'\left(\frac{a+b}{2}\right)\right]\\ &\qquad + \frac{4}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(u)\,du - f\left(\frac{3a+b}{4}\right)\\ &\le f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\\ &\le \frac{1}{4}\left[\frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(u)\,du - f(a) + \frac{1}{4}(b-a) f'\left(\frac{a+b}{2}\right)\right]\\ &\qquad + \frac{4}{b-a}\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(u)\,du - f\left(\frac{a+3b}{4}\right).\end{aligned} \tag{1.9}$$

COROLLARY 1.4. *Assume that $f:[a,b]\to\mathbb{R}$ is as in Theorem 1.2. Then*

$$\begin{aligned}&\frac{8}{5(b-a)}\int_{\frac{a+b}{2}}^b f(u)\,du - \frac{1}{5}f(b) - \frac{1}{10}(b-a) f'(a) - \frac{4}{5}f\left(\frac{a+b}{2}\right)\\ &\le \frac{4}{5}\left[\frac{f(a)+f(b)}{2}\right] - \frac{1}{b-a}\int_a^b f(u)\,du\\ &\le \frac{8}{5(b-a)}\int_a^{\frac{a+b}{2}} f(u)\,du - \frac{1}{5}f(a) + \frac{1}{10}(b-a) f'(b) - \frac{4}{5}f\left(\frac{a+b}{2}\right).\end{aligned} \tag{1.10}$$

PROOF. At $x=a$, we have from (1.3)

$$\begin{aligned}&\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(u)\,du - f(b) - \frac{1}{2}(b-a) f'(a)\right]\\ &\le f(a) - \frac{1}{b-a}\int_a^b f(u)\,du\\ &\le 2\left[\frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(u)\,du - f\left(\frac{a+b}{2}\right)\right].\end{aligned} \tag{1.11}$$

At $x = b$,

$$2\left[\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} f(u)\,du - f\left(\frac{a+b}{2}\right)\right] \le f(b) - \frac{1}{b-a}\int_a^b f(u)\,du \le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(u)\,du - f(a) + \frac{1}{2}(b-a)f'(b)\right]. \tag{1.12}$$

Manipulating (1.11) and (1.12) we arrive at (1.10). ∎

2. OSTROWSKI FOR DERIVATIVES THAT ARE CONVEX IN MODULUS

2.1. A Representation Result. In [**8**], the author pointed out the following identity in representing an absolutely continuous function.

LEMMA 2.1 (Dragomir, 2002 [**8**]). *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then for any $x \in [a,b]$, one has the equality:*

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b (x-t)\left(\int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda\right)dt. \tag{2.1}$$

We have the following results.

2.2. Some Refinements. Using the above lemma the following result can be pointed out improving Ostrowski's inequality.

THEOREM 2.2 (Barnett et al., 2001 [**3**]). *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$ so that $|f'|$ is convex on (a,b). If $f' \in L_\infty[a,b]$, then for any $x \in [a,b]$,*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{1}{2}\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[|f'(x)| + \|f'\|_\infty\right]. \tag{2.2}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

PROOF. Using (2.1) and taking the modulus, we have

$$\begin{aligned}\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| &= \frac{1}{b-a}\left|\int_a^b\int_0^1 (x-t)f'[(1-\lambda)x+\lambda t]\,d\lambda dt\right| \\ &\le \frac{1}{b-a}\int_a^b\int_0^1 |x-t|\,|f'[(1-\lambda)x+\lambda t]|\,d\lambda dt \\ &:= K\end{aligned}$$

Utilizing the convexity of $|f'|$ we have

$$
\begin{aligned}
K &\leq \frac{1}{b-a}\int_a^b\int_0^1 |x-t|\left[(1-\lambda)\left|f'(x)\right| + \lambda\left|f'(t)\right|\right] d\lambda dt \\
&= \frac{1}{b-a}\int_a^b |x-t|\left[\left|f'(x)\right|\int_0^1 (1-\lambda)\, d\lambda + \left|f'(t)\right|\int_0^1 \lambda d\lambda\right] dt \\
&= \frac{1}{b-a}\int_a^b |x-t|\left[\frac{\left|f'(x)\right| + \left|f'(t)\right|}{2}\right] dt := M(x) \\
&\leq \frac{1}{2}\frac{1}{b-a} ess.\sup_{t\in[a,b]}\left[\left|f'(x)\right| + \left|f'(t)\right|\right]\int_a^b |x-t|\, dt \\
&= \frac{1}{2}\left[\frac{(x-a)^2 + (b-x)^2}{2(b-a)}\right]\left[\left|f'(x)\right| + \left\|f'\right\|_\infty\right] \\
&= \frac{1}{2}\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[\left|f'(x)\right| + \left\|f'\right\|_\infty\right],
\end{aligned}
$$

for any $x \in [a,b]$, and the inequality (2.2) is proved.
Assume that (2.2) holds with a constant $C > 0$, that is,

$$
\begin{aligned}
&\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq C\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[\left|f'(x)\right| + \left\|f'\right\|_\infty\right]
\end{aligned} \tag{2.3}
$$

for any $x \in [a,b]$ with f as in the hypothesis of the theorem.
Consider the function

$$f_0 : [a,b] \to \mathbb{R}, f_0(t) = k\left|t - \frac{a+b}{2}\right|,\ k > 0, t \in [a,b].$$

Since $|f_0'(t)| = k$, for any $t \in [a,b]$ and

$$\frac{1}{b-a}\int_a^b f_0(t)dt = \frac{k}{4}(b-a),\ \left\|f_0'\right\|_\infty = k$$

then choosing $f = f_0$ and $x = \frac{a+b}{2}$ in (2.3), we get

$$\frac{k}{4}(b-a) \leq \frac{Ck(b-a)}{2}$$

giving $C \geq \frac{1}{2}$, which proves the sharpness of the constant $\frac{1}{2}$. ∎

The following particular case is interesting.

COROLLARY 2.3. *With the assumptions of Theorem 2.4, we have the inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{8}(b-a)\left[\left|f'\left(\frac{a+b}{2}\right)\right| + \left\|f'\right\|_\infty\right] \tag{2.4}$$

and the constant $\frac{1}{8}$ is the best possible.

The following result in terms of the p-norms also holds:

THEOREM 2.4. *Let $f : [a,b] \to \mathbb{C}$ be as in Theorem 2.2. If $f' \in L_p[a,b]$, $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in [a,b]$,*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{2(q+1)^{\frac{1}{q}}} \left[\left(\frac{b-x}{b-a}\right)^{q+1} + \left(\frac{x-a}{b-a}\right)^{q+1} \right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} \left\| |f'(x)| + |f'| \right\|_p . \tag{2.5}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

PROOF. According to the proof of Theorem 2.2, we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{b-a}\int_a^b |x-t| \left[\frac{|f'(x)| + |f'(t)|}{2} \right] dt := M(x).$$

Using Hölder's integral inequality for $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, we get that

$$\begin{aligned} M(x) &\leq \frac{1}{2(b-a)} \left(\int_a^b |x-t|^q \, dt \right)^{\frac{1}{q}} \left(\int_a^b (|f'(x)| + |f'(t)|)^p \, dt \right)^{\frac{1}{p}} \\ &= \frac{1}{2(b-a)} \left[\frac{(b-x)^{q+1} + (x-a)^{q+1}}{q+1} \right]^{\frac{1}{q}} \left\| |f'(x)| + |f'| \right\|_p \end{aligned}$$

and the inequality (2.5) is proved.
Reconsider the function utilised in Theorem 2.2,

$$f_0 : [a,b] \to \mathbb{R}, \; f_0(t) = k \left| t - \frac{a+b}{2} \right|, \; k > 0, \; t \in [a,b]$$

which has $|f_0'(t)| \; (= k)$ convex in $[a,b]$. If we assume that (2.5) holds with a constant $D > 0$ instead of $\frac{1}{2}$, so that

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ &\leq \frac{D}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{b-x}{b-a}\right)^{q+1} + \left(\frac{x-a}{b-a}\right)^{q+1} \right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} \left\| |f'(x)| + |f'| \right\|_p , \end{aligned}$$

then taking $f = f_0$ over $x = \frac{a+b}{2}$, we get,

$$\frac{k}{4}(b-a) \leq \frac{D}{(q+1)^{\frac{1}{q}}} \left(\frac{1}{2^q} \right)^{\frac{1}{q}} (b-a)^{\frac{1}{q}} k (b-a)^{\frac{1}{p}} ,$$

$q > 1, p > 1, \frac{1}{p} + \frac{1}{q} = 1$ giving, on simplification,

$$D \geq \frac{1}{2}(q+1)^{\frac{1}{q}}, q > 1.$$

Taking the limit as $q \to \infty$ and since,

$$\lim_{q\to\infty} (q+1)^{\frac{1}{q}} = \exp\left\{ \lim_{q\to\infty} \left[\frac{\ln(1+q)}{q} \right] \right\} = \exp 0 = 1,$$

we deduce that $D \geq \frac{1}{2}$, which proves the sharpness of the constant. ∎

A particular case is the following mid-point inequality:

COROLLARY 2.5. *With the assumptions of Theorem 2.4, we have,*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \tag{2.6}$$
$$\leq \frac{1}{4(q-a)^{\frac{1}{q}}}(b-a)^{\frac{1}{q}}\left(\int_a^b\left[\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(t)|\right]^p dt\right)^{\frac{1}{p}}$$

$\left(p>1, \frac{1}{p}+\frac{1}{q}=1\right)$. *The constant* $\frac{1}{4}$ *is sharp in the previous sense.*

Finally, the case involving the 1-norm is embodied in the following theorem:

THEOREM 2.6. *Let* $f:[a,b]\to\mathbb{R}$ *be as in Theorem 2.2. If* $f'\in L_1[a,b]$*, then, for any* $x\in[a,b]$*,*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{1}{2}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[(b-a)\left|f'(x)\right|+\left\|f'\right\|_1\right]. \tag{2.7}$$

PROOF. We have, from the proof of Theorem 2.2, that

$$\begin{aligned}M(x)&\leq\sup_{t\in[a,b]}|x-t|\frac{1}{b-a}\int_a^b\left[\frac{|f'(x)|+|f'(t)|}{2}\right]dt\\&=\frac{1}{2(b-a)}\max(x-a,b-x)\left[(b-a)\left|f'(x)\right|+\int_a^b\left|f'(t)\right|dt\right]\\&=\frac{1}{2}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[(b-a)\left|f'(x)\right|+\left\|f'\right\|_1\right]\end{aligned}$$

and the inequality (2.7) is proved. ∎

In particular, we have the mid-point inequality:

COROLLARY 2.7. *Assume that* f *is as in Theorem 2.6. Then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{1}{4}\left[(b-a)\left|f'\left(\frac{a+b}{2}\right)\right|+\int_a^b\left|f'(t)\right|dt\right]. \tag{2.8}$$

Another way to estimate the difference

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

is presented in the following theorem.

THEOREM 2.8. *Let* $f:[a,b]\to\mathbb{R}$ *be an absolutely continuous function on* $[a,b]$ *so that* $|f'|$ *is convex on* (a,b)*. Then, for any* $x\in[a,b]$*,*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \tag{2.9}$$
$$\leq\frac{1}{2}\left\{\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left|f'(x)\right|(b-a)\right.$$
$$\left.+\frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\|f'\right\|_p\right\},$$

where $p>1, \frac{1}{p}+\frac{1}{q}=1$.

PROOF. With the notation of Theorem 2.2, we have,

$$\begin{aligned}M(x) &= \frac{1}{2(b-a)}\left[|f'(x)|\int_a^b |x-t|\,dt + \int_a^b |x-t|\,|f'(t)|\,dt\right]\\ &= \frac{1}{2(b-a)}\left[|f'(x)|\,\frac{(x-a)^2+(b-x)^2}{2} + \int_a^b |x-t|\,|f'(t)|\,dt\right]\\ &= \frac{1}{2}\left[|f'(x)|\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a) + \frac{1}{b-a}\int_a^b |x-t|\,|f'(t)|\,dt\right].\end{aligned}$$

Using Hölder's inequality,

$$\begin{aligned}&\frac{1}{b-a}\int_a^b |x-t|\,|f'(t)|\,dt\\ &\le \frac{1}{b-a}\left(\int_a^b |x-t|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b |f'(t)|^p\,dt\right)^{\frac{1}{p}}\\ &= \frac{1}{b-a}\left[\frac{(b-x)^{q+1}+(x-a)^{q+1}}{q+1}\right]^{\frac{1}{q}}\|f'\|_p\\ &= \frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\|f'\|_p,\end{aligned}$$

and the theorem is proved. ∎

The following particular corollary is of interest providing a bound for the midpoint.

COROLLARY 2.9. *Let f be as in the previous theorem. Then one has the inequality:*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\\ &\le \frac{1}{4}\left\{\frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right|(b-a)+\frac{1}{(q+1)^{\frac{1}{q}}}(b-a)^{\frac{1}{q}}\|f'\|_p\right\}.\end{aligned} \tag{2.10}$$

3. OSTROWSKI FOR DERIVATIVES THAT HAVE CERTAIN CONVEXITY PROPERTIES

3.1. Some Inequalities for $|f'|$ Convex.

The following representation holds:

LEMMA 3.1 (Cerone & Dragomir, 2004 [7]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then we have the representation*

$$\begin{aligned}f(x) &= \frac{1}{b-a}\int_a^b f(t)\,dt + (x-a)^2\cdot\frac{1}{b-a}\int_0^1 \lambda f'[(1-\lambda)a+\lambda x]\,d\lambda\\ &\quad - (b-x)^2\cdot\frac{1}{b-a}\int_0^1 \lambda f'[\lambda x+(1-\lambda)b]\,d\lambda\end{aligned} \tag{3.1}$$

for any $x\in[a,b]$.

PROOF. We start with the following Montgomery identity

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{b-a}\int_a^x (t-a)\,f'(t)\,dt \\ +\frac{1}{b-a}\int_x^b (t-b)\,f'(t)\,dt \tag{3.2}$$

for any $x\in[a,b]$.

If we make the change of variable $t=(1-\lambda)\,a+\lambda x$, $\lambda\in[0,1]$, then we get

$$\int_a^x (t-a)\,f'(t)\,dt=(x-a)^2\int_0^1 \lambda f'\left[(1-\lambda)\,a+\lambda x\right]d\lambda.$$

Also, the change of variable $t=\mu x+(1-\mu)\,b$, $\mu\in[0,1]$, will provide the equality

$$\int_x^b (t-b)\,f'(t)\,dt=-(b-x)^2\int_0^1 \mu f'\left[\mu x+(1-\mu)\,b\right]d\mu.$$

Using (3.2), we then deduce the desired identity (3.1). ∎

The following Ostrowski-type inequality holds for $|f'|$ convex.

THEOREM 3.2 (Cerone & Dragomir, 2004 [**7**]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ and $x\in[a,b]$. If $|f'|$ is convex on $[a,x]$ and $[x,b]$, then one has the inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{6}\left[\left|f'(a)\right|\left(\frac{x-a}{b-a}\right)^2+\left|f'(b)\right|\left(\frac{b-x}{b-a}\right)^2\right. \\ \left.+\left[1+2\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left|f'(x)\right|\right](b-a). \tag{3.3}$$

The constant $\frac{1}{6}$ is best possible in the sense that it cannot be replaced by a smaller value.

PROOF. Taking the modulus in (3.1) we have

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le (x-a)^2\cdot\frac{1}{b-a}\int_0^1 \lambda\left|f'\left[(1-\lambda)\,a+\lambda x\right]\right|d\lambda \\ &+(b-x)^2\cdot\frac{1}{b-a}\int_0^1 \lambda\left|f'\left[\lambda x+(1-\lambda)\,b\right]\right|d\lambda \\ &:=M(x).\end{aligned} \tag{3.4}$$

Since $|f'|$ is convex on $[a,x]$ and $[x,b]$, then obviously

$$\begin{aligned}\int_0^1 \lambda\left|f'\left[(1-\lambda)\,a+\lambda x\right]\right|d\lambda &\le \left|f'(a)\right|\int_0^1 \lambda(1-\lambda)\,d\lambda+\left|f'(x)\right|\int_0^1 \lambda^2 d\lambda \\ &=\frac{1}{6}\left|f'(a)\right|+\frac{1}{3}\left|f'(x)\right|\end{aligned}$$

and

$$\int_0^1 \lambda \left|f'\left[\lambda x + (1-\lambda) b\right]\right| d\lambda \le \left|f'(x)\right| \int_0^1 \lambda^2 d\lambda + \left|f'(b)\right| \int_0^1 \lambda (1-\lambda)\, d\lambda$$
$$= \frac{1}{3}\left|f'(x)\right| + \frac{1}{6}\left|f'(b)\right|.$$

Thus from (3.4)

$$\begin{aligned} &M(x) \\ &\le \frac{1}{3}\left[\frac{1}{2}\left|f'(a)\right| + \left|f'(x)\right|\right](x-a)^2 + \frac{1}{3}\left[\left|f'(x)\right| + \frac{1}{2}\left|f'(b)\right|\right](b-x)^2 \\ &= \frac{1}{3}\left[\frac{\left|f'(a)\right|(x-a)^2 + \left|f'(b)\right|(b-x)^2}{2} + \left[(x-a)^2 + (b-x)^2\right]\left|f'(x)\right|\right] \\ &= \frac{1}{6}\left[\left|f'(a)\right|(x-a)^2 + \left|f'(b)\right|(b-x)^2 + \left[(b-a)^2 + 2\left(x - \frac{a+b}{2}\right)^2\right]\left|f'(x)\right|\right] \end{aligned}$$

and the inequality (3.3) is proved.

To prove the sharpness of the constant $\frac{1}{6}$, assume that (3.3) holds with a constant $C > 0$. Namely,

$$\begin{aligned} \left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le C\Bigg[&\left|f'(a)\right|\left(\frac{x-a}{b-a}\right)^2 + \left|f'(b)\right|\left(\frac{b-x}{b-a}\right)^2 \\ &+ \left[1 + 2\left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right]\left|f'(x)\right|\Bigg](b-a), \end{aligned} \tag{3.5}$$

provided that $|f'|$ is convex on $[a,x]$ and $[x,b]$.

If we choose $x = \frac{a+b}{2}$, then by (3.5) we deduce

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le C\left[\frac{\left|f'(a)\right| + \left|f'(b)\right|}{4} + \left|f'\left(\frac{a+b}{2}\right)\right|\right](b-a), \end{aligned} \tag{3.6}$$

where $|f'|$ is convex on $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$.

Consider the function $f_0 : [a,b] \to \mathbb{R}$, given by

$$f_0(x) = \begin{cases} \frac{a+b}{2} - x, & \text{if } x \in \left[a, \frac{a+b}{2}\right], \\ \\ x - \frac{a+b}{2}, & \text{if } x \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

The function is absolutely continuous on $[a,b]$ and, obviously, $|f'| = 1$ on $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ showing that it is convex on $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$.

On the other hand, we have

$$\begin{aligned} f_0\left(\frac{a+b}{2}\right) &= 0, \quad \frac{1}{b-a}\int_a^b f_0(x)\,dx = \frac{b-a}{4}, \\ \left|f_0'(a)\right| &= \left|f_0'(b)\right| = \left|f_0'\left(\frac{a+b}{2}\right)\right| = 1 \end{aligned}$$

and so from (3.6) we get

$$\frac{b-a}{4} \le C\left(\frac{1}{2}+1\right)(b-a)$$

giving $C \ge \frac{1}{6}$. ∎

The following corollary is a natural consequence.

COROLLARY 3.3. *Assume that $f : [a,b] \to \mathbb{R}$ is absolutely continuous such that $|f'|$ is a convex function on $\left[a, \frac{a+b}{2}\right]$ and $\left(\frac{a+b}{2}, b\right]$. Then we have the inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{6}\left[\frac{|f'(a)|+|f'(b)|}{4} + \left|f'\left(\frac{a+b}{2}\right)\right|\right](b-a). \tag{3.7}$$

The $\frac{1}{6}$ is best possible in (3.7) in the sense that it cannot be replaced by a smaller constant.

3.2. Inequalities for $|f'|$ Quasi-Convex.

Firstly, let us recall the definition of quasi-convex functions.

DEFINITION 3.1. The function $h : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is said to be *quasi-convex* (QC) on the interval I if

$$h(\lambda x + (1-\lambda) y) \le \max\{h(x), h(y)\} \tag{3.8}$$

for any $x, y \in I$ and $\lambda \in [0,1]$.

Following [**15**], we say that for an interval $I \subseteq \mathbb{R}$, the mapping $h : I \to \mathbb{R}$ is *quasi-monotone* on I if it is either monotone on $I = [c,d]$ or monotone nonincreasing on a proper subinterval $[c,c'] \subset I$ and monotone nondecreasing on $[c',d]$.

The class $QM(I)$ of quasi-monotone functions on I provides an immediate characterization of quasi-convex functions [**15**].

PROPOSITION 3.4. *Suppose $I \subseteq \mathbb{R}$. Then the following statements are equivalent for a function $h : I \to \mathbb{R}$:*

(a) $h \in QM(I)$;
(b) *On any subinterval of I, h achieves its supremum at an end point;*
(c) $h \in QC(I)$.

As examples of quasi-convex functions we may consider the class of monotonic functions on an interval I for the class of convex functions on that interval.

The following Ostrowski type inequality for absolutely continuous functions for which $|f'|$ is quasi-convex holds.

THEOREM 3.5 (Cerone & Dragomir, 2004 [**7**]). *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x \in [a,b]$. If $|f'|$ is quasi-convex on $[a,x]$ and $[x,b]$, then one has the inequality*

$$\begin{aligned}\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| &\le \frac{1}{4}\left\{\left(\frac{x-a}{b-a}\right)^2 [|f'(a)| + |f'(x)| + ||f'(x)| - |f'(a)||]\right. \\ &\quad \left. + \left(\frac{b-x}{b-a}\right)^2 [|f'(x)| + |f'(b)| + ||f'(x)| - |f'(b)||]\right\}.\end{aligned} \tag{3.9}$$

The constant $\frac{1}{4}$ is sharp in (3.9) in the sense that it cannot be replaced by a smaller value.

PROOF. Since $|f'|$ is quasi-convex on $[a,x]$ and $[x,b]$, then from (3.8)

$$\begin{aligned}\int_0^1 \lambda \left|f'\left((1-\lambda)a+\lambda x\right)\right| d\lambda &\le \max\left\{\left|f'(a)\right|,\left|f'(x)\right|\right\}\int_0^1 \lambda d\lambda \\ &= \frac{1}{2}\left[\frac{\left|f'(a)\right|+\left|f'(x)\right|}{2}+\frac{1}{2}\left|\left|f'(x)\right|-\left|f'(a)\right|\right|\right] \\ &:= M_1(x)\end{aligned}$$

and, similarly

$$\begin{aligned}\int_0^1 \lambda \left|f'\left(\lambda x+(1-\lambda)b\right)\right| d\lambda &\le \frac{1}{2}\left[\frac{\left|f'(x)\right|+\left|f'(b)\right|}{2}+\frac{1}{2}\left|\left|f'(x)\right|-\left|f'(b)\right|\right|\right] \\ &:= M_2(x).\end{aligned}$$

Using (3.4) and the notation $M(x)$ for the right hand side of that inequality, we deduce that

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| &\le M(x) \\ &\le \left(\frac{x-a}{b-a}\right)^2 M_1(x)+\left(\frac{b-x}{b-a}\right)^2 M_2(x)\end{aligned}$$

and the result (3.9) is thus proved.

The fact that $\frac{1}{4}$ is the best possible constant will be shown in the following. ∎

COROLLARY 3.6. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $|f'|$ is quasi-convex on $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$, then one has the inequality:*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le \frac{1}{16}\left\{\left|f'(a)\right|+2\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'(b)\right|+\left|\left|f'\left(\frac{a+b}{2}\right)\right|-\left|f'(a)\right|\right|\right. \\ &\left.+\left|\left|f'(b)\right|-\left|f'\left(\frac{a+b}{2}\right)\right|\right|\right\}(b-a).\end{aligned} \tag{3.10}$$

The constant $\frac{1}{16}$ is best possible.

PROOF. The inequality follows by (3.9) on choosing $x=\frac{a+b}{2}$. To prove the sharpness of the constant $\frac{1}{16}$, assume that (3.10) holds with a constant $C>0$. That is,

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le C\left\{\left|f'(a)\right|+2\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'(b)\right|+\left|\left|f'\left(\frac{a+b}{2}\right)\right|-\left|f'(a)\right|\right|\right. \\ &\left.+\left|\left|f'(b)\right|-\left|f'\left(\frac{a+b}{2}\right)\right|\right|\right\}(b-a).\end{aligned} \tag{3.11}$$

Consider the function $f_0:[a,b]\to\mathbb{R}$, $f_0(t)=\left|t-\frac{a+b}{2}\right|$. Then f_0 is absolutely continuous and $\left|f_0'(t)\right|=1$, $t\in[a,b]$. Thus, from (3.11), we deduce

$$\frac{b-a}{4}\le C\cdot 4(b-a)$$

giving $C \geq \frac{1}{2}$ and the corollary is proved. ∎

3.3. Inequalities for $|f'|$ Log-Convex. In what follows, I will denote an interval of real numbers. A function $f : I \to (0, \infty)$ is said to be *log-convex* or *multiplicatively convex* if $\log f$ is convex, or, equivalently, if for any $x, y \in I$ and $t \in [0, 1]$ one has the inequality

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}. \tag{3.12}$$

We note that if f and g are convex and g is increasing, then $g \circ f$ is convex, moreover, since $f = \exp[\log f]$, it follows that a log-convex function is convex, but the converse may not necessarily be true. This follows directly from (3.12) since, by the arithmetic-geometric mean inequality we have

$$[f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1-t)f(y) \tag{3.13}$$

for all $x, y \in I$ and $t \in [0, 1]$.

The following result holds.

THEOREM 3.7 (Cerone & Dragomir, 2004 [**7**]). *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$ and $x \in [a, b]$. If $|f'|$ is log-convex on $[a, x]$ and $[x, b]$, then one has the inequality*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\leq \left\{ \left(\frac{x-a}{b-a}\right)^2 |f'(a)| \frac{A \ln A + 1 - A}{(\ln A)^2} \right. \\
&\left. + \left(\frac{b-x}{b-a}\right)^2 |f'(b)| \frac{B \ln B + 1 - B}{(\ln B)^2} \right\} (b-a),
\end{aligned} \tag{3.14}$$

where

$$A := \left| \frac{f'(x)}{f'(a)} \right|, \quad B := \left| \frac{f'(x)}{f'(b)} \right|.$$

PROOF. Using the representation (3.1) and the definition of log-convexity (3.12), we have successively:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.15}$$
$$\leq (b-a)\left\{\left(\frac{x-a}{b-a}\right)^2\int_0^1 \lambda\left|f'((1-\lambda)a+\lambda x)\right|d\lambda\right.$$
$$\left.+\left(\frac{b-x}{b-a}\right)^2\int_0^1 \lambda\left|f'(\lambda x+(1-\lambda)b)\right|d\lambda\right\}$$
$$\leq (b-a)\left\{\left(\frac{x-a}{b-a}\right)^2\int_0^1 \lambda\left|f'(a)\right|^{1-\lambda}\left|f'(x)\right|^{\lambda}d\lambda\right.$$
$$\left.+\left(\frac{b-x}{b-a}\right)^2\int_0^1 \lambda\left|f'(x)\right|^{\lambda}\left|f'(b)\right|^{1-\lambda}d\lambda\right\}$$
$$=(b-a)\left\{\left(\frac{x-a}{b-a}\right)^2\left|f'(a)\right|\int_0^1 \lambda A^{\lambda}d\lambda\right.$$
$$\left.+\left(\frac{b-x}{b-a}\right)^2\left|f'(b)\right|\int_0^1 \lambda B^{\lambda}d\lambda\right\}.$$

Since, a simple calculation shows that for any $C>0,$ one has

$$\int_0^1 \lambda C^{\lambda}d\lambda=\frac{C\ln C+1-C}{(\ln C)^2},$$

then from (3.15) we deduce the desired result (3.14). ∎

The following corollary holds.

COROLLARY 3.8. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $|f'|$ is log-convex on $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$, then one has the inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.16}$$
$$\leq\frac{1}{4}\left[\left|f'(a)\right|\frac{\alpha\ln\alpha+1-\alpha}{(\ln\alpha)^2}+\left|f'(b)\right|\frac{\beta\ln\beta+1-\beta}{(\ln\beta)^2}\right](b-a),$$

where

$$\alpha:=\left|\frac{f'\left(\frac{a+b}{2}\right)}{f'(a)}\right|,\quad \beta:=\left|\frac{f'\left(\frac{a+b}{2}\right)}{f'(b)}\right|.$$

4. OSTROWSKI FOR FUNCTIONS WHOSE DERIVATIVES ARE h-CONVEX IN MODULUS

4.1. Some Classes of Functions. We recall here some concepts of convexity that are well known in the literature. Let I be an interval in $\mathbb{R}$.

DEFINITION 4.1 (Godunova-Levin, 1985 [**18**]). We say that $f:I\to\mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x,y\in I$ and $t\in(0,1)$ we have

$$f(tx+(1-t)y)\leq\frac{1}{t}f(x)+\frac{1}{1-t}f(y).$$

Some further properties of this class of functions can be found in [**12**], [**13**], [**16**], [**22**], [**24**] and [**25**]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

DEFINITION 4.2 (Dragomir et al., 1995 [**16**]). We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [**16**] and [**23**] while for quasi convex functions, the reader can consult [**14**].

DEFINITION 4.3 (Breckner, 1978 [**5**]). Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [**1**], [**2**], [**5**], [**6**], [**10**], [**11**], [**19**], [**20**] and [**27**].

In order to unify the above concepts, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}, (0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

DEFINITION 4.4 (Varošanec, 2007 [**30**]). Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if for all $x, y \in I$ we have

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y)$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [**30**], [**4**], [**21**], [**28**], [**26**] and [**29**].

4.2. Inequalities of Hermite-Hadamard Type. In [**26**] the authors proved the following Hermite-Hadamard type inequality for integrable h-convex functions.

THEOREM 4.1 (Sarikaya et al., 2008 [**26**]). *Assume that $f : I \to [0,\infty)$ is an h-convex function, $h \in L[0,1]$ and $f \in L[a,b]$ where $a, b \in I$ with $a < b$. Then*

$$\text{(HH)} \qquad \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq [f(a) + f(b)]\int_0^1 h(t)\,dt.$$

If we write (HH) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions.

If we write it for the case of P-type functions, i.e., $h(t) = 1$, then we get the inequality

$$\frac{1}{2} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq f(a) + f(b), \tag{4.1}$$

provided $f \in L[a,b]$, that has been obtained in [**16**].

If f is integrable on $[a,b]$ and Breckner s-convex on $[a,b]$, for $s\in(0,1)$, then by taking $h(t)=t^s$ in (HH) we get

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)\,dt\le\frac{f(a)+f(b)}{s+1} \tag{4.2}$$

that was obtained in [**10**].

Since for the case of Godunova-Levin class of function we have $h(t)=\frac{1}{t}$, which is not Lebesgue integrable on $(0,1)$, we cannot apply the left inequality in (HH).

We can introduce now another class of functions.

DEFINITION 4.5 (Dragomir, 2013 [**9**]). We say that the function $f:I\to[0,\infty)$ is of s-Godunova-Levin type, with $s\in[0,1]$, if

$$f(tx+(1-t)y)\le\frac{1}{t^s}f(x)+\frac{1}{(1-t)^s}f(y), \tag{4.3}$$

for all $t\in(0,1)$ and $x,y\in I$.

We observe that for $s=0$ we obtain the class of P-functions while for $s=1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(I)$ the class of s-Godunova-Levin functions defined on I, then we obviously have

$$P(I)=Q_0(I)\subseteq Q_{s_1}(I)\subseteq Q_{s_2}(I)\subseteq Q_1(I)=Q(I)$$

for $0\le s_1\le s_2\le 1$.

We have the following Hermite-Hadamard type inequality.

THEOREM 4.2 (Dragomir, 2013 [**9**]). *Assume that the function $f:I\to[0,\infty)$ is of s-Godunova-Levin type, with $s\in[0,1)$. If $f\in L[a,b]$ where $a,b\in I$ and $a<b$, then*

$$\frac{1}{2^{s+1}}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)\,dt\le\frac{f(a)+f(b)}{1-s}. \tag{4.4}$$

We notice that for $s=1$ the first inequality in (4.4) still holds and was obtained for the first time in [**16**].

4.3. Inequalities for Functions Whose Derivatives are h-Convex in Modulus. In [**8**], the author pointed out the following identity in representing an absolutely continuous function.

LEMMA 4.3 (Dragomir, 2002 [**8**]). *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then for any $x\in[a,b]$, one has the equality:*

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{b-a}\int_a^b(x-t)\left(\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right)dt. \tag{4.5}$$

The following result holds:

THEOREM 4.4 (Dragomir, 2013 [**9**]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ so that $|f'|$ is h-convex on (a,b) with $h\in L[0,1]$.*

(i) *If $f'\in L_\infty[a,b]$, then for any $x\in[a,b]$,*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\\&\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[|f'(x)|+\|f'\|_\infty\right]\int_0^1 h(t)\,dt.\end{aligned} \tag{4.6}$$

(ii) *If* $f' \in L_p[a,b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *then for any* $x \in [a,b]$,

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \\ &\leq \frac{1}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{b-x}{b-a} \right)^{q+1} + \left(\frac{x-a}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} \\ &\times (b-a)^{\frac{1}{q}} \left\| |f'(x)| + |f'| \right\|_p \int_0^1 h(t)\,dt. \end{aligned} \tag{4.7}$$

(iii) *If* $f' \in L_1[a,b]$*, then for any* $x \in [a,b]$*,*

$$\begin{aligned} \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| &\leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \\ &\times \left[(b-a) |f'(x)| + \|f'\|_1 \right] \int_0^1 h(t)\,dt. \end{aligned} \tag{4.8}$$

PROOF. (i). Using (4.5) and taking the modulus, we have

$$\begin{aligned} \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| &= \frac{1}{b-a} \left| \int_a^b \int_0^1 (x-t) f'[(1-\lambda)x + \lambda t]\, d\lambda dt \right| \\ &\leq \frac{1}{b-a} \int_a^b \int_0^1 |x-t| \left| f'[(1-\lambda)x + \lambda t] \right| d\lambda dt := K \end{aligned}$$

Utilizing the h-convexity of $|f'|$ we have

$$\begin{aligned} K &\leq \frac{1}{b-a} \int_a^b \int_0^1 |x-t| \left[h(1-\lambda) |f'(x)| + h(\lambda) |f'(t)| \right] d\lambda dt \\ &= \frac{1}{b-a} \int_a^b |x-t| \left[|f'(x)| \int_0^1 h(1-\lambda)\, d\lambda + |f'(t)| \int_0^1 h(\lambda)\, d\lambda \right] dt \\ &= \frac{1}{b-a} \int_0^1 h(\lambda)\, d\lambda \int_a^b |x-t| \left[|f'(x)| + |f'(t)| \right] dt := M(x) \int_0^1 h(\lambda)\, d\lambda \\ &\leq \frac{1}{b-a} \int_0^1 h(\lambda)\, d\lambda \; ess \sup_{t\in[a,b]} \left[|f'(x)| + |f'(t)| \right] \int_a^b |x-t|\, dt \\ &= \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \left[|f'(x)| + \|f'\|_\infty \right] \int_0^1 h(\lambda)\, d\lambda, \end{aligned}$$

for any $x \in [a,b]$, and the inequality (4.6) is proved.

(ii). As above, we have

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \\ &\leq \frac{1}{b-a} \int_a^b |x-t| \left[|f'(x)| + |f'(t)| \right] dt := M(x) \int_0^1 h(\lambda)\, d\lambda. \end{aligned}$$

Using Hölder's integral inequality for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, we get that

$$\begin{aligned} M(x) &\le \frac{1}{b-a}\left(\int_a^b |x-t|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b (|f'(x)|+|f'(t)|)^p\,dt\right)^{\frac{1}{p}} \\ &= \frac{1}{b-a}\left[\frac{(b-x)^{q+1}+(x-a)^{q+1}}{q+1}\right]^{\frac{1}{q}} \||f'(x)|+|f'|\|_p \end{aligned}$$

and the inequality (4.7) is proved.

(iii). We also have that

$$\begin{aligned} M(x) &\le \sup_{t\in[a,b]} |x-t|\,\frac{1}{b-a}\int_a^b [|f'(x)|+|f'(t)|]\,dt \\ &= \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right][(b-a)\,|f'(x)|+\|f'\|_1] \end{aligned}$$

and the inequality (4.8) is proved. ∎

The following particular case is interesting.

COROLLARY 4.5. *With the assumptions of Theorem 4.4, we have the midpoint inequality*

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\le \frac{1}{4}(b-a)\left[\left|f'\left(\frac{a+b}{2}\right)\right|+\|f'\|_\infty\right]\int_0^1 h(t)\,dt, \end{aligned} \tag{4.9}$$

provided $f'\in L_\infty[a,b]$.

If $f'\in L_p[a,b]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, *then, we have,*

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\le \frac{1}{2}(b-a)^{\frac{1}{q}}\left(\int_a^b \left[\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(t)|\right]^p dt\right)^{\frac{1}{p}}\int_0^1 h(t)\,dt. \end{aligned} \tag{4.10}$$

If $f'\in L_1[a,b]$, *then*

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\le \frac{1}{2}\left[(b-a)\left|f'\left(\frac{a+b}{2}\right)\right|+\int_a^b |f'(t)|\,dt\right]\int_0^1 h(t)\,dt. \end{aligned} \tag{4.11}$$

REMARK 4.1. Assume that $|f'|$ is Breckner s-convex on $[a,b]$, for $s\in(0,1)$.

(a) If $f'\in L_\infty[a,b]$, then for any $x\in[a,b]$,

$$\begin{aligned} &\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\le \frac{1}{s+1}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\,[|f'(x)|+\|f'\|_\infty]. \end{aligned} \tag{4.12}$$

(aa) If $f' \in L_p[a,b]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then for any $x \in [a,b]$,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{(s+1)(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{q}}\left\| |f'(x)|+|f'| \right\|_p . \tag{4.13}$$

(aaa) If $f' \in L_1[a,b]$, then for any $x \in [a,b]$,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{s+1}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \times \left[(b-a)|f'(x)|+\|f'\|_1\right]. \tag{4.14}$$

Assume that $|f'|$ is of s-Godunova-Levin type, with $s \in [0,1)$.

(b) If $f' \in L_\infty[a,b]$, then for any $x \in [a,b]$,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{1-s}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left[|f'(x)|+\|f'\|_\infty\right]. \tag{4.15}$$

(bb) If $f' \in L_p[a,b]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then for any $x \in [a,b]$,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{(1-s)(q+1)^{\frac{1}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{q}}\left\| |f'(x)|+|f'| \right\|_p . \tag{4.16}$$

(bbb) If $f' \in L_1[a,b]$, then for any $x \in [a,b]$,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{1-s}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \times \left[(b-a)|f'(x)|+\|f'\|_1\right]. \tag{4.17}$$

The following result also holds:

THEOREM 4.6 (Dragomir, 2013 [**9**]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ so that $|f'|^p$ with $p>1$ is h-convex on (a,b) and $h \in L[0,1]$.*

(i) *If* $f' \in L_\infty[a,b]$*, then for any* $x \in [a,b]$*,*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a) \times \left[|f'(x)|^p + \|f'\|_\infty^p\right]^{1/p}\left(\int_0^1 h(t)\,dt\right)^{1/p}. \tag{4.18}$$

(ii) *If* $f' \in L_p[a,b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *then for any* $x \in [a,b]$,

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{1/q}}\left[\left(\frac{b-x}{b-a}\right)^{q+1} + \left(\frac{x-a}{b-a}\right)^{q+1}\right]^{1/q} \times \left[(b-a)\,|f'(x)|^p + \|f'\|_p^p\right]^{1/p}\left(\int_0^1 h(t)\,dt\right)^{1/p}. \tag{4.19}$$

(iii) *If* $f' \in L_p[a,b]$*, then for any* $x \in [a,b]$*,*

$$\begin{aligned}\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| &\leq \left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right] \times \left\||f'(x)|^p + |f'|^p\right\|^p\left(\int_0^1 h(t)\,dt\right)^{1/p} \\ &\leq \left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right] \times \left((b-a)\,|f'(x)|^p + \|f'\|_p^p\right)^{1/p}\left(\int_0^1 h(t)\,dt\right)^{1/p}.\end{aligned} \tag{4.20}$$

PROOF. As in the proof of Theorem 4.4 we have

$$\begin{aligned}\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| &= \frac{1}{b-a}\left|\int_a^b\int_0^1 (x-t)\,f'\left[(1-\lambda)\,x + \lambda t\right]d\lambda dt\right| \\ &\leq \frac{1}{b-a}\int_a^b |x-t|\left(\int_0^1 \left|f'\left[(1-\lambda)\,x + \lambda t\right]\right|d\lambda\right)dt \\ &:= K\end{aligned}$$

for any $x \in [a,b]$.

By Hölder's integral inequality we have

$$\begin{aligned}\int_0^1 \left|f'\left[(1-\lambda)\,x + \lambda t\right]\right|d\lambda &\leq \left(\int_0^1 1^q d\lambda\right)^{1/q}\left(\int_0^1 \left|f'\left[(1-\lambda)\,x + \lambda t\right]\right|^p d\lambda\right)^{1/p} \\ &= \left(\int_0^1 \left|f'\left[(1-\lambda)\,x + \lambda t\right]\right|^p d\lambda\right)^{1/p}\end{aligned}$$

for any $x \in [a,b]$, where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$.

Since $|f'|^p$ is h-convex on (a,b) with $h \in L[0,1]$, then

$$\int_0^1 |f'[(1-\lambda)x+\lambda t]|^p \, d\lambda \le \left[|f'(x)|^p + |f'(t)|^p\right] \int_0^1 h(\lambda)\, d\lambda,$$

for any $x \in [a,b]$.

Therefore

$$K \le \frac{1}{b-a}\left(\int_0^1 h(\lambda)\, d\lambda\right)^{1/p} \int_a^b |x-t| \left[|f'(x)|^p + |f'(t)|^p\right]^{1/p} dt \tag{4.21}$$

for any $x \in [a,b]$.

(i). Now, if $f' \in L_\infty[a,b]$ then

$$\begin{aligned}
&\int_a^b |x-t| \left[|f'(x)|^p + |f'(t)|^p\right]^{1/p} dt \\
&\le ess \sup_{t\in[a,b]} \left[|f'(x)|^p + |f'(t)|^p\right]^{1/p} \int_a^b |x-t|\, dt \\
&= \left[|f'(x)|^p + \|f'\|_\infty^p\right]^{1/p} \frac{1}{2}\left[(x-a)^2 + (b-x)^2\right]
\end{aligned}$$

for any $x \in [a,b]$, and utilizing (4.21), the inequality (4.18) is proved.

(ii). If $f' \in L_p[a,b]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$\begin{aligned}
&\int_a^b |x-t| \left[|f'(x)|^p + |f'(t)|^p\right]^{1/p} dt \\
&\le \left(\int_a^b |x-t|^q\, dt\right)^{1/q} \left(\int_a^b \left(\left[|f'(x)|^p + |f'(t)|^p\right]^{1/p}\right)^p dt\right)^{1/p} \\
&= \left[\frac{(b-x)^{q+1} + (x-a)^{q+1}}{q+1}\right]^{1/q} \left[(b-a)|f'(x)|^p + \|f'\|_p^p\right]^{1/p} \\
&= \frac{(b-a)^{1+\frac{1}{q}}}{(q+1)^{1/q}} \left[\left(\frac{b-x}{b-a}\right)^{q+1} + \left(\frac{x-a}{b-a}\right)^{q+1}\right]^{1/q} \\
&\times \left[(b-a)|f'(x)|^p + \|f'\|_1^p\right]^{1/p}
\end{aligned}$$

for any $x \in [a,b]$, and by (4.21) we deduce the desired inequality (4.19).

(iii). If $f' \in L_p[a,b]$, then by Hölder's inequality we also have

$$\begin{aligned}
&\int_a^b |x-t| \left[|f'(x)|^p + |f'(t)|^p\right]^{1/p} dt \\
&\leq \sup_{t\in[a,b]} |x-t| \int_a^b \left[|f'(x)|^p + |f'(t)|^p\right]^{1/p} dt \\
&= \max\{x-a, b-x\} \int_a^b \left[|f'(x)|^p + |f'(t)|^p\right]^{1/p} dt \\
&= (b-a)\left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \left\| |f'(x)|^p + |f'|^p \right\|^p \\
&\leq (b-a)\left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \left(\int_a^b \left[|f'(x)|^p + |f'(t)|^p\right] dt\right)^{1/p} \\
&= (b-a)\left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \left((b-a)|f'(x)|^p + \|f'\|_p^p\right)^{1/p}
\end{aligned}$$

for any $x \in [a,b]$. ∎

The following midpoint type inequalities are of interest.

COROLLARY 4.7. *With the assumptions of Theorem 4.6, we have the inequality*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq \frac{1}{4}(b-a)\left[\left|f'\left(\frac{a+b}{2}\right)\right|^p + \|f'\|_\infty^p\right]^{1/p} \left(\int_0^1 h(t)\,dt\right)^{1/p},
\end{aligned} \tag{4.22}$$

provided $f' \in L_\infty[a,b]$.

If $f' \in L_p[a,b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *then we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq \frac{1}{2(q+1)^{1/q}}(b-a)^{\frac{1}{q}} \\
&\times \left[(b-a)\left|f'\left(\frac{a+b}{2}\right)\right|^p + \|f'\|_p^p\right]^{1/p} \left(\int_0^1 h(t)\,dt\right)^{1/p}.
\end{aligned} \tag{4.23}$$

If $f' \in L_p[a,b]$*, then*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq \frac{1}{2}\left\|\left|f'\left(\frac{a+b}{2}\right)\right|^p + |f'|^p\right\|^p \left(\int_0^1 h(t)\,dt\right)^{1/p} \\
&\leq \frac{1}{2}\left((b-a)\left|f'\left(\frac{a+b}{2}\right)\right|^p + \|f'\|_p^p\right)^{1/p} \left(\int_0^1 h(t)\,dt\right)^{1/p}.
\end{aligned} \tag{4.24}$$

REFERENCES

[1] M. Alomari and M. Darus, The Hadamard's inequality for s-convex function. *Int. J. Math. Anal.* (Ruse) **2** (2008), no. 13-16, 639–646.

[2] M. Alomari and M. Darus, Hadamard-type inequalities for s-convex functions. *Int. Math. Forum* **3** (2008), no. 37-40, 1965–1975.

[3] N. S. Barnett, P. Cerone, S. S. Dragomir, M. R. Pinheiro,and A. Sofo, Ostrowski type inequalities for functions whose modulus of the derivatives are convex and applications. *Inequality Theory and Applications*, **Vol. 2** (Chinju/Masan, 2001), 19–32, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint: *RGMIA Res. Rep. Coll.* **5** (2002), No. 2, Art. 1 [Online http://rgmia.org/papers/v5n2/Paperwapp2q.pdf].

[4] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities. *Comput. Math. Appl.* **58** (2009), no. 9, 1869–1877.

[5] W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. (German) *Publ. Inst. Math.* (Beograd) (N.S.) **23(37)** (1978), 13–20.

[6] W. W. Breckner and G. Orbán, Continuity properties of rationally s-convex mappings with values in an ordered topological linear space. Universitatea "Babeş-Bolyai", Facultatea de Matematica, Cluj-Napoca, 1978. viii+92 pp.

[7] P. Cerone and S. S. Dragomir, Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions. *Demonstratio Math.* **37** (2004), No. 2, 299–308.

[8] S. S. Dragomir, Ostrowski type inequalities for isotonic linear functionals, *J. Ineq. Pure & Appl. Math.*, **3** (5) (2002), Art. 68. [Online http://www.emis.de/journals/JIPAM/article220.html?sid=220].

[9] S. S. Dragomir, Ostrowski type inequalities for functions whose derivatives are h-convex in absolute value, Preprint *RGMIA Res. Rep. Coll.* **16** (2013), Art. 71.

[10] S.S. Dragomir and S. Fitzpatrick, The Hadamard inequalities for s-convex functions in the second sense. *Demonstratio Math.* **32** (1999), no. 4, 687–696.

[11] S.S. Dragomir and S. Fitzpatrick,The Jensen inequality for s-Breckner convex functions in linear spaces. *Demonstratio Math.* **33** (2000), no. 1, 43–49.

[12] S. S. Dragomir and B. Mond, On Hadamard's inequality for a class of functions of Godunova and Levin. *Indian J. Math.* **39** (1997), no. 1, 1–9.

[13] S. S. Dragomir and C. E. M. Pearce, On Jensen's inequality for a class of functions of Godunova and Levin. *Period. Math. Hungar.* **33** (1996), no. 2, 93–100.

[14] S. S. Dragomir and C. E. M. Pearce, Quasi-convex functions and Hadamard's inequality, *Bull. Austral. Math. Soc.* **57** (1998), 377-385.

[15] S. S. Dragomir and C. E. M. Pearce, Quasi-convex functions and Hadamard's inequality, *Bull. Australian Math. Soc.*, **57**(1998), 377-385.

[16] S. S. Dragomir, J. Pečarić and L. Persson, Some inequalities of Hadamard type. *Soochow J. Math.* **21** (1995), no. 3, 335–341.

[17] S. S. Dragomir and A. Sofo, Ostrowski type inequalities for functions whose derivatives are convex, *Proc. 4th Int. Conf. on Modelling and Simulation*, 2002, 369-374. Preprint: *RGMIA Res. Rep. Coll.* **5** (2002), Supplement [Online http://rgmia.org/papers/v5e/OTIDC2_col.pdf].

[18] E. K. Godunova and V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. (Russian) *Numerical mathematics and mathematical physics* (Russian), 138–142, 166, Moskov. Gos. Ped. Inst., Moscow, 1985

[19] H. Hudzik and L. Maligranda, Some remarks on s-convex functions. *Aequationes Math*. **48** (1994), no. 1, 100–111.

[20] U. S. Kirmaci, M. Klaričić Bakula, M. E Özdemir and J. Pečarić, Hadamard-type inequalities for s-convex functions. *Appl. Math. Comput.* **193** (2007), no. 1, 26–35.

[21] M. A. Latif, On some inequalities for h-convex functions. *Int. J. Math. Anal.* (Ruse) **4** (2010), no. 29-32, 1473–1482.

[22] D. S. Mitrinović and J. E. Pečarić, Note on a class of functions of Godunova and Levin. *C. R. Math. Rep. Acad. Sci. Canada* **12** (1990), no. 1, 33–36.

[23] C. E. M. Pearce and A. M. Rubinov, P-functions, quasi-convex functions, and Hadamard-type inequalities. *J. Math. Anal. Appl.* **240** (1999), no. 1, 92–104.

[24] J. E. Pečarić and S. S. Dragomir, On an inequality of Godunova-Levin and some refinements of Jensen integral inequality. *Itinerant Seminar on Functional Equations, Approximation and Convexity* (Cluj-Napoca, 1989), 263–268, Preprint, 89-6, Univ. "Babeş-Bolyai”, Cluj-Napoca, 1989.

[25] M. Radulescu, S. Radulescu and P. Alexandrescu, On the Godunova-Levin-Schur class of functions. *Math. Inequal. Appl.* **12** (2009), no. 4, 853–862.

[26] M. Z. Sarikaya, A. Saglam, and H. Yildirim, On some Hadamard-type inequalities for h-convex functions. *J. Math. Inequal.* **2** (2008), no. 3, 335–341.

[27] E. Set, M. E. Özdemir and M. Z. Sarıkaya, New inequalities of Ostrowski's type for s-convex functions in the second sense with applications. *Facta Univ. Ser. Math. Inform.* **27** (2012), no. 1, 67–82.

[28] M. Z. Sarikaya, E. Set and M. E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions. *Acta Math. Univ. Comenian.* (N.S.) **79** (2010), no. 2, 265–272.

[29] M. Tunç, Ostrowski-type inequalities via h-convex functions with applications to special means. *J. Inequal. Appl.* **2013**, 2013:326.

[30] S. Varošanec, On h-convexity. *J. Math. Anal. Appl.* **326** (2007), no. 1, 303–311.

CHAPTER 6

Other Ostrowski Type Inequalities

1. Ostrowski for Products

1.1. An Identity for RS-Integral. The following identity is of interest in itself:

LEMMA 1.1 (Dragomir, 2013 [**2**]). *Let $f, g : [a,b] \to \mathbb{C}$ be two functions of bounded variation and such that the Riemann-Stieltjes integral $\int_a^b f(t)\,dg(t)$ exists. If $h : [a,b] \to \mathbb{C}$ is continuous, then the Riemann-Stieltjes integral $\int_a^b h(t)\,d(f(t)\,g(t))$ exists and*

$$\int_a^b h(t)\,d(f(t)\,g(t)) = \int_a^b h(t)\,f(t)\,d(g(t)) + \int_a^b h(t)\,g(t)\,d(f(t)). \tag{1.1}$$

PROOF. Since $f, g : [a,b] \to \mathbb{C}$ are of bounded variation, then fg is of bounded variation and since $h : [a,b] \to \mathbb{C}$ is continuous, it follows that the Riemann-Stieltjes integral $\int_a^b h(t)\,d(f(t)\,g(t))$ exists.

Observe that, since the integral $\int_a^b f(t)\,dg(t)$ exists, then for any $s \in [a,b]$ the integral $\ell(s) := \int_a^s f(t)\,dg(t)$ exists and the function ℓ is of bounded variation on $[a,b]$.

Indeed, let

$$a = s_0 < s_1 < \ldots < s_{n-1} < s_n = b$$

a division of the interval $[a,b]$. Then we have

$$\begin{aligned}
\sum_{i=0}^{n-1} |\ell(s_{i+1}) - \ell(s_i)| &= \sum_{i=0}^{n-1} \left| \int_a^{s_{i+1}} f(t)\,dg(t) - \int_a^{s_i} f(t)\,dg(t) \right| \\
&= \sum_{i=0}^{n-1} \left| \int_{s_i}^{s_{i+1}} f(t)\,dg(t) \right| \le \sum_{i=0}^{n-1} \left(\sup_{t\in[s_i,s_{i+1}]} |f(t)| \bigvee_{s_i}^{s_{i+1}} (g) \right) \\
&\le \sup_{t\in[a,b]} |f(t)| \sum_{i=0}^{n-1} \left(\bigvee_{s_i}^{s_{i+1}} (g) \right) = \sup_{t\in[a,b]} |f(t)| \bigvee_a^b (g) < \infty,
\end{aligned}$$

which shows that ℓ is of bounded variation on $[a,b]$ and

$$\bigvee_a^b (\ell) \le \|f\|_\infty \bigvee_a^b (g),$$

where $\|f\|_\infty := \sup_{t\in[a,b]} |f(t)|$.

Now, by the integration by parts theorem, since $\int_a^s f(t)\,dg(t)$ exists for any $s \in [a,b]$, then $\int_a^s g(t)\,df(t)$ also exists and we have the equality

$$f(s)\,g(s) = f(a)\,g(a) + \int_a^s f(t)\,dg(t) + \int_a^s g(t)\,df(t) \tag{1.2}$$

for any $s \in [a,b]$.

Since the functions $\int_a^{\cdot} f(t)\,dg(t)$ and $\int_a^{\cdot} g(t)\,df(t)$ are of bounded variation, then the Riemann-Stieltjes integrals

$$\int_a^b h(s)\,d\left(\int_a^s f(t)\,dg(t)\right) \text{ and } \int_a^b h(s)\,d\left(\int_a^s g(t)\,df(t)\right)$$

exist and

$$\int_a^b h(s)\,d\left(\int_a^s f(t)\,dg(t)\right) = \int_a^b h(s)\,f(s)\,dg(s)\,, \tag{1.3}$$

and

$$\int_a^b h(s)\,d\left(\int_a^s g(t)\,df(t)\right) = \int_a^b h(s)\,g(s)\,df(s)\,. \tag{1.4}$$

Now, on utilizing (1.2), (1.3) and (1.4) we have

$$\begin{aligned}\int_a^b h(s)\,d(f(s)\,g(s)) &= \int_a^b h(s)\,d(f(a)\,g(a)) \\ &\quad + \int_a^b h(s)\,d\left(\int_a^s f(t)\,dg(t)\right) + \int_a^b h(s)\left(\int_a^s g(t)\,df(t)\right) \\ &= \int_a^b h(s)\,f(s)\,dg(s) + \int_a^b h(s)\,g(s)\,df(s)\end{aligned}$$

and the equality (1.1) is proved. ∎

REMARK 1.1. The dual case also holds, namely, when the functions $f, g : [a,b] \to \mathbb{C}$ are continuous and such that the Riemann-Stieltjes integral $\int_a^b f(t)\,dg(t)$ exists, then for any function $h : [a,b] \to \mathbb{C}$ of bounded variation the Riemann-Stieltjes integral $\int_a^b h(t)\,d(f(t)\,g(t))$ exists and the equality (1.1) is satisfied.

COROLLARY 1.2. *Let $f : [a,b] \to \mathbb{C}$ be a functions of bounded variation and such that the Riemann-Stieltjes integral $\int_a^b f(t)\,df(t)$ exists. If $h : [a,b] \to \mathbb{C}$ is continuous, then the Riemann-Stieltjes integral $\int_a^b h(t)\,d(f^2(t))$ exists and*

$$\int_a^b h(t)\,d\left(f^2(t)\right) = 2\int_a^b h(t)\,f(t)\,d(f(t))\,. \tag{1.5}$$

If $\int_a^b f(t)\,d\overline{f(t)}$ exists, then for any continuous function $h : [a,b] \to \mathbb{C}$, the Riemann-Stieltjes integral $\int_a^b h(t)\,d\left(|f(t)|^2\right)$ exists and

$$\int_a^b h(t)\,d\left(|f(t)|^2\right) = \int_a^b h(t)\,f(t)\,d\left(\overline{f(t)}\right) + \int_a^b h(t)\,\overline{f(t)}d(f(t))\,. \tag{1.6}$$

In particular, if $h : [a,b] \to \mathbb{R}$, then

$$\int_a^b h(t)\,d\left(|f(t)|^2\right) = 2\,\mathrm{Re}\left(\int_a^b h(t)\,\overline{f(t)}d(f(t))\right). \tag{1.7}$$

1.2. Inequalities for Product Integrators. The first bound for the Riemann-Stieltjes integral of product integrators is as follows:

THEOREM 1.3 (Dragomir, 2013 [**2**]). *Let $f,g:[a,b]\to\mathbb{C}$ be two functions of bounded variation and such that the Riemann-Stieltjes integral $\int_a^b f(t)\,dg(t)$ exists. If $h:[a,b]\to\mathbb{C}$ is continuous, then*

$$\left|\int_a^b h(t)\,d(f(t)\,g(t))\right| \le \|hf\|_\infty \bigvee_a^b (g) + \|hg\|_\infty \bigvee_a^b (f) \le \|h\|_\infty \left[\|f\|_\infty \bigvee_a^b (g) + \|g\|_\infty \bigvee_a^b (f)\right]. \tag{1.8}$$

Both inequalities in (1.8) are sharp.

PROOF. We know that if $p:[a,b]\to\mathbb{C}$ is bounded, $v:[a,b]\to\mathbb{C}$ is of bounded variation and the Riemann-Stieltjes integral $\int_a^b p(s)\,dv(s)$ exists, then we have the inequality

$$\left|\int_a^b p(s)\,dv(s)\right| \le \|p\|_\infty \bigvee_a^b (v), \tag{1.9}$$

where $\|p\|_\infty = \sup_{t\in[a,b]} |p(t)| < \infty$.

Taking the modulus in (1.1) and using the property (1.9) we have

$$\begin{aligned}\left|\int_a^b h(t)\,d(f(t)\,g(t))\right| &\le \left|\int_a^b h(t)\,f(t)\,d(g(t))\right| + \left|\int_a^b h(t)\,g(t)\,d(f(t))\right| \\ &\le \|hf\|_\infty \bigvee_a^b (g) + \|hg\|_\infty \bigvee_a^b (f) \\ &\le \|h\|_\infty \|f\|_\infty \bigvee_a^b (g) + \|h\|_\infty \|g\|_\infty \bigvee_a^b (f) \\ &= \|h\|_\infty \left[\|f\|_\infty \bigvee_a^b (g) + \|g\|_\infty \bigvee_a^b (f)\right]\end{aligned}$$

and the inequality (1.8) is proved.

Now, to prove the sharpness of the inequalities (1.8) we consider the functions $f,g:[a,b]\to\mathbb{R}$ given by

$$f(t) := \begin{cases} 0 & \text{if } t=a \\ 1 & \text{if } t\in(a,b], \end{cases} \qquad g(t) := \begin{cases} 1 & \text{if } t\in[a,b) \\ 0 & \text{if } t=b. \end{cases}$$

The functions f and g are of bounded variation, $\bigvee_a^b (f) = \bigvee_a^b (g) = 1$ and $\|f\|_\infty = \|g\|_\infty = 1$.

We have

$$f(t)\,g(t) := \begin{cases} 0 & \text{if } t=a \\ 1 & \text{if } t\in(a,b) \\ 0 & \text{if } t=b. \end{cases}$$

The function fg is of bounded variation and for a continuous function $h:[a,b]\to\mathbb{C}$ the Riemann-Stieltjes integral $\int_a^b h(t)\,d(f(t)\,g(t))$ exists and integrating by parts we have

$$\begin{aligned}\int_a^b h(t)\,d(f(t)\,g(t)) \\ &= f(b)\,g(b)\,h(b)-f(a)\,g(a)\,h(a)-\int_a^b f(t)\,g(t)\,d(h(t)) \\ &= -\int_a^b f(t)\,g(t)\,d(h(t)).\end{aligned}\tag{1.10}$$

Consider the following sequence of divisions and intermediate points

$$a=x_0^{(n)}<\xi_0^{(n)}<x_1^{(n)}<\dots<x_{n-1}^{(n)}<\xi_{n-1}^{(n)}<x_n^{(n)}=b$$

such that the norm of the division $\Delta_n:=\max_{i\in\{0,\dots,n-1\}}\left(x_{i+1}^{(n)}-x_i^{(n)}\right)\to 0$ as $n\to\infty$. By the definition of the Riemann-Stieltjes integral we have

$$\begin{aligned}\int_a^b f(t)\,g(t)\,d(h(t)) &= \lim_{n\to\infty}\sum_{i=0}^{n-1} f\left(\xi_i^{(n)}\right)g\left(\xi_i^{(n)}\right)\left(h\left(x_{i+1}^{(n)}\right)-h\left(x_i^{(n)}\right)\right) \\ &= \lim_{n\to\infty}\sum_{i=0}^{n-1}\left(h\left(x_{i+1}^{(n)}\right)-h\left(x_i^{(n)}\right)\right)=h(b)-h(a),\end{aligned}$$

and then, by (1.10) we have

$$\int_a^b h(t)\,d(f(t)\,g(t))=-h(b)+h(a).$$

We also have

$$h(t)\,f(t):=\begin{cases}0 & \text{if } t=a\\ h(t) & \text{if } t\in(a,b],\end{cases}\qquad h(t)\,g(t):=\begin{cases}h(t) & \text{if } t\in[a,b)\\ 0 & \text{if } t=b,\end{cases}$$

which implies that

$$\|hf\|_\infty=\|hg\|_\infty=\|h\|_\infty.$$

Therefore the inequality (1.8) reduces to

$$|h(b)-h(a)|\le 2\|h\|_\infty\le 2\|h\|_\infty.\tag{1.11}$$

We observe that, this inequality is sharp since for continuous functions $h:[a,b]\to\mathbb{R}$ for which

$$0<h(b)=-h(a)=\sup_{t\in[a,b]}h(t),$$

we get equality in (1.11).

For instance, if we take

$$h(t)=t-\frac{a+b}{2},t\in[a,b],$$

then

$$|h(b)-h(a)|=b-a,\|h\|_\infty=\frac{b-a}{2}$$

and the equality in (1.11) is realized. ∎

We say that the function $f:[a,b]\to\mathbb{C}$ is *Lipschitzian* with the constant $L>0$ if

$$|f(t)-f(s)|\le L|t-s|$$

for any $t,s\in[a,b]$.

THEOREM 1.4 (Dragomir, 2013 [**2**]). *Assume that the function $f:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, $g:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $K>0$ and $h:[a,b]\to\mathbb{C}$ a continuous function on $[a,b]$. Then we have the inequality*

$$\left|\int_a^b h(t)\,d(f(t)g(t))\right|\le K\int_a^b|h(t)f(t)|\,dt+L\int_a^b|h(t)g(t)|\,dt \tag{1.12}$$
$$\le\max\{K,L\}\int_a^b|h(t)|(|f(t)|+|g(t)|)\,dt.$$

The inequalities in (1.12) are sharp.

PROOF. It is known that, if $p:[a,b]\to\mathbb{C}$ is continuous and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, then the Riemann-Stieltjes integral $\int_a^b p(s)\,dv(s)$ exists and we have the inequality

$$\left|\int_a^b p(s)\,dv(s)\right|\le L\int_a^b|p(s)|\,ds. \tag{1.13}$$

Taking the modulus in (1.1) and using the property (1.9) we have

$$\left|\int_a^b h(t)\,d(f(t)g(t))\right|\le\left|\int_a^b h(t)f(t)\,d(g(t))\right|+\left|\int_a^b h(t)g(t)\,d(f(t))\right|$$
$$\le K\int_a^b|h(t)f(t)|\,dt+L\int_a^b|h(t)g(t)|\,dt$$
$$\le\max\{K,L\}\int_a^b|h(t)|(|f(t)|+|g(t)|)\,dt,$$

and the inequality (1.12) is proved.

Consider now the functions $f,g:[a,b]\to\mathbb{R}$, $f(t)=g(t)=\left|t-\frac{a+b}{2}\right|$. We observe that f and g are Lipschitzian with the constant $L=1$.

Indeed, for any $t,s\in[a,b]$ we have

$$|f(t)-f(s)|=\left|\left|t-\frac{a+b}{2}\right|-\left|s-\frac{a+b}{2}\right|\right|$$
$$\le|t-s|,$$

which shows that the function f is Lipschitzian with the constant $L=1$.

Now

$$\left|\int_a^b h(t)\,d(f(t)g(t))\right|=\int_a^b h(t)\,d\left(\left(t-\frac{a+b}{2}\right)^2\right)$$
$$=2\left|\int_a^b h(t)\left(t-\frac{a+b}{2}\right)dt\right|$$

and

$$K\int_a^b|h(t)f(t)|\,dt+L\int_a^b|h(t)g(t)|\,dt=2\int_a^b|h(t)|\left|t-\frac{a+b}{2}\right|dt$$

and the first inequality in (1.12) becomes

$$\left|\int_a^b h(t)\left(t-\frac{a+b}{2}\right)dt\right| \le \int_a^b |h(t)|\left|t-\frac{a+b}{2}\right|dt.$$

We observe that the equality case holds if we take $h:[a,b]\to\mathbb{R}$, $h(t)=t-\frac{a+b}{2}$. ∎

THEOREM 1.5 (Dragomir, 2013 [**2**]). *Assume that $f,g:[a,b]\to\mathbb{R}$ are monotonic nondecreasing on $[a,b]$ and such that the Riemann-Stieltjes integral $\int_a^b f(t)\,dg(t)$ exists, and $h:[a,b]\to\mathbb{C}$ is continuous on $[a,b]$. Then we have the inequality*

$$\left|\int_a^b h(t)\,d(f(t)g(t))\right| \le \int_a^b |f(t)h(t)|\,dg(t)+\int_a^b |g(t)h(t)|\,df(t). \tag{1.14}$$

The inequality (1.14) is sharp.

PROOF. It is well known that if $p:[a,b]\to\mathbb{C}$ is continuous and $v:[a,b]\to\mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dv(t).$$

Taking the modulus in (1.1) we have

$$\begin{aligned}\left|\int_a^b h(t)\,d(f(t)g(t))\right| &\le \left|\int_a^b f(t)h(t)\,dg(t)\right|+\left|\int_a^b g(t)h(t)\,df(t)\right|\\ &\le \int_a^b |f(t)h(t)|\,dg(t)+\int_a^b |g(t)h(t)|\,df(t).\end{aligned}$$

Consider the functions $f,g:[a,b]\to\mathbb{R}$ defined by

$$f(t):=\begin{cases}0 & \text{if } t=a\\ 1 & \text{if } t\in(a,b],\end{cases}\qquad g(t):=\begin{cases}-1 & \text{if } t\in[a,b)\\ 0 & \text{if } t=b.\end{cases}$$

The functions f and g are monotonic nondecreasing. We will show that the Riemann-Stieltjes integral $\int_a^b f(t)\,dg(t)$ exists.

Take the sequence of divisions and intermediate points

$$d_n: a=x_0^{(n)}<\xi_0^{(n)}<x_1^{(n)}<\ldots<x_{n-1}^{(n)}<\xi_{n-1}^{(n)}<x_n^{(n)}=b$$

such that $\Delta(d_n):=\max_{i\in\{0,\ldots,n-1\}}\left\{x_{i+1}^{(n)}-x_i^{(n)}\right\}\to 0$ as $n\to\infty$.

By the definition of the Riemann-Stieltjes integral $\int_a^b f(t)\,dg(t)$ we have

$$\begin{aligned}\int_a^b f(t)\,dg(t) &= \lim_{n\to\infty}\sum_{i=0}^{n-1} f\left(\xi_i^{(n)}\right)\left[g\left(x_{i+1}^{(n)}\right)-g\left(x_i^{(n)}\right)\right]\\ &= \lim_{n\to\infty}\sum_{i=0}^{n-2} f\left(\xi_i^{(n)}\right)\left[g\left(x_{i+1}^{(n)}\right)-g\left(x_i^{(n)}\right)\right]\\ &\quad+\lim_{n\to\infty} f\left(\xi_{n-1}^{(n)}\right)\left[g(b)-g\left(x_{n-1}^{(n)}\right)\right]\\ &= 0+1=1.\end{aligned}$$

Now, define the function $\ell : [a,b] \to \mathbb{R}$ by

$$\ell(t) := f(t)\, g(t) = \begin{cases} 0 & \text{if } t = a \\ -1 & \text{if } t \in (a,b) \\ 0 & \text{if } t = a. \end{cases}$$

For a continuous function $h : [a,b] \to \mathbb{R}$, since ℓ is of bounded variation, then the integral $\int_a^b h(t)\, d\ell(t)$ exists.

Take the sequence of divisions and intermediate points

$$d_n : a = x_0^{(n)} < \xi_0^{(n)} < x_1^{(n)} < \ldots < x_{n-1}^{(n)} < \xi_{n-1}^{(n)} < x_n^{(n)} = b$$

such that $\Delta(d_n) := \max_{i\in\{0,\ldots,n-1\}} \left\{ x_{i+1}^{(n)} - x_i^{(n)} \right\} \to 0$ as $n \to \infty$.Then we have

$$\begin{aligned}
\int_a^b h(t)\, d\ell(t) &= \lim_{n\to\infty} \sum_{i=0}^{n-1} h\left(\xi_i^{(n)}\right) \left[\ell\left(x_{i+1}^{(n)}\right) - \ell\left(x_i^{(n)}\right)\right] \\
&= \lim_{n\to\infty} h\left(\xi_0^{(n)}\right) \left[\ell\left(x_1^{(n)}\right) - \ell(a)\right] \\
&+ \lim_{n\to\infty} \sum_{i=1}^{n-2} h\left(\xi_i^{(n)}\right) \left[\ell\left(x_{i+1}^{(n)}\right) - \ell\left(x_i^{(n)}\right)\right] \\
&+ \lim_{n\to\infty} h\left(\xi_{n-1}^{(n)}\right) \left[\ell(b) - \ell\left(x_{n-1}^{(n)}\right)\right] \\
&= \lim_{n\to\infty} h\left(\xi_0^{(n)}\right)(-1-0) + 0 + \lim_{n\to\infty} h\left(\xi_{n-1}^{(n)}\right)[0-(-1)] \\
&= h(b) - h(a).
\end{aligned}$$

Consider the functions $u, v : [a,b] \to \mathbb{R}$ given by

$$u(t) := |f(t)\, h(t)| = \begin{cases} 0 & \text{if } t = a \\ |h(t)| & \text{if } t \in (a,b], \end{cases}$$

and

$$v(t) := |g(t)\, h(t)| = \begin{cases} |h(t)| & \text{if } t \in [a,b) \\ 0 & \text{if } t = b. \end{cases}$$

Take the sequence of divisions and intermediate points

$$d_n : a = x_0^{(n)} < \xi_0^{(n)} < x_1^{(n)} < \ldots < x_{n-1}^{(n)} < \xi_{n-1}^{(n)} < x_n^{(n)} = b$$

such that $\Delta(d_n) := \max_{i\in\{0,\ldots,n-1\}} \left\{ x_{i+1}^{(n)} - x_i^{(n)} \right\} \to 0$ as $n \to \infty$.Then we have

$$\begin{aligned}
\int_a^b u(t)\, dg(t) &= \lim_{n\to\infty} \sum_{i=0}^{n-1} u\left(\xi_i^{(n)}\right) \left[g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right] \\
&= \lim_{n\to\infty} u\left(\xi_{n-1}^{(n)}\right) \left[g(b) - g\left(x_{n-1}^{(n)}\right)\right] \\
&= \lim_{n\to\infty} \left| h\left(\xi_{n-1}^{(n)}\right) \right| [0-(-1)] = |h(b)|
\end{aligned}$$

and

$$\begin{aligned}\int_a^b v(t)\,df(t) &= \lim_{n\to\infty}\sum_{i=0}^{n-1} v\left(\xi_i^{(n)}\right)\left[f\left(x_{i+1}^{(n)}\right)-f\left(x_i^{(n)}\right)\right]\\ &= \lim_{n\to\infty} v\left(\xi_0^{(n)}\right)\left[f\left(x_1^{(n)}\right)-f(a)\right]\\ &= \lim_{n\to\infty}\left|h\left(\xi_0^{(n)}\right)\right|(1-0)=|h(a)|\,.\end{aligned}$$

Replacing these values in (1.14) we have

$$|h(b)-h(a)|\le|h(b)|+|h(a)|\,. \tag{1.15}$$

This inequality reduces to an equality if we choose a continuous function $h:[a,b]\to\mathbb{R}$ such that $h(b)=-h(a)$. For instance, for $h:[a,b]\to\mathbb{R}$, $h(t)=t-\frac{a+b}{2}$, we get in both sides of (1.15) the same quantity $b-a$. ∎

1.3. Ostrowski Type Inequalities for Products.

The following result holds:

THEOREM 1.6 (Dragomir, 2013 [**2**]). *Let $f,g:[a,b]\to\mathbb{C}$ be two functions of bounded variation and such that for $x\in(a,b)$ the Riemann-Stieltjes integrals $\int_a^x f(t)\,dg(t)$ and $\int_x^b f(t)\,dg(t)$ exist. Then we have*

$$\begin{aligned}&\left|f(x)g(x)(b-a)-\int_a^b f(t)g(t)\,dt\right| \\ &\le (x-a)\sup_{t\in[a,x]}\{|f(t)|\}\bigvee_a^x(g)+(x-a)\sup_{t\in[a,x]}\{|g(t)|\}\bigvee_a^x(f)\\ &+(b-x)\sup_{t\in[x,b]}\{|f(t)|\}\bigvee_x^b(g)+(b-x)\sup_{t\in[x,b]}\{|g(t)|\}\bigvee_x^b(f)\\ &\le\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\|g\|_\infty\bigvee_a^b(f)+\|f\|_\infty\bigvee_a^b(g)\right].\end{aligned} \tag{1.16}$$

In particular, if the Riemann-Stieltjes integrals $\int_a^{\frac{a+b}{2}} f(t)\,dg(t)$ and $\int_{\frac{a+b}{2}}^b f(t)\,dg(t)$ exist, then we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)(b-a)-\int_a^b f(t)g(t)\,dt\right|\\ &\le\frac{b-a}{2}\left[\sup_{t\in\left[a,\frac{a+b}{2}\right]}\{|f(t)|\}\bigvee_a^{\frac{a+b}{2}}(g)+\sup_{t\in\left[a,\frac{a+b}{2}\right]}\{|g(t)|\}\bigvee_a^{\frac{a+b}{2}}(f)\right.\\ &\left.+\sup_{t\in\left[\frac{a+b}{2},b\right]}\{|f(t)|\}\bigvee_{\frac{a+b}{2}}^b(g)+\sup_{t\in\left[\frac{a+b}{2},b\right]}\{|g(t)|\}\bigvee_{\frac{a+b}{2}}^b(f)\right]\\ &\le\frac{1}{2}(b-a)\left[\|g\|_\infty\bigvee_a^b(f)+\|f\|_\infty\bigvee_a^b(g)\right].\end{aligned} \tag{1.17}$$

The inequalities in (1.17) are sharp.

PROOF. We use the following identity (see for instance [**1**])

$$F(x)(b-a)-\int_a^b F(t)\,dt=\int_a^x (t-a)\,dF(t)+\int_x^b (t-b)\,dF(t) \tag{1.18}$$

that holds for any function of bounded variation $F:[a,b]\to\mathbb{C}$ and any $x\in[a,b]$.

If we write the equality (1.18) for $F=fg$ we get

$$\begin{aligned} &f(x)\,g(x)\,(b-a)-\int_a^b f(t)\,g(t)\,dt \\ &=\int_a^x (t-a)\,d(f(t)\,g(t))+\int_x^b (t-b)\,d(f(t)\,g(t)), \end{aligned} \tag{1.19}$$

for any functions $f,g:[a,b]\to\mathbb{C}$ of bounded variation and any $x\in[a,b]$.

Taking the modulus on (1.19) and utilizing Theorem 1.3 on the intervals $[a,x]$ and $[x,b]$ we have successively that

$$\begin{aligned} &\left|f(x)\,g(x)\,(b-a)-\int_a^b f(t)\,g(t)\,dt\right| \\ &\le\left|\int_a^x (t-a)\,d(f(t)\,g(t))\right|+\left|\int_x^b (t-b)\,d(f(t)\,g(t))\right| \\ &\le\sup_{t\in[a,x]}\{(t-a)\,|f(t)|\}\bigvee_a^x(g)+\sup_{t\in[a,x]}\{(t-a)\,|g(t)|\}\bigvee_a^x(f) \\ &+\sup_{t\in[x,b]}\{(b-t)\,|f(t)|\}\bigvee_x^b(g)+\sup_{t\in[x,b]}\{(b-t)\,|g(t)|\}\bigvee_x^b(f) \\ &\le(x-a)\sup_{t\in[a,x]}\{|f(t)|\}\bigvee_a^x(g)+(x-a)\sup_{t\in[a,x]}\{|g(t)|\}\bigvee_a^x(f) \\ &+(b-x)\sup_{t\in[x,b]}\{|f(t)|\}\bigvee_x^b(g)+(b-x)\sup_{t\in[x,b]}\{|g(t)|\}\bigvee_x^b(f) \\ &\le\max\{x-a,b-x\}\sup_{t\in[a,b]}\{|f(t)|\}\bigvee_a^b(g) \\ &+\max\{x-a,b-x\}\sup_{t\in[a,b]}\{|g(t)|\}\bigvee_a^b(f) \\ &=\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\|g\|_\infty\bigvee_a^b(f)+\|f\|_\infty\bigvee_a^b(g)\right], \end{aligned} \tag{1.20}$$

which proves the desired result (1.16).

The inequality (1.17) is obvious from (1.16).

Consider now the functions $f,g:[a,b]\to\mathbb{R}$ defined by

$$f(t):=\begin{cases}0 & \text{if } t\in\left[a,\frac{a+b}{2}\right) \\ 1 & \text{if } t\in\left[\frac{a+b}{2},b\right]\end{cases}\quad g(t):=\begin{cases}1 & \text{if } t\in\left[a,\frac{a+b}{2}\right] \\ 0 & \text{if } t\in\left(\frac{a+b}{2},b\right].\end{cases}$$

We observe that f and g are of bounded variation and

$$\bigvee_a^b (f) = \bigvee_a^b (g) = 1.$$

The Riemann-Stieltjes integrals $\int_a^{\frac{a+b}{2}} f(t)\,dg(t)$ and $\int_{\frac{a+b}{2}}^b f(t)\,dg(t)$ exist since one function is continuous while the other is of bounded variation on those intervals.

We observe that for these functions we have

$$f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f(t)\,g(t)\,dt = b-a,$$

$$\begin{aligned}
&\sup_{t\in\left[a,\frac{a+b}{2}\right]} \{|f(t)|\} \bigvee_a^{\frac{a+b}{2}} (g) + \sup_{t\in\left[a,\frac{a+b}{2}\right]} \{|g(t)|\} \bigvee_a^{\frac{a+b}{2}} (f) \\
&+ \sup_{t\in\left[\frac{a+b}{2},b\right]} \{|f(t)|\} \bigvee_{\frac{a+b}{2}}^b (g) + \sup_{t\in\left[\frac{a+b}{2},b\right]} \{|g(t)|\} \bigvee_{\frac{a+b}{2}}^b (f) \\
&= 2
\end{aligned}$$

and

$$\|g\|_\infty \bigvee_a^b (f) + \|f\|_\infty \bigvee_a^b (g) = 2.$$

Replacing these values in (1.17) we obtain in all terms the same quantity $b-a$, which proves the sharpness of the inequalities. ∎

In a similar way we can prove the following results as well:

THEOREM 1.7 (Dragomir, 2013 [**2**]). *Let $f:[a,b]\to\mathbb{C}$ be Lipschitzian with the constant $L>0$ and $g:[a,b]\to\mathbb{C}$ be Lipschitzian with the constant $K>0$. Then for any $x\in[a,b]$ we have*

$$\begin{aligned}
&\left| f(x)\,g(x)\,(b-a) - \int_a^b f(t)\,g(t)\,dt \right| \\
&\le L\left(\int_a^x (t-a)\,|g(t)|\,dt + \int_x^b (b-t)\,|g(t)|\,dt \right) \\
&+ K\left(\int_a^x (t-a)\,|f(t)|\,dt + \int_x^b (b-t)\,|f(t)|\,dt \right) \\
&\le \left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2 \right] \left(L\,\|g\|_\infty + K\,\|f\|_\infty \right).
\end{aligned} \tag{1.21}$$

In particular, we have

$$\begin{aligned}(1.22)\qquad &\left|f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)(b-a)-\int_a^b f(t)\,g(t)\,dt\right|\\ &\le L\left(\int_a^{\frac{a+b}{2}}(t-a)\,|g(t)|\,dt+\int_{\frac{a+b}{2}}^b (b-t)\,|g(t)|\,dt\right)\\ &+K\left(\int_a^{\frac{a+b}{2}}(t-a)\,|f(t)|\,dt+\int_{\frac{a+b}{2}}^b (b-t)\,|f(t)|\,dt\right)\\ &\le \frac{1}{4}(b-a)^2\left(L\,\|g\|_\infty+K\,\|f\|_\infty\right).\end{aligned}$$

We also have:

THEOREM 1.8 (Dragomir, 2013 [**2**]). *Assume that $f,g:[a,b]\to\mathbb{R}$ are monotonic nondecreasing on $[a,b]$ and such that for $x\in(a,b)$ the Riemann-Stieltjes integrals $\int_a^x f(t)\,dg(t)$ and $\int_x^b f(t)\,dg(t)$ exist. Then we have*

$$\begin{aligned}(1.23)\qquad &\left|f(x)\,g(x)\,(b-a)-\int_a^b f(t)\,g(t)\,dt\right|\\ &\le \int_a^x (t-a)\,|g(t)|\,df(t)+\int_x^b (b-t)\,|g(t)|\,df(t)\\ &+\int_a^x (t-a)\,|f(t)|\,dg(t)+\int_x^b (b-t)\,|f(t)|\,dg(t)\\ &\le \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left(\int_a^b |g(t)|\,df(t)+\int_a^b |f(t)|\,dg(t)\right).\end{aligned}$$

In particular, if the Riemann-Stieltjes integrals $\int_a^{\frac{a+b}{2}} f(t)\,dg(t)$ and $\int_{\frac{a+b}{2}}^b f(t)\,dg(t)$ exist, then we have

$$\begin{aligned}(1.24)\qquad &\left|f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)(b-a)-\int_a^b f(t)\,g(t)\,dt\right|\\ &\le \int_a^{\frac{a+b}{2}}(t-a)\,|g(t)|\,df(t)+\int_{\frac{a+b}{2}}^b (b-t)\,|g(t)|\,df(t)\\ &+\int_a^{\frac{a+b}{2}}(t-a)\,|f(t)|\,dg(t)+\int_{\frac{a+b}{2}}^b (b-t)\,|f(t)|\,dg(t)\\ &\le \frac{1}{2}(b-a)\left(\int_a^b |g(t)|\,df(t)+\int_a^b |f(t)|\,dg(t)\right).\end{aligned}$$

2. OSTROWSKI FOR S-DOMINATED FUNCTIONS

2.1. S-Dominated Functions. We start with the following definition:

DEFINITION 2.1 (Dragomir, 2013 [**4**]). Assume that $u,v:[a,b]\to\mathbb{R}$ are *monotonic nondecreasing* on the interval $[a,b]$. We say that the complex-valued function $h:[a,b]\to\mathbb{C}$ is *S-dominated* by the pair (u,v) if

$$\text{(S)}\qquad |h(y)-h(x)|^2\le[u(y)-u(x)]\,[v(y)-v(x)]$$

for any $x,y\in[a,b]$.

We observe that by the monotonicity of the functions u and v and by the symmetry of the inequality (S) over x and y we can assume that (S) is satisfied only for $y > x$ with $x, y \in [a, b]$.

We can give numerous examples of such functions.

For instance, if we take $f, g \in L_2[a, b]$ the Hilbert space of all complex-valued functions that are square-Lebesgue integrable and denote

$$h(x) := \int_a^x f(t)\, g(t)\, dt,\ u(x) := \int_a^x |f(t)|^2\, dt \text{ and } v(x) := \int_a^x |g(t)|^2\, dt,$$

then we observe that u and v are monotonic nondecreasing on $[a, b]$ and by Cauchy-Bunyakovsky-Schwarz integral inequality we have for any $y > x$ with $x, y \in [a, b]$ that

$$\begin{aligned} |h(y) - h(x)|^2 &= \left| \int_x^y f(t)\, g(t)\, dt \right|^2 \le \int_x^y |f(t)|^2\, dt \int_x^y |g(t)|^2\, dt \\ &\le [u(y) - u(x)]\,[v(y) - v(x)]. \end{aligned}$$

Now, for $p, q > 0$ if we consider $f(t) := t^p$ and $g(t) := t^q$ for $t \ge 0$, then

$$h_{p,q}(x) := \int_0^x t^{p+q} dt = \frac{1}{p+q+1} x^{p+q+1}$$

and

$$u_p(x) := \int_0^x t^{2p} dt = \frac{1}{2p+1} x^{2p+1},\ v_q(x) := \int_0^x t^{2q} dt = \frac{1}{2q+1} x^{2q+1}.$$

Taking into account the above comments we observe that the function $h_{p,q}$ is *S-dominated* by the pair (u_p, v_q) on any subinterval of $[0, \infty)$.

PROPOSITION 2.1 (Dragomir, 2013 [**4**]). *If $h : [a, b] \to \mathbb{C}$ is S-dominated by the pair (u, v), then h is of bounded variation on any subinterval $[c, d] \subset [a, b]$ and*

$$\left[\bigvee_c^d (h) \right]^2 \le [u(d) - u(c)]\,[v(d) - v(c)]. \tag{2.1}$$

PROOF. Consider a division δ of the interval $[c, d]$ given by

$$\delta : c = x_0 < x_1 < ... < x_{n-1} < x_n = b.$$

Since $h : [a, b] \to \mathbb{C}$ is S-dominated by the pair (u, v) then we have

$$|h(x_{i+1}) - h(x_i)| \le [u(x_{i+1}) - u(x_i)]^{1/2}\,[v(x_{i+1}) - v(x_i)]^{1/2}$$

for any $i \in \{0, ..., n-1\}$.

Summing this inequality over i from 0 to $n-1$ and utilizing the Cauchy-Bunyakovsky-Schwarz discrete inequality we have

$$\begin{aligned} &\sum_{i=1}^{n-1} |h(x_{i+1}) - h(x_i)| \\ &\le \sum_{i=1}^{n-1} [u(x_{i+1}) - u(x_i)]^{1/2}\,[v(x_{i+1}) - v(x_i)]^{1/2} \\ &\le \left(\sum_{i=1}^{n-1} [u(x_{i+1}) - u(x_i)] \right)^{1/2} \left(\sum_{i=1}^{n-1} [v(x_{i+1}) - v(x_i)] \right)^{1/2} \\ &= [u(d) - u(c)]^{1/2}\,[v(d) - v(c)]^{1/2}. \end{aligned} \tag{2.2}$$

Taking the supremum over δ we deduce the desired result (2.1). ∎

COROLLARY 2.2. *If $h:[a,b]\to\mathbb{C}$ is S-dominated by the pair (u,v), then the cumulative variation function $V:[a,b]\to[0,\infty)$ defined by*

$$V(x):=\bigvee_a^x(h)$$

is also S-dominated by the pair (u,v).

THEOREM 2.3 (Dragomir, 2013 [**4**]). *Assume that $u,v:[a,b]\to\mathbb{R}$ are monotonic nondecreasing on the interval $[a,b]$. If $h:[a,b]\to\mathbb{C}$ is S-dominated by the pair (u,v) and $f:[a,b]\to\mathbb{C}$ is a continuous function on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)\,dh(t)$ exists and*

$$\left|\int_a^b f(t)\,dh(t)\right|^2\leq\int_a^b|f(t)|\,du(t)\int_a^b|f(t)|\,dv(t). \tag{2.3}$$

PROOF. Since the Riemann-Stieltjes integral $\int_a^b f(t)\,dh(t)$ exists, then for any sequence of partitions

$$I_n^{(n)}:a=t_0^{(n)}<t_1^{(n)}<\cdots<t_{n-1}^{(n)}<t_n^{(n)}=b$$

with the norm

$$v\left(I_n^{(n)}\right):=\max_{i\in\{0,\ldots,n-1\}}\left(t_{i+1}^{(n)}-t_i^{(n)}\right)\to 0$$

as $n\to\infty$, and for any intermediate points $\xi_i^{(n)}\in[t_i^{(n)},t_{i+1}^{(n)}]$, $i\in\{0,\ldots,n-1\}$ we have:

$$\begin{aligned}\left|\int_a^b f(t)\,dh(t)\right| &= \left|\lim_{v\left(I_n^{(n)}\right)\to0}\sum_{i=0}^{n-1}f\left(\xi_i^{(n)}\right)\left[h\left(t_{i+1}^{(n)}\right)-h\left(t_i^{(n)}\right)\right]\right| \\ &\leq \lim_{v\left(I_n^{(n)}\right)\to0}\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left|h\left(t_{i+1}^{(n)}\right)-h\left(t_i^{(n)}\right)\right| \\ &\leq \lim_{v\left(I_n^{(n)}\right)\to0}\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left[u\left(t_{i+1}^{(n)}\right)-u\left(t_i^{(n)}\right)\right]^{1/2} \\ &\times\left[v\left(t_{i+1}^{(n)}\right)-v\left(t_i^{(n)}\right)\right]^{1/2} \\ &\leq \lim_{v\left(I_n^{(n)}\right)\to0}\left(\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left[u\left(t_{i+1}^{(n)}\right)-u\left(t_i^{(n)}\right)\right]\right)^{1/2} \\ &\times\lim_{v\left(I_n^{(n)}\right)\to0}\left(\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left[v\left(t_{i+1}^{(n)}\right)-v\left(t_i^{(n)}\right)\right]\right)^{1/2} \\ &= \int_a^b|f(t)|\,du(t)\int_a^b|f(t)|\,dv(t),\end{aligned} \tag{2.4}$$

where for the last inequality we employed the Cauchy-Bunyakovsky-Schwarz weighted discrete inequality

$$\sum_{k=1}^n m_ka_kb_k\leq\left(\sum_{k=1}^n m_ka_k^2\right)^{1/2}\left(\sum_{k=1}^n m_kb_k^2\right)^{1/2},$$

where m_k, a_k, $b_k\geq0$ for $k\in\{1,...,n\}$. ∎

2.2. Ostrowski Type Inequality. We can state the following new result.

THEOREM 2.4 (Dragomir, 2013 [**4**]). *Assume that $u, v : [a, b] \to \mathbb{R}$ are monotonic nondecreasing on the interval $[a, b]$. If $h : [a, b] \to \mathbb{C}$ is S-dominated by the pair (u, v), then*

$$\begin{aligned}(2.5)\qquad & \left|h(x)(b-a) - \int_a^b h(t)\,dt\right|^2 \\ & \leq \left[[2x-(a+b)]u(x) + \int_a^b sgn(t-x)u(t)\,dt\right] \\ & \times \left[[2x-(a+b)]v(x) + \int_a^b sgn(t-x)v(t)\,dt\right] \\ & \leq [(b-x)[u(b)-u(x)] + (x-a)[u(x)-u(a)]] \\ & \times [(b-x)[v(b)-v(x)] + (x-a)[v(x)-v(a)]] \\ & \leq \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^2 [u(b)-u(a)][v(b)-v(a)],\end{aligned}$$

for any $x \in [a, b]$.

PROOF. Integrating by parts on the Riemann-Stieltjes integral we have [**1**]

$$(2.6)\qquad h(x)(b-a) - \int_a^b h(t)\,dt = \int_a^x (t-a)\,dh(t) + \int_x^b (b-t)\,dh(t),$$

for any $x \in [a, b]$.

Taking the modulus in (2.6) we have

$$\begin{aligned}(2.7)\qquad & \left|h(x)(b-a) - \int_a^b h(t)\,dt\right| \\ & \leq \left|\int_a^x (t-a)\,dh(t)\right| + \left|\int_x^b (b-t)\,dh(t)\right|.\end{aligned}$$

Applying the inequality (2.3) twice, we have

$$\left|\int_a^x (t-a)\,dh(t)\right| \leq \left(\int_a^x (t-a)\,du(t)\right)^{1/2} \left(\int_a^x (t-a)\,dv(t)\right)^{1/2}$$

and

$$\left|\int_x^b (b-t)\,dh(t)\right| \leq \left(\int_x^b (b-t)\,du(t)\right)^{1/2} \left(\int_x^b (b-t)\,dv(t)\right)^{1/2}.$$

Summing these inequalities and utilizing the elementary result

$$\alpha\beta + \lambda\delta \leq \left(\alpha^2 + \lambda^2\right)^{1/2} \left(\beta^2 + \delta^2\right)^{1/2}$$

where $\alpha, \beta, \lambda, \delta \geq 0$, we have

$$\begin{aligned} &\left| \int_a^x (t-a)\, dh(t) \right| + \left| \int_x^b (b-t)\, dh(t) \right| \\ &\leq \left(\int_a^x (t-a)\, du(t) \right)^{1/2} \left(\int_a^x (t-a)\, dv(t) \right)^{1/2} \\ &+ \left(\int_x^b (b-t)\, du(t) \right)^{1/2} \left(\int_x^b (b-t)\, dv(t) \right)^{1/2} \\ &\leq \left(\int_a^x (t-a)\, du(t) + \int_x^b (b-t)\, du(t) \right)^{1/2} \\ &+ \left(\int_a^x (t-a)\, dv(t) + \int_x^b (b-t)\, dv(t) \right)^{1/2}. \end{aligned} \tag{2.8}$$

Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned} &\int_a^x (t-a) du(t) + \int_x^b (b-t) du(t) \\ &= (t-a)u(t)\Big|_a^x - \int_a^x u(t)\, dt - (b-t)u(t)\Big|_x^b + \int_x^b u(t)\, dt \\ &= [2x-(a+b)]u(x) - \int_a^x u(t)\, dt + \int_x^b u(t)\, dt \\ &= [2x-(a+b)]u(x) + \int_a^b sgn(t-x)u(t)\, dt. \end{aligned} \tag{2.9}$$

The same equality holds for v as well.

Making use of (2.7) and (2.8) we deduce the first inequality in (2.5).

As u is monotonic nondecreasing on $[a,b]$, we can state that

$$\int_a^x u(t)\, dt \geq (x-a)u(a) \text{ and } \int_x^b u(t)\, dt \leq (b-x)u(b)$$

so that

$$\int_a^b sgn(t-x)u(t)\, dt \leq (b-x)u(b) - (x-a)u(a).$$

Consequently

$$\begin{aligned} &[2x-(a+b)]u(x) + \int_a^b sgn(t-x)u(t)\, dt \\ &\leq [2x-(a+b)]u(x) + (b-x)u(b) - (x-a)u(a) \\ &= (b-x)[u(b)-u(x)] + (x-a)[u(x)-u(a)] \end{aligned}$$

and a similar inequality for v.

Finally, let us observe that

$$\begin{aligned} &(b-x)[u(b)-u(x)] + (x-a)[u(x)-u(a)] \\ &\leq \max\{b-x, x-a\}[u(b)-u(x)+u(x)-u(a)] \\ &= \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] [u(b)-u(a)] \end{aligned}$$

and a similar inequality for v.

Utilising these inequalities we deduce the last part of (2.5). ∎

As a particular case of interest, we can state the following midpoint inequality obtained in [**4**].

COROLLARY 2.5 (Dragomir, 2013 [**4**]). *With the assumptions of Theorem 2.4 we have the midpoint inequality*

$$\begin{aligned} &\left|h\left(\frac{a+b}{2}\right)(b-a)-\int_a^b h(t)\,dt\right|^2 \\ &\leq \int_a^b sgn(t-\frac{a+b}{2})u(t)\,dt\int_a^b sgn(t-\frac{a+b}{2})v(t)\,dt \\ &\leq \frac{1}{4}(b-a)^2[u(b)-u(a)][v(b)-v(a)]. \end{aligned} \tag{2.10}$$

3. OSTROWSKI FOR H-DOMINATED FUNCTIONS

3.1. H-Dominated Functions. Assume that $u,v:[a,b]\to\mathbb{R}$ are *monotonic nondecreasing* on the interval $[a,b]$. Assume everywhere in what follows that $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

We can introduce the following concept:

DEFINITION 3.1 (Dragomir, 2013 [**3**]). We say that the complex-valued function $h:[a,b]\to\mathbb{C}$ is (p,q)-H-*dominated* by the pair (u,v) if

$$|h(y)-h(x)|\leq[u(y)-u(x)]^{1/p}[v(y)-v(x)]^{1/q} \tag{H}$$

for any $x,y\in[a,b]$ with $y\geq x$.

We can give numerous examples of such functions.

For instance, if we take f,g two measurable complex-valued functions such that $|f|^p$ and $|g|^q$ are Lebesgue integrable and denote

$$h(x):=\int_a^x f(t)\,g(t)\,dt,\ u(x):=\int_a^x|f(t)|^p\,dt \text{ and } v(x):=\int_a^x|g(t)|^q\,dt,$$

then we observe that u and v are monotonic nondecreasing on $[a,b]$ and by *Hölder integral inequality* we have for any $y\geq x$ with $x,y\in[a,b]$ that

$$\begin{aligned} |h(y)-h(x)| &= \left|\int_x^y f(t)\,g(t)\,dt\right|\leq\left(\int_x^y|f(t)|^p\,dt\right)^{1/p}\left(\int_x^y|g(t)|^q\,dt\right)^{1/q} \\ &\leq[u(y)-u(x)]^{1/p}[v(y)-v(x)]^{1/q}. \end{aligned}$$

Now, for $m,n>0$ if we consider $f(t):=t^m$ and $g(t):=t^n$ for $t\geq 0$, then

$$h_{m,n}(x):=\int_0^x t^{m+n}dt=\frac{1}{m+n+1}x^{m+n+1}$$

and

$$\begin{aligned} u_{m,p}(x) &:= \int_0^x t^{pm}dt=\frac{1}{2pm+1}x^{2pm+1}, \\ v_{n,q}(x) &:= \int_0^x t^{qn}dt=\frac{1}{2qn+1}x^{2qn+1}, \end{aligned}$$

for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Taking into account the above comments we observe that the function $h_{m,n}$ is (p,q)-H-*dominated* by the pair $(u_{m,p},v_{n,q})$ on any subinterval of $[0,\infty)$.

PROPOSITION 3.1 (Dragomir, 2013 [**3**]). *If* $h : [a,b] \to \mathbb{C}$ *is* (p,q)*-H-dominated by the pair* (u,v) *, then* h *is of bounded variation on any subinterval* $[c,d] \subset [a,b]$ *and*

$$\bigvee_c^d (h) \leq [u(d) - u(c)]^{1/p} [v(d) - v(c)]^{1/q} . \tag{3.1}$$

for $p,q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

PROOF. Consider a division δ of the interval $[c,d]$ given by

$$\delta : c = x_0 < x_1 < ... < x_{n-1} < x_n = b.$$

Since $h : [a,b] \to \mathbb{C}$ is (p,q)-H-*dominated* by the pair (u,v) then we have

$$|h(x_{i+1}) - h(x_i)| \leq [u(x_{i+1}) - u(x_i)]^{1/p} [v(x_{i+1}) - v(x_i)]^{1/q}$$

for any $i \in \{0, ..., n-1\}$.

Summing this inequality over i from 0 to $n-1$ and utilizing the Hölder discrete inequality we have

$$\begin{aligned} &\sum_{i=1}^{n-1} |h(x_{i+1}) - h(x_i)| \\ &\leq \sum_{i=1}^{n-1} [u(x_{i+1}) - u(x_i)]^{1/p} [v(x_{i+1}) - v(x_i)]^{1/q} \\ &\leq \left(\sum_{i=1}^{n-1} [u(x_{i+1}) - u(x_i)] \right)^{1/p} \left(\sum_{i=1}^{n-1} [v(x_{i+1}) - v(x_i)] \right)^{1/q} \\ &= [u(d) - u(c)]^{1/p} [v(d) - v(c)]^{1/q} . \end{aligned} \tag{3.2}$$

Taking the supremum over δ we deduce the desired result (3.1). ∎

COROLLARY 3.2. *If* $h : [a,b] \to \mathbb{C}$ *is* (p,q)*-H-dominated by the pair* (u,v) *, then the cumulative variation function* $V : [a,b] \to [0,\infty)$ *defined by*

$$V(x) := \bigvee_a^x (h)$$

is also (p,q)*-H-dominated by the pair* (u,v) *.*

The following result is a kind of Hölder integral inequality for the Riemann-Stieltjes integral:

THEOREM 3.3 (Dragomir, 2013 [**3**]). *Assume that* $u,v : [a,b] \to \mathbb{R}$ *are monotonic nondecreasing on the interval* $[a,b]$ *. If* $h : [a,b] \to \mathbb{C}$ *is* (p,q)*-H-dominated by the pair* (u,v) *and* $f : [a,b] \to \mathbb{C}$ *is a continuous function on* $[a,b]$ *, then the Riemann-Stieltjes integral* $\int_a^b f(t)\,dh(t)$ *exists and*

$$\left| \int_a^b f(t)\,dh(t) \right| \leq \left(\int_a^b |f(t)|\,du(t) \right)^{1/p} \left(\int_a^b |f(t)|\,dv(t) \right)^{1/q} . \tag{3.3}$$

PROOF. Since the Riemann-Stieltjes integral $\int_a^b f(t)\,dh(t)$ exists, then for any sequence of partitions

$$I_n^{(n)} : a = t_0^{(n)} < t_1^{(n)} < \cdots < t_{n-1}^{(n)} < t_n^{(n)} = b$$

with the norm

$$v\left(I_n^{(n)}\right) := \max_{i\in\{0,\dots,n-1\}}\left(t_{i+1}^{(n)} - t_i^{(n)}\right) \to 0$$

as $n \to \infty$, and for any intermediate points $\xi_i^{(n)} \in [t_i^{(n)}, t_{i+1}^{(n)}]$, $i \in \{0, \dots, n-1\}$ we have:

$$\begin{aligned}
(3.4)\qquad \left|\int_a^b f(t)\,dh(t)\right| &= \left|\lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1} f\left(\xi_i^{(n)}\right)\left[h\left(t_{i+1}^{(n)}\right) - h\left(t_i^{(n)}\right)\right]\right| \\
&\le \lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left|h\left(t_{i+1}^{(n)}\right) - h\left(t_i^{(n)}\right)\right| \\
&\le \lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left[u\left(t_{i+1}^{(n)}\right) - u\left(t_i^{(n)}\right)\right]^{1/p} \\
&\times \left[v\left(t_{i+1}^{(n)}\right) - v\left(t_i^{(n)}\right)\right]^{1/q} \\
&\le \lim_{v\left(I_n^{(n)}\right)\to 0}\left(\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left[u\left(t_{i+1}^{(n)}\right) - u\left(t_i^{(n)}\right)\right]\right)^{1/p} \\
&\times \lim_{v\left(I_n^{(n)}\right)\to 0}\left(\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left[v\left(t_{i+1}^{(n)}\right) - v\left(t_i^{(n)}\right)\right]\right)^{1/q} \\
&= \left(\int_a^b |f(t)|\,du(t)\right)^{1/p}\left(\int_a^b |f(t)|\,dv(t)\right)^{1/q},
\end{aligned}$$

where for the last inequality we employed the Hölder weighted discrete inequality

$$\sum_{k=1}^n m_k a_k b_k \le \left(\sum_{k=1}^n m_k a_k^p\right)^{1/p}\left(\sum_{k=1}^n m_k b_k^q\right)^{1/q},$$

where m_k, a_k, $b_k \ge 0$ for $k \in \{1, ..., n\}$. ∎

3.2. Ostrowski Type Inequality.

We have the following new result:

THEOREM 3.4 (Dragomir, 2013 [**3**]). *Assume that $u, v : [a, b] \to \mathbb{R}$ are monotonic nondecreasing on the interval $[a, b]$. If $h : [a, b] \to \mathbb{C}$ is (p, q)-H-dominated by the pair (u, v) for*

$p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\left|h(x)(b-a)-\int_a^b h(t)\,dt\right| \le \left[[2x-(a+b)]u(x)+\int_a^b sgn(t-x)u(t)\,dt\right]^{1/p} \times \left[[2x-(a+b)]v(x)+\int_a^b sgn(t-x)v(t)\,dt\right]^{1/q} \le \left[(b-x)\left[u(b)-u(x)\right]+(x-a)\left[u(x)-u(a)\right]\right]^{1/p} \times \left[(b-x)\left[v(b)-v(x)\right]+(x-a)\left[v(x)-v(a)\right]\right]^{1/q} \le \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\left[u(b)-u(a)\right]^{1/p}\left[v(b)-v(a)\right]^{1/q} \tag{3.5}$$

for any $x\in[a,b]$.

PROOF. Integrating by parts on the Riemann-Stieltjes integral we have [**1**]

$$h(x)(b-a)-\int_a^b h(t)\,dt=\int_a^x (t-a)\,dh(t)+\int_x^b (b-t)\,dh(t). \tag{3.6}$$

Taking the modulus in (3.6) we have

$$\left|h(x)(b-a)-\int_a^b h(t)\,dt\right|\le\left|\int_a^x (t-a)\,dh(t)\right|+\left|\int_x^b (b-t)\,dh(t)\right|. \tag{3.7}$$

Applying the inequality (3.3) twice, we have

$$\left|\int_a^x (t-a)\,dh(t)\right|\le\left(\int_a^x (t-a)\,du(t)\right)^{1/p}\left(\int_a^x (t-a)\,dv(t)\right)^{1/q}$$

and

$$\left|\int_x^b (b-t)\,dh(t)\right|\le\left(\int_x^b (b-t)\,du(t)\right)^{1/p}\left(\int_x^b (b-t)\,dv(t)\right)^{1/q}.$$

Summing these inequalities and utilizing the elementary result

$$\alpha\beta+\lambda\delta\le\left(\alpha^p+\lambda^p\right)^{1/p}\left(\beta^q+\delta^q\right)^{1/q}$$

for $\alpha,\beta,\lambda,\delta\ge 0$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\left|\int_a^x (t-a)\,dh(t)\right|+\left|\int_x^b (b-t)\,dh(t)\right| \le\left(\int_a^x (t-a)\,du(t)\right)^{1/p}\left(\int_a^x (t-a)\,dv(t)\right)^{1/q} +\left(\int_x^b (b-t)\,du(t)\right)^{1/p}\left(\int_x^b (b-t)\,dv(t)\right)^{1/q} \le\left(\int_a^x (t-a)\,du(t)+\int_x^b (b-t)\,du(t)\right)^{1/p} +\left(\int_a^x (t-a)\,dv(t)+\int_x^b (b-t)\,dv(t)\right)^{1/q}. \tag{3.8}$$

Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned}(3.9)\qquad &\int_a^x (t-a)du\,(t) + \int_x^b (b-t)du\,(t)\\ &= (t-a)u\,(t)|_a^x - \int_a^x u\,(t)\,dt - (b-t)u\,(t)|_x^b + \int_x^b u\,(t)\,dt\\ &= [2x-(a+b)]u\,(x) - \int_a^x u\,(t)\,dt + \int_x^b u\,(t)\,dt\\ &= [2x-(a+b)]u\,(x) + \int_a^b sgn(t-x)u\,(t)\,dt.\end{aligned}$$

The same equality holds for v as well.

Making use of (3.7) and (3.8) we deduce the first inequality in (3.5).

As u is monotonic nondecreasing on $[a,b]$, we can state that

$$\int_a^x u\,(t)\,dt \geq (x-a)u\,(a) \text{ and } \int_x^b u\,(t)\,dt \leq (b-x)u\,(b)$$

so that

$$\int_a^b sgn(t-x)u\,(t)\,dt \leq (b-x)u\,(b) - (x-a)u\,(a)\,.$$

Consequently

$$\begin{aligned}&[2x-(a+b)]u\,(x) + \int_a^b sgn(t-x)u\,(t)\,dt\\ &\leq [2x-(a+b)]u\,(x) + (b-x)u\,(b) - (x-a)u\,(a)\\ &= (b-x)\,[u\,(b)-u\,(x)] + (x-a)\,[u\,(x)-u\,(a)]\end{aligned}$$

and a similar inequality for v.

Finally, let us observe that

$$\begin{aligned}&(b-x)\,[u\,(b)-u\,(x)] + (x-a)\,[u\,(x)-u\,(a)]\\ &\leq \max\{b-x, x-a\}[u\,(b)-u\,(x)+u\,(x)-u\,(a)]\\ &= \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right][u\,(b)-u\,(a)]\end{aligned}$$

and a similar inequality for v.

Utilising these inequalities we deduce the last part of (3.5). ∎

As a particular case of interest, we can state the following midpoint inequality obtained in [**3**].

COROLLARY 3.5 (Dragomir, 2013 [**3**]). *With the assumptions of Theorem 3.4 we have the midpoint inequality*

$$\begin{aligned}(3.10)\qquad &\left|h\left(\frac{a+b}{2}\right)(b-a) - \int_a^b h\,(t)\,dt\right|\\ &\leq \left[\int_a^b sgn(t-\frac{a+b}{2})u\,(t)\,dt\right]^{1/p}\left[\int_a^b sgn(t-\frac{a+b}{2})v\,(t)\,dt\right]^{1/q}\\ &\leq \frac{1}{2}\,(b-a)\,[u\,(b)-u\,(a)]^{1/p}\,[v\,(b)-v\,(a)]^{1/q}\,.\end{aligned}$$

REFERENCES

[1] S. S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. Appl.* **4** (2001), No. 1, 59-66. Preprint: *RGMIA Res. Rep. Coll.* **2** (1999), Art. 7, [Online: http://rgmia.org/papers/v2n1/v2n1-7.pdf].

[2] S. S. Dragomir, Inequalities for the Riemann-Stieltjes integral of product integrators with applications, *J. Korean Math. Soc.* **51** (2014), no. 4, 791–815. Preprint, *RGMIA Res. Rep. Coll.* **16** (2013), Article 37.[Online http://rgmia.org/papers/v16/v16a37.pdf].

[3] S. S. Dragomir, Inequalities for the Riemann-Stieltjes Integral of $(p;q)$-H-Dominated Integrators with Applications, *Appl. Math. E-Notes* **15** (2015), 243–260. Preprint *RGMIA Res. Rep. Coll.* **16** (2013), Art. 33 [Online http://rgmia.org/papers/v16/v16a33.pdf].

[4] S. S. Dragomir, Inequalities for the Riemann-Stieltjes integral of S-dominated integrators with applications (I), *Probl. Anal. Issues Anal.* **4** (22) (2015), no. 1, 11–37. Preprint *RGMIA Res. Rep. Coll.* **16** (2013), Art. 31 [Online http://rgmia.org/papers/v16/v16a31.pdf].

CHAPTER 7

Perturbed Ostrowski Type Inequalities

1. Perturbed Ostrowski Type Inequalities for Functions of Bounded Variation

1.1. Some Identities. We start with the following identity that will play an important role in the following:

LEMMA 1.1 (Dragomir, 2013 [**10**]). *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x\in[a,b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have*

$$
\begin{aligned}
f(x)&+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \\
&=\frac{1}{b-a}\int_a^x(t-a)\,d\left[f(t)-\lambda_1(x)t\right]+\frac{1}{b-a}\int_x^b(t-b)\,d\left[f(t)-\lambda_2(x)t\right],
\end{aligned}
\tag{1.1}
$$

where the integrals in the right hand side are taken in the Riemann-Stieltjes sense.

PROOF. Utilising the integration by parts formula in the Riemann-Stieltjes integral, we have

$$
\begin{aligned}
&\int_a^x(t-a)\,d\left[f(t)-\lambda_1(x)t\right] \\
&=(t-a)\left[f(t)-\lambda_1(x)t\right]\Big|_a^x-\int_a^x\left[f(t)-\lambda_1(x)t\right]dt \\
&=(x-a)\left[f(x)-\lambda_1(x)x\right]-\int_a^x f(t)\,dt+\frac{1}{2}\lambda_1(x)\left(x^2-a^2\right) \\
&=(x-a)f(x)-\lambda_1(x)x(x-a)-\int_a^x f(t)\,dt+\frac{1}{2}\lambda_1(x)\left(x^2-a^2\right) \\
&=(x-a)f(x)-\int_a^x f(t)\,dt-\frac{1}{2}(x-a)^2\lambda_1(x)
\end{aligned}
\tag{1.2}
$$

and

$$
\begin{aligned}
&\int_x^b(t-b)\,d\left[f(t)-\lambda_2(x)t\right] \\
&=(t-b)\left[f(t)-\lambda_2(x)t\right]\Big|_x^b-\int_x^b\left[f(t)-\lambda_2(x)t\right]dt \\
&=(b-x)\left[f(x)-\lambda_2(x)x\right]-\int_x^b f(t)\,dt+\frac{1}{2}\lambda_2(x)\left(b^2-x^2\right) \\
&=(b-x)f(x)-\int_x^b f(t)\,dt-(b-x)\lambda_2(x)x+\frac{1}{2}\lambda_2(x)\left(b^2-x^2\right) \\
&=(b-x)f(x)-\int_x^b f(t)\,dt+\frac{1}{2}(b-x)^2\lambda_2(x).
\end{aligned}
\tag{1.3}
$$

By adding the equalities (1.2) and (1.3) and dividing by $b-a$ we get the desired representation (1.1). ∎

COROLLARY 1.2. *With the assumption in Lemma 1.1, we have for any $\lambda(x) \in \mathbb{C}$ that*

$$f(x) + \left(\frac{a+b}{2} - x\right)\lambda(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^x (t-a)\,d[f(t) - \lambda(x)t] + \frac{1}{b-a}\int_x^b (t-b)\,d[f(t) - \lambda(x)t]. \tag{1.4}$$

We have the following midpoint representation:

COROLLARY 1.3. *With the assumption in Lemma 1.1, we have for any $\lambda_1, \lambda_2 \in \mathbb{C}$ that*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &+ \frac{1}{8}(b-a)(\lambda_2 - \lambda_1) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\,d[f(t) - \lambda_1 t] \\ &+ \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\,d[f(t) - \lambda_2 t]. \end{aligned} \tag{1.5}$$

In particular, if $\lambda_1 = \lambda_2 = \lambda$, then we have the equality

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\,d[f(t) - \lambda t] + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\,d[f(t) - \lambda t]. \tag{1.6}$$

REMARK 1.1. If we take $\lambda(x) = 0$ in (1.4) we recapture the Montgomery type identity established in [**2**].

1.2. Inequalities for Functions of Bounded Variation. The following lemma will be used in the sequel and is of interest in itself as well [**1**, p. 177]. For a simple proof see also [**6**].

LEMMA 1.4. *Let $f, u : [a,b] \to \mathbb{C}$. If f is continuous on $[a,b]$ and u is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and*

$$\left|\int_a^b f(t)\,du(t)\right| \le \int_a^b |f(t)|\,d\left(\bigvee_a^t (u)\right) \le \max_{t\in[a,b]} |f(t)| \bigvee_a^b (u). \tag{1.7}$$

We denote by $\ell : [a,b] \to [a,b]$ the identity function, namely $\ell(t) = t$ for any $t \in [a,b]$.
We have the following result:

THEOREM 1.5 (Dragomir, 2013 [**10**]). *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x \in [a,b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have the inequality*

$$\left|f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 \lambda_2(x) - (x-a)^2 \lambda_1(x)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.8}$$
$$\leq \frac{1}{b-a}\left[\int_a^x \left(\bigvee_t^x (f - \lambda_1(x)\ell)\right) dt + \int_x^b \left(\bigvee_x^t (f - \lambda_2(x)\ell)\right) dt\right]$$
$$\leq \frac{1}{b-a}\left[(x-a)\bigvee_a^x (f - \lambda_1(x)\ell) + (b-x)\bigvee_x^b (f - \lambda_2(x)\ell)\right]$$
$$\leq \begin{cases} \max\left\{\bigvee_a^x (f - \lambda_1(x)\ell), \bigvee_x^b (f - \lambda_2(x)\ell)\right\} \\ \left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right]\left(\bigvee_a^x (f - \lambda_1(x)\ell) + \bigvee_x^b (f - \lambda_2(x)\ell)\right), \end{cases}$$

where $\bigvee_c^d (g)$ denotes the total variation of g on the interval $[c,d]$.

PROOF. Taking the modulus in (1.1) and using the property (1.7) we have

$$\left|f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 \lambda_2(x) - (x-a)^2 \lambda_1(x)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.9}$$
$$\leq \frac{1}{b-a}\left|\int_a^x (t-a)\,d\left[f(t) - \lambda_1(x)t\right]\right|$$
$$+ \frac{1}{b-a}\left|\int_x^b (t-b)\,d\left[f(t) - \lambda_2(x)t\right]\right|$$
$$\leq \frac{1}{b-a}\int_a^x (t-a)\,d\left(\bigvee_a^t (f - \lambda_1(x)\ell)\right)$$
$$+ \frac{1}{b-a}\int_x^b (b-t)\,d\left(\bigvee_a^t (f - \lambda_2(x)\ell)\right).$$

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^x (t-a)\, d\left(\bigvee_a^t (f-\lambda_1(x)\ell)\right) \\
&= (t-a)\bigvee_a^t (f-\lambda_1(x)\ell)\Big|_a^x - \int_a^x \left(\bigvee_a^t (f-\lambda_1(x)\ell)\right) dt \\
&= (x-a)\bigvee_a^x (f-\lambda_1(x)\ell) - \int_a^x \left(\bigvee_a^t (f-\lambda_1(x)\ell)\right) dt \\
&= \int_a^x \left(\bigvee_t^x (f-\lambda_1(x)\ell)\right) dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b (b-t)\, d\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right) \\
&= (b-t)\bigvee_a^t (f-\lambda_2(x)\ell)\Big|_x^b + \int_x^b \left(\bigvee_a^t (f-\lambda_2(x)\ell)\right) dt \\
&= \int_x^b \left(\bigvee_a^t (f-\lambda_2(x)\ell)\right) dt - (b-x)\bigvee_a^x (f-\lambda_2(x)\ell) \\
&= \int_x^b \left(\bigvee_x^t (f-\lambda_2(x)\ell)\right) dt.
\end{aligned}$$

Using (1.9) we deduce the first inequality in (1.8).

We also have

$$\int_a^x \left(\bigvee_t^x (f-\lambda_1(x)\ell)\right) dt \le (x-a)\bigvee_a^x (f-\lambda_1(x)\ell)$$

and

$$\int_x^b \left(\bigvee_x^t (f-\lambda_2(x)\ell)\right) dt \le (b-x)\bigvee_x^b (f-\lambda_2(x)\ell),$$

which prove the second inequality in (1.8).

The last part is obvious. ∎

The following result holds:

COROLLARY 1.6. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x \in [a,b]$. Then for any $\lambda(x)$ a complex number, we have the inequality*

$$\begin{aligned}
&\left|f(x) + \left(\frac{a+b}{2} - x\right)\lambda(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{b-a}\left[\int_a^x \left(\bigvee_t^x (f - \lambda(x)\ell)\right)dt + \int_x^b \left(\bigvee_x^t (f - \lambda(x)\ell)\right)dt\right] \\
&\leq \frac{1}{b-a}\left[(x-a)\bigvee_a^x (f - \lambda(x)\ell) + (b-x)\bigvee_x^b (f - \lambda(x)\ell)\right] \\
&\leq \begin{cases} \frac{1}{2}\bigvee_a^b (f - \lambda(x)\ell) + \frac{1}{2}\left|\bigvee_x^b (f - \lambda(x)\ell) - \bigvee_a^x (f - \lambda(x)\ell)\right| \\[2ex] \left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b (f - \lambda(x)\ell). \end{cases}
\end{aligned} \tag{1.10}$$

REMARK 1.2. Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$. Then for any $\lambda \in \mathbb{C}$ we have the inequalities

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} \left(\bigvee_t^{\frac{a+b}{2}} (f - \lambda\ell)\right)dt + \int_{\frac{a+b}{2}}^b \left(\bigvee_{\frac{a+b}{2}}^t (f - \lambda\ell)\right)dt\right] \\
&\leq \frac{1}{2}\bigvee_a^b (f - \lambda\ell).
\end{aligned} \tag{1.11}$$

This is equivalent to

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{b-a}\inf_{\lambda\in\mathbb{C}}\left[\int_a^{\frac{a+b}{2}} \left(\bigvee_t^{\frac{a+b}{2}} (f - \lambda\ell)\right)dt + \int_{\frac{a+b}{2}}^b \left(\bigvee_{\frac{a+b}{2}}^t (f - \lambda\ell)\right)dt\right] \\
&\leq \frac{1}{2}\inf_{\lambda\in\mathbb{C}}\left[\bigvee_a^b (f - \lambda\ell)\right].
\end{aligned} \tag{1.12}$$

1.3. Inequalities for Lipshitzian Functions.

We can state the following result:

THEOREM 1.7 (Dragomir, 2013 [**10**]). *Let $f : [a,b] \to \mathbb{C}$ be a bounded function on $[a,b]$ and $x \in (a,b)$. If $\lambda_1(x)$ and $\lambda_2(x)$ are complex numbers and there exist the positive numbers $L_1(x)$ and $L_2(x)$ such that $f - \lambda_1(x)\ell$ is Lipschitzian with the constant $L_1(x)$ on the interval*

$[a,x]$ *and* $f-\lambda_2(x)\ell$ *is Lipschitzian with the constant* $L_2(x)$ *on the interval* $[x,b]$ *, then*

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.13}$$

$$\leq \frac{1}{2}\left[\left(\frac{x-a}{b-a}\right)^2 L_1(x)+\left(\frac{b-x}{b-a}\right)^2 L_2(x)\right](b-a)$$

$$\leq \begin{cases} \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\max\{L_1(x),L_2(x)\}(b-a), \\ \frac{1}{2}\left[\left(\frac{x-a}{b-a}\right)^{2q}+\left(\frac{b-x}{b-a}\right)^{2q}\right]^{1/q}\left(L_1^p(x)+L_2^p(x)\right)^{1/p}(b-a), \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\frac{L_1(x)+L_2(x)}{2}(b-a). \end{cases}$$

PROOF. It is known that, if $g:[c,d]\to\mathbb{C}$ is Riemann integrable and $u:[c,d]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, then the Riemann-Stieltjes integral $\int_c^d g(t)\,du(t)$ exists and

$$\left|\int_c^d g(t)\,du(t)\right|\leq L\int_c^d |g(t)|\,dt. \tag{1.14}$$

Taking the modulus in (1.1) and using the property (1.14) we have

$$\begin{aligned}
&\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{b-a}\left|\int_a^x (t-a)\,d\left[f(t)-\lambda_1(x)t\right]\right| \\
&+\frac{1}{b-a}\left|\int_x^b (t-b)\,d\left[f(t)-\lambda_2(x)t\right]\right| \\
&\leq \frac{1}{b-a}\left[L_1(x)\int_a^x (t-a)\,dt+L_2(x)\int_x^b (b-t)\,dt\right] \\
&=\frac{L_1(x)(x-a)^2+L_2(x)(b-x)^2}{2(b-a)} \\
&=\frac{1}{2}\left[L_1(x)\left(\frac{x-a}{b-a}\right)^2+L_2(x)\left(\frac{b-x}{b-a}\right)^2\right](b-a),
\end{aligned}$$

and the first inequality in (1.13) is proved.

By Hölder's inequality we have

$$L_1(x)\left(\frac{x-a}{b-a}\right)^2+L_2(x)\left(\frac{b-x}{b-a}\right)^2$$

$$\leq\begin{cases}\left[\left(\frac{x-a}{b-a}\right)^2+\left(\frac{b-x}{b-a}\right)^2\right]\max\{L_1(x),L_2(x)\}\\ \\ \left[\left(\frac{x-a}{b-a}\right)^{2q}+\left(\frac{b-x}{b-a}\right)^{2q}\right]^{1/q}\left(L_1^p(x)+L_2^p(x)\right)^{1/p},\\ p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ \\ \max\left\{\left(\frac{x-a}{b-a}\right)^2,\left(\frac{b-x}{b-a}\right)^2\right\}[L_1(x)+L_2(x)],\end{cases}$$

which proves, upon simple calculations, the last part of the inequality (1.13). ∎

COROLLARY 1.8. *Let $f:[a,b]\to\mathbb{C}$ be a bounded function on $[a,b]$ and $x\in(a,b)$. If $\lambda(x)$ is a complex number and there exist the positive number $L(x)$ such that $f-\lambda(x)\ell$ is Lipschitzian with the constant $L(x)$ on the interval $[a,b]$, then*

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)\lambda(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]L(x)(b-a). \tag{1.15}$$

REMARK 1.3. If λ is a complex number and there exist the positive number L such that $f-\lambda\ell$ is Lipschitzian with the constant L on the interval $[a,b]$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{4}L(b-a). \tag{1.16}$$

1.4. Inequalities for Monotonic Functions. Now, the case of monotonic integrators is as follows:

THEOREM 1.9 (Dragomir, 2013 [**10**]). *Let $f:[a,b]\to\mathbb{R}$ be a bounded function on $[a,b]$ and $x\in(a,b)$. If $\lambda_1(x)$ and $\lambda_2(x)$ are real numbers such that $f-\lambda_1(x)\ell$ is monotonic nondecreasing on the interval $[a,x]$ and $f-\lambda_2(x)\ell$ is monotonic nondecreasing on the interval*

$[x,b]$, *then*

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.17}$$

$$\le \frac{1}{b-a}\left[(2x-a-b)f(x)+\int_a^b sgn(t-x)f(t)\,dt \right.$$
$$\left. -\frac{1}{2}\left[\lambda_1(x)(x-a)^2+\lambda_2(x)(b-x)^2\right]\right]$$

$$\le \frac{1}{b-a}\left\{(x-a)\left[f(x)-f(a)-\lambda_1(x)(x-a)\right]\right.$$
$$\left.+(b-x)\left[f(b)-f(x)-\lambda_2(x)(b-x)\right]\right\}$$

$$\le \begin{cases} \frac{1}{2}\left[f(b)-f(a)-\lambda_1(x)(x-a)-\lambda_2(x)(b-x)\right] \\ +\left|f(x)-\frac{f(a)+f(b)}{2}-\frac{1}{2}\lambda_1(x)(x-a)+\frac{1}{2}\lambda_2(x)(b-x)\right|, \\ \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \\ \times\left[f(b)-f(a)-\lambda_1(x)(x-a)-\lambda_2(x)(b-x)\right]. \end{cases}$$

PROOF. It is known that, if $g:[c,d]\to\mathbb{C}$ is continuous and $u:[c,d]\to\mathbb{C}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_c^d g(t)\,du(t)$ exists and

$$\left|\int_c^d g(t)\,du(t)\right|\le\int_c^d |g(t)|\,du(t). \tag{1.18}$$

Taking the modulus in (1.1) and using the property (1.18) we have

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.19}$$
$$\le\frac{1}{b-a}\left|\int_a^x (t-a)\,d\left[f(t)-\lambda_1(x)t\right]\right|$$
$$+\frac{1}{b-a}\left|\int_x^b (t-b)\,d\left[f(t)-\lambda_2(x)t\right]\right|$$
$$\le\frac{1}{b-a}\int_a^x (t-a)\,d\left[f(t)-\lambda_1(x)t\right]$$
$$+\frac{1}{b-a}\int_x^b (b-t)\,d\left[f(t)-\lambda_2(x)t\right].$$

Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned}
&\int_a^x (t-a)\, d\left[f(t)-\lambda_1(x)\, t\right] \\
&= (t-a)\left[f(t)-\lambda_1(x)\, t\right]\Big|_a^x - \int_a^x \left[f(t)-\lambda_1(x)\, t\right] dt \\
&= (x-a)\left[f(x)-\lambda_1(x)\, x\right] - \int_a^x \left[f(t)-\lambda_1(x)\, t\right] dt \\
&= (x-a)\, f(x) - \lambda_1(x)\, x\,(x-a) - \int_a^x f(t)\, dt + \lambda_1(x)\, \frac{x^2-a^2}{2} \\
&= (x-a)\, f(x) - \int_a^x f(t)\, dt - \frac{1}{2}\lambda_1(x)\,(x-a)^2
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b (b-t)\, d\left[f(t)-\lambda_2(x)\, t\right] \\
&= (b-t)\left[f(t)-\lambda_2(x)\, t\right]\Big|_x^b + \int_x^b \left[f(t)-\lambda_2(x)\, t\right] dt \\
&= \int_x^b f(t)\, dt - \lambda_2(x)\int_x^b t dt - (b-x)\left[f(x)-\lambda_2(x)\, x\right] \\
&= \int_x^b f(t)\, dt - \lambda_2(x)\, \frac{b^2-x^2}{2} - (b-x)\, f(x) + (b-x)\,\lambda_2(x)\, x \\
&= \int_x^b f(t)\, dt - (b-x)\, f(x) - \frac{1}{2}\lambda_2(x)\,(b-x)^2 .
\end{aligned}$$

If we add these equalities, we get

$$\begin{aligned}
&\int_a^x (t-a)\, d\left[f(t)-\lambda_1(x)\, t\right] + \int_x^b (b-t)\, d\left[f(t)-\lambda_2(x)\, t\right] \\
&= (x-a)\, f(x) - \int_a^x f(t)\, dt - \frac{1}{2}\lambda_1(x)\,(x-a)^2 \\
&+ \int_x^b f(t)\, dt - (b-x)\, f(x) - \frac{1}{2}\lambda_2(x)\,(b-x)^2 \\
&= (2x-a-b)\, f(x) + \int_a^b sgn\,(t-x)\, f(t)\, dt \\
&- \frac{1}{2}\left[\lambda_1(x)\,(x-a)^2 + \lambda_2(x)\,(b-x)^2\right]
\end{aligned}$$

and by (1.19) we get the first inequality in (1.17).

Now, since $f-\lambda_1(x)\,\ell$ is monotonic nondecreasing on the interval $[a,x]$, then

$$\begin{aligned}
&\int_a^x (t-a)\, d\left[f(t)-\lambda_1(x)\, t\right] \\
&\le (x-a)\left[f(x)-\lambda_1(x)\, x - f(a) + \lambda_1(x)\, a\right] \\
&= (x-a)\left[f(x)-f(a)-\lambda_1(x)\,(x-a)\right]
\end{aligned}$$

and, since $f-\lambda_2(x)\ell$ is monotonic nondecreasing on the interval $[x,b]$, then also

$$\begin{aligned}&\int_x^b (b-t)\,d[f(t)-\lambda_2(x)t]\\&\le (b-x)[f(b)-\lambda_2(x)b-f(x)+\lambda_2(x)x]\\&=(b-x)[f(b)-f(x)-\lambda_2(x)(b-x)].\end{aligned}$$

These prove the second inequality in (1.17).

The last part follows by the properties of maximum and the details are omitted. ∎

COROLLARY 1.10. *Let $f:[a,b]\to\mathbb{R}$ be a bounded function on $[a,b]$ and $x\in(a,b)$. If $\lambda(x)$ is a real number such that $f-\lambda(x)\ell$ is monotonic nondecreasing on the interval $[a,b]$, then*

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)\lambda(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.20}$$

$$\le \frac{1}{b-a}\left[(2x-a-b)f(x)+\int_a^b sgn(t-x)f(t)\,dt - \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2\lambda(x)\right]$$

$$\le \frac{1}{b-a}\{(x-a)[f(x)-f(a)-\lambda(x)(x-a)] + (b-x)[f(b)-f(x)-\lambda(x)(b-x)]\}$$

$$\le \begin{cases}\frac{f(b)-f(a)}{2}-\frac{1}{2}\lambda(x)(b-a)\\ \quad +\left|f(x)-\frac{f(a)+f(b)}{2}-\frac{1}{2}\lambda(x)(2x-a-b)\right|,\\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\\ \quad\times[f(b)-f(a)-\lambda(x)(b-a)].\end{cases}$$

REMARK 1.4. If λ is a real number such that $f-\lambda\ell$ is monotonic nondecreasing on the interval $[a,b]$, then

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\&\le\frac{1}{b-a}\left[\int_a^b sgn\left(t-\frac{a+b}{2}\right)f(t)\,dt-\frac{1}{4}\lambda(b-a)^2\right]\\&\le\frac{1}{2}[f(b)-f(a)-\lambda(b-a)].\end{aligned}\tag{1.21}$$

2. SOME PERTURBED OSTROWSKI TYPE INEQUALITIES FOR ABSOLUTELY CONTINUOUS FUNCTIONS

2.1. Some Identities. We start with the following identity that will play an important role in the following:

LEMMA 2.1 (Dragomir, 2013 [7]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x\in[a,b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have*

$$\begin{aligned}(2.1)\qquad & f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\lambda_2(x)\right]dt,\end{aligned}$$

where the integrals in the right hand side are taken in the Lebesgue sense.

PROOF. Utilising the integration by parts formula in the Lebesgue integral, we have

$$\begin{aligned}(2.2)\qquad & \int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt\\ &= (t-a)\left[f(t)-\lambda_1(x)t\right]\Big|_a^x-\int_a^x\left[f(t)-\lambda_1(x)t\right]dt\\ &=(x-a)\left[f(x)-\lambda_1(x)x\right]-\int_a^x f(t)\,dt+\frac{1}{2}\lambda_1(x)\left(x^2-a^2\right)\\ &=(x-a)f(x)-\lambda_1(x)x(x-a)-\int_a^x f(t)\,dt+\frac{1}{2}\lambda_1(x)\left(x^2-a^2\right)\\ &=(x-a)f(x)-\int_a^x f(t)\,dt-\frac{1}{2}(x-a)^2\lambda_1(x)\end{aligned}$$

and

$$\begin{aligned}(2.3)\qquad & \int_x^b (t-b)\left[f'(t)-\lambda_2(x)\right]dt\\ &= (t-b)\left[f(t)-\lambda_2(x)t\right]\Big|_x^b-\int_x^b\left[f(t)-\lambda_2(x)t\right]dt\\ &=(b-x)\left[f(x)-\lambda_2(x)x\right]-\int_x^b f(t)\,dt+\frac{1}{2}\lambda_2(x)\left(b^2-x^2\right)\\ &=(b-x)f(x)-\int_x^b f(t)\,dt-(b-x)\lambda_2(x)x+\frac{1}{2}\lambda_2(x)\left(b^2-x^2\right)\\ &=(b-x)f(x)-\int_x^b f(t)\,dt+\frac{1}{2}(b-x)^2\lambda_2(x).\end{aligned}$$

If we add the identities (2.2) and (2.3) and divide by $b-a$ we deduce the desired identity (2.1). ∎

COROLLARY 2.2. *With the assumption in Lemma 2.1, we have for any $\lambda(x)\in\mathbb{C}$ that*

$$\begin{aligned}(2.4)\qquad & f(x)+\left(\frac{a+b}{2}-x\right)\lambda(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\lambda(x)\right]dt.\end{aligned}$$

REMARK 2.1. If we take $\lambda(x) = 0$ in (2.4), then we get Montgomery's identity for absolutely continuous functions, i.e.

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^x (t-a) f'(t)\,dt + \frac{1}{b-a}\int_x^b (t-b) f'(t)\,dt, \tag{2.5}$$

for $x \in [a,b]$.

We have the following midpoint representation:

COROLLARY 2.3. *With the assumption in Lemma 2.1, we have for any $\lambda_1, \lambda_2 \in \mathbb{C}$ that*

$$f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)(\lambda_2 - \lambda_1) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)[f'(t) - \lambda_1]\,dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)[f'(t) - \lambda_2]\,dt. \tag{2.6}$$

In particular, if $\lambda_1 = \lambda_2 = \lambda$, then we have the equality

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)[f'(t) - \lambda]\,dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)[f'(t) - \lambda]\,dt. \tag{2.7}$$

REMARK 2.2. The identity (2.1) has many particular cases of interest.

If we assume that the derivatives $f'_+(a)$, $f'_-(b)$ and $f'(x)$ exist and are finite, then by taking

$$\lambda_1(x) = \frac{f'_+(a) + f'(x)}{2} \text{ and } \lambda_2(x) = \frac{f'(x) + f'_-(b)}{2}$$

in (2.1) we get

$$\begin{aligned} f(x) &+ \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &+ \frac{1}{4(b-a)}\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right] \\ &= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - \frac{f'_+(a) + f'(x)}{2}\right] dt \\ &+ \frac{1}{b-a}\int_x^b (t-b)\left[f'(t) - \frac{f'(x) + f'_-(b)}{2}\right] dt. \end{aligned} \tag{2.8}$$

In particular, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &+ \frac{1}{16}(b-a)\left[f'_-(b) - f'_+(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - \frac{f'_+(a) + f'\left(\frac{a+b}{2}\right)}{2}\right] dt \\ &+ \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t) - \frac{f'\left(\frac{a+b}{2}\right) + f'_-(b)}{2}\right] dt. \end{aligned} \tag{2.9}$$

2.2. Inequalities for Bounded Derivatives. Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\begin{aligned}&\bar{U}_{[a,b]}(\gamma,\Gamma)\\&:=\left\{f:[a,b]\to\mathbb{C}|\operatorname{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right]\geq 0 \text{ for almost every } t\in[a,b]\right\}\end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}|\ \left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma| \text{ for a.e. } t\in[a,b]\right\}.$$

The following representation result may be stated.

PROPOSITION 2.4 (Dragomir, 2013 [7]). *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma)=\bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{2.10}$$

PROOF. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left|z-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|$$

if and only if

$$\operatorname{Re}[(\Gamma-z)(\bar{z}-\bar{\gamma})]\geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma-\gamma|^2-\left|z-\frac{\gamma+\Gamma}{2}\right|^2=\operatorname{Re}[(\Gamma-z)(\bar{z}-\bar{\gamma})]$$

that holds for any $z \in \mathbb{C}$.

The equality (2.10) is thus a simple consequence of this fact. ∎

On making use of the complex numbers field properties we can also state that:

COROLLARY 2.5. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$,we have that*

$$\begin{aligned}\bar{U}_{[a,b]}(\gamma,\Gamma)=\{f:[a,b]\to\mathbb{C}\mid\ &(\operatorname{Re}\Gamma-\operatorname{Re}f(t))(\operatorname{Re}f(t)-\operatorname{Re}\gamma)\\&+(\operatorname{Im}\Gamma-\operatorname{Im}f(t))(\operatorname{Im}f(t)-\operatorname{Im}\gamma)\geq 0 \textit{ for a.e. } t\in[a,b]\}.\end{aligned} \tag{2.11}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{[a,b]}(\gamma,\Gamma):=\{f:[a,b]\to\mathbb{C}\mid\ &\operatorname{Re}(\Gamma)\geq\operatorname{Re}f(t)\geq\operatorname{Re}(\gamma)\\&\text{and } \operatorname{Im}(\Gamma)\geq\operatorname{Im}f(t)\geq\operatorname{Im}(\gamma) \text{ for a.e. } t\in[a,b]\}.\end{aligned} \tag{2.12}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset\neq\bar{S}_{[a,b]}(\gamma,\Gamma)\subseteq\bar{U}_{[a,b]}(\gamma,\Gamma). \tag{2.13}$$

THEOREM 2.6 (Dragomir, 2013 [7]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x\in(a,b)$. Suppose that $\gamma_i,\Gamma_i\in\mathbb{C}$ with $\gamma_i\neq\Gamma_i$, $i=1,2$ and $f'\in\bar{U}_{[a,x]}(\gamma_1,\Gamma_1)\cap$*

$\bar{U}_{[x,b]}(\gamma_2,\Gamma_2)$, *then we have*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right.\\
&\left.+\frac{1}{2(b-a)}\left[(b-x)^2\frac{\Gamma_2+\gamma_2}{2}-(x-a)^2\frac{\Gamma_1+\gamma_1}{2}\right]\right|\\
&\le\frac{1}{4}\left[|\Gamma_1-\gamma_1|\left(\frac{x-a}{b-a}\right)^2+|\Gamma_2-\gamma_2|\left(\frac{b-x}{b-a}\right)^2\right](b-a)\\
&\le\frac{1}{4}(b-a)\\
&\times\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\max\{|\Gamma_1-\gamma_1|,|\Gamma_2-\gamma_2|\}.\\[2ex]
\left[\left(\frac{x-a}{b-a}\right)^{2p}+\left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[|\Gamma_1-\gamma_1|^q+|\Gamma_2-\gamma_2|^q\right]^{1/q}\\
p>1,\frac{1}{p}+\frac{1}{q}=1,\\[2ex]
\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[|\Gamma_1-\gamma_1|+|\Gamma_2-\gamma_2|\right].\end{cases}
\end{aligned}\tag{2.14}$$

PROOF. Since $f'\in\bar{U}_{[a,x]}(\gamma_1,\Gamma_1)\cap\bar{U}_{[x,b]}(\gamma_2,\Gamma_2)$, then by taking the modulus in (2.1) for $\lambda_1(x)=\frac{\Gamma_1+\gamma_1}{2}$ and $\lambda_2(x)=\frac{\Gamma_2+\gamma_2}{2}$ we get

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right.\\
&\left.+\frac{1}{2(b-a)}\left[(b-x)^2\frac{\Gamma_2+\gamma_2}{2}-(x-a)^2\frac{\Gamma_1+\gamma_1}{2}\right]\right|\\
&\le\frac{1}{b-a}\left|\int_a^x(t-a)\left[f'(t)-\frac{\Gamma_1+\gamma_1}{2}\right]dt\right|\\
&+\frac{1}{b-a}\left|\int_x^b(t-b)\left[f'(t)-\frac{\Gamma_2+\gamma_2}{2}\right]dt\right|\\
&\le\frac{1}{b-a}\int_a^x(t-a)\left|f'(t)-\frac{\Gamma_1+\gamma_1}{2}\right|dt\\
&+\frac{1}{b-a}\int_x^b(t-b)\left|f'(t)-\frac{\Gamma_2+\gamma_2}{2}\right|dt\\
&\le\frac{1}{b-a}\frac{|\Gamma_1-\gamma_1|}{2}\int_a^x(t-a)\,dt+\frac{1}{b-a}\frac{|\Gamma_2-\gamma_2|}{2}\int_x^b(b-t)\,dt\\
&=\frac{1}{4}\left[|\Gamma_1-\gamma_1|\left(\frac{x-a}{b-a}\right)^2+|\Gamma_2-\gamma_2|\left(\frac{b-x}{b-a}\right)^2\right](b-a)
\end{aligned}$$

and the first inequality in (2.14) is proved.

The last part follows by Hölder's inequality

$$mn+pq\le(m^\alpha+p^\alpha)^{1/\alpha}\left(n^\beta+q^\beta\right)^{1/\beta},$$

where $m,n,p,q\ge0$ and $\alpha>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$. ∎

COROLLARY 2.7. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x \in (a,b)$. Suppose that $\gamma, \Gamma \in \mathbb{C}$ with $\gamma \neq \Gamma$, and $f' \in \bar{U}_{[a,b]}(\gamma, \Gamma)$, then we have*

$$\left| f(x) + \left(\frac{a+b}{2} - x\right) \frac{\Gamma + \gamma}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \tag{2.15}$$
$$\leq \frac{1}{2} |\Gamma - \gamma| \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a).$$

In particular, we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{1}{8} |\Gamma - \gamma| (b-a). \tag{2.16}$$

REMARK 2.3. If the derivative $f' : [a,b] \to \mathbb{R}$ is bounded above and below, that is, there exists the constants $M > m$ such that

$$-\infty < m \leq f'(t) \leq M < \infty \text{ for a.e. } t \in [a,b],$$

then we recapture from (2.15) the inequality [**5**]

$$\left| f(x) + \left(\frac{a+b}{2} - x\right) \frac{M + m}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt \right|$$
$$\leq \frac{1}{2} (M - m) \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a).$$

REMARK 2.4. Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous on $[a,b]$. Suppose that $\gamma_i, \Gamma_i \in \mathbb{C}$ with $\gamma_i \neq \Gamma_i$, $i = 1, 2$ and $f' \in \bar{U}_{\left[a, \frac{a+b}{2}\right]}(\gamma_1, \Gamma_1) \cap \bar{U}_{\left[\frac{a+b}{2}, b\right]}(\gamma_2, \Gamma_2)$, then we have from (2.14) that

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\, dt + \frac{1}{8}(b-a) \left(\frac{\Gamma_2 + \gamma_2}{2} - \frac{\Gamma_1 + \gamma_1}{2} \right) \right| \tag{2.17}$$
$$\leq \frac{1}{16} \left[|\Gamma_1 - \gamma_1| + |\Gamma_2 - \gamma_2| \right] (b-a).$$

2.3. Inequalities for Derivatives of Bounded Variation. Assume that the function $f : I \to \mathbb{C}$ is differentiable on the interior of I, denoted $\mathring{I}$, and $[a,b] \subset \mathring{I}$. Then, as in (2.8), we have the equality

$$f(x) + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \tag{2.18}$$
$$+ \frac{1}{4(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right]$$
$$= \frac{1}{b-a} \int_a^x (t-a) \left[f'(t) - \frac{f'(a) + f'(x)}{2} \right] dt$$
$$+ \frac{1}{b-a} \int_x^b (t-b) \left[f'(t) - \frac{f'(x) + f'(b)}{2} \right] dt,$$

for any $x \in [a,b]$.

THEOREM 2.8 (Dragomir, 2013 [**7**]). *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\begin{aligned}
(2.19)\qquad & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \right. \\
& \left. + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \right| \\
& \le \frac{1}{4}\left[\left(\frac{x-a}{b-a}\right)^2 \bigvee_a^x (f') + \left(\frac{b-x}{b-a}\right)^2 \bigvee_x^b (f')\right](b-a) \\
& \le \frac{1}{4}(b-a) \\
& \times \begin{cases}
\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{2}\bigvee_a^b (f') + \frac{1}{2}\left|\bigvee_a^x (f') - \bigvee_x^b (f')\right|\right], \\[2ex]
\left[\left(\frac{x-a}{b-a}\right)^{2p} + \left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[\left[\bigvee_a^x (f')\right]^q + \left[\bigvee_x^b (f')\right]^q\right]^{1/q} \\
p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\[2ex]
\left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b (f'),
\end{cases}
\end{aligned}$$

for any $x \in [a,b]$.

PROOF. Taking the modulus in (2.18) we have

$$\begin{aligned}
(2.20)\qquad & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \right. \\
& \left. + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \right| \\
& \le \frac{1}{b-a}\left|\int_a^x (t-a)\left[f'(t) - \frac{f'(a) + f'(x)}{2}\right] dt\right| \\
& + \frac{1}{b-a}\left|\int_x^b (t-b)\left[f'(t) - \frac{f'(x) + f'(b)}{2}\right] dt\right| \\
& \le \frac{1}{b-a}\int_a^x (t-a)\left|f'(t) - \frac{f'(a) + f'(x)}{2}\right| dt \\
& + \frac{1}{b-a}\int_x^b (b-t)\left|f'(t) - \frac{f'(x) + f'(b)}{2}\right| dt.
\end{aligned}$$

Since $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,x]$ and $[x,b]$, then

$$\begin{aligned}\left|f'(t) - \frac{f'(a)+f'(x)}{2}\right| &= \frac{|f'(t)-f'(a)+f'(t)-f'(x)|}{2} \\ &\le \frac{1}{2}\left[|f'(t)-f'(a)| + |f'(x)-f'(t)|\right] \\ &\le \frac{1}{2}\bigvee_a^x (f')\end{aligned}$$

for any $t \in [a,x]$ and, similarly,

$$\left|f'(t) - \frac{f'(x)+f'(b)}{2}\right| \le \frac{1}{2}\bigvee_x^b (f')$$

for any $t \in [x,b]$.

Then

$$\begin{aligned}\int_a^x (t-a)\left|f'(t) - \frac{f'(a)+f'(x)}{2}\right| dt &\le \frac{1}{2}\bigvee_a^x (f') \int_a^x (t-a)\,dt \\ &= \frac{1}{4}(x-a)^2 \bigvee_a^x (f')\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-t)\left|f'(t) - \frac{f'(x)+f'(b)}{2}\right| dt &\le \frac{1}{2}\bigvee_x^b (f') \int_x^b (b-t)\,dt \\ &= \frac{1}{4}(b-x)^2 \bigvee_x^b (f')\end{aligned}$$

and by (2.20) we get the desired inequality (2.19).

The last part follows by Hölder's inequality

$$mn + pq \le (m^\alpha + p^\alpha)^{1/\alpha}\left(n^\beta + q^\beta\right)^{1/\beta},$$

where $m,n,p,q \ge 0$ and $\alpha > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. ∎

COROLLARY 2.9. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{16}(b-a)\left[f'(b)-f'(a)\right]\right| \\ &\le \frac{1}{16}(b-a)\bigvee_a^b (f').\end{aligned} \tag{2.21}$$

REMARK 2.5. If $p \in (a,b)$ is a median point in bounded variation for the derivative, i.e. $\bigvee_a^p (f') = \bigvee_p^b (f')$, then under the assumptions of Theorem 2.8, we have

$$\begin{aligned} (2.22) \quad & \left| f(p) - \frac{1}{b-a} \int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - p\right) f'(p) \right. \\ & \left. + \frac{1}{4(b-a)} \left[(b-p)^2 f'(b) - (p-a)^2 f'(a) \right] \right| \\ & \leq \frac{1}{8}(b-a)\left[\frac{1}{4} + \left(\frac{p - \frac{a+b}{2}}{b-a}\right)^2\right] \bigvee_a^b (f'). \end{aligned}$$

2.4. Inequalities for Lipschitzian Derivatives. We say that $v : [a,b] \to \mathbb{C}$ is *Lipschitzian* with the constant $L > 0$, if it satisfies the condition

$$|v(t) - v(s)| \leq L|t-s| \text{ for any } t, s \in [a,b].$$

THEOREM 2.10 (Dragomir, 2013 [7]). *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Let $x \in (a,b)$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is Lipschitzian with the constant $K_1(x)$ on $[a,x]$ and constant $K_2(x)$ on $[x,b]$, then*

$$\begin{aligned} (2.23) \quad & \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \right. \\ & \left. + \frac{1}{4(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] \right| \\ & \leq \frac{1}{8}\left[\left(\frac{x-a}{b-a}\right)^3 K_1(x) + \left(\frac{b-x}{b-a}\right)^3 K_2(x)\right](b-a)^2 \\ & \leq \frac{1}{8}(b-a)^2 \\ & \times \begin{cases} \left[\left(\frac{x-a}{b-a}\right)^3 + \left(\frac{b-x}{b-a}\right)^3\right] \max\{K_1(x), K_2(x)\}, \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{2p} + \left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p} \left[K_1^q(x) + K_2^q(x)\right]^{1/q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right]^3 \left[K_1(x) + K_2(x)\right]. \end{cases} \end{aligned}$$

PROOF. Since $f' : \mathring{I} \to \mathbb{C}$ is Lipschitzian with the constant $K_1(x)$ on $[a,x]$ and constant $K_2(x)$ on $[x,b]$, then

$$\begin{aligned} \left| f'(t) - \frac{f'(a) + f'(x)}{2} \right| &= \frac{|f'(t) - f'(a) + f'(t) - f'(x)|}{2} \\ &\leq \frac{1}{2}\left[|f'(t) - f'(a)| + |f'(x) - f'(t)|\right] \\ &\leq \frac{1}{2} K_1(x)\left[|t-a| + |x-t|\right] \\ &= \frac{1}{2} K_1(x)(x-a) \end{aligned}$$

for any $t \in [a, x]$ and, similarly,

$$\begin{aligned}\left|f'(t) - \frac{f'(x) + f'(b)}{2}\right| &\leq \frac{1}{2} K_2(x)\left[|t - x| + |b - t|\right] \\ &= \frac{1}{2} K_2(x)(b - x)\end{aligned}$$

for any $t \in [x, b]$.

Then

$$\begin{aligned}\int_a^x (t-a)\left|f'(t) - \frac{f'(a) + f'(x)}{2}\right| dt &\leq \frac{1}{2} K_1(x)(x-a)\int_a^x (t-a)\,dt \\ &= \frac{1}{8}(x-a)^3 K_1(x)\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-t)\left|f'(t) - \frac{f'(x) + f'(b)}{2}\right| dt &\leq \frac{1}{2} K_2(x)(b-x)\int_x^b (b-t)\,dt \\ &= \frac{1}{8}(b-x)^3 K_2(x).\end{aligned}$$

Making use of the inequality (2.20) we deduce the first bound in (2.23).

The second part is obvious. ∎

COROLLARY 2.11. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a, b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is Lipschitzian with the constant K on $[a, b]$ then*

(2.24)
$$\begin{aligned}&\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x)\right. \\ &\left. + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right]\right| \\ &\leq \frac{1}{8}\left[\left(\frac{x-a}{b-a}\right)^3 + \left(\frac{b-x}{b-a}\right)^3\right] K (b-a)^2\end{aligned}$$

for any $x \in [a, b]$.

In particular, we have

(2.25)
$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) + \frac{1}{16}(b-a)\left[f'(b) - f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{32} K (b-a)^2.\end{aligned}$$

3. OTHER PERTURBED OSTROWSKI TYPE INEQUALITIES FOR ABSOLUTELY CONTINUOUS FUNCTION

3.1. Inequalities for Derivatives of Bounded Variation. Assume that the function $f : I \to \mathbb{C}$ is differentiable on the interior of I, denoted $\mathring{I}$, and $[a, b] \subset \mathring{I}$. Then, we have the equality [**8**]

(3.1)
$$\begin{aligned}&f(x) + \left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - f'(x)\right] dt + \frac{1}{b-a}\int_x^b (t-b)\left[f'(t) - f'(x)\right] dt\end{aligned}$$

for any $x \in [a,b]$.

We have the following result:

THEOREM 3.1 (Dragomir, 2013 [**8**]). *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \tag{3.2}$$

$$\leq \frac{1}{b-a} \left[\int_a^x (t-a) \bigvee_t^x (f')\, dt + \int_x^b (b-t) \bigvee_x^t (f')\, dt \right]$$

$$\leq \frac{1}{2}(b-a) \left[\left(\frac{x-a}{b-a} \right)^2 \bigvee_a^x (f')\, dt + \left(\frac{b-x}{b-a} \right)^2 \bigvee_x^b (f') \right]$$

$$\leq \frac{1}{2}(b-a) \times \begin{cases} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \left[\frac{1}{2} \bigvee_a^b (f') + \frac{1}{2} \left| \bigvee_a^x (f') - \bigvee_x^b (f') \right| \right], \\ \left[\left(\frac{x-a}{b-a} \right)^{2p} + \left(\frac{b-x}{b-a} \right)^{2p} \right]^{1/p} \left[\left[\bigvee_a^x (f') \right]^q + \left[\bigvee_x^b (f') \right]^q \right]^{1/q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f'), \end{cases}$$

for any $x \in [a,b]$.

PROOF. Taking the modulus in (3.1) we have

$$\begin{aligned} &\left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\leq \frac{1}{b-a} \left| \int_a^x (t-a) \left[f'(t) - f'(x) \right] dt \right| \\ &+ \frac{1}{b-a} \left| \int_x^b (t-b) \left[f'(t) - f'(x) \right] dt \right| \\ &\leq \frac{1}{b-a} \int_a^x (t-a) \left| f'(t) - f'(x) \right| dt \\ &+ \frac{1}{b-a} \int_x^b (b-t) \left| f'(t) - f'(x) \right| dt. \end{aligned} \tag{3.3}$$

Since the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,x]$ and $[x,b]$, then

$$\left| f'(t) - f'(x) \right| \leq \bigvee_t^x (f') \text{ for } t \in [a,x]$$

and

$$|f'(t) - f'(x)| \leq \bigvee_x^t (f') \text{ for } t \in [x, b].$$

Therefore

$$\begin{aligned}\int_a^x (t-a)|f'(t) - f'(x)|\,dt &\leq \int_a^x (t-a)\bigvee_t^x (f')\,dt \\ &\leq \frac{1}{2}(x-a)^2 \bigvee_a^x (f')\,dt\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-t)|f'(t) - f'(x)|\,dt &\leq \int_x^b (b-t)\bigvee_x^t (f')\,dt \\ &\leq \frac{1}{2}(b-x)^2 \bigvee_x^b (f'),\end{aligned}$$

which, by (3.3) produce the first two inequalities in (3.2).

The last part follows by Hölder's inequality

$$mn + pq \leq (m^\alpha + p^\alpha)^{1/\alpha}\left(n^\beta + q^\beta\right)^{1/\beta},$$

where $m, n, p, q \geq 0$ and $\alpha > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. ∎

COROLLARY 3.2. *With the assumptions of Theorem 3.1, we have*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} (t-a)\bigvee_t^{\frac{a+b}{2}} (f')\,dt + \int_{\frac{a+b}{2}}^b (b-t)\bigvee_{\frac{a+b}{2}}^t (f')\,dt\right] \\ &\leq \frac{1}{8}(b-a)\bigvee_a^b (f')\,dt.\end{aligned} \tag{3.4}$$

REMARK 3.1. If $p \in (a,b)$ is a median point in bounded variation for the derivative, i.e. $\bigvee_a^p (f') = \bigvee_p^b (f')$, then under the assumptions of Theorem 3.1 we have

$$\begin{aligned}&\left|f(p) + \left(\frac{a+b}{2} - p\right)f'(p) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{b-a}\left[\int_a^p (t-a)\bigvee_t^p (f')\,dt + \int_p^b (b-t)\bigvee_p^t (f')\,dt\right] \\ &\leq \frac{1}{4}(b-a)\left[\frac{1}{4} + \left(\frac{p - \frac{a+b}{2}}{b-a}\right)^2\right]\bigvee_a^b (f').\end{aligned} \tag{3.5}$$

3.2. Inequalities for Lipschitzian Derivatives. We start with the following result.

THEOREM 3.3 (Dragomir, 2013 [**8**]). *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Let $x \in (a,b)$. If $\alpha_i > -1$ and $L_{\alpha_i} > 0$ with $i = 1,2$ are such that*

$$|f'(t) - f'(x)| \leq L_{\alpha_1} (x-t)^{\alpha_1} \text{ for any } t \in [a,x) \tag{3.6}$$

and

$$|f'(t) - f'(x)| \leq L_{\alpha_2} (t-x)^{\alpha_2} \text{ for any } t \in (x,b], \tag{3.7}$$

then we have

$$\begin{aligned} &\left| f(x) + \left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \\ &\leq \frac{1}{b-a} \left[\frac{L_{\alpha_1}}{(\alpha_1+1)(\alpha_1+2)} (x-a)^{\alpha_1+2} + \frac{L_{\alpha_2}}{(\alpha_2+1)(\alpha_2+2)} (b-x)^{\alpha_2+2} \right]. \end{aligned} \tag{3.8}$$

PROOF. Taking the modulus in (3.1) we have

$$\begin{aligned} &\left| f(x) + \left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \\ &\leq \frac{1}{b-a} \int_a^x (t-a) |f'(t) - f'(x)|\,dt \\ &+ \frac{1}{b-a} \int_x^b (b-t) |f'(t) - f'(x)|\,dt. \end{aligned} \tag{3.9}$$

Using the properties (3.6) and (3.7) we have

$$\begin{aligned} \int_a^x (t-a) |f'(t) - f'(x)|\,dt &\leq L_{\alpha_1} \int_a^x (t-a)(x-t)^{\alpha_1}\,dt \\ &= L_{\alpha_1} (x-a)^{\alpha_1+2} \int_0^1 u(1-u)^{\alpha_1}\,du \\ &= L_{\alpha_1} (x-a)^{\alpha_1+2} \int_0^1 u^{\alpha_1}(1-u)\,du \\ &= \frac{1}{(\alpha_1+1)(\alpha_1+2)} L_{\alpha_1} (x-a)^{\alpha_1+2} \end{aligned}$$

and

$$\begin{aligned} \int_x^b (b-t) |f'(t) - f'(x)|\,dt &\leq L_{\alpha_2} \int_x^b (b-t)(t-x)^{\alpha_2}\,dt \\ &= \frac{1}{(\alpha_2+1)(\alpha_2+2)} L_{\alpha_2} (b-x)^{\alpha_2+2}. \end{aligned}$$

Utilising (3.9) we get the desired result (3.8). ∎

COROLLARY 3.4. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative is f' of r-H-Hölder type on $[a,b]$, i.e. we have the condition*

$$|f'(t) - f'(s)| \leq H |t-s|^r$$

for any $t, s \in [a, b]$, *where* $r \in (0, 1]$ *and* $H > 0$ *are given, then*

$$\begin{aligned}(3.10)\qquad &\left|f(x) + \left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le \frac{H}{(r+1)(r+2)}\left[\left(\frac{x-a}{b-a}\right)^{r+2} + \left(\frac{b-x}{b-a}\right)^{r+2}\right](b-a)^{r+1},\end{aligned}$$

for any $x \in [a, b]$.

In particular, if f' *is Lipschitzian with the constant* $L > 0$, *then*

$$\begin{aligned}(3.11)\qquad &\left|f(x) + \left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le \frac{1}{6}L\left[\left(\frac{x-a}{b-a}\right)^{3} + \left(\frac{b-x}{b-a}\right)^{3}\right](b-a)^{2},\end{aligned}$$

for any $x \in [a, b]$.

3.3. Inequalities for Differentiable Convex Functions. The case of convex functions is as follows:

THEOREM 3.5 (Dragomir, 2013 [**8**]). *Let* $f : I \to \mathbb{C}$ *be a differentiable convex function on* $\mathring{I}$ *and* $[a, b] \subset \mathring{I}$. *Then for any* $x \in [a, b]$ *we have*

$$(3.12)\qquad 0 \le \frac{1}{b-a}\int_a^b f(t)\,dt - f(x) - \left(\frac{a+b}{2} - x\right) f'(x) \le \begin{cases} I_1(x) \\ I_2(x) \\ I_3(x) \end{cases}$$

where

$$I_1(x) := \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - f(x) - 2f'(x)\left(\frac{a+b}{2} - x\right),$$

$$I_2(x) := \frac{1}{2}\frac{f'(b)(b-x)^2 - f'(a)(x-a)^2}{b-a} - f'(x)\left(\frac{a+b}{2} - x\right)$$

and

$$I_3(x) := \frac{1}{2}\left[\frac{f(b)(b-x) + f(a)(x-a)}{b-a} - f(x)\right] - f'(x)\left(\frac{a+b}{2} - x\right)$$

PROOF. We have the equality

$$\begin{aligned}(3.13)\qquad &\frac{1}{b-a}\int_a^b f(t)\,dt - f(x) - \left(\frac{a+b}{2} - x\right) f'(x) \\ &= \frac{1}{b-a}\int_a^x (t-a)\left[f'(x) - f'(t)\right]dt + \frac{1}{b-a}\int_x^b (b-t)\left[f'(t) - f'(x)\right]dt\end{aligned}$$

for any $x \in [a, b]$.

Since f is a differentiable convex function on $\mathring{I}$, then f' is monotonic nondecreasing on $\mathring{I}$ and then

$$\int_a^x (t-a)\left[f'(x) - f'(t)\right]dt \ge 0$$

and

$$\int_x^b (b-t)\,[f'(t)-f'(x)]\,dt \geq 0,$$

which proves the first inequality in (3.12).

We have

$$\begin{aligned}\int_a^x (t-a)\,[f'(x)-f'(t)]\,dt &\leq (x-a)\int_a^x [f'(x)-f'(t)]\,dt\\ &= (x-a)\,[f'(x)\,(x-a)-f(x)+f(a)]\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-t)\,[f'(t)-f'(x)]\,dt &\leq (b-x)\int_x^b [f'(t)-f'(x)]\,dt\\ &= (b-x)\,[f(b)-f(x)-f'(x)\,(b-x)]\,.\end{aligned}$$

Adding these inequalities we get

$$\begin{aligned}&\int_a^x (t-a)\,[f'(x)-f'(t)]\,dt+\int_x^b (b-t)\,[f'(t)-f'(x)]\,dt\\ &\leq (x-a)\,[f'(x)\,(x-a)-f(x)+f(a)]\\ &+(b-x)\,[f(b)-f(x)-f'(x)\,(b-x)]\\ &=(b-x)\,f(b)+(x-a)\,f(a)-(b-a)\,f(x)\\ &+f'(x)\,[2x-(a+b)]\,(b-a)\end{aligned}$$

and by (3.13) we get the second inequality for $I_1(x)$.

We also have

$$\begin{aligned}\int_a^x (t-a)\,[f'(x)-f'(t)]\,dt &\leq \int_a^x (t-a)\,[f'(x)-f'(a)]\,dt\\ &=\frac{1}{2}\,[f'(x)-f'(a)]\,(x-a)^2\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-t)\,[f'(t)-f'(x)]\,dt &\leq \int_x^b (b-t)\,[f'(b)-f'(x)]\,dt\\ &=\frac{1}{2}\,[f'(b)-f'(x)]\,(b-x)^2\,.\end{aligned}$$

Adding these inequalities we get

$$\begin{aligned}&\int_a^x (t-a)\,[f'(x)-f'(t)]\,dt+\int_x^b (b-t)\,[f'(t)-f'(x)]\,dt\\ &\leq \frac{1}{2}\,[f'(x)-f'(a)]\,(x-a)^2+\frac{1}{2}\,[f'(b)-f'(x)]\,(b-x)^2\\ &=\frac{1}{2}\left[f'(b)\,(b-x)^2-f'(a)\,(x-a)^2+f'(x)\,(b-a)\,[2x-(a+b)]\right]\end{aligned}$$

and by (3.13) we get the second inequality for $I_2(x)$.

Further, we use the Čebyšev inequality for asynchronous functions (functions of opposite monotonicity), namely

$$\frac{1}{d-c}\int_c^d g(t)\,h(t)\,dt \leq \frac{1}{d-c}\int_c^d g(t)\,dt\cdot\frac{1}{d-c}\int_c^d h(t)\,dt.$$

Therefore

$$\begin{aligned}
&\frac{1}{x-a}\int_a^x (t-a)\left[f'(x)-f'(t)\right]dt \\
&\leq \frac{1}{x-a}\int_a^x (t-a)\,dt\cdot\frac{1}{x-a}\int_a^x \left[f'(x)-f'(t)\right]dt \\
&= \frac{(x-a)^2}{2(x-a)}\cdot\frac{f'(x)(x-a)-f(x)+f(a)}{x-a} \\
&= \frac{1}{2}\left[f'(x)(x-a)-f(x)+f(a)\right]
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{b-x}\int_x^b (b-t)\left[f'(t)-f'(x)\right]dt \\
&\leq \frac{1}{b-x}\int_x^b (b-t)\,dt\cdot\frac{1}{b-x}\int_x^b \left[f'(t)-f'(x)\right]dt \\
&= \frac{(b-x)^2}{2(b-x)}\cdot\frac{f(b)-f(x)-f'(x)(b-x)}{b-x} \\
&= \frac{1}{2}\left[f(b)-f(x)-f'(x)(b-x)\right].
\end{aligned}$$

Adding these inequalities, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^x (t-a)\left[f'(x)-f'(t)\right]dt+\frac{1}{b-a}\int_x^b (b-t)\left[f'(t)-f'(x)\right]dt \\
&\leq \frac{1}{2}\frac{\left[f'(x)(x-a)-f(x)+f(a)\right](x-a)}{b-a} \\
&+\frac{1}{2}\frac{\left[f(b)-f(x)-f'(x)(b-x)\right](b-x)}{b-a} \\
&= \frac{1}{2(b-a)}\left[\left[f'(x)(x-a)-f(x)+f(a)\right](x-a)\right] \\
&+\frac{1}{2(b-a)}\left[\left[f(b)-f(x)-f'(x)(b-x)\right](b-x)\right] \\
&= \frac{1}{2}\left[\frac{f(b)(b-x)+f(a)(x-a)}{b-a}-f(x)\right]+f'(x)\left(x-\frac{a+b}{2}\right)
\end{aligned}$$

which proves the inequality for $I_3(x)$. ∎

REMARK 3.2. From the first inequality in (3.12) we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{(b-x)f(b)+(x-a)f(a)}{b-a}-f'(x)\left(\frac{a+b}{2}-x\right) \tag{3.14}$$

for any $x\in[a,b]$.

From the second inequality in (3.12) we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt-f(x) \leq \frac{1}{2}\cdot\frac{f'(b)(b-x)^2-f'(a)(x-a)^2}{b-a} \tag{3.15}$$

for any $x\in[a,b]$.

From the third inequality in (3.12) we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{2}\left[\frac{f(b)(b-x)+f(a)(x-a)}{b-a}+f(x)\right] \tag{3.16}$$

for any $x\in[a,b]$.

3.4. Inequalities for Absolutely Continuous Derivatives. We use the *Lebesgue p-norms* defined as follows:

$$\|g\|_{[c,d],p} := \left(\int_c^d |g(s)|^p\,dt\right)^{1/p},\ g\in L_p[c,d],\ p\ge 1$$

and

$$\|g\|_{[c,d],\infty} := ess \sup_{s\in[c,d]} |g(s)|,\ g\in L_\infty[c,d].$$

The case of absolutely continuous derivatives is as follows:

THEOREM 3.6 (Dragomir, 2013 [**8**]). *Let $f: I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b]\subset\mathring{I}$. If the derivative f' is absolutely continuous on $[a,b]$, then for any $x\in[a,b]$*

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.17}$$

$$\le \frac{1}{b-a}\times\begin{cases}\frac{1}{6}(x-a)^3\|f''\|_{[a,x],\infty}, \\ \frac{q}{(q+1)(q+2)}(x-a)^{1/q+2}\|f''\|_{[a,x],p}, \\ \frac{1}{2}(x-a)^2\|f''\|_{[a,x],1},\end{cases}$$

$$+\frac{1}{b-a}\times\begin{cases}\frac{1}{6}(b-x)^3\|f''\|_{[x,b],\infty}, \\ \frac{q}{(q+1)(q+2)}(b-x)^{1/q+2}\|f''\|_{[x,b],p}, \\ \frac{1}{2}(b-x)^2\|f''\|_{[x,b],1},\end{cases}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

PROOF. Taking the modulus in (3.1) we have

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt+\left(\frac{a+b}{2}-x\right)f'(x)\right| \\ &\le \frac{1}{b-a}\int_a^x (t-a)\left|f'(t)-f'(x)\right|dt+\frac{1}{b-a}\int_x^b (b-t)\left|f'(t)-f'(x)\right|dt \\ &= \frac{1}{b-a}\int_a^x (t-a)\left|\int_x^t f''(s)\,ds\right|+\frac{1}{b-a}\int_x^b (b-t)\left|\int_x^t f''(s)\,ds\right| \\ &\le \frac{1}{b-a}\int_a^x (t-a)\int_t^x |f''(s)|\,ds+\frac{1}{b-a}\int_x^b (b-t)\int_x^t |f''(s)|\,ds.\end{aligned} \tag{3.18}$$

Using Hölder's integral inequality we have for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$,

$$\int_a^x (t-a)\int_t^x |f''(s)|\,ds \leq \begin{cases} \int_a^x (t-a)(x-t)\|f''\|_{[t,x],\infty}\,dt \\ \\ \int_a^x (t-a)(x-t)^{1/q}\|f''\|_{[t,x],p}\,dt \\ \\ \int_a^x (t-a)\|f''\|_{[t,x],1}\,dt \end{cases}$$

$$\leq \begin{cases} \|f''\|_{[a,x],\infty}\int_a^x (t-a)(x-t)\,dt \\ \\ \|f''\|_{[a,x],p}\int_a^x (t-a)(x-t)^{1/q}\,dt \\ \\ \|f''\|_{[a,x],1}\int_a^x (t-a)\,dt \end{cases}$$

$$= \begin{cases} \frac{1}{6}(x-a)^3\|f''\|_{[a,x],\infty} \\ \\ \frac{q}{(q+1)(q+2)}(x-a)^{1/q+2}\|f''\|_{[a,x],p} \\ \\ \frac{1}{2}(x-a)^2\|f''\|_{[a,x],1} \end{cases}$$

and, similarly

$$\int_x^b (b-t)\int_x^t |f''(s)|\,ds \leq \begin{cases} \frac{1}{6}(b-x)^3\|f''\|_{[x,b],\infty} \\ \\ \frac{q}{(q+1)(q+2)}(b-x)^{1/q+2}\|f''\|_{[x,b],p} \\ \\ \frac{1}{2}(b-x)^2\|f''\|_{[x,b],1}\,. \end{cases}$$

Utilizing the inequality (3.18) we get the desired result (3.17). ∎

REMARK 3.3. Since

$$\begin{aligned} &\frac{1}{6}(x-a)^3\|f''\|_{[a,x],\infty}+\frac{1}{6}(b-x)^3\|f''\|_{[x,b],\infty} \\ &\leq \frac{1}{6}\left[(x-a)^3+(b-x)^3\right]\max\left\{\|f''\|_{[a,x],\infty},\|f''\|_{[x,b],\infty}\right\} \\ &= \frac{1}{6}(b-a)\left[(x-a)^2-(x-a)(b-x)+(b-x)^2\right]\|f''\|_{[a,b],\infty}\,, \end{aligned}$$

then by (3.17) we get

$$\begin{aligned} &\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{6}\left[\left(\frac{x-a}{b-a}\right)^2-\left(\frac{x-a}{b-a}\right)\left(\frac{b-x}{b-a}\right)+\left(\frac{b-x}{b-a}\right)^2\right] \\ &\times (b-a)^2\|f''\|_{[a,b],\infty}\,, \end{aligned} \tag{3.19}$$

for any $x\in[a,b]$.

Since

$$\begin{aligned}
&(x-a)^{1/q+2}\|f''\|_{[a,x],p}+(b-x)^{1/q+2}\|f''\|_{[x,b],p}\\
&\le\left[(x-a)^{2q+1}+(b-x)^{2q+1}\right]^{1/q}\left[\|f''\|^p_{[a,x],p}+\|f''\|_{[x,b],p}\right]^{1/p}\\
&=\left[(x-a)^{2q+1}+(b-x)^{2q+1}\right]^{1/q}\|f''\|_{[a,b],p}\,,
\end{aligned}$$

then by (3.17) we get

$$\begin{aligned}
&\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\
&\le\frac{q}{(q+1)(q+2)}\left[\left(\frac{x-a}{b-a}\right)^{2q+1}+\left(\frac{b-x}{b-a}\right)^{2q+1}\right]^{1/q}\\
&\times(b-a)^{1+1/q}\|f''\|_{[a,b],p}\,,
\end{aligned}\tag{3.20}$$

for any $x\in[a,b]$.

Since

$$\begin{aligned}
&(x-a)^2\|f''\|_{[a,x],1}+(b-x)^2\|f''\|_{[x,b],1}\\
&\le\max\left\{(x-a)^2,(b-x)^2\right\}\left[\|f''\|_{[a,x],1}+\|f''\|_{[x,b],1}\right]\\
&=\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^2\|f''\|_{[a,b],1}\,,
\end{aligned}$$

then by (3.17) we get

$$\begin{aligned}
&\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\
&\le\frac{1}{2}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^2(b-a)\|f''\|_{[a,b],1}
\end{aligned}$$

for any $x\in[a,b]$.

4. More Perturbed Ostrowski Type Inequalities for Absolutely Continuous Functions

4.1. Inequalities for Derivatives of Bounded Variation. Assume that the function $f:I\to\mathbb{C}$ is differentiable on the interior of I, denoted $\mathring{I}$, and $[a,b]\subset\mathring{I}$. Then, we have the equality

$$\begin{aligned}
&f(x)+\frac{1}{2(b-a)}\left[(b-x)^2f'(b)-(x-a)^2f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\\
&=\frac{1}{b-a}\int_a^x(t-a)\left[f'(t)-f'(a)\right]dt+\frac{1}{b-a}\int_x^b(b-t)\left[f'(b)-f'(t)\right]dt,
\end{aligned}\tag{4.1}$$

for any $x\in[a,b]$.

In particular, for $x = \frac{a+b}{2}$, we have

$$
\begin{aligned}
(4.2) \qquad & f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)\left[f'(b) - f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\
&= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - f'(a)\right] dt \\
&+ \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (b-t)\left[f'(b) - f'(t)\right] dt.
\end{aligned}
$$

THEOREM 4.1. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then for any $x \in [a,b]$*

$$
\begin{aligned}
(4.3) \qquad & \left| f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\le \frac{1}{b-a}\left[\int_a^x (t-a)\bigvee_a^t (f')\,dt + \int_x^b (b-t)\bigvee_t^b (f')\,dt\right] \\
&\le \frac{1}{b-a}\begin{cases} \frac{1}{2}(x-a)^2 \bigvee_a^x (f'), \\[2ex] \frac{1}{(q+1)^{1/q}}(x-a)^{1+1/q}\left(\int_a^x \left(\bigvee_a^t (f')\right)^p dt\right)^{1/p}, \\[2ex] (x-a)\int_a^x \left(\bigvee_a^t (f')\right) dt \end{cases} \\
&\quad + \frac{1}{b-a}\begin{cases} \frac{1}{2}(b-x)^2 \bigvee_x^b (f'), \\[2ex] \frac{1}{(q+1)^{1/q}}(b-x)^{1+1/q}\left(\int_x^b \left(\bigvee_t^b (f')\right)^p dt\right)^{1/p}, \\[2ex] (b-x)\int_x^b \left(\bigvee_t^b (f')\right) dt. \end{cases}
\end{aligned}
$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

PROOF. Taking the modulus in (4.1) we have

$$
\begin{aligned}
(4.4) \qquad & \left| f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\le \frac{1}{b-a}\int_a^x (t-a)\left|f'(t) - f'(a)\right| dt + \frac{1}{b-a}\int_x^b (b-t)\left|f'(b) - f'(t)\right| dt,
\end{aligned}
$$

for any $x \in [a,b]$.

Since the derivative $f':\mathring{I}\to\mathbb{C}$ is of bounded variation on $[a,b]$, then

$$\left|f'(t)-f'(a)\right|\leq\bigvee_a^t(f')\text{ for any }t\in[a,x]$$

and

$$\left|f'(b)-f'(t)\right|\leq\bigvee_t^b(f')\text{ for any }t\in[x,b].$$

Therefore

$$\int_a^x(t-a)\left|f'(t)-f'(a)\right|dt\leq\int_a^x(t-a)\bigvee_a^t(f')\,dt$$

and

$$\int_x^b(b-t)\left|f'(b)-f'(t)\right|dt\leq\int_x^b(b-t)\bigvee_t^b(f')\,dt$$

for any $x\in[a,b]$.

Adding these two inequalities and dividing by $b-a$ we get the first inequality in (4.3).

Using Hölder's integral inequality we have

$$\int_a^x(t-a)\bigvee_a^t(f')\,dt\leq\begin{cases}\bigvee_a^x(f')\int_a^x(t-a)\,dt,\\[2ex] \left(\int_a^x(t-a)^q\,dt\right)^{1/q}\left(\int_a^x\left(\bigvee_a^t(f')\right)^p dt\right)^{1/p},\\[2ex] (x-a)\int_a^x\left(\bigvee_a^t(f')\right)dt,\end{cases}$$
$$=\begin{cases}\frac{1}{2}(x-a)^2\bigvee_a^x(f'),\\[2ex] \frac{1}{(q+1)^{1/q}}(x-a)^{1+1/q}\left(\int_a^x\left(\bigvee_a^t(f')\right)^p dt\right)^{1/p},\\[2ex] (x-a)\int_a^x\left(\bigvee_a^t(f')\right)dt\end{cases}$$

and

$$\int_x^b (b-t) \bigvee_t^b (f') \, dt \le \begin{cases} \frac{1}{2}(b-x)^2 \bigvee_x^b (f'), \\ \\ \frac{1}{(q+1)^{1/q}} (b-x)^{1+1/q} \left(\int_x^b \left(\bigvee_x^b (f') \right)^p dt \right)^{1/p}, \\ \\ (b-x) \int_x^b \left(\bigvee_x^b (f') \right) dt. \end{cases}$$

∎

REMARK 4.1. From the first branch in (4.3) we have the sequence of inequalities

(4.5)
$$\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t) \, dt \right|$$
$$\le \frac{1}{b-a} \left[\int_a^x (t-a) \bigvee_a^t (f') \, dt + \int_x^b (b-t) \bigvee_t^b (f') \, dt \right]$$
$$\le \frac{1}{2}(b-a) \left[\left(\frac{x-a}{b-a} \right)^2 \bigvee_a^x (f') + \left(\frac{b-x}{b-a} \right)^2 \bigvee_x^b (f') \right]$$

$$\le \frac{1}{2}(b-a)$$
$$\times \begin{cases} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \left[\frac{1}{2} \bigvee_a^b (f') + \frac{1}{2} \left| \bigvee_a^x (f') - \bigvee_x^b (f') \right| \right], \\ \\ \left[\left(\frac{x-a}{b-a} \right)^{2p} + \left(\frac{b-x}{b-a} \right)^{2p} \right]^{1/p} \left[\left[\bigvee_a^x (f') \right]^q + \left[\bigvee_x^b (f') \right]^q \right]^{1/q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f'), \end{cases}$$

for any $x \in [a, b]$.

From the second branch in (4.3) we have

$$
\begin{aligned}
&\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{b-a}\left[\int_a^x (t-a)\bigvee_a^t (f')\,dt+\int_x^b (b-t)\bigvee_t^b (f')\,dt\right] \\
&\leq \frac{1}{(q+1)^{1/q}}\left\{\left(\frac{x-a}{b-a}\right)^{1+1/q}\left(\int_a^x\left(\bigvee_a^t (f')\right)^p dt\right)^{1/p}\right. \\
&\left.+\left(\frac{b-x}{b-a}\right)^{1+1/q}\left(\int_x^b\left(\bigvee_t^b (f')\right)^p dt\right)^{1/p}\right\}(b-a)^{1/q} \\
&\qquad\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/p} \\
&\qquad\times\left[\int_a^x\left(\bigvee_a^t (f')\right)^p dt+\int_x^b\left(\bigvee_t^b (f')\right)^p dt\right]^{1/p}(b-a)^{1/q} \\
&\qquad\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/p} \\
&\qquad\times\left[(x-a)\left(\bigvee_a^x (f')\right)^p+(b-x)\left(\bigvee_x^b (f')\right)^p\right]^{1/p}(b-a)^{1/q}
\end{aligned}
\tag{4.6}
$$

for any $x\in[a,b]$ and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

From the third branch in (4.3) we have

$$
\begin{aligned}
&\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{b-a}\left[\int_a^x (t-a)\bigvee_a^t (f')\,dt+\int_x^b (b-t)\bigvee_t^b (f')\,dt\right] \\
&\leq \left(\frac{x-a}{b-a}\right)\int_a^x\left(\bigvee_a^t (f')\right)dt+\left(\frac{b-x}{b-a}\right)\int_x^b\left(\bigvee_t^b (f')\right)dt \\
&\leq \begin{cases}
\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\int_a^x\left(\bigvee_a^t (f')\right)dt+\int_x^b\left(\bigvee_t^b (f')\right)dt\right] \\[2ex]
\left[\left(\frac{x-a}{b-a}\right)^q+\left(\frac{b-x}{b-a}\right)^q\right]^{1/q} \\
\times\left[\left[\int_a^x\left(\bigvee_a^t (f')\right)dt\right]^p+\left[\int_x^b\left(\bigvee_t^b (f')\right)dt\right]^p\right]^{1/p} \\[2ex]
\max\left\{\int_a^x\left(\bigvee_a^t (f')\right)dt,\int_x^b\left(\bigvee_t^b (f')\right)dt\right\}
\end{cases}
\end{aligned}
\tag{4.7}
$$

for any $x \in [a,b]$ and $p > 1, \frac{1}{p} + \frac{1}{q} = 1.$

REMARK 4.2. We observe that, if we take $x = \frac{a+b}{2}$ in (4.5) then we get the perturbed midpoint inequality

$$\left| f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)\left[f'(b) - f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{4.8}$$
$$\leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} (t-a)\bigvee_a^t (f')\,dt + \int_{\frac{a+b}{2}}^b (b-t)\bigvee_t^b (f')\,dt\right]$$
$$\leq \frac{1}{8}(b-a)\bigvee_a^b (f').$$

4.2. Inequalities for Lipschitzian Derivatives. We start with the following result.

THEOREM 4.2. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Let $x \in (a,b)$. If $\alpha_i > -1$ and $L_{\alpha_i} > 0$ with $i = 1,2$ are such that*

$$|f'(t) - f'(a)| \leq L_{\alpha_1}(t-a)^{\alpha_1} \text{ for any } t \in [a,x) \tag{4.9}$$

and

$$|f'(b) - f'(t)| \leq L_{\alpha_2}(b-t)^{\alpha_2} \text{ for any } t \in (x,b], \tag{4.10}$$

then we have

$$\left| f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{4.11}$$
$$\leq \frac{1}{b-a}\left[\frac{L_{\alpha_1}}{\alpha_1+2}(x-a)^{\alpha_1+2} + \frac{L_{\alpha_2}}{\alpha_2+2}(b-x)^{\alpha_2+2}\right].$$

PROOF. Using the conditions (4.9) and (4.10) we have

$$\left| f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$
$$\leq \frac{1}{b-a}\int_a^x (t-a)\,|f'(t) - f'(a)|\,dt + \frac{1}{b-a}\int_x^b (b-t)\,|f'(b) - f'(t)|\,dt$$
$$\leq \frac{1}{b-a}L_{\alpha_1}\int_a^x (t-a)^{\alpha_1+1}\,dt + \frac{1}{b-a}L_{\alpha_2}\int_x^b (b-t)^{\alpha_2+1}\,dt$$
$$= \frac{1}{b-a}L_{\alpha_1}\frac{(x-a)^{\alpha_1+2}}{\alpha_1+2} + \frac{1}{b-a}L_{\alpha_2}\frac{(b-x)^{\alpha_2+2}}{\alpha_2+2}$$
$$= \frac{1}{b-a}\left[\frac{L_{\alpha_1}}{\alpha_1+2}(x-a)^{\alpha_1+2} + \frac{L_{\alpha_2}}{\alpha_2+2}(b-x)^{\alpha_2+2}\right]$$

and the inequality (4.11) is obtained. ∎

COROLLARY 4.3. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative is f' of r-H-Hölder type on $[a,b]$, i.e. we have the condition*

$$|f'(t) - f'(s)| \leq H\,|t-s|^r$$

for any $t, s \in [a,b]$, *where* $r \in (0,1]$ *and* $H > 0$ *are given, then*

$$\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \frac{H}{r+2} \left[\left(\frac{x-a}{b-a} \right)^{r+2} + \left(\frac{b-x}{b-a} \right)^{r+2} \right] (b-a)^{r+1}, \tag{4.12}$$

for any $x \in [a,b]$.

In particular, if f' *is Lipschitzian with the constant* $L > 0$, *then*

$$\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \frac{1}{3} L \left[\left(\frac{x-a}{b-a} \right)^3 + \left(\frac{b-x}{b-a} \right)^3 \right] (b-a)^2, \tag{4.13}$$

for any $x \in [a,b]$.

REMARK 4.3. With the assumptions of Corollary 4.3 we have the midpoint inequality

$$\left| f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)\left[f'(b) - f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{H}{2^{r+1}(r+2)}(b-a)^{r+1}. \tag{4.14}$$

If if f' is Lipschitzian with the constant $L > 0$, then

$$\left| f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)\left[f'(b) - f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{12} L (b-a)^2. \tag{4.15}$$

4.3. Inequalities for Differentiable Functions with the Property (S)**.** Let $f : I \to \mathbb{C}$ be a differentiable convex function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Then f' is monotonic nondecreasing and by the equality (4.1) we have

$$f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \geq 0 \tag{4.16}$$

or, equivalently

$$\frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] \geq \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \tag{4.17}$$

for any $x \in [a,b]$.

We observe that the inequalities (4.16) and (4.17) remain valid for the larger class of differentiable functions f that that satisfy the *property* (S) on the interval $[a,b]$, namely

$$f'(a) \leq f'(t) \leq f'(b) \tag{S}$$

for any $t \in [a,b]$.

THEOREM 4.4. *Let* $f : I \to \mathbb{C}$ *be a differentiable function on* $\mathring{I}$ *and* $[a,b] \subset \mathring{I}$.

(i) Let $x \in [a,b]$. *If* f *satisfies the property* (S) *on the interval* $[a,x]$ *and* $[x,b]$, *then*

$$f'(x)\left(\frac{a+b}{2} - x\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt - f(x). \tag{4.18}$$

(ii) If f satisfies the property (S) on the interval $[a,b]$, then for any $x \in [a,b]$

$$\frac{f(a)(x-a)+f(b)(b-x)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]. \tag{4.19}$$

PROOF. (i) Since f satisfies the property (S) on the interval $[a,x]$ and $[x,b]$, then

$$\begin{aligned}
& f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \\
&= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'(a)\right]dt+\frac{1}{b-a}\int_x^b (b-t)\left[f'(b)-f'(t)\right]dt \\
&\le \frac{1}{b-a}\int_a^x (t-a)\left[f'(x)-f'(a)\right]dt+\frac{1}{b-a}\int_x^b (b-t)\left[f'(b)-f'(x)\right]dt \\
&= \frac{f'(x)-f'(a)}{b-a}\int_a^x (t-a)\,dt+\frac{f'(b)-f'(x)}{b-a}\int_x^b (b-t)\,dt \\
&= \frac{f'(x)-f'(a)}{b-a}\cdot\frac{(x-a)^2}{2}+\frac{f'(b)-f'(x)}{b-a}\cdot\frac{(b-x)^2}{2} \\
&= \frac{1}{2(b-a)}\left[(f'(x)-f'(a))(x-a)^2+(f'(b)-f'(x))(b-x)^2\right] \\
&= \frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-f'(x)\left(\frac{a+b}{2}-x\right),
\end{aligned}$$

which proves the inequality (4.18).

(ii) If f satisfies the property (S) on the interval $[a,b]$, then for any $x\in[a,b]$

$$\begin{aligned}
& \frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'(a)\right]dt+\frac{1}{b-a}\int_x^b (b-t)\left[f'(b)-f'(t)\right]dt \\
&\le \frac{x-a}{b-a}\int_a^x \left[f'(t)-f'(a)\right]dt+\frac{b-x}{b-a}\int_x^b \left[f'(b)-f'(t)\right]dt \\
&= \frac{1}{b-a}(x-a)\left[f(x)-f(a)-f'(a)(x-a)\right] \\
&+ \frac{1}{b-a}(b-x)\left[f'(b)(b-x)-f(b)+f(x)\right] \\
&= \frac{1}{b-a}\left[f(x)(x-a)-f(a)(x-a)-f'(a)(x-a)^2\right] \\
&+ \frac{1}{b-a}\left[f'(b)(b-x)^2-f(b)(b-x)+f(x)(b-x)\right] \\
&= \frac{1}{b-a}\left\{f'(b)(b-x)^2-f'(a)(x-a)^2-f(a)(x-a)-f(b)(b-x)\right. \\
&\left.+f(x)(b-a)\right\} \\
&= \frac{f'(b)(b-x)^2-f'(a)(x-a)^2}{b-a}+f(x)-\frac{f(a)(x-a)+f(b)(b-x)}{b-a},
\end{aligned}$$

which proves the inequality (4.19). ∎

REMARK 4.4. The inequality (4.18) was obtained for the case of convex functions in [**3**] while (4.19) was established for convex functions in [**4**] with different proofs.

Further, we use the Čebyšev inequality for synchronous functions (functions with same monotonicity), namely

$$\frac{1}{d-c}\int_c^d g(t)\,h(t)\,dt \geq \frac{1}{d-c}\int_c^d g(t)\,dt \cdot \frac{1}{d-c}\int_c^d h(t)\,dt. \tag{4.20}$$

THEOREM 4.5. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Let $x \in [a,b]$. If f is convex on the interval $[a,x]$ and $[x,b]$, then*

$$\frac{1}{2}\left[f(x) + \frac{f(a)(x-a) + f(b)(b-x)}{b-a}\right] \geq \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{4.21}$$

PROOF. We have

$$\begin{aligned} & f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ & = \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - f'(a)\right]dt + \frac{1}{b-a}\int_x^b (b-t)\left[f'(b) - f'(t)\right]dt \end{aligned} \tag{4.22}$$

for any $x \in [a,b]$.

Since f' is monotonic nondecreasing on $[a,x]$, then by Čebyšev inequality (4.12) we have

$$\begin{aligned} \int_a^x (t-a)\left[f'(t) - f'(a)\right]dt &\geq \frac{1}{x-a}\int_a^x (t-a)\,dt \cdot \int_a^x \left[f'(t) - f'(a)\right]dt \\ &= \frac{1}{2}(x-a)\left[f(x) - f(a) - f'(a)(x-a)\right] \\ &= \frac{1}{2}\left[f(x)(x-a) - f(a)(x-a) - f'(a)(x-a)^2\right] \end{aligned}$$

and, by the same inequality,

$$\begin{aligned} \int_x^b (b-t)\left[f'(b) - f'(t)\right]dt &\geq \frac{1}{b-x}\int_x^b (b-t)\,dt \cdot \int_x^b \left[f'(b) - f'(t)\right]dt \\ &= \frac{1}{2}(b-x)\left[f'(b)(b-x) - f(b) + f(x)\right] \\ &= \frac{1}{2}\left[f'(b)(b-x)^2 - f(b)(b-x) + f(x)(b-x)\right]. \end{aligned}$$

If we add these two inequalities, then we get

$$\begin{aligned} & \int_a^x (t-a)\left[f'(t) - f'(a)\right]dt + \int_x^b (b-t)\left[f'(b) - f'(t)\right]dt \\ & \geq \frac{1}{2}\left[f(x)(x-a) - f(a)(x-a) - f'(a)(x-a)^2\right] \\ & + \frac{1}{2}\left[f'(b)(b-x)^2 - f(b)(b-x) + f(x)(b-x)\right] \\ & = \frac{1}{2}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] + \frac{1}{2}f(x)(b-a) \\ & - \frac{1}{2}\left[f(a)(x-a) + f(b)(b-x)\right]. \end{aligned}$$

Dividing by $b-a$ and utilizing the equality (4.22) we deduce the inequality (4.21). ∎

REMARK 4.5. If the function is convex on the whole interval $[a,b]$, then the inequality (4.21) is true for any $x \in [a,b]$.

REFERENCES

[1] T. M. Apostol, *Mathematical Analysis*, Second Edition, Addison-Wesley Publishing Company, 1975

[2] S. S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. & Appl.*, **4**(1) (2001), 33-40. Preprint, *RGMIA Res. Rep. Coll.* **2** (1999), No. 1, Article 7. [http://rgmia.org/papers/v2n1/v2n1-7.pdf].

[3] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 31, 8 pp.

[4] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 3, Article 35, 8 pp.

[5] S. S. Dragomir, Improvements of Ostrowski and generalised trapezoid inequality in terms of the upper and lower bounds of the first derivative. *Tamkang J. Math.* **34** (2003), no. 3, 213–222.

[6] S. S. Dragomir, Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation. *Arch. Math.* (Basel) **91** (2008), no. 5, 450–460.

[7] S. S. Dragomir, Some perturbed Ostrowski type inequalities for absolutely continuous functions (I), *Acta Univ. M. Belii Ser. Math.* **2015**, 1–16. Preprint *RGMIA Res. Rep. Coll.* **16** (2013), Art. 94. [Online http://rgmia.org/papers/v16/v16a94.pdf].

[8] S. S. Dragomir, Some perturbed Ostrowski type inequalities for absolutely continuous functions (II), *Acta Univ. Apulensis Math. Inform.* No. **43** (2015), 209–228. Preprint *RGMIA Res. Rep. Coll.* **16** (2013), Art. 95. [Online http://rgmia.org/papers/v16/v16a95.pdf].

[9] S. S. Dragomir, Some perturbed Ostrowski type inequalities for absolutely continuous functions (III), *Transylv. J. Math. Mech.* **7** (2015), no. 1, 31–43. Preprint *RGMIA Res. Rep. Coll.* **16** (2013), Art. 96. [Online http://rgmia.org/papers/v16/v16a96.pdf].

[10] S. S. Dragomir, Some perturbed Ostrowski type inequalities for functions of bounded variation, *Asian-Eur. J. Math.* **8** (2015), no. 4, 1550069, 14 pp. Preprint *RGMIA Res. Rep. Coll.* **16** (2013), Art. 93. [Online http://rgmia.org/papers/v16/v16a93.pdf].

CHAPTER 8

Companions of Ostrowski's Inequality

1. A Companion of Ostrowski Inequality for Functions of Bounded Variation

1.1. Introduction. In [**13**], Guessab and Schmeisser have proved among others, the following companion of Ostrowski's inequality.

THEOREM 1.1 (Guessab & Schmeisser, 2002 [**13**]). *Let $f:[a,b]\to\mathbb{R}$ be such that*

$$|f(t)-f(s)|\le H|t-s|^k,\quad \textit{for any } t,s\in[a,b] \tag{1.1}$$

with $k\in(0,1]$, i.e., $f\in Lip_H(k)$. Then, for each $x\in\left[a,\frac{a+b}{2}\right]$, we have the inequality

$$\begin{aligned}\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ \le\left[\frac{2^{k+1}(x-a)^{k+1}+(a+b-2x)^{k+1}}{2^k(k+1)(b-a)}\right]H.\end{aligned} \tag{1.2}$$

This inequality is sharp for each admissible x. Equality is obtained if and only if $f=\pm Hf_*+c$ with $c\in\mathbb{R}$ and

$$f_*(t)=\begin{cases}(x-t)^k & \textit{for}\quad a\le t\le x\\[2mm] (t-x)^k & \textit{for}\quad x\le t\le\frac{1}{2}(a+b)\\[2mm] f_*(a+b-t) & \textit{for}\quad \frac{1}{2}(a+b)\le t\le b.\end{cases} \tag{1.3}$$

We remark that for $k=1$, i.e., $f\in Lip_H$, since

$$\frac{4(x-a)^2+(a+b-2x)^2}{4(b-a)}=\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)$$

then we have the inequality

$$\begin{aligned}\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ \le\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)H\end{aligned} \tag{1.4}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

The inequality $\frac{1}{8}$ is best possible in (1.4) in the sense that it cannot be replaced by a smaller constant.

We must also observe that the best inequality in (1.4) is obtained for $x = \frac{a+3b}{4}$, giving the trapezoid type inequality

$$\left| \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{8}(b-a)H. \tag{1.5}$$

The constant $\frac{1}{8}$ is sharp in (1.5) in the sense mentioned above.

1.2. Some Integral Inequalities. The following identity holds.

LEMMA 1.2 (Dragomir, 2002 [**5**]). *Assume that the function* $f : [a,b] \to \mathbb{R}$ *is of bounded variation on* $[a,b]$. *Then we have the equality*

$$\begin{aligned} &\frac{1}{2}[f(x) + f(a+b-x)] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\left[\int_a^x (t-a)\,df(t) + \int_x^{a+b-x}\left(t - \frac{a+b}{2}\right)df(t) + \int_{a+b-x}^b (t-b)\,df(t)\right] \end{aligned} \tag{1.6}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

PROOF. Obviously, all the Riemann-Stieltjes integrals from the right hand side of (1.6) exist because the functions $(\cdot - a)$, $\left(\cdot - \frac{a+b}{2}\right)$ and $(\cdot - b)$ are continuous on those intervals and f is of bounded variation.

Using the integration by parts formula for Riemann-Stieltjes integrals, we have, for any $x \in \left[a, \frac{a+b}{2}\right]$, that

$$\int_a^x (t-a)\,df(t) = f(x)(x-a) - \int_a^x f(t)\,dt,$$

$$\begin{aligned} &\int_x^{a+b-x}\left(t - \frac{a+b}{2}\right)df(t) \\ &= f(a+b-x)\left(\frac{a+b}{2} - x\right) - f(x)\left(x - \frac{a+b}{2}\right) - \int_x^{a+b-x} f(t)\,dt \end{aligned}$$

and

$$\int_{a+b-x}^b (t-b)\,df(t) = (x-a)f(a+b-x) - \int_{a+b-x}^b f(t)\,dt.$$

Summing the above equalities we deduce (1.6). ∎

REMARK 1.1. A version of this identity for piecewise continuously differentiable functions has been obtained in [**13**, Lemma 3.2].

The following companion of Ostrowski's inequality holds.

THEOREM 1.3 (Dragomir, 2002 [**5**]). *Assume that the function* $f : [a,b] \to \mathbb{R}$ *is of bounded variation on* $[a,b]$. *Then we have the inequalities:*

$$\begin{aligned} &\left| \frac{1}{2}[f(x) + f(a+b-x)] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\le \frac{1}{b-a}\left[(x-a)\bigvee_a^x (f) + \left(\frac{a+b}{2} - x\right)\bigvee_x^{a+b-x}(f) + (x-a)\bigvee_{a+b-x}^b (f)\right] \end{aligned} \tag{1.7}$$

$$\leq \begin{cases} \left[\dfrac{1}{4}+\left|\dfrac{x-\frac{3a+b}{4}}{b-a}\right|\right]\bigvee_a^b(f) \\ \\ \left[2\left(\dfrac{x-a}{b-a}\right)^\alpha+\left(\dfrac{\frac{a+b}{2}-x}{b-a}\right)^\alpha\right]^{\frac{1}{\alpha}} \\ \times\left[\left[\bigvee_a^x(f)\right]^\beta+\left[\bigvee_x^{a+b-x}(f)\right]^\beta+\left[\bigvee_{a+b-x}^b(f)\right]^\beta\right]^{\frac{1}{\beta}}, \ \textit{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \\ \left(\dfrac{x+\frac{b-3a}{2}}{b-a}\right)\max\left\{\bigvee_a^x(f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^b(f)\right\} \end{cases}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ *, where* $\bigvee_c^d(f)$ *denotes the total variation of* f *on* $[c,d]$ *. The constant* $\frac{1}{4}$ *is best possible in the first branch of the second inequality in (1.7).*

PROOF. We use the fact that for a continuous function $p:[c,d]\to\mathbb{R}$ and a function $v:[a,b]\to\mathbb{R}$ of bounded variation, one has the inequality

$$\left|\int_c^d p(t)\,dv(t)\right|\leq\sup_{t\in[c,d]}|p(t)|\bigvee_c^d(v). \tag{1.8}$$

Taking the modulus in (1.6) we have

$$\begin{aligned}&\left|\frac{1}{2}[f(x)+f(a+b-x)]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ \leq\ &\frac{1}{b-a}\left[\left|\int_a^x(t-a)\,df(t)\right|+\left|\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)df(t)\right|+\left|\int_{a+b-x}^b(t-b)\,df(t)\right|\right]\\ \leq\ &\frac{1}{b-a}\left[(x-a)\bigvee_a^x(f)+\left(\frac{a+b}{2}-x\right)\bigvee_x^{a+b-x}(f)+(x-a)\bigvee_{a+b-x}^b(f)\right]=:M(x)\end{aligned}$$

and the first inequality in (1.7) is obtained.

Now, observe that

$$\begin{aligned}M(x)&\leq\frac{1}{b-a}\max\left\{x-a,\frac{a+b}{2}-x\right\}\left[\bigvee_a^x(f)+\bigvee_x^{a+b-x}(f)+\bigvee_{a+b-x}^b(f)\right]\\ &=\frac{1}{b-a}\left[\frac{1}{4}(b-a)+\left|x-\frac{3a+b}{4}\right|\right]\bigvee_a^b(f)\end{aligned}$$

and the first branch in the second inequality in (1.7) is proved.

Using Hölder's discrete inequality we have (for $\alpha>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$) that

$$\begin{aligned}M(x)\leq\frac{1}{b-a}&\left[(x-a)^\alpha+\left(\frac{a+b}{2}-x\right)^\alpha+(x-a)^\alpha\right]^{\frac{1}{\alpha}}\\ &\times\left[\left[\bigvee_a^x(f)\right]^\beta+\left[\bigvee_x^{a+b-x}(f)\right]^\beta+\left[\bigvee_{a+b-x}^b(f)\right]^\beta\right]^{\frac{1}{\beta}}\end{aligned}$$

giving the second branch in the second inequality.

Finally, we have

$$M(x) \leq \frac{1}{b-a} \max\left\{ \bigvee_a^x (f), \bigvee_x^{a+b-x} (f), \bigvee_{a+b-x}^b (f) \right\}$$
$$\times \left[(x-a) + \left(\frac{a+b}{2} - x \right) + (x-a) \right],$$

which is equivalent with the last inequality in (1.7).

The sharpness of the constant $\frac{1}{4}$ in the first branch of the second inequality in (1.7) will be proved in a particular case later. ∎

COROLLARY 1.4. *With the assumptions in Theorem 1.3, one has the trapezoid inequality*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \frac{1}{2} \bigvee_a^b (f). \tag{1.9}$$

The constant $\frac{1}{2}$ is best possible in (1.9).

PROOF. Follows from the first inequality in (1.7) on choosing $x = a$.

For the sharpness of the constant, assume that (1.9) holds with a constant $A > 0$, i.e.,

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq A \bigvee_a^b (f). \tag{1.10}$$

If we choose $f : [a,b] \to \mathbb{R}$ with

$$f(x) = \begin{cases} 1 & \text{if} \quad x = a, \\ 0 & \text{if} \quad x \in (a,b), \\ 1 & \text{if} \quad x = b, \end{cases}$$

then f is of bounded variation on $[a,b]$ and

$$\frac{f(a)+f(b)}{2} = 1, \quad \int_a^b f(t)\,dt = 0, \quad \text{and} \quad \bigvee_a^b (f) = 2,$$

giving in (1.10) $1 \leq 2A$, thus $A \geq \frac{1}{2}$ and the corollary is proved. ∎

REMARK 1.2. The inequality (1.9) was first proved in a different manner in [**2**].

COROLLARY 1.5. *With the assumptions in Theorem 1.3, one has the midpoint inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \frac{1}{2} \bigvee_a^b (f). \tag{1.11}$$

The constant $\frac{1}{2}$ is best possible in (1.11).

PROOF. Follows from the first inequality in (1.7) on choosing $x = \frac{a+b}{2}$.

For the sharpness of the constant, assume that (1.11) holds with a constant $B > 0$, i.e.,

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq B \bigvee_a^b (f). \tag{1.12}$$

If we choose $f:[a,b]\to\mathbb{R}$ with

$$f(x)=\begin{cases}0 & \text{if} \quad x\in\left[a,\frac{a+b}{2}\right),\\ 1 & \text{if} \quad x=\frac{a+b}{2},\\ 0 & \text{if} \quad x\in\left(\frac{a+b}{2},b\right],\end{cases}$$

then f is of bounded variation on $[a,b]$, and

$$f\left(\frac{a+b}{2}\right)=1,\quad \int_a^b f(t)\,dt=0,\quad \text{and}\quad \bigvee_a^b(f)=2,$$

giving in (1.12), $1\le 2B$, thus $B\ge\frac{1}{2}$. ∎

REMARK 1.3. The inequality (1.11) was firstly proved in a different manner in [**3**].

The best inequality we may get from Theorem 1.3 on using the bound provided by the first branch in the second inequality in (1.7) is incorporated in the following corollary.

COROLLARY 1.6 (Dragomir, 2002 [**5**]). *With the assumptions in Theorem 1.3 , one has the inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{4}\bigvee_a^b(f). \tag{1.13}$$

The constant $\frac{1}{4}$ is best possible.

PROOF. Follows by Theorem 1.3 on choosing $x=\frac{3a+b}{4}$.

To prove the sharpness of the constant $\frac{1}{4}$, assume that (1.13) holds with a constant $C>0$, i.e.,

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le C\bigvee_a^b(f). \tag{1.14}$$

Consider the function $f:[a,b]\to\mathbb{R}$, given by

$$f(x)=\begin{cases}1 & \text{if} \quad x\in\left\{\frac{3a+b}{4},\frac{a+3b}{4}\right\},\\ 0 & \text{if} \quad x\in[a,b]\setminus\left\{\frac{3a+b}{4},\frac{a+3b}{4}\right\}.\end{cases}$$

Then f is of bounded variation on $[a,b]$,

$$\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}=1,\quad \int_a^b f(t)\,dt=0$$

and

$$\bigvee_a^b(f)=4,$$

giving in (1.14) $4C\ge 1$, thus $C\ge\frac{1}{4}$.

This example can be used to prove the sharpness of the constant $\frac{1}{4}$ in (1.7) as well. ∎

2. A Companion of Ostrowski Inequality for Absolutely Continuous Functions

2.1. An Identity. The following identity holds.

LEMMA 2.1 (Dragomir, 2002 [**6**]). *Assume that* $f:[a,b]\to\mathbb{R}$ *is an absolutely continuous function on* $[a,b]$. *Then we have the equality*

$$\begin{aligned}(2.1)\qquad &\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\int_a^x (t-a)\,f'(t)\,dt+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)f'(t)\,dt\\ &+\frac{1}{b-a}\int_{a+b-x}^b (t-b)\,f'(t)\,dt,\end{aligned}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

PROOF. Using the integration by parts formula for Lebesgue integrals, we have

$$\int_a^x (t-a)\,f'(t)\,dt=f(x)(x-a)-\int_a^x f(t)\,dt,$$

$$\begin{aligned}&\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)f'(t)\,dt\\ &=f(a+b-x)\left(\frac{a+b}{2}-x\right)-f(x)\left(x-\frac{a+b}{2}\right)-\int_x^{a+b-x} f(t)\,dt\end{aligned}$$

and

$$\int_{a+b-x}^b (t-b)\,f'(t)\,dt=(x-a)\,f(a+b-x)-\int_{a+b-x}^b f(t)\,dt.$$

Summing the above equalities, we deduce the desired identity (2.1). ∎

REMARK 2.1. The identity (2.1) was obtained in [**13**, Lemma 3.2] for the case of piecewise continuously differentiable functions on $[a,b]$.

2.2. The case of Sup-Norm. The following result holds.

THEOREM 2.2 (Dragomir, 2002 [**6**]). *Let* $f:[a,b]\to\mathbb{R}$ *be an absolutely continuous function on* $[a,b]$. *Then we have the inequality*

$$\begin{aligned}(2.2)\qquad &\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ &\le\frac{1}{b-a}\left[\int_a^x (t-a)\,|f'(t)|\,dt+\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\,|f'(t)|\,dt\right.\\ &\qquad\left.+\int_{a+b-x}^b (b-t)\,|f'(t)|\,dt\right]\\ &:=M(x)\end{aligned}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

If $f' \in L_\infty [a,b]$, then we have the inequalities

$$M(x) \le \frac{1}{b-a}\left[\frac{(x-a)^2}{2}\|f'\|_{[a,x],\infty} + \left(\frac{a+b}{2}-x\right)^2 \|f'\|_{[x,a+b-x],\infty} + \frac{(x-a)^2}{2}\|f'\|_{[a+b-x,b],\infty}\right] \tag{2.3}$$

$$\le \begin{cases} \left[\frac{1}{8} + 2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f'\|_{[a,b],\infty} \\ \\ \left[\frac{1}{2^{\alpha-1}}\left(\frac{x-a}{b-a}\right)^{2\alpha} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{2\alpha}\right]^{\frac{1}{\alpha}} \\ \times\left[\|f'\|^{\beta}_{[a,x],\infty} + \|f'\|^{\beta}_{[x,a+b-x],\infty} + \|f'\|^{\beta}_{[a+b-x,b],\infty}\right]^{\frac{1}{\beta}}(b-a) \\ \hfill \text{if } \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \max\left\{\frac{1}{2}\left(\frac{x-a}{b-a}\right)^2, \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right\} \\ \quad \times\left[\|f'\|_{[a,x],\infty} + \|f'\|_{[x,a+b-x],\infty} + \|f'\|_{[a+b-x,b],\infty}\right](b-a) \end{cases}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

The inequality (2.2), the first inequality in (2.3) and the constant $\frac{1}{8}$ are sharp.

PROOF. The inequality (2.2) follows by Lemma 2.1 on taking the modulus and using it properties.

If $f' \in L_\infty [a,b]$, then

$$\int_a^x (t-a)|f'(t)|\,dt \le \frac{(x-a)^2}{2}\|f'\|_{[a,x],\infty},$$

$$\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right||f'(t)|\,dt \le \left(\frac{a+b}{2}-x\right)^2\|f'\|_{[x,a+b-x],\infty},$$

$$\int_{a+b-x}^b (b-t)|f'(t)|\,dt \le \frac{(x-a)^2}{2}\|f'\|_{[a+b-x,b],\infty}$$

and the first inequality in (2.3) is proved.

Denote

$$\tilde{M}(x) := \frac{(x-a)^2}{2}\|f'\|_{[a,x],\infty} + \left(\frac{a+b}{2}-x\right)^2\|f'\|_{[x,a+b-x],\infty} + \frac{(x-a)^2}{2}\|f'\|_{[a+b-x,b],\infty}$$

for $x \in \left[a, \frac{a+b}{2}\right]$.

Firstly, observe that

$$\begin{aligned}\tilde{M}(x) &\leq \max\left\{\|f'\|_{[a,x],\infty}, \|f'\|_{[x,a+b-x],\infty}, \|f'\|_{[a+b-x,b],\infty}\right\} \\ &\quad\times\left[\frac{(x-a)^2}{2}+\left(\frac{a+b}{2}-x\right)^2+\frac{(x-a)^2}{2}\right] \\ &= \|f'\|_{[a,b],\infty}\left[\frac{1}{8}(b-a)^2+2\left(x-\frac{3a+b}{4}\right)^2\right]\end{aligned}$$

and the first inequality in (2.3) is proved.

Using Hölder's inequality for $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, we also have

$$\begin{aligned}\tilde{M}(x) &\leq \left\{\left[\frac{(x-a)^2}{2}\right]^{\alpha}+\left(x-\frac{a+b}{2}\right)^{2\alpha}+\left[\frac{(x-a)^2}{2}\right]^{\alpha}\right\}^{\frac{1}{\alpha}} \\ &\quad\times\left[\|f'\|^{\beta}_{[a,x],\infty}+\|f'\|^{\beta}_{[x,a+b-x],\infty}+\|f'\|^{\beta}_{[a+b-x,b],\infty}\right]^{\frac{1}{\beta}}\end{aligned}$$

giving the second inequality in (2.3).

Finally, we also observe that

$$\begin{aligned}\tilde{M}(x) &\leq \max\left\{\frac{(x-a)^2}{2}, \left(x-\frac{a+b}{2}\right)^2\right\} \\ &\quad\times\left[\|f'\|_{[a,x],\infty}+\|f'\|_{[x,a+b-x],\infty}+\|f'\|_{[a+b-x,b],\infty}\right].\end{aligned}$$

The sharpness of the inequalities follows easily. The details are omitted. ∎

REMARK 2.2. If in Theorem 2.2 we choose $x = a$, then we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{4}(b-a)\|f'\|_{[a,b],\infty} \tag{2.4}$$

with $\frac{1}{4}$ as a sharp constant (see for example [**11**, p. 25]).

If in the same theorem we now choose $x = \frac{a+b}{2}$, then we get

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{8}(b-a)\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}+\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right] \\ &\leq \frac{1}{4}(b-a)\|f'\|_{[a,b],\infty}\end{aligned} \tag{2.5}$$

with the constants $\frac{1}{8}$ and $\frac{1}{4}$ being sharp. This result was obtained in [**4**].

It is natural to consider the following corollary.

COROLLARY 2.3. *With the assumptions in Theorem 2.2, one has the inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{8}(b-a)\|f'\|_{[a,b],\infty}. \tag{2.6}$$

The constant $\frac{1}{8}$ is best possible in the sense that it cannot be replaced by a smaller constant.

2.3. The Case of p-Norm. The case when $f' \in L_p[a,b]$, $p>1$ is embodied in the following theorem.

THEOREM 2.4 (Dragomir, 2002 [**6**]). *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ so that $f'\in L_p[a,b]$, $p>1$. If $M(x)$ is as defined in (2.2), then we have the bounds:*

$$\begin{aligned} M(x) &\le \frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{q}}\|f'\|_{[a,x],p}\right. \\ &\left. +2^{\frac{1}{q}}\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{1+\frac{1}{q}}\|f'\|_{[x,a+b-x],p}+\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{q}}\|f'\|_{[a+b-x,b],p}\right](b-a)^{\frac{1}{q}} \\ &\le \frac{1}{(q+1)^{\frac{1}{q}}}\begin{cases} \left[2\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{q}}+2^{\frac{1}{q}}\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{1+\frac{1}{q}}\right] \\ \times\max\left\{\|f'\|_{[a,x],p},\|f'\|_{[x,a+b-x],p},\|f'\|_{[a+b-x,b],p}\right\}(b-a)^{\frac{1}{q}} \\ \\ \left[2\left(\frac{x-a}{b-a}\right)^{\alpha+\frac{\alpha}{q}}+2^{\frac{\alpha}{q}}\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{\alpha+\frac{\alpha}{q}}\right]^{\frac{1}{\alpha}} \\ \times\left[\|f'\|^{\beta}_{[a,x],p}+\|f'\|^{\beta}_{[x,a+b-x],p}+\|f'\|^{\beta}_{[a+b-x,b],p}\right]^{\frac{1}{\beta}}(b-a)^{\frac{1}{q}} \\ \qquad\qquad \textit{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \max\left\{\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{q}},2^{\frac{1}{q}}\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{1+\frac{1}{q}}\right\} \\ \quad\times\left[\|f'\|_{[a,x],p}+\|f'\|_{[x,a+b-x],p}+\|f'\|_{[a+b-x,b],p}\right](b-a)^{\frac{1}{q}} \end{cases} \end{aligned} \tag{2.7}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

PROOF. Using Hölder's integral inequality for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\int_a^x (t-a)|f'(t)|\,dt \le \left(\int_a^x (t-a)^q\,dt\right)^{\frac{1}{q}}\|f'\|_{[a,x],p} = \frac{(x-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[a,x],p},$$

$$\begin{aligned}\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right||f'(t)|\,dt &\le \left(\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}}\|f'\|_{[x,a+b-x],p} \\ &= \frac{2^{\frac{1}{q}}\left(\frac{a+b}{2}-x\right)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[x,a+b-x],p}\end{aligned}$$

and

$$\begin{aligned}\int_{a+b-x}^{b}(b-t)|f'(t)|\,dt &\le \left(\int_{a+b-x}^{b}(b-t)^q\,dt\right)^{\frac{1}{q}}\|f'\|_{[a+b-x,b],p} \\ &= \frac{(x-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[a+b-x,b],p}.\end{aligned}$$

Summing the above inequalities, we deduce the first bound in (2.7).

The last part may be proved in a similar fashion to the one in Theorem 2.2, and we omit the details. ∎

REMARK 2.3. If in (2.7) we choose $\alpha = q$, $\beta = p$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then we get the inequality

$$M(x) \le \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} + \left(\frac{\frac{a+b}{2} - x}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} \|f'\|_{[a,b],p} \tag{2.8}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

REMARK 2.4. If in Theorem 2.4 we choose $x = a$, then we get the trapezoid inequality

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \frac{1}{2} \cdot \frac{(b-a)^{\frac{1}{q}} \|f'\|_{[a,b],p}}{(q+1)^{\frac{1}{q}}}, \tag{2.9}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Indeed, if we assume that (2.9) holds with a constant $C > 0$, instead of $\frac{1}{2}$, i.e.,

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le C \cdot \frac{(b-a)^{\frac{1}{q}} \|f'\|_{[a,b],p}}{(q+1)^{\frac{1}{q}}}, \tag{2.10}$$

then for the function $f : [a,b] \to \mathbb{R}$, $f(x) = k \left| x - \frac{a+b}{2} \right|$, $k > 0$, we have

$$\begin{aligned} \frac{f(a) + f(b)}{2} &= k \cdot \frac{b-a}{2}, \\ \frac{1}{b-a} \int_a^b f(t)\, dt &= k \cdot \frac{b-a}{4}, \\ \|f'\|_{[a,b],p} &= k (b-a)^{\frac{1}{p}}; \end{aligned}$$

and by (2.10) we deduce

$$\left| \frac{k(b-a)}{2} - \frac{k(b-a)}{4} \right| \le \frac{C \cdot k(b-a)}{(q+1)^{\frac{1}{q}}},$$

giving $C \ge \frac{(q+1)^{\frac{1}{q}}}{4}$. Letting $q \to 1+$, we deduce $C \ge \frac{1}{2}$, and the sharpness of the constant is proved.

REMARK 2.5. If in Theorem 2.4 we choose $x = \frac{a+b}{2}$, then we get the midpoint inequality

$$\begin{aligned} &\left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\le \frac{1}{2} \cdot \frac{(b-a)^{\frac{1}{q}}}{2^{\frac{1}{q}} (q+1)^{\frac{1}{q}}} \left[\|f'\|_{\left[a, \frac{a+b}{2}\right],p} + \|f'\|_{\left[\frac{a+b}{2}, b\right],p} \right] \\ &\le \frac{1}{2} \cdot \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|f'\|_{[a,b],p}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1. \end{aligned} \tag{2.11}$$

In both inequalities the constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

To show this fact, assume that (2.11) holds with $C, D > 0$, i.e.,

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.12}$$
$$\le C\cdot\frac{(b-a)^{\frac{1}{q}}}{2^{\frac{1}{q}}(q+1)^{\frac{1}{q}}}\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],p}+\|f'\|_{\left[\frac{a+b}{2},b\right],p}\right]$$
$$\le D\cdot\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[a,b],p}.$$

For the function $f:[a,b]\to\mathbb{R}$, $f(x)=k\left|x-\frac{a+b}{2}\right|$, $k>0$, we have

$$f\left(\frac{a+b}{2}\right)=0,\quad \frac{1}{b-a}\int_a^b f(t)\,dt=\frac{k(b-a)}{4},$$

$$\|f'\|_{\left[a,\frac{a+b}{2}\right],p}+\|f'\|_{\left[\frac{a+b}{2},b\right],p}=2\left(\frac{b-a}{2}\right)^{\frac{1}{p}}k=2^{\frac{1}{q}}(b-a)^{\frac{1}{p}}k,$$

$$\|f'\|_{[a,b],p}=(b-a)^{\frac{1}{p}}k;$$

and then by (2.12) we deduce

$$\frac{k(b-a)}{4}\le C\cdot\frac{k(b-a)}{(q+1)^{\frac{1}{q}}}\le D\cdot\frac{k(b-a)}{(q+1)^{\frac{1}{q}}},$$

giving $C, D\ge\frac{(q+1)^{\frac{1}{q}}}{4}$ for any $q>1$. Letting $q\to 1+$, we deduce $C, D\ge\frac{1}{2}$ and the sharpness of the constants in (2.11) are proved.

The following result is useful in providing the best quadrature rule in the class for approximating the integral of an absolutely continuous function whose derivative is in $L_p[a,b]$.

COROLLARY 2.5. *Assume that $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function so that $f'\in L_p[a,b]$, $p>1$. Then one has the inequality*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{4}\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[a,b],p}, \tag{2.13}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

PROOF. The inequality follows by Theorem 2.4 and Remark 2.3 on choosing $x=\frac{3a+b}{4}$.

To prove the sharpness of the constant, assume that (2.13) holds with a constant $E>0$, i.e.,

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le E\cdot\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[a,b],p}. \tag{2.14}$$

Consider the function $f:[a,b]\to\mathbb{R}$,

$$f(x)=\begin{cases}\left|x-\dfrac{3a+b}{4}\right| & \text{if}\quad x\in\left[a,\frac{a+b}{2}\right]\\[2ex] \left|x-\dfrac{a+3b}{4}\right| & \text{if}\quad x\in\left(\frac{a+b}{2},b\right].\end{cases}$$

Then f is absolutely continuous and $f' \in L_p[a,b]$, $p>1$. We also have

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]=0,\quad \frac{1}{b-a}\int_a^b f(t)\,dt=\frac{b-a}{8}$$

$$\|f'\|_{[a,b],p}=(b-a)^{\frac{1}{p}},$$

and then, by (2.14), we obtain:

$$\frac{b-a}{8}\le E\frac{(b-a)}{(q+1)^{\frac{1}{q}}}$$

giving $E\ge\frac{(q+1)^{\frac{1}{q}}}{8}$ for any $q>1$, i.e., $E\ge\frac{1}{4}$, and the corollary is proved. ∎

2.4. The Case of 1-Norm. If one is interested in obtaining bounds in terms of the 1-norm for the derivative, then the following result may be useful.

THEOREM 2.6 (Dragomir, 2002 [**6**]). *Assume that the function $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. If $M(x)$ is as in equation (2.2), then we have the bounds*

$$M(x) \tag{2.15}$$
$$\le\left(\frac{x-a}{b-a}\right)\|f'\|_{[a,x],1}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)\|f'\|_{[x,a+b-x],1}+\left(\frac{x-a}{b-a}\right)\|f'\|_{[a+b-x,b],1}$$
$$\le\begin{cases}\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\|f'\|_{[a,b],1}\\[2ex] \left[2\left(\frac{x-a}{b-a}\right)^{\alpha}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{\alpha}\right]^{\frac{1}{\alpha}}\\ \times\left[\|f'\|^{\beta}_{[a,x],1}+\|f'\|^{\beta}_{[x,a+b-x],1}+\|f'\|^{\beta}_{[a+b-x,b],1}\right]^{\frac{1}{\beta}}\\ \qquad\qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\\[2ex] \left[\frac{x+\frac{b-3a}{2}}{b-a}\right]\max\left[\|f'\|_{[a,x],1},\|f'\|_{[x,a+b-x],1},\|f'\|_{[a+b-x,b],1}\right].\end{cases}$$

The proof is as in Theorem 2.2 and we omit it.

REMARK 2.6. By the use of Theorem 2.4, for $x=a$, we get the trapezoid inequality (see for example [**11**, p. 55])

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{2}\|f'\|_{[a,b],1}. \tag{2.16}$$

If in (2.15) we also choose $x=\frac{a+b}{2}$, then we get the mid point inequality (see for example [**11**, p. 56])

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{2}\|f'\|_{[a,b],1}. \tag{2.17}$$

The following corollary also holds.

COROLLARY 2.7. *With the assumption in Theorem 2.4, one has the inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{4}\left\|f'\right\|_{[a,b],1}. \tag{2.18}$$

3. PERTURBED COMPANIONS OF OSTROWSKI INEQUALITY FOR FUNCTIONS OF BOUNDED VARIATION

3.1. Some Identities. The following identity holds.

LEMMA 3.1 (Dragomir, 2014 [**9**]). *Assume that $f:[a,b]\to\mathbb{C}$ is a function of bounded variation on $[a,b]$. Then we have the equality*

$$\begin{aligned} &\frac{1}{2}[f(x)+f(a+b-x)]+\frac{1}{2}(x-a)^2\frac{\lambda_3(x)-\lambda_1(x)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &=\frac{1}{b-a}\int_a^x (t-a)\,d[f(t)-\lambda_1(x)t] \\ &+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)d[f(t)-\lambda_2(x)t] \\ &+\frac{1}{b-a}\int_{a+b-x}^b (t-b)\,d[f(t)-\lambda_3(x)t], \end{aligned} \tag{3.1}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ and $\lambda_i(x), i=1,2,3$ complex numbers.

PROOF. Using the integration by parts formula for Riemann-Stieltjes integrals, we have

$$\begin{aligned} &\int_a^x (t-a)\,d[f(t)-\lambda_1(x)t] \\ &=\int_a^x (t-a)\,df(t)-\lambda_1(x)\int_a^x (t-a)\,dt \\ &=(x-a)f(x)-\int_a^x f(t)\,dt-\frac{1}{2}\lambda_1(x)(x-a)^2, \end{aligned}$$

$$\begin{aligned} &\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)d[f(t)-\lambda_2(x)t] \\ &=\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)df(t)-\lambda_2(x)\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)dt \\ &=f(a+b-x)\left(\frac{a+b}{2}-x\right)-f(x)\left(x-\frac{a+b}{2}\right) \\ &-\int_x^{a+b-x}f(t)\,dt-\lambda_2(x)\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)dt \\ &=f(a+b-x)\left(\frac{a+b}{2}-x\right)-f(x)\left(x-\frac{a+b}{2}\right)-\int_x^{a+b-x}f(t)\,dt \end{aligned}$$

since, by symmetry

$$\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)dt=0$$

and

$$\begin{aligned}
&\int_{a+b-x}^{b} (t-b)\, d\left[f(t)-\lambda_3(x)\, t\right] \\
&= \int_{a+b-x}^{b} (t-b)\, df(t) - \lambda_3(x) \int_{a+b-x}^{b} (t-b)\, dt \\
&= (x-a)\, f(a+b-x) - \int_{a+b-x}^{b} f(t)\, dt + \frac{1}{2}\lambda_3(x)\,(x-a)^2 .
\end{aligned}$$

Summing the above equalities, we deduce

$$\begin{aligned}
&\int_{a}^{x} (t-a)\, d\left[f(t)-\lambda_1(x)\, t\right] + \int_{x}^{a+b-x} \left(t-\frac{a+b}{2}\right) d\left[f(t)-\lambda_2(x)\, t\right] \\
&+ \int_{a+b-x}^{b} (t-b)\, d\left[f(t)-\lambda_3(x)\, t\right] \\
&= (b-a)\, \frac{f(x)+f(a+b-x)}{2} - \int_{a}^{b} f(t)\, dt + \frac{1}{2}\left[\lambda_3(x)-\lambda_1(x)\right](x-a)^2 ,
\end{aligned}$$

which is equivalent with the desired identity (3.1). ∎

The following particular cases are of interest:

COROLLARY 3.2 (Dragomir, 2014 [**9**]). *With the assumption of Lemma 3.1 we have the equalities*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_{a}^{b} f(t)\, dt = \frac{1}{b-a}\int_{a}^{b} \left(t-\frac{a+b}{2}\right) d\left[f(t)-\lambda_2 t\right], \tag{3.2}$$

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)(\lambda_3-\lambda_1) - \frac{1}{b-a}\int_{a}^{b} f(t)\, dt \\
&= \frac{1}{b-a}\int_{a}^{\frac{a+b}{2}} (t-a)\, d\left[f(t)-\lambda_1 t\right] + \frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} (t-b)\, d\left[f(t)-\lambda_3 t\right],
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] + \frac{1}{32}(b-a)(\lambda_3-\lambda_1) - \frac{1}{b-a}\int_{a}^{b} f(t)\, dt \\
&= \frac{1}{b-a}\int_{a}^{\frac{3a+b}{4}} (t-a)\, d\left[f(t)-\lambda_1 t\right] \\
&+ \frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \left(t-\frac{a+b}{2}\right) d\left[f(t)-\lambda_2 t\right] \\
&+ \frac{1}{b-a}\int_{\frac{a+3b}{4}}^{b} (t-b)\, d\left[f(t)-\lambda_3 t\right],
\end{aligned} \tag{3.4}$$

for any $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$.

The following particular result with no parameter in the left hand term holds:

COROLLARY 3.3 (Dragomir, 2014 [**9**]). *Assume that $f : [a, b] \to \mathbb{C}$ is a function of bounded variation on $[a, b]$. Then we have the equality*

$$\begin{aligned}(3.5)\qquad & \frac{1}{2}[f(x) + f(a+b-x)] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^x (t-a)\,d[f(t) - \lambda_1(x)t] \\ &+ \frac{1}{b-a}\int_x^{a+b-x}\left(t - \frac{a+b}{2}\right)d[f(t) - \lambda_2(x)t] \\ &+ \frac{1}{b-a}\int_{a+b-x}^b (t-b)\,d[f(t) - \lambda_1(x)t],\end{aligned}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$ and $\lambda_i(x), i = 1, 2$ complex numbers.

REMARK 3.1. We get from (3.3) the following particular case:

$$\begin{aligned}(3.6)\qquad & f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\,d[f(t) - \lambda_1 t] + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\,d[f(t) - \lambda_1 t],\end{aligned}$$

for any $\lambda_1 \in \mathbb{C}$, while from (3.4) we get

$$\begin{aligned}(3.7)\qquad & \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^{\frac{3a+b}{4}} (t-a)\,d[f(t) - \lambda_1 t] \\ &+ \frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \left(t - \frac{a+b}{2}\right)d[f(t) - \lambda_2 t] \\ &+ \frac{1}{b-a}\int_{\frac{a+3b}{4}}^b (t-b)\,d[f(t) - \lambda_1 t],\end{aligned}$$

for any $\lambda_1, \lambda_2 \in \mathbb{C}$.

3.2. Inequalities for Functions of Bounded Variation. The following lemma will be used in the sequel and is of interest in itself as well [**1**, p. 177]. For a simple proof see [**7**].

LEMMA 3.4. *Let $f, u : [a, b] \to \mathbb{C}$. If f is continuous on $[a, b]$ and u is of bounded variation on $[a, b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and*

$$(3.8)\qquad \left|\int_a^b f(t)\,du(t)\right| \le \int_a^b |f(t)|\,d\left(\bigvee_a^t (u)\right) \le \max_{t\in[a,b]} |f(t)| \bigvee_a^b (u).$$

We denote by $\ell : [a, b] \to [a, b]$ the *identity function*, namely $\ell(t) = t$ for any $t \in [a, b]$. We have the following result:

THEOREM 3.5 (Dragomir, 2014 [**9**]). *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then we have the inequalities*

$$\begin{aligned}(3.9)\quad &\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^2\frac{\lambda_3(x)-\lambda_1(x)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\
&\leq\frac{1}{b-a}\left[\int_a^x\left(\bigvee_t^x(f-\lambda_1(x)\ell)\right)dt+\int_x^{\frac{a+b}{2}}\left(\bigvee_x^t(f-\lambda_2(x)\ell)\right)dt\right.\\
&\left.+\int_{\frac{a+b}{2}}^{a+b-x}\left(\bigvee_t^{a+b-x}(f-\lambda_2(x)\ell)\right)dt+\int_{a+b-x}^b\left(\bigvee_{a+b-x}^t(f-\lambda_3(x)\ell)\right)dt\right]\\
&\leq\frac{1}{b-a}\left[(x-a)\bigvee_a^x(f-\lambda_1(x)\ell)+\left(\frac{a+b}{2}-x\right)\bigvee_x^{a+b-x}(f-\lambda_2(x)\ell)\right.\\
&\left.+(x-a)\bigvee_{a+b-x}^b(f-\lambda_3(x)\ell)\right]\end{aligned}$$

$$\leq\begin{cases}\left[\dfrac{1}{4}+\left|\dfrac{x-\frac{3a+b}{4}}{b-a}\right|\right]\\ \times\left[\bigvee_a^x(f-\lambda_1(x)\ell)+\bigvee_x^{a+b-x}(f-\lambda_2(x)\ell)+\bigvee_{a+b-x}^b(f-\lambda_3(x)\ell)\right]\\[2ex] \dfrac{x+\frac{b-3a}{2}}{b-a}\\ \times\max\left\{\bigvee_a^x(f-\lambda_1(x)\ell),\bigvee_x^{a+b-x}(f-\lambda_2(x)\ell),\bigvee_{a+b-x}^b(f-\lambda_3(x)\ell)\right\}\end{cases}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ and $\lambda_i(x), i=1,2,3$ complex numbers.

PROOF. Taking the modulus on (3.1) and making use of (3.8), we have

$$\begin{aligned}(3.10)\quad &\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^2\frac{\lambda_3(x)-\lambda_1(x)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\
&\leq\frac{1}{b-a}\int_a^x(t-a)\,d\left(\bigvee_a^t(f-\lambda_1(x)\ell)\right)\\
&+\frac{1}{b-a}\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|d\left(\bigvee_a^t(f-\lambda_2(x)\ell)\right)\\
&+\frac{1}{b-a}\int_{a+b-x}^b(b-t)\,d\left(\bigvee_a^t(f-\lambda_3(x)\ell)\right).\end{aligned}$$

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^x (t-a)\,d\left(\bigvee_a^t (f-\lambda_1(x)\ell)\right)\\
&= (t-a)\bigvee_a^t (f-\lambda_1(x)\ell)\Big|_a^x - \int_a^x \bigvee_a^t (f-\lambda_1(x)\ell)\,dt\\
&= (x-a)\bigvee_a^x (f-\lambda_1(x)\ell) - \int_a^x \bigvee_a^t (f-\lambda_1(x)\ell)\,dt\\
&= \int_a^x \left(\bigvee_t^x (f-\lambda_1(x)\ell)\right) dt.
\end{aligned}$$

Also

$$\begin{aligned}
&\int_x^{a+b-x} \left|t-\frac{a+b}{2}\right| d\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)\\
&= \int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right) d\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)\\
&+ \int_{\frac{a+b}{2}}^{a+b-x} \left(t-\frac{a+b}{2}\right) d\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)\\
&= \left(\frac{a+b}{2}-t\right)\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)\Bigg|_x^{\frac{a+b}{2}} + \int_x^{\frac{a+b}{2}} \left(\bigvee_a^t (f-\lambda_2(x)\ell)\right) dt\\
&+ \left(t-\frac{a+b}{2}\right)\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)\Bigg|_{\frac{a+b}{2}}^{a+b-x} - \int_{\frac{a+b}{2}}^{a+b-x} \left(\bigvee_a^t (f-\lambda_2(x)\ell)\right) dt
\end{aligned}$$

$$\begin{aligned}
&= \int_x^{\frac{a+b}{2}} \left(\bigvee_a^t (f-\lambda_2(x)\ell)\right) dt - \left(\frac{a+b}{2}-x\right)\left(\bigvee_a^x (f-\lambda_2(x)\ell)\right)\\
&+ \left(\frac{a+b}{2}-x\right)\left(\bigvee_a^{a+b-x} (f-\lambda_2(x)\ell)\right) - \int_{\frac{a+b}{2}}^{a+b-x} \left(\bigvee_a^t (f-\lambda_2(x)\ell)\right) dt\\
&= \int_{\frac{a+b}{2}}^{a+b-x} \left(\bigvee_t^{a+b-x} (f-\lambda_2(x)\ell)\right) dt + \int_x^{\frac{a+b}{2}} \left(\bigvee_x^t (f-\lambda_2(x)\ell)\right) dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_{a+b-x}^{b} (b-t)\, d\left(\bigvee_{a}^{t} (f-\lambda_3(x)\,\ell)\right) \\
&= (b-t)\left(\bigvee_{a}^{t} (f-\lambda_3(x)\,\ell)\right)\Bigg|_{a+b-x}^{b} + \int_{a+b-x}^{b} \left(\bigvee_{a}^{t} (f-\lambda_3(x)\,\ell)\right) dt \\
&= \int_{a+b-x}^{b} \left(\bigvee_{a}^{t} (f-\lambda_3(x)\,\ell)\right) dt - (b-(a+b-x))\left(\bigvee_{a}^{a+b-x} (f-\lambda_3(x)\,\ell)\right) \\
&= \int_{a+b-x}^{b} \left(\bigvee_{a+b-x}^{t} (f-\lambda_3(x)\,\ell)\right) dt.
\end{aligned}$$

Making use of (3.10) we deduce the first inequality in (3.9).

Since

$$\int_{a}^{x} \left(\bigvee_{t}^{x} (f-\lambda_1(x)\,\ell)\right) dt \le (x-a) \bigvee_{a}^{x} (f-\lambda_1(x)\,\ell),$$

$$\begin{aligned}
&\int_{\frac{a+b}{2}}^{a+b-x} \left(\bigvee_{t}^{a+b-x} (f-\lambda_2(x)\,\ell)\right) dt + \int_{x}^{\frac{a+b}{2}} \left(\bigvee_{x}^{t} (f-\lambda_2(x)\,\ell)\right) dt \\
&\le \left(\frac{a+b}{2}-x\right) \bigvee_{\frac{a+b}{2}}^{a+b-x} (f-\lambda_2(x)\,\ell) + \left(\frac{a+b}{2}-x\right) \bigvee_{x}^{\frac{a+b}{2}} (f-\lambda_2(x)\,\ell) \\
&= \left(\frac{a+b}{2}-x\right) \bigvee_{x}^{a+b-x} (f-\lambda_2(x)\,\ell)
\end{aligned}$$

and

$$\int_{a+b-x}^{b} \left(\bigvee_{a+b-x}^{t} (f-\lambda_3(x)\,\ell)\right) dt \le (x-a) \bigvee_{a+b-x}^{b} (f-\lambda_3(x)\,\ell),$$

the second inequality is also proved.

The last inequality is obvious by the maximum properties. ∎

The following midpoint and trapezoid type inequalities hold:

COROLLARY 3.6. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then we have the inequalities*

$$\begin{aligned}
(3.11)\qquad &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\le \frac{1}{b-a}\left[\int_{a}^{\frac{a+b}{2}} \left(\bigvee_{a}^{t} (f-\lambda_2\ell)\right) dt + \int_{\frac{a+b}{2}}^{b} \left(\bigvee_{t}^{b} (f-\lambda_2\ell)\right) dt\right] \\
&\le \frac{1}{2}\bigvee_{a}^{b} (f-\lambda_2\ell),
\end{aligned}$$

$$\left|f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.12}$$
$$\leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(f-\lambda_1\ell)\right)dt+\int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(f-\lambda_3\ell)\right)dt\right]$$
$$\leq \frac{1}{2}\left[\bigvee_a^{\frac{a+b}{2}}(f-\lambda_1\ell)+\bigvee_{\frac{a+b}{2}}^b(f-\lambda_3\ell)\right]$$

and

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]+\frac{1}{32}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.13}$$
$$\leq \frac{1}{b-a}\left[\int_a^{\frac{3a+b}{4}}\left(\bigvee_t^{\frac{3a+b}{4}}(f-\lambda_1\ell)\right)dt+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left(\bigvee_{\frac{3a+b}{4}}^t(f-\lambda_2\ell)\right)dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left(\bigvee_t^{\frac{a+3b}{4}}(f-\lambda_2\ell)\right)dt+\int_{\frac{a+3b}{4}}^b\left(\bigvee_{\frac{a+3b}{4}}^t(f-\lambda_3\ell)\right)dt\right]$$
$$\leq \frac{1}{4}\left[\bigvee_a^{\frac{3a+b}{4}}(f-\lambda_1\ell)+\bigvee_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}(f-\lambda_2\ell)+\bigvee_{\frac{a+3b}{4}}^b(f-\lambda_3\ell)\right]$$

for any $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$.

COROLLARY 3.7. *Assume that* $f:[a,b]\to\mathbb{C}$ *is a function of bounded variation on* $[a,b]$. *Then we have the inequalities*

$$\left|\frac{1}{2}[f(x)+f(a+b-x)]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.14}$$
$$\leq \frac{1}{b-a}\left[\int_a^x\left(\bigvee_t^x(f-\lambda_1(x)\ell)\right)dt+\int_x^{\frac{a+b}{2}}\left(\bigvee_x^t(f-\lambda_2(x)\ell)\right)dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^{a+b-x}\left(\bigvee_t^{a+b-x}(f-\lambda_2(x)\ell)\right)dt+\int_{a+b-x}^b\left(\bigvee_{a+b-x}^t(f-\lambda_1(x)\ell)\right)dt\right]$$
$$\leq \frac{1}{b-a}\left[(x-a)\bigvee_a^x(f-\lambda_1(x)\ell)+\left(\frac{a+b}{2}-x\right)\bigvee_x^{a+b-x}(f-\lambda_2(x)\ell)\right.$$
$$\left.+(x-a)\bigvee_{a+b-x}^b(f-\lambda_1(x)\ell)\right]$$

$$\leq \begin{cases} \left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right] \\ \times\left[\bigvee_a^x\left(f-\lambda_1(x)\ell\right)+\bigvee_x^{a+b-x}\left(f-\lambda_2(x)\ell\right)+\bigvee_{a+b-x}^b\left(f-\lambda_1(x)\ell\right)\right] \\ \\ \frac{x+\frac{b-3a}{2}}{b-a} \\ \times\max\left\{\bigvee_a^x\left(f-\lambda_1(x)\ell\right),\bigvee_x^{a+b-x}\left(f-\lambda_2(x)\ell\right),\bigvee_{a+b-x}^b\left(f-\lambda_1(x)\ell\right)\right\} \end{cases}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ *and* $\lambda_i(x), i=1,2,$ *complex numbers.*

REMARK 3.2. We have the particular inequalities of interest

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(f-\lambda_1\ell)\right)dt+\int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(f-\lambda_1\ell)\right)dt\right] \leq \frac{1}{2}\bigvee_a^b(f-\lambda_1\ell) \tag{3.15}$$

and

$$\begin{aligned} &\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{b-a}\left[\int_a^{\frac{3a+b}{4}}\left(\bigvee_t^{\frac{3a+b}{4}}(f-\lambda_1\ell)\right)dt+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left(\bigvee_{\frac{3a+b}{4}}^t(f-\lambda_2\ell)\right)dt\right. \\ &\left.+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left(\bigvee_t^{\frac{a+3b}{4}}(f-\lambda_2\ell)\right)dt+\int_{\frac{a+3b}{4}}^b\left(\bigvee_{\frac{a+3b}{4}}^t(f-\lambda_1\ell)\right)dt\right] \\ &\leq \frac{1}{4}\left[\bigvee_a^{\frac{3a+b}{4}}(f-\lambda_1\ell)+\bigvee_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}(f-\lambda_2\ell)+\bigvee_{\frac{a+3b}{4}}^b(f-\lambda_1\ell)\right] \end{aligned} \tag{3.16}$$

for any $\lambda_1,\lambda_2\in\mathbb{C}$.

If we take $\lambda_1 = \lambda_2 = \lambda$, then we get

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.17}$$
$$\leq \frac{1}{b-a}\left[\int_a^{\frac{3a+b}{4}}\left(\bigvee_t^{\frac{3a+b}{4}}(f-\lambda\ell)\right)dt+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left(\bigvee_{\frac{3a+b}{4}}^{t}(f-\lambda\ell)\right)dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left(\bigvee_t^{\frac{a+3b}{4}}(f-\lambda\ell)\right)dt+\int_{\frac{a+3b}{4}}^{b}\left(\bigvee_{\frac{a+3b}{4}}^{t}(f-\lambda\ell)\right)dt\right]$$
$$\leq \frac{1}{4}\left[\bigvee_a^b(f-\lambda\ell)\right]$$

for any $\lambda_1, \lambda_2 \in \mathbb{C}$.

From (3.15) we deduce the simpler inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.18}$$
$$\leq \frac{1}{b-a}\int_a^b\left|\bigvee_t^{\frac{a+b}{2}}(f-\lambda\ell)\right|dt \leq \frac{1}{2}\bigvee_a^b(f-\lambda\ell)$$

while from (3.17) we get

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.19}$$
$$\leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}\left|\bigvee_t^{\frac{3a+b}{4}}(f-\lambda\ell)\right|dt+\int_{\frac{a+b}{2}}^{b}\left(\left|\bigvee_t^{\frac{a+3b}{4}}(f-\lambda\ell)\right|\right)dt\right]$$
$$\leq \frac{1}{4}\bigvee_a^b(f-\lambda\ell)$$

for any $\lambda \in \mathbb{C}$.

We can state the following result.

PROPOSITION 3.8 (Dragomir, 2014 [**9**]). *Assume that $f : [a,b] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$. If there exists the constants $\gamma, \Gamma \in \mathbb{C}$ such that*

$$\bigvee_a^b\left(f-\frac{\gamma+\Gamma}{2}\ell\right) \leq \frac{1}{2}|\Gamma-\gamma|,$$

then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{4}|\Gamma-\gamma|$$

and

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{8}|\Gamma-\gamma|.$$

The inequalities follow by (3.18) and (3.19).

PROPOSITION 3.9 (Dragomir, 2014 [**9**]). *Assume that* $f : [a,b] \to \mathbb{C}$ *is a function of bounded variation on* $[a,b]$. *If for some* $\lambda \in \mathbb{C}$ *the cumulative variation function* $V_\lambda : [a,b] \to [0,\infty)$,

$$V_\lambda(t) := \bigvee_a^t (f - \lambda\ell)$$

is Lipschitzian with the constant $L_\gamma > 0$, *then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{4} L_\gamma (b-a) \tag{3.20}$$

and

$$\left| \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{8} L_\gamma (b-a). \tag{3.21}$$

PROOF. From (3.18) we have

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\le \frac{1}{b-a}\int_a^b \left| \bigvee_t^{\frac{a+b}{2}} (f-\lambda\ell) \right| dt = \frac{1}{b-a}\int_a^b \left| V_\lambda\left(\frac{a+b}{2}\right) - V_\lambda(t) \right| dt \\
&\le \frac{L_\gamma}{b-a}\int_a^b \left| \frac{a+b}{2} - t \right| dt = \frac{1}{4} L_\gamma (b-a)
\end{aligned}$$

and the inequality (3.20) is proved.

From (3.19) we have

$$\begin{aligned}
&\left| \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} \left| \bigvee_t^{\frac{3a+b}{4}} (f-\lambda\ell) \right| dt + \int_{\frac{a+b}{2}}^b \left(\left| \bigvee_t^{\frac{a+3b}{4}} (f-\lambda\ell) \right| \right) dt \right] \\
&= \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} \left| V_\lambda\left(\frac{3a+b}{4}\right) - V_\lambda(t) \right| dt \right. \\
&\left. + \int_{\frac{a+b}{2}}^b \left(\left| V_\lambda\left(\frac{a+3b}{4}\right) - V_\lambda(t) \right| \right) dt \right] \\
&\le \frac{L_\gamma}{b-a}\left[\int_a^{\frac{a+b}{2}} \left| \frac{3a+b}{4} - t \right| dt + \int_{\frac{a+b}{2}}^b \left(\left| \frac{a+3b}{4} - t \right| \right) dt \right] \\
&= \frac{L_\gamma}{b-a}\left[\int_a^{\frac{a+b}{2}} \left| \frac{3a+b}{4} - t \right| dt + \int_{\frac{a+b}{2}}^b \left(\left| \frac{a+3b}{4} - t \right| \right) dt \right] \\
&= \frac{L_\gamma}{b-a}\left[\frac{1}{16}(b-a)^2 + \frac{1}{16}(b-a)^2 \right] = \frac{L_\gamma}{8}(b-a)
\end{aligned}$$

and the inequality (3.21) is proved. ∎

3.3. Inequalities for Lipschitzian Functions. We say that a function $f:[c,d]\to\mathbb{C}$ is *Lipschitzian* with a constant $L>0$ on the interval $[c,d]$ if

$$|f(t)-f(s)|\le L|t-s|$$

for any $t,s\in[c,d]$.

THEOREM 3.10 (Dragomir, 2014 [**9**]). *Let $f:[a,b]\to\mathbb{C}$ be a bounded function on $[a,b]$. For $x\in\left[a,\frac{a+b}{2}\right]$ and $\lambda_i(x), i=1,2,3$ complex numbers, assume that $f-\lambda_1(x)\ell$ is Lipschitzian with the constant $L_1(x)>0$ on $[a,x]$, $f-\lambda_2(x)\ell$ with the constant $L_2(x)>0$ on $[x,a+b-x]$ and $f-\lambda_3(x)\ell$ with the constant $L_3(x)>0$ on $[a+b-x,b]$, then*

$$\begin{aligned}
&\left|\frac{1}{2}[f(x)+f(a+b-x)]+\frac{1}{2}(x-a)^2\frac{\lambda_3(x)-\lambda_1(x)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\le\frac{1}{(b-a)}\left[\frac{1}{2}(x-a)^2L_1(x)+\left(x-\frac{a+b}{2}\right)^2L_2(x)+\frac{1}{2}(x-a)^2L_3(x)\right] \\
&\le\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\max\{L_1(x),L_2(x),L_3(x)\}.
\end{aligned}\tag{3.22}$$

PROOF. It is known that if $g:[c,d]\to\mathbb{C}$ is Riemann integrable on $[c,d]$ and $u:[c,d]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and we have the inequality

$$\left|\int_a^b f(t)\,du(t)\right|\le L\int_a^b|f(t)|\,dt.\tag{3.23}$$

Taking the modulus in (3.1) and using the property (3.23) we have

$$\begin{aligned}
&\left|\frac{1}{2}[f(x)+f(a+b-x)]+\frac{1}{2}(x-a)^2\frac{\lambda_3(x)-\lambda_1(x)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\le\frac{1}{b-a}\left|\int_a^x(t-a)\,d[f(t)-\lambda_1(x)t]\right| \\
&+\frac{1}{b-a}\left|\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)d[f(t)-\lambda_2(x)t]\right| \\
&+\frac{1}{b-a}\left|\int_{a+b-x}^b(t-b)\,d[f(t)-\lambda_3(x)t]\right| \\
&\le\frac{1}{b-a}L_1(x)\int_a^x(t-a)\,dt+\frac{1}{b-a}L_2(x)\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|dt \\
&+\frac{1}{b-a}L_3(x)\int_{a+b-x}^b(b-t)\,dt \\
&=\frac{1}{(b-a)}\left[\frac{1}{2}(x-a)^2L_1(x)+\left(x-\frac{a+b}{2}\right)^2L_2(x)+\frac{1}{2}(x-a)^2L_3(x)\right],
\end{aligned}$$

which proves the first inequality in (3.22).

Since

$$\begin{aligned}
&\frac{1}{2}(x-a)^2 L_1(x) + \left(x-\frac{a+b}{2}\right)^2 L_2(x) + \frac{1}{2}(x-a)^2 L_3(x)\\
&\le \left[\frac{1}{2}(x-a)^2 + \left(x-\frac{a+b}{2}\right)^2 + \frac{1}{2}(x-a)^2\right]\max\{L_1(x), L_2(x), L_3(x)\}\\
&= \left[\frac{1}{8} + 2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\max\{L_1(x), L_2(x), L_3(x)\},
\end{aligned}$$

the last part of (3.22) is also proved. ∎

COROLLARY 3.11 (Dragomir, 2014 [**9**]). *Let $f:[a,b]\to\mathbb{C}$ be a bounded function on $[a,b]$.*

(i) If for $\lambda_2\in\mathbb{C}$ the function $f-\lambda_2\ell$ is Lipschitzian with the constant $L_2>0$ on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{4}(b-a)L_2. \tag{3.24}$$

(ii) If for $\lambda_1,\lambda_2\in\mathbb{C}$ the function $f-\lambda_1\ell$ is Lipschitzian with the constant $L_1>0$ on $\left[a,\frac{a+b}{2}\right]$ and $f-\lambda_3\ell$ is Lipschitzian with the constant $L_3>0$ on $\left[\frac{a+b}{2},b\right]$, then

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)(\lambda_3-\lambda_1) - \frac{1}{b-a}\int_a^b f(t)\,dt\right|\\
&\le \frac{1}{4}\left(\frac{L_1+L_3}{2}\right)(b-a).
\end{aligned} \tag{3.25}$$

(iii) If for $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$ the function $f-\lambda_1\ell$ is Lipschitzian with the constant $L_1>0$ on $\left[a,\frac{3a+b}{4}\right]$, $f-\lambda_2\ell$ is Lipschitzian with the constant $L_2>0$ on $\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]$ and $f-\lambda_3\ell$ is Lipschitzian with the constant $L_3>0$ on $\left[\frac{a+3b}{4},b\right]$, then

$$\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] + \frac{1}{32}(b-a)(\lambda_3-\lambda_1) - \frac{1}{b-a}\int_a^b f(t)\,dt\right|\\
&\le \frac{1}{16}\left(\frac{1}{2}L_1 + L_2 + \frac{1}{2}L_3\right)(b-a).
\end{aligned} \tag{3.26}$$

REMARK 3.3. We have the following particular cases of interest.

If for some $\lambda\in\mathbb{C}$ the function $f-\lambda\ell$ is Lipschitzian with the constant $L_\lambda>0$ on $[a,b]$, then

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{4}L_\lambda(b-a) \tag{3.27}$$

and

$$\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right|\\
&\le \frac{1}{8}L_\lambda(b-a).
\end{aligned} \tag{3.28}$$

The following lemma may be stated:

LEMMA 3.12. *Let $u:[a,b]\to\mathbb{R}$ and $l,L\in\mathbb{R}$ with $L>l$. The following statements are equivalent:*

(i) *The function $u-\frac{l+L}{2}\cdot e$, where $e(t)=t$, $t\in[a,b]$ is $\frac{1}{2}(L-l)$-Lipschitzian;*

(ii) *We have the inequalities*

$$l \le \frac{u(t) - u(s)}{t - s} \le L \quad \text{for each } t, s \in [a, b] \quad \text{with } t \neq s; \tag{3.29}$$

(iii) *We have the inequalities*

$$l(t - s) \le u(t) - u(s) \le L(t - s) \quad \text{for each } t, s \in [a, b] \quad \text{with } t > s. \tag{3.30}$$

Following [**12**], we can introduce the definition of (l, L)-Lipschitzian functions:

DEFINITION 3.1. The function $u : [a, b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) from Lemma 3.12 is said to be (l, L)-Lipschitzian on $[a, b]$.

If $L > 0$ and $l = -L$, then $(-L, L)$ $-$Lipschitzian means L-Lipschitzian in the classical sense.

Utilising *Lagrange's mean value theorem,* we can state the following result that provides examples of (l, L)-Lipschitzian functions.

PROPOSITION 3.13. *Let* $u : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b). *If* $-\infty < l = \inf_{t\in(a,b)} u'(t)$ *and* $\sup_{t\in(a,b)} u'(t) = L < \infty$, *then* u *is* (l, L)*-Lipschitzian on* $[a, b]$.

As consequences of the inequalities (3.27) and (3.28) for real valued functions we can state the following result.

PROPOSITION 3.14 (Dragomir, 2014 [**9**]). *Let* $l, L \in \mathbb{R}$ *with* $L > l$ *and* $f : [a, b] \to \mathbb{R}$ *an* (l, L)*-Lipschitzian function on* $[a, b]$, *then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{8}(L - l)(b - a) \tag{3.31}$$

and

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{16}(L - l)(b - a). \tag{3.32}$$

4. PERTURBED COMPANIONS OF OSTROWSKI'S INEQUALITY FOR ABSOLUTELY CONTINUOUS FUNCTIONS

4.1. Some Identities. The following identity holds.

LEMMA 4.1 (Dragomir, 2014 [**10**]). *Assume that* $f : [a, b] \to \mathbb{C}$ *is an absolutely continuous function on* $[a, b]$. *Then we have the equality*

$$\begin{aligned} &\frac{1}{2}[f(x) + f(a + b - x)] + \frac{1}{2}(x - a)^2 \frac{\lambda_3(x) - \lambda_1(x)}{b - a} - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^x (t - a)[f'(t) - \lambda_1(x)]\,dt \\ &+ \frac{1}{b-a}\int_x^{a+b-x}\left(t - \frac{a+b}{2}\right)[f'(t) - \lambda_2(x)]\,dt \\ &+ \frac{1}{b-a}\int_{a+b-x}^b (t - b)[f'(t) - \lambda_3(x)]\,dt, \end{aligned} \tag{4.1}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$ *and* $\lambda_j(x)$, $j = 1, 2, 3$ *complex numbers.*

PROOF. Using the integration by parts formula for Lebesgue integral, we have

$$\begin{aligned}
&\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt \\
&= \int_a^x (t-a) f'(t)\,dt - \lambda_1(x)\int_a^x (t-a)\,dt \\
&= (x-a) f(x) - \int_a^x f(t)\,dt - \frac{1}{2}\lambda_1(x)(x-a)^2,
\end{aligned}$$

$$\begin{aligned}
&\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2(x)\right]dt \\
&= \int_x^{a+b-x}\left(t-\frac{a+b}{2}\right) f'(t)\,dt - \lambda_2(x)\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)dt \\
&= f(a+b-x)\left(\frac{a+b}{2}-x\right) - f(x)\left(x-\frac{a+b}{2}\right) - \int_x^{a+b-x} f(t)\,dt \\
&- \lambda_2(x)\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)dt \\
&= f(a+b-x)\left(\frac{a+b}{2}-x\right) - f(x)\left(x-\frac{a+b}{2}\right) - \int_x^{a+b-x} f(t)\,dt,
\end{aligned}$$

since, by symmetry

$$\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)dt = 0$$

and

$$\begin{aligned}
&\int_{a+b-x}^{b} (t-b)\left[f'(t)-\lambda_3(x)\right]dt \\
&= \int_{a+b-x}^{b} (t-b) f'(t)\,dt - \lambda_3(x)\int_{a+b-x}^{b} (t-b)\,dt \\
&= (x-a) f(a+b-x) - \int_{a+b-x}^{b} f(t)\,dt + \frac{1}{2}\lambda_3(x)(x-a)^2.
\end{aligned}$$

Summing the above equalities, we deduce

$$\begin{aligned}
&\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt + \int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2(x)\right]dt \\
&+ \int_{a+b-x}^{b} (t-b)\left[f'(t)-\lambda_3(x)\right]dt \\
&= (b-a)\frac{f(x)+f(a+b-x)}{2} - \int_a^b f(t)\,dt + \frac{1}{2}\left[\lambda_3(x)-\lambda_1(x)\right](x-a)^2,
\end{aligned}$$

which is equivalent with the desired identity (4.1). ∎

The following particular cases are of interest:

COROLLARY 4.2 (Dragomir, 2014 [**10**]). *With the assumption of Lemma 4.1 we have the equalities*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2\right]dt, \tag{4.2}$$

$$
\begin{aligned}
(4.3)\qquad & f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt \\
&=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\left[f'(t)-\lambda_1\right]dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\lambda_3\right]dt,
\end{aligned}
$$

and

$$
\begin{aligned}
(4.4)\qquad & \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]+\frac{1}{32}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt \\
&=\frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(t-a)\left[f'(t)-\lambda_1\right]dt \\
&+\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2\right]dt \\
&+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^b (t-b)\left[f'(t)-\lambda_3\right]dt
\end{aligned}
$$

for any $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$.

The following particular result with no parameter in the left hand term holds:

COROLLARY 4.3 (Dragomir, 2014 [**10**]). *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous on* $[a,b]$. *Then we have the equality*

$$
\begin{aligned}
(4.5)\qquad & \frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \\
&=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt \\
&+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2(x)\right]dt \\
&+\frac{1}{b-a}\int_{a+b-x}^b (t-b)\left[f'(t)-\lambda_1(x)\right]dt,
\end{aligned}
$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ *and* $\lambda_i(x), i=1,2$ *complex numbers.*

REMARK 4.1. We get from (4.3) the following particular case:

$$
\begin{aligned}
(4.6)\qquad & f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt \\
&=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\left[f'(t)-\lambda_1\right]dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\lambda_1\right]dt,
\end{aligned}
$$

for any $\lambda_1 \in \mathbb{C}$, while from (4.4) we get

$$\begin{aligned}(4.7)\quad & \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(t-a)\left[f'(t)-\lambda_1\right]dt \\ &+ \frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2\right]dt \\ &+ \frac{1}{b-a}\int_{\frac{a+3b}{4}}^{b}(t-b)\left[f'(t)-\lambda_1\right]dt\end{aligned}$$

for any $\lambda_1, \lambda_2 \in \mathbb{C}$.

4.2. Inequalities for Bounded Derivatives. Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\begin{aligned}&\bar{U}_{[a,b]}(\gamma,\Gamma) \\ &:= \left\{f:[a,b]\to\mathbb{C} \mid \operatorname{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right] \geq 0 \text{ for almost every } t\in[a,b]\right\}\end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{f:[a,b]\to\mathbb{C} \mid \left|f(t)-\frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}\left|\Gamma-\gamma\right| \text{ for a.e. } t\in[a,b]\right\}.$$

The following representation result may be stated.

PROPOSITION 4.4. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{4.8}$$

PROOF. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left|z-\frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}\left|\Gamma-\gamma\right|$$

if and only if

$$\operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right] \geq 0.$$

This follows by the equality

$$\frac{1}{4}\left|\Gamma-\gamma\right|^2-\left|z-\frac{\gamma+\Gamma}{2}\right|^2 = \operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]$$

that holds for any $z \in \mathbb{C}$.

The equality (4.8) is thus a simple consequence of this fact. ∎

On making use of the complex numbers field properties we can also state that:

COROLLARY 4.5. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$,we have that*

$$\begin{aligned}\bar{U}_{[a,b]}(\gamma,\Gamma) = \{f:[a,b]\to\mathbb{C} \mid\ & (\operatorname{Re}\Gamma-\operatorname{Re}f(t))(\operatorname{Re}f(t)-\operatorname{Re}\gamma) \\ &+ (\operatorname{Im}\Gamma-\operatorname{Im}f(t))(\operatorname{Im}f(t)-\operatorname{Im}\gamma) \geq 0 \textit{ for a.e. } t\in[a,b]\}.\end{aligned} \tag{4.9}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{[a,b]}(\gamma,\Gamma) := \{f : [a,b] \to \mathbb{C} \mid\ & \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \\ & \text{and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for a.e. } t \in [a,b]\}.\end{aligned} \tag{4.10}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma,\Gamma) \subseteq \bar{U}_{[a,b]}(\gamma,\Gamma). \tag{4.11}$$

THEOREM 4.6 (Dragomir, 2014 [**10**]). *Assume that $f : [a,b] \to \mathbb{C}$ is an absolutely continuous function on $[a,b]$ and $x \in \left[a, \frac{a+b}{2}\right]$. If there exists the complex numbers $\gamma_j(x) \neq \Gamma_j(x)$, $j = 1,2,3$ such that*

$$\begin{aligned}f' \in\ & \bar{\Delta}_{[a,x]}(\gamma_1(x),\Gamma_1(x)) \cap \bar{\Delta}_{[x,a+b-x]}(\gamma_2(x),\Gamma_2(x)) \\ & \cap \bar{\Delta}_{[a+b-x,b]}(\gamma_3(x),\Gamma_3(x)),\end{aligned} \tag{4.12}$$

then we have the inequality

$$\begin{aligned}&\left|\frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{4}(x-a)^2 \frac{\gamma_3(x)+\Gamma_3(x)-\gamma_1(x)-\Gamma_1(x)}{b-a}\right. \\ &\left.-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{4(b-a)}\left[|\Gamma_1(x)-\gamma_1(x)|(x-a)^2 + 2|\Gamma_2(x)-\gamma_2(x)|\left(\frac{a+b}{2}-x\right)^2\right. \\ &\left.+ |\Gamma_3(x)-\gamma_3(x)|(x-a)^2\right].\end{aligned} \tag{4.13}$$

PROOF. Taking the modulus in the equality (4.1) written for

$$\begin{aligned}\lambda_1(x) &= \frac{\gamma_1(x)+\Gamma_1(x)}{2}, \lambda_2(x) = \frac{\gamma_2(x)+\Gamma_2(x)}{2}, \\ \lambda_3(x) &= \frac{\gamma_3(x)+\Gamma_3(x)}{2}\end{aligned}$$

and utilizing the condition (4.12) we have

$$\begin{aligned}
&\left|\frac{1}{2}\left[f\left(x\right)+f\left(a+b-x\right)\right]+\frac{1}{4}\left(x-a\right)^{2}\frac{\gamma_{3}\left(x\right)+\Gamma_{3}\left(x\right)-\gamma_{1}\left(x\right)-\Gamma_{1}\left(x\right)}{b-a}\right.\\
&\left.-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right|\\
&\leq\frac{1}{b-a}\int_{a}^{x}\left(t-a\right)\left|f'\left(t\right)-\frac{\gamma_{1}\left(x\right)+\Gamma_{1}\left(x\right)}{2}\right|dt\\
&+\frac{1}{b-a}\int_{x}^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'\left(t\right)-\frac{\gamma_{2}\left(x\right)+\Gamma_{2}\left(x\right)}{2}\right|dt\\
&+\frac{1}{b-a}\int_{a+b-x}^{b}\left(b-t\right)\left|f'\left(t\right)-\frac{\gamma_{3}\left(x\right)+\Gamma_{3}\left(x\right)}{2}\right|dt\\
&\leq\frac{1}{4\left(b-a\right)}\left|\Gamma_{1}\left(x\right)-\gamma_{1}\left(x\right)\right|\left(x-a\right)^{2}\\
&+\frac{2}{4\left(b-a\right)}\left|\Gamma_{2}\left(x\right)-\gamma_{2}\left(x\right)\right|\left(\frac{a+b}{2}-x\right)^{2}\\
&+\frac{1}{4\left(b-a\right)}\left|\Gamma_{3}\left(x\right)-\gamma_{3}\left(x\right)\right|\left(x-a\right)^{2}
\end{aligned}$$

and the inequality (4.13) is proved. ∎

COROLLARY 4.7. *Assume that* $f:[a,b]\rightarrow\mathbb{C}$ *is an absolutely continuous function on* $[a,b]$ *and* $x\in\left[a,\frac{a+b}{2}\right]$. *If there exists the complex numbers* $\gamma_{j}\left(x\right)\neq\Gamma_{j}\left(x\right), j=1,2$ *such that*

$$f'\in\bar{\Delta}_{[a,x]}\left(\gamma_{1}\left(x\right),\Gamma_{1}\left(x\right)\right)\cap\bar{\Delta}_{[x,a+b-x]}\left(\gamma_{2}\left(x\right),\Gamma_{2}\left(x\right)\right)\cap\bar{\Delta}_{[a+b-x,b]}\left(\gamma_{1}\left(x\right),\Gamma_{1}\left(x\right)\right), \tag{4.14}$$

then we have the inequality

$$\begin{aligned}
&\left|\frac{1}{2}\left[f\left(x\right)+f\left(a+b-x\right)\right]-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right|\\
&\leq\frac{1}{2\left(b-a\right)}\left[\left|\Gamma_{1}\left(x\right)-\gamma_{1}\left(x\right)\right|\left(x-a\right)^{2}+\left|\Gamma_{2}\left(x\right)-\gamma_{2}\left(x\right)\right|\left(\frac{a+b}{2}-x\right)^{2}\right].
\end{aligned}\tag{4.15}$$

REMARK 4.2 (Dragomir, 2014 [**10**]). Assume that $f:[a,b]\rightarrow\mathbb{C}$ is an absolutely continuous function on $[a,b]$.

If there exists the complex numbers $\gamma_{2}\neq\Gamma_{2}$ such that $f'\in\bar{\Delta}_{[a,b]}\left(\gamma_{2},\Gamma_{2}\right)$, then

$$\left|\frac{1}{2}\left[f\left(a\right)+f\left(b\right)\right]-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right|\leq\frac{1}{8}\left(b-a\right)\left|\Gamma_{2}-\gamma_{2}\right|. \tag{4.16}$$

If there exists the complex numbers $\gamma_{j}\neq\Gamma_{j}, j=1,3$ such that

$$f'\in\bar{\Delta}_{\left[a,\frac{a+b}{2}\right]}\left(\gamma_{1},\Gamma_{1}\right)\cap\bar{\Delta}_{\left[\frac{a+b}{2},b\right]}\left(\gamma_{3},\Gamma_{3}\right),$$

then we have the inequality

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)+\frac{1}{16}\left(b-a\right)\left(\Gamma_{3}+\gamma_{3}-\Gamma_{1}-\gamma_{1}\right)-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right|\\
&\leq\frac{1}{16}\left(b-a\right)\left[\left|\Gamma_{1}-\gamma_{1}\right|+\left|\Gamma_{3}-\gamma_{3}\right|\right].
\end{aligned}\tag{4.17}$$

In particular, if $f' \in \bar{\Delta}_{[a,b]}(\gamma_1, \Gamma_1)$ then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{8}(b-a)|\Gamma_1 - \gamma_1|. \tag{4.18}$$

If there exists the complex numbers $\gamma_j \neq \Gamma_j$, $j = 1, 2, 3$ such that

$$f' \in \bar{\Delta}_{\left[a, \frac{3a+b}{4}\right]}(\gamma_1, \Gamma_1) \cap \bar{\Delta}_{\left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]}(\gamma_2, \Gamma_2) \cap \bar{\Delta}_{\left[\frac{a+3b}{4}, b\right]}(\gamma_3, \Gamma_3),$$

then

$$\begin{aligned} &\left| \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] + \frac{1}{64}(b-a)(\Gamma_3 + \gamma_3 - \Gamma_1 - \gamma_1) \right. \\ &\left. - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\leq \frac{1}{64}(b-a)\left[|\Gamma_1 - \gamma_1| + 2|\Gamma_2 - \gamma_2| + |\Gamma_3 - \gamma_3| \right]. \end{aligned} \tag{4.19}$$

In particular, if $\gamma_3 = \gamma_1$ and $\Gamma_3 = \Gamma_1$, then

$$\begin{aligned} &\left| \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\leq \frac{1}{32}(b-a)\left[|\Gamma_1 - \gamma_1| + |\Gamma_2 - \gamma_2| \right], \end{aligned} \tag{4.20}$$

provided

$$f' \in \bar{\Delta}_{\left[a, \frac{3a+b}{4}\right]}(\gamma_1, \Gamma_1) \cap \bar{\Delta}_{\left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]}(\gamma_2, \Gamma_2) \cap \bar{\Delta}_{\left[\frac{a+3b}{4}, b\right]}(\gamma_1, \Gamma_1).$$

Moreover, if $f' \in \bar{\Delta}_{[a,b]}(\gamma_1, \Gamma_1)$ then

$$\begin{aligned} &\left| \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\leq \frac{1}{16}(b-a)|\Gamma_1 - \gamma_1|. \end{aligned} \tag{4.21}$$

The case of real-valued functions is of interest.

REMARK 4.3. If the function $f : [a,b] \to \mathbb{R}$ is absolutely continuous and if there exists the constants $l < L$ such that $l \leq f'(t) \leq L$ for almost every $t \in [a,b]$, then we have the inequalities

$$\left| \frac{1}{2}[f(a) + f(b)] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{8}(b-a)(L-l), \tag{4.22}$$

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{8}(b-a)(L-l) \tag{4.23}$$

and

$$\begin{aligned} &\left| \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\leq \frac{1}{16}(b-a)(L-l). \end{aligned} \tag{4.24}$$

These results improve the corresponding inequalities from Introduction.

4.3. Inequalities for Derivatives of Bounded Variation. Assume that $f : I \to \mathbb{C}$ is an absolutely continuous function on $[a,b] \subset \mathring{I}$, the interior of I. Then from (4.1) we have for $\lambda_1(x) = f'(a)$, $\lambda_2(x) = \frac{f'(x)+f'(a+b-x)}{2}$ and $\lambda_3(x) = f'(b)$ the equality

$$\begin{aligned}
&\frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{2}(x-a)^2\frac{f'(b)-f'(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt \\
&= \frac{1}{b-a}\int_a^x (t-a)[f'(t)-f'(a)]\,dt \\
&+ \frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\frac{f'(x)+f'(a+b-x)}{2}\right]dt \\
&+ \frac{1}{b-a}\int_{a+b-x}^b (t-b)[f'(t)-f'(b)]\,dt,
\end{aligned} \tag{4.25}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

We can state the following result.

THEOREM 4.8 (Dragomir, 2014 [**10**]). *Assume that $f : I \to \mathbb{C}$ is an absolutely continuous function on $[a,b] \subset \mathring{I}$. If the derivative f' is of bounded variation on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{2}(x-a)^2\frac{f'(b)-f'(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\le \frac{1}{b-a}\int_a^x (t-a)\bigvee_a^t(f')\,dt + \frac{1}{2(b-a)}\left(x-\frac{a+b}{2}\right)^2\bigvee_x^{a+b-x}(f') \\
&+ \frac{1}{b-a}\int_{a+b-x}^b (b-t)\bigvee_t^b(f')\,dt
\end{aligned} \tag{4.26}$$

$$\le \frac{1}{2(b-a)}\left[(x-a)^2\bigvee_a^x(f') + \left(x-\frac{a+b}{2}\right)^2\bigvee_x^{a+b-x}(f') + (x-a)^2\bigvee_{a+b-x}^b(f')\right]$$

$$\le \begin{cases} \frac{1}{2(b-a)}\max\left\{(x-a)^2, \left(x-\frac{a+b}{2}\right)^2\right\}\bigvee_a^b(f') \\ \\ \frac{1}{2(b-a)}\max\left\{\bigvee_a^x(f'), \bigvee_x^{a+b-x}(f'), \bigvee_{a+b-x}^b(f')\right\}\left[2(x-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right] \end{cases}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

PROOF. If we take the modulus in (4.25) we get

$$\begin{aligned} &\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^2\frac{f'(b)-f'(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le \frac{1}{b-a}\int_a^x (t-a)\left|f'(t)-f'(a)\right|dt \\ &+\frac{1}{b-a}\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)-\frac{f'(x)+f'(a+b-x)}{2}\right|dt \\ &+\frac{1}{b-a}\int_{a+b-x}^b (b-t)\left|f'(t)-f'(b)\right|dt := K, \end{aligned} \tag{4.27}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

Let $x\in\left(a,\frac{a+b}{2}\right)$. Since f' is of bounded variation on $[a,b]$, then

$$\left|f'(t)-f'(a)\right|\le\bigvee_a^t(f'),$$

for any $t\in[a,x]$ and

$$\begin{aligned} &\left|f'(t)-\frac{f'(x)+f'(a+b-x)}{2}\right| \\ &=\left|\frac{f'(t)-f'(x)+f'(t)-f'(a+b-x)}{2}\right| \\ &\le\frac{1}{2}\left[\left|f'(t)-f'(x)\right|+\left|f'(a+b-x)-f'(t)\right|\right]\le\frac{1}{2}\bigvee_x^{a+b-x}(f') \end{aligned}$$

for any $t\in[x,a+b-x]$.

We also have

$$\left|f'(t)-f'(b)\right|\le\bigvee_t^b(f'),\ t\in[a+b-x,b].$$

Then we get

$$\begin{aligned} K&\le\frac{1}{b-a}\int_a^x(t-a)\bigvee_a^t(f')\,dt+\frac{1}{2(b-a)}\bigvee_x^{a+b-x}(f')\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|dt \\ &+\frac{1}{b-a}\int_{a+b-x}^b(b-t)\bigvee_t^b(f')\,dt \\ &\le\frac{1}{b-a}\bigvee_a^x(f')\int_a^x(t-a)\,dt+\frac{1}{2(b-a)}\bigvee_x^{a+b-x}(f')\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|dt \\ &+\frac{1}{b-a}\bigvee_{a+b-x}^b(f')\int_{a+b-x}^b(b-t)\,dt \end{aligned}$$

$$= \frac{1}{2(b-a)}(x-a)^2 \bigvee_a^x (f') + \frac{1}{2(b-a)}\left(x - \frac{a+b}{2}\right)^2 \bigvee_x^{a+b-x} (f')$$
$$+ \frac{1}{2(b-a)}(x-a)^2 \bigvee_{a+b-x}^b (f'),$$

which proves the first two inequalities in (4.26).

The last part is obvious by the maximum properties. ∎

COROLLARY 4.9 (Dragomir, 2014 [**10**]). *With the assumptions of Theorem 4.8 we have*

$$\left|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{8}(b-a)\bigvee_a^b (f'), \tag{4.28}$$

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)[f'(b)-f'(a)] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\le \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\bigvee_a^t (f')\,dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (b-t)\bigvee_t^b (f')\,dt \\
&\le \frac{1}{8}(b-a)\bigvee_a^b (f')
\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] + \frac{1}{32}(b-a)[f'(b)-f'(a)]\right. \\
&\left. - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\le \frac{1}{b-a}\int_a^{\frac{3a+b}{4}} (t-a)\bigvee_a^t (f')\,dt + \frac{1}{32}(b-a)\bigvee_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} (f') \\
&+ \frac{1}{b-a}\int_{\frac{a+3b}{4}}^b (b-t)\bigvee_t^b (f')\,dt \\
&\le \frac{1}{32}(b-a)\bigvee_a^b (f').
\end{aligned} \tag{4.30}$$

5. More Perturbed Companions of Ostrowski's Inequality for Absolutely Continuous Functions

5.1. Some Identities. In the recent paper [**10**] we established the following identity:

LEMMA 5.1 (Dragomir, 2014 [**10**]). *Assume that $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function on $[a,b]$. Then we have the equality*

$$\begin{aligned}(5.1)\quad &\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^2\frac{\lambda_3(x)-\lambda_1(x)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt\\ &+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2(x)\right]dt\\ &+\frac{1}{b-a}\int_{a+b-x}^b (t-b)\left[f'(t)-\lambda_3(x)\right]dt,\end{aligned}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ and $\lambda_j(x)$, $j=1,2,3$ complex numbers.

The following particular cases are of interest:

COROLLARY 5.2 (Dragomir, 2014 [**10**]). *With the assumption of Lemma 5.1 we have the equalities*

$$(5.2)\quad \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2\right]dt,$$

$$\begin{aligned}(5.3)\quad &f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\left[f'(t)-\lambda_1\right]dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\lambda_3\right]dt,\end{aligned}$$

and

$$\begin{aligned}(5.4)\quad &\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]+\frac{1}{32}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(t-a)\left[f'(t)-\lambda_1\right]dt\\ &+\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2\right]dt\\ &+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^b (t-b)\left[f'(t)-\lambda_3\right]dt\end{aligned}$$

for any $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$.

The following particular result with no parameter in the left hand term holds:

COROLLARY 5.3 (Dragomir, 2014 [**10**]). *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous on* $[a,b]$*. Then we have the equality*

$$\begin{aligned}(5.5)\qquad &\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt\\ &+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2(x)\right]dt\\ &+\frac{1}{b-a}\int_{a+b-x}^b (t-b)\left[f'(t)-\lambda_1(x)\right]dt,\end{aligned}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ *and* $\lambda_i(x)\,,i=1,2$ *complex numbers.*

REMARK 5.1. We get from (5.3) the following particular case:

$$\begin{aligned}(5.6)\qquad &f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\left[f'(t)-\lambda_1\right]dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\lambda_1\right]dt,\end{aligned}$$

for any $\lambda_1\in\mathbb{C}$, while from (5.4) we get

$$\begin{aligned}(5.7)\qquad &\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(t-a)\left[f'(t)-\lambda_1\right]dt\\ &+\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2\right]dt\\ &+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^b (t-b)\left[f'(t)-\lambda_1\right]dt\end{aligned}$$

for any $\lambda_1,\lambda_2\in\mathbb{C}$.

5.2. Inequalities for Lipschitzian Derivatives. We say that the function $g:[a,b]\to\mathbb{C}$ is *Lipschitzian* with the constant $L>0$ if

$$|g(t)-g(s)|\le L\,|t-s|$$

for any $t,s\in[a,b]$.

THEOREM 5.4 (Dragomir, 2014 [**10**]). *Assume that* $f:I\to\mathbb{C}$ *is an absolutely continuous function on* $[a,b]\subset\mathring{I}$*, the interior of* I*. If the derivative* f' *is Lipschitzian with the constant* $K>0$*, then*

$$\begin{aligned}(5.8)\qquad &\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^2\frac{f'(b)-f'(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ &\le\frac{2K}{3(b-a)}\left[(x-a)^3+\left(\frac{a+b}{2}-x\right)^3\right]\end{aligned}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

PROOF. If we take in the equality (5.1) $\lambda_1(x) = f'(a)$, $\lambda_2(x) = f'\left(\frac{a+b}{2}\right)$ and $\lambda_3(x) = f'(b)$ then we have

$$\begin{aligned} &\frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{2}(x-a)^2\frac{f'(b)-f'(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^x (t-a)[f'(t)-f'(a)]\,dt \\ &+ \frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt \\ &+ \frac{1}{b-a}\int_{a+b-x}^b (t-b)[f'(t)-f'(b)]\,dt, \end{aligned} \tag{5.9}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Taking the modulus in (5.9) we have

$$\begin{aligned} &\left|\frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{2}(x-a)^2\frac{f'(b)-f'(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le \frac{1}{b-a}\int_a^x (t-a)\left|f'(t)-f'(a)\right|dt \\ &+ \frac{1}{b-a}\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)-f'\left(\frac{a+b}{2}\right)\right|dt \\ &+ \frac{1}{b-a}\int_{a+b-x}^b (b-t)\left|f'(b)-f'(t)\right|dt \\ &\le \frac{K}{b-a}\left[\int_a^x (t-a)^2\,dt + \int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 dt + \int_{a+b-x}^b (b-t)^2\,dt\right] \\ &= \frac{2}{3}\frac{K}{b-a}\left[(x-a)^3 + \left(\frac{a+b}{2}-x\right)^3\right] \end{aligned}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$ and the inequality (5.8) is proved. ∎

COROLLARY 5.5 (Dragomir, 2014 [**10**]). *With the assumptions of Theorem 5.4 we have the inequalities*

$$\left|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{12}K(b-a)^2, \tag{5.10}$$

$$\left|f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)[f'(b)-f'(a)] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{12}K(b-a)^2 \tag{5.11}$$

and

$$\begin{aligned} &\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] + \frac{1}{32}(b-a)[f'(b)-f'(a)] \right. \\ &\left. -\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le \frac{1}{48}K(b-a)^2 \end{aligned} \tag{5.12}$$

The following dual result also holds:

THEOREM 5.6 (Dragomir, 2014 [**10**]). *With the assumptions of Theorem 5.4 we have*

$$\begin{aligned}
&\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^{2}\frac{f'(a+b-x)-f'(x)}{b-a}\right.\\
&\left.-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|\\
&\leq\frac{K}{3(b-a)}\left[(x-a)^{3}+2\left(\frac{a+b}{2}-x\right)^{3}\right]
\end{aligned}\tag{5.13}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

PROOF. If we take in the equality (5.1) $\lambda_1(x)=f'(x)$, $\lambda_2(x)=f'\left(\frac{a+b}{2}\right)$ and $\lambda_3(x)=f'(a+b-x)$ then we have

$$\begin{aligned}
&\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^{2}\frac{f'(a+b-x)-f'(x)}{b-a}\\
&-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\\
&=\frac{1}{b-a}\int_{a}^{x}(t-a)\left[f'(t)-f'(x)\right]dt\\
&+\frac{1}{b-a}\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt\\
&+\frac{1}{b-a}\int_{a+b-x}^{b}(t-b)\left[f'(t)-f'(a+b-x)\right]dt,
\end{aligned}\tag{5.14}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

Taking the modulus in (5.14) we have

$$\begin{aligned}
&\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^{2}\frac{f'(a+b-x)-f'(x)}{b-a}\right.\\
&\left.-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|\\
&\leq\frac{1}{b-a}\int_{a}^{x}(t-a)\left|f'(t)-f'(x)\right|dt\\
&+\frac{1}{b-a}\int_{x}^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)-f'\left(\frac{a+b}{2}\right)\right|dt\\
&+\frac{1}{b-a}\int_{a+b-x}^{b}(b-t)\left|f'(t)-f'(a+b-x)\right|dt\\
&\leq\frac{K}{b-a}\left[\int_{a}^{x}(t-a)(x-t)\,dt+\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)^{2}dt\right.\\
&\left.+\int_{a+b-x}^{b}(b-t)(t-a-b+x)\,dt\right]\\
&:=J,
\end{aligned}\tag{5.15}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

However

$$\int_a^x (t-a)(x-t)\,dt = \frac{1}{6}(x-a)^3,$$

$$\int_{a+b-x}^b (b-t)(t-a-b+x)\,dt = \frac{1}{6}(x-a)^3$$

and

$$\int_x^{a+b-x} \left(t-\frac{a+b}{2}\right)^2 dt = \frac{2}{3}\left(\frac{a+b}{2}-x\right)^3$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Then

$$J = \frac{K}{3(b-a)}\left[(x-a)^3 + 2\left(\frac{a+b}{2}-x\right)^3\right]$$

and by (5.15) we get the desired result (5.13). ∎

COROLLARY 5.7 (Dragomir, 2014 [**10**]). *With the assumptions of Theorem 5.4 we have*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{24}K(b-a)^2 \tag{5.16}$$

and

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right. \\ &\left.+\frac{1}{32}(b-a)\left[f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le \frac{1}{64}K(b-a)^2.\end{aligned} \tag{5.17}$$

5.3. Inequalities for Convex Functions. Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_-(x) \le f'_+(x) \le f'_-(y) \le f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \ge f(a) + (x-a)\varphi(a) \text{ for any } x, a \in I.$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \le \varphi(x) \le f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

THEOREM 5.8 (Dragomir, 2014 [**10**]). *Let $f : [a,b] \to \mathbb{R}$ be a convex function with the lateral derivatives $f'_-(b)$ and $f'_+(a)$ finite. Then we have the inequality*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{2}(x-a)^2\frac{f'_-(b)-f'_+(a)}{b-a} \tag{5.18}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

PROOF. If we take in the equality (5.1) $\lambda_1(x) = f'_+(a)$, $\lambda_2(x) = f'_+\left(\frac{a+b}{2}\right)$ and $\lambda_3(x) = f'_-(b)$ then we have

$$\begin{aligned}(5.19)\qquad &\frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{2}(x-a)^2 \frac{f'_-(b)-f'_+(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^x (t-a)\left[\varphi(t) - f'_+(a)\right]dt \\ &+ \frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[\varphi(t) - f'_+\left(\frac{a+b}{2}\right)\right]dt \\ &+ \frac{1}{b-a}\int_{a+b-x}^b (t-b)\left[\varphi(t) - f'_-(b)\right]dt\end{aligned}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$ and $\varphi \in \partial f$, since $f' = \varphi$ almost everywhere on $[a,b]$.

Let $x \in \left(a, \frac{a+b}{2}\right)$. We have

$$(t-a)\left[\varphi(t) - f'_+(a)\right] \ge 0 \text{ for any } t \in [a,x]$$

and

$$(t-b)\left[\varphi(t) - f'_-(b)\right] = (b-t)\left[f'_-(b) - \varphi(t)\right] \ge 0 \text{ for any } t \in [a+b-x, b].$$

Also

$$\left(t-\frac{a+b}{2}\right)\left[\varphi(t) - f'_+\left(\frac{a+b}{2}\right)\right] \ge 0 \text{ for any } t \in [a, a+b-x].$$

Therefore the right hand side of (5.19) is nonnegative and the inequality (5.18) is proved. ∎

COROLLARY 5.9 (Dragomir, 2014 [**10**]). *With the assumptions in Theorem 5.8 we have*

$$(5.20)\qquad 0 \le \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \le \frac{1}{8}(b-a)\left[f'_-(b) - f'_+(a)\right]$$

and

$$\begin{aligned}(5.21)\qquad \frac{1}{b-a}\int_a^b f(t)\,dt &\le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \\ &+ \frac{1}{32}(b-a)\left[f'_-(b) - f'_+(a)\right].\end{aligned}$$

REMARK 5.2. If the $f : [a,b] \to \mathbb{R}$ is a twice differentiable convex function with the lateral derivatives $f'_-(b)$ and $f'_+(a)$ finite, then we have the inequality

$$\begin{aligned}(5.22)\qquad 0 &\le \frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{2}(x-a)^2\frac{f'_-(b)-f'_+(a)}{b-a} \\ &- \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le \frac{2S}{3(b-a)}\left[(x-a)^3 + \left(\frac{a+b}{2} - x\right)^3\right]\end{aligned}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$, provided that $0 \le f''(t) \le S$ for any $t \in (a,b)$.

In particular we have

$$0 \leq f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)\left[f'(b) - f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{12}S(b-a)^2 \tag{5.23}$$

and

$$0 \leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] + \frac{1}{32}(b-a)\left[f'(b) - f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{48}S(b-a)^2. \tag{5.24}$$

REFERENCES

[1] T. M. Apostol, *Mathematical Analysis*, Second Edition, Addison-Wesley Publishing Company, 1975.

[2] S. S. Dragomir, The Ostrowski integral inequality for mappings of bounded variation, *Bull. Austral. Math. Soc.*, **60**(1) (1999), 495-508.

[3] S. S. Dragomir, On the midpoint quadrature formula for mappings with bounded variation and applications, *Kragujevac J. Math.*, **22** (2000), 13-18.

[4] S. S. Dragomir, A refinement of Ostrowski's inequality for absolutely continuous functions whose derivatives belong to L_∞ and applications, *Libertas Math.* **22** (2002), 49–63.

[5] S. S. Dragomir, A companion of Ostrowski's inequality for functions of bounded variation and applications, Preprint *RGMIA Res. Rep. Coll.* **5** (2002), Suppl. Art. 28. [Online http://rgmia.org/papers/v5e/COIFBVApp.pdf].

[6] S. S. Dragomir, Some companions of Ostrowski's inequality for absolutely continuous functions and applications. *Bull. Korean Math. Soc.* **42** (2005), no. 2, 213–230. Preprint *RGMIA Res. Rep. Coll.* **5** (2002), Suppl. Art. 29. [Online http://rgmia.org/papers/v5e/COIACFApp.pdf].

[7] S. S. Dragomir, Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation. *Arch. Math.* (Basel) **91** (2008), no. 5, 450–460.

[8] S. S. Dragomir, Perturbed companions of Ostrowski's inequality for absolutely continuous functions (II), Preprint *RGMIA Res. Rep. Coll.* **17** (2014), Art 19. [Online http://rgmia.org/papers/v17/v17a19.pdf].

[9] S. S. Dragomir, Perturbed companions of Ostrowski's inequality for functions of bounded variation, *Asian-Eur. J. Math.* **8** (2015), no. 4, 155-169, 14 pp. Preprint *RGMIA Res. Rep. Coll.* **17** (2014), Art. 1. [Online http://rgmia.org/papers/v17/v17a01.pdf].

[10] S. S. Dragomir, Perturbed companions of Ostrowski's inequality for absolutely continuous functions (I), *An. Univ. Vest Timiş. Ser. Mat.-Inform.* **54** (2016), no. 1, 119–138. Preprint *RGMIA Res. Rep. Coll.* **17** (2014), Art. 7. [Online http://rgmia.org/papers/v17/v17a07.pdf].

[11] S. S. Dragomir and Th.M. Rassias (Ed.), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2002.

[12] Z. Liu, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral, *Soochow J. Math.*, **30**(4) (2004), 483-489.

[13] A. Guessab and G. Schmeisser, Sharp integral inequalities of the Hermite-Hadamard type, *J. Approx. Th.*, **115** (2002), 260-288.

CHAPTER 9

Ostrowski's Inequality for General Lebesgue Integral

1. Ostrowski-Jensen Type Inequalities

1.1. Introduction. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set $\Omega,$ a σ – algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1.$ Consider the Lebesgue space

$$L(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)| \, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t) \, d\mu(t)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S.S. Dragomir obtained in 2002 [**1**] the following result:

THEOREM 1.1 (Dragomir 2002, [**1**]). *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) \cdot f \in L(\Omega, \mu)$. Then we have the inequality:*

$$
\begin{aligned}
0 &\leq \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\
&\leq \int_\Omega f \cdot (\Phi' \circ f) \, d\mu - \int_\Omega \Phi' \circ f d\mu \int_\Omega f d\mu \\
&\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu.
\end{aligned} \tag{1.1}
$$

In the case of discrete measure, we have:

COROLLARY 1.2. *Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable convex function on (m, M) . If $x_i \in [m, M]$ and $w_i \geq 0\ \ (i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^n w_i = 1,$ then one has the counterpart of Jensen's weighted discrete inequality:*

$$
\begin{aligned}
0 &\leq \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n w_i x_i\right) \\
&\leq \sum_{i=1}^n w_i \Phi'(x_i) \, x_i - \sum_{i=1}^n w_i \Phi'(x_i) \sum_{i=1}^n w_i x_i \\
&\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \sum_{i=1}^n w_i \left| x_i - \sum_{j=1}^n w_j x_j \right| .
\end{aligned} \tag{1.2}
$$

REMARK 1.1. We notice that the inequality between the first and the second term in (1.2) was proved in 1994 by Dragomir & Ionescu, see [**6**].

If $f, g : \Omega \to \mathbb{R}$ are $\mu-$measurable functions and $f, g, fg \in L(\Omega, \mu)$, then we may consider the *Čebyšev functional*

$$T(f, g) := \int_\Omega f g d\mu - \int_\Omega f d\mu \int_\Omega g d\mu. \tag{1.3}$$

The following result is known in the literature as the *Grüss inequality*

$$|T(f,g)| \leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta), \tag{1.4}$$

provided

$$-\infty<\gamma\leq f(t)\leq\Gamma<\infty, \quad -\infty<\delta\leq g(t)\leq\Delta<\infty \tag{1.5}$$

for μ – a.e. $t\in\Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If we assume that $-\infty<\gamma\leq f(x)\leq\Gamma<\infty$ for μ – a.e. $x\in\Omega$, then by the Grüss inequality for $g=f$ and by the Schwarz's integral inequality, we have

$$\int_\Omega \left|f-\int_\Omega f d\mu\right| d\mu \leq \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}} \leq \frac{1}{2}(\Gamma-\gamma). \tag{1.6}$$

On making use of the results (1.1) and (1.6), we can state the following string of reverse inequalities

$$\begin{aligned} 0 &\leq \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\leq \int_\Omega f\cdot(\Phi'\circ f) d\mu - \int_\Omega \Phi'\circ f d\mu \int_\Omega f d\mu \\ &\leq \frac{1}{2}[\Phi'(M)-\Phi'(m)] \int_\Omega \left|f-\int_\Omega f d\mu\right| d\mu \\ &\leq \frac{1}{2}[\Phi'(M)-\Phi'(m)] \left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}[\Phi'(M)-\Phi'(m)](M-m), \end{aligned} \tag{1.7}$$

provided that $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ is a differentiable convex function on (m,M) and $f:\Omega\to[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ f\cdot(\Phi'\circ f)\in L(\Omega,\mu)$, with $\int_\Omega d\mu=1$.

The following reverse of the Jensen's inequality also holds [**2**]:

THEOREM 1.3. *Let $\Phi:I\to\mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m,M\in\mathbb{R}$, $m<M$ with $[m,M]\subset\mathring{I}$, where $\mathring{I}$ is the interior of I. If $f:\Omega\to\mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty<m\leq f(t)\leq M<\infty \text{ for } \mu\text{-a.e. } t\in\Omega$$

and such that $f,\Phi\circ f\in L(\Omega,\mu)$, then

$$\begin{aligned} 0 &\leq \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\leq \left(M-\int_\Omega f d\mu\right)\left(\int_\Omega f d\mu - m\right)\frac{\Phi'_-(M)-\Phi'_+(m)}{M-m} \\ &\leq \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right], \end{aligned} \tag{1.8}$$

where Φ'_- is the left and Φ'_+ is the right derivative of the convex function Φ.

For other reverse of Jensen inequality and applications to divergence measures see [**2**].

In 1938, A. Ostrowski [**8**], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b \Phi(t)\,dt$ and the value $\Phi(x)$, $x\in[a,b]$.

THEOREM 1.4. *Let $\Phi : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $\Phi' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|\Phi'\|_\infty := \sup_{t\in(a,b)} |\Phi'(t)| < \infty$. Then*

$$\left|\Phi(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right]\|\Phi'\|_\infty (b-a), \tag{1.9}$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

Motivated by the above results, in this section we investigate the magnitude of the quantity

$$\int_\Omega \Phi\circ g d\mu - \Phi(x) - \lambda\left(\int_\Omega g d\mu - x\right),\ x\in[a,b],$$

for various assumptions on the absolutely continuous function Φ, which in the particular case of $x = \int_\Omega g d\mu$ provides some results connected with Jensen's inequality while in the case $\lambda = 0$ provides some generalizations of Ostrowski's inequality. Applications for divergence measures are provided as well.

1.2. Some Identities.

The following result holds:

LEMMA 1.5 (Dragomir, 2014 [**4**]). *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi\circ g,\ g \in L(\Omega,\mu)$, then we have the equality*

$$\begin{aligned}\int_\Omega \Phi\circ g d\mu - \Phi(x) - \lambda\left(\int_\Omega g d\mu - x\right) \\ = \int_\Omega\left[(g-x)\int_0^1 (\Phi'((1-s)x+sg)-\lambda)\,ds\right]d\mu\end{aligned} \tag{1.10}$$

for any $\lambda\in\mathbb{C}$ and $x\in[a,b]$.

In particular, we have

$$\int_\Omega \Phi\circ g d\mu - \Phi(x) = \int_\Omega\left[(g-x)\int_0^1 \Phi'((1-s)x+sg)\,ds\right]d\mu, \tag{1.11}$$

for any $x\in[a,b]$.

PROOF. Since Φ is absolutely continuous on $[a,b]$, then for any $u,v\in[a,b]$ we have

$$\Phi(u)-\Phi(v) = (u-v)\int_0^1 \Phi'((1-s)v+su)\,ds. \tag{1.12}$$

This implies that

$$\Phi(g(t)) - \Phi(x) = (g(t)-x)\int_0^1 \Phi'((1-s)x+sg(t))\,ds$$

for any $t\in\Omega$, or equivalently,

$$\Phi\circ g - \Phi(x) = (g-x)\int_0^1 \Phi'((1-s)x+sg)\,ds. \tag{1.13}$$

Since $\Phi : I\to\mathbb{C}$ is an absolutely continuous functions on $[a,b]$ the Lebesgue integral over μ in the right side of (1.10) exists for any $\lambda\in\mathbb{C}$ and $x\in[a,b]$.

Integrating (1.13) over the measure μ on Ω and since $\int_\Omega d\mu = 1$, then we have

$$\int_\Omega \Phi\circ g d\mu - \Phi(x) = \int_\Omega\left[(g-x)\int_0^1 \Phi'((1-s)x+sg)\,ds\right]d\mu. \tag{1.14}$$

Now, observe that for $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}(1.15)\qquad &\int_{\Omega}\left[(g-x)\int_0^1\left(\Phi'\left((1-s)x+sg\right)-\lambda\right)ds\right]d\mu\\ &=\int_{\Omega}\left[(g-x)\left(\int_0^1\Phi'\left((1-s)x+sg\right)ds-\lambda\right)\right]d\mu\\ &=\int_{\Omega}\left[(g-x)\int_0^1\Phi'\left((1-s)x+sg\right)ds\right]d\mu-\lambda\int_{\Omega}(g-x)\,d\mu\\ &=\int_{\Omega}\left[(g-x)\int_0^1\Phi'\left((1-s)x+sg\right)ds\right]d\mu-\lambda\left(\int_{\Omega}gd\mu-x\right).\end{aligned}$$

Making use of (1.14) and (1.15) we deduce the desired result (1.10). ∎

REMARK 1.2. With the assumptions of Lemma 1.5 we have

$$\begin{aligned}(1.16)\qquad &\int_{\Omega}\Phi\circ gd\mu-\Phi\left(\frac{a+b}{2}\right)\\ &=\int_{\Omega}\left[\left(g-\frac{a+b}{2}\right)\int_0^1\Phi'\left((1-s)\frac{a+b}{2}+sg\right)ds\right]d\mu.\end{aligned}$$

COROLLARY 1.6 (Dragomir, 2014 [**4**]). *With the assumptions of Lemma 1.5 we have*

$$\begin{aligned}(1.17)\qquad &\int_{\Omega}\Phi\circ gd\mu-\Phi\left(\int_{\Omega}gd\mu\right)\\ &=\int_{\Omega}\left[\left(g-\int_{\Omega}gd\mu\right)\int_0^1\Phi'\left((1-s)\int_{\Omega}gd\mu+sg\right)ds\right]d\mu.\end{aligned}$$

PROOF. We observe that since $g:\Omega\to[a,b]$ and $\int_{\Omega}d\mu=1$ then $\int_{\Omega}gd\mu\in[a,b]$ and by taking $x=\int_{\Omega}gd\mu$ in (1.11) we get (1.17). ∎

COROLLARY 1.7 (Dragomir, 2014 [**4**]). *With the assumptions of Lemma 1.5 we have*

$$\begin{aligned}(1.18)\qquad &\int_{\Omega}\Phi\circ gd\mu-\frac{1}{b-a}\int_a^b\Phi(x)\,dx-\lambda\left(\int_{\Omega}gd\mu-\frac{a+b}{2}\right)\\ &=\int_{\Omega}\left\{\frac{1}{b-a}\int_a^b\left[(g-x)\int_0^1\left(\Phi'\left((1-s)x+sg\right)-\lambda\right)ds\right]dx\right\}d\mu.\end{aligned}$$

PROOF. Follows by integrating the identity (1.10) over $x\in[a,b]$, dividing by $b-a>0$ and using Fubini's theorem. ∎

COROLLARY 1.8 (Dragomir, 2014 [**4**]). *Let $\Phi:I\to\mathbb{C}$ be an absolutely continuous functions on $[a,b]\subset\mathring{I}$, the interior of I. If $g,h:\Omega\to[a,b]$ are Lebesgue μ-measurable on Ω and such that $\Phi\circ g,\ \Phi\circ h,\ g,\ h\in L(\Omega,\mu)$, then we have the equality*

$$\begin{aligned}(1.19)\qquad &\int_{\Omega}\Phi\circ gd\mu-\int_{\Omega}\Phi\circ hd\mu-\lambda\left(\int_{\Omega}gd\mu-\int_{\Omega}hd\mu\right)\\ &=\int_{\Omega}\int_{\Omega}\left[(g(t)-h(\tau))\int_0^1\left(\Phi'\left((1-s)h(\tau)+sg(t)\right)-\lambda\right)ds\right]\\ &\quad\times d\mu(t)\,d\mu(\tau)\end{aligned}$$

for any $\lambda\in\mathbb{C}$ and $x\in[a,b]$.

In particular, we have

$$\int_\Omega \Phi \circ g d\mu - \int_\Omega \Phi \circ h d\mu = \int_\Omega \int_\Omega \left[(g(t) - h(\tau)) \int_0^1 \Phi'((1-s)h(\tau) + sg(t))\,ds \right] d\mu(t)\,d\mu(\tau), \tag{1.20}$$

for any $x \in [a,b]$.

PROOF. From (1.10) we have for any $\tau \in \Omega$ that

$$\int_\Omega \Phi \circ g d\mu - \Phi(h(\tau)) - \lambda \left(\int_\Omega g d\mu - \Phi(h(\tau)) \right) = \int_\Omega \left[(g - \Phi(h(\tau))) \int_0^1 (\Phi'((1-s)\Phi(h(\tau)) + sg) - \lambda)\,ds \right] d\mu$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$.

Integrating on Ω over $d\mu(\tau)$ and using Fubini's theorem we get the desired result (1.19). ∎

REMARK 1.3. The above inequality (1.19) can be extended for two measures as follows

$$\begin{aligned} &\int_{\Omega_1} \Phi \circ g d\mu_1 - \int_{\Omega_2} \Phi \circ h d\mu_2 - \lambda \left(\int_{\Omega_1} g d\mu_1 - \int_{\Omega_2} h d\mu_2 \right) \\ &= \int_{\Omega_1} \int_{\Omega_2} \left[(g(t) - h(\tau)) \int_0^1 (\Phi'((1-s)h(\tau) + sg(t)) - \lambda)\,ds \right] \\ &\times d\mu_1(t)\,d\mu_2(\tau), \end{aligned} \tag{1.21}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$ and provided that $\Phi \circ g, g \in L(\Omega_1, \mu_1)$ while $\Phi \circ h, h \in L(\Omega_2, \mu_2)$.

REMARK 1.4. If $w \geq 0$ μ-almost everywhere (μ-a.e.) on Ω with $\int_\Omega w d\mu > 0$, then by replacing $d\mu$ with $\frac{w d\mu}{\int_\Omega w d\mu}$ in (1.10) we have the weighted equality

$$\frac{1}{\int_\Omega w d\mu} \int_\Omega w (\Phi \circ g)\,d\mu - \Phi(x) - \lambda \left(\frac{1}{\int_\Omega w d\mu} \int_\Omega w g d\mu - x \right) = \frac{1}{\int_\Omega w d\mu} \int_\Omega w \cdot \left[(g - x) \int_0^1 (\Phi'((1-s)x + sg) - \lambda)\,ds \right] d\mu \tag{1.22}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$, provided $\Phi \circ g, g \in L_w(\Omega, \mu)$ where

$$L_w(\Omega, \mu) := \left\{ g \middle| \int_\Omega w |g|\,d\mu < \infty \right\}.$$

The other equalities have similar weighted versions. However the details are omitted.

If we use the discrete measure, then for $\Phi : I \to \mathbb{C}$ an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I, $x_j \in [a,b]$ and $p_j \geq 0$ with $\sum_{j=1}^n p_j = 1$ we can state the following identity

$$\sum_{j=1}^n p_j \Phi(x_j) - \Phi(x) - \lambda \left(\sum_{j=1}^n p_j x_j - x \right) = \sum_{j=1}^n p_j \left[(x_j - x) \int_0^1 (\Phi'((1-s)x + sx_j) - \lambda)\,ds \right] \tag{1.23}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$.

In particular, we have

$$\sum_{j=1}^{n} p_j \Phi(x_j) - \Phi(x) = \sum_{j=1}^{n} p_j \left[(x_j - x) \int_0^1 \Phi'((1-s)x + sx_j)\, ds \right] \tag{1.24}$$

for any $x \in [a,b]$ and

$$\begin{aligned} &\sum_{j=1}^{n} p_j \Phi(x_j) - \Phi\left(\frac{a+b}{2}\right) \\ &= \sum_{j=1}^{n} p_j \left[\left(x_j - \frac{a+b}{2} \right) \int_0^1 \Phi'\left((1-s)\frac{a+b}{2} + sx_j \right) ds \right] \end{aligned} \tag{1.25}$$

and

$$\begin{aligned} &\sum_{j=1}^{n} p_j \Phi(x_j) - \Phi\left(\sum_{k=1}^{n} p_k x_k\right) \\ &= \sum_{j=1}^{n} p_j \left[\left(x_j - \sum_{k=1}^{n} p_k x_k \right) \int_0^1 \Phi'\left((1-s)\sum_{k=1}^{n} p_k x_k + sx_j \right) ds \right]. \end{aligned} \tag{1.26}$$

If $x_j \in [a,b]$ and $p_j \geq 0$, $j \in \{1,...,n\}$ with $\sum_{j=1}^{n} p_j = 1$ and if $y_k \in [a,b]$ and $q_k \geq 0$, $k \in \{1,...,m\}$ with $\sum_{k=1}^{m} q_k = 1$, then we can state the following identity as well:

$$\begin{aligned} &\sum_{j=1}^{n} p_j \Phi(x_j) - \sum_{k=1}^{m} q_k \Phi(y_k) - \lambda \left(\sum_{j=1}^{n} p_j x_j - \sum_{k=1}^{m} q_k y_k \right) \\ &= \sum_{j=1}^{n} p_j \sum_{k=1}^{m} q_k \left[(x_j - y_k) \int_0^1 (\Phi'((1-s)y_k + sx_j) - \lambda)\, ds \right]. \end{aligned} \tag{1.27}$$

In particular, we have

$$\begin{aligned} &\sum_{j=1}^{n} p_j \Phi(x_j) - \sum_{k=1}^{m} q_k \Phi(y_k) \\ &= \sum_{j=1}^{n} p_j \sum_{k=1}^{m} q_k \left[(x_j - y_k) \int_0^1 \Phi'((1-s)y_k + sx_j)\, ds \right]. \end{aligned} \tag{1.28}$$

1.3. Bounds in Terms of p-Norms.

We use the notations

$$\|k\|_{\Omega,p} := \begin{cases} \left(\int_\Omega |k(t)|^p \, d\mu(t) \right)^{1/p} < \infty, \ p \geq 1, \ k \in L_p(\Omega,\mu); \\ ess\sup_{t\in\Omega} |k(t)| < \infty, \ p = \infty, \ k \in L_\infty(\Omega,\mu); \end{cases}$$

and

$$\|\Phi\|_{[0,1],p} := \begin{cases} \left(\int_0^1 |\Phi(s)|^p \, ds \right)^{1/p} < \infty, \ p \geq 1, \ \Phi \in L_p(0,1); \\ ess\sup_{s\in[0,1]} |\Phi(s)| < \infty, \ p = \infty, \ \Phi \in L_\infty(0,1). \end{cases}$$

If we consider the identity function $\ell : [0,1] \to [0,1]$, $\ell(s) = s$ we have

$$\int_0^1 |\Phi'((1-s)x + sg(t)) - \lambda|^p \, ds = \|\Phi'((1-\ell)x + \ell g(t)) - \lambda\|_{[0,1],p}^p$$

and

$$ess \sup_{s\in[0,1]} |\Phi'((1-s)x + sg(t)) - \lambda| = \|\Phi'((1-\ell)x + \ell g(t)) - \lambda\|_{[0,1],\infty}$$

for $t \in \Omega$.

THEOREM 1.9 (Dragomir, 2014 [**4**]). *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega,\mu)$, then*

$$\begin{aligned}
(1.29)\qquad & \left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \lambda \left(\int_\Omega g d\mu - x \right) \right| \\
& \leq \int_\Omega |g - x| \, \|\Phi'((1-\ell)x + \ell g) - \lambda\|_{[0,1],1} \, d\mu \\
& \leq \begin{cases} \|g-x\|_{\Omega,\infty} \left\| \|\Phi'((1-\ell)x + \ell g) - \lambda\|_{[0,1],1} \right\|_{\Omega,1}; \\ \\ \|g-x\|_{\Omega,p} \left\| \|\Phi'((1-\ell)x + \ell g) - \lambda\|_{[0,1],1} \right\|_{\Omega,q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \|g-x\|_{\Omega,1} \left\| \|\Phi'((1-\ell)x + \ell g) - \lambda\|_{[0,1],1} \right\|_{\Omega,\infty}; \end{cases}
\end{aligned}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$.

In particular, we have

$$\begin{aligned}
(1.30)\qquad & \left| \int_\Omega \Phi \circ g d\mu - \Phi(x) \right| \\
& \leq \int_\Omega |g - x| \, \|\Phi'((1-\ell)x + \ell g)\|_{[0,1],1} \, d\mu \\
& \leq \begin{cases} \|g-x\|_{\Omega,\infty} \left\| \|\Phi'((1-\ell)x + \ell g)\|_{[0,1],1} \right\|_{\Omega,1}; \\ \\ \|g-x\|_{\Omega,p} \left\| \|\Phi'((1-\ell)x + \ell g)\|_{[0,1],1} \right\|_{\Omega,q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \|g-x\|_{\Omega,1} \left\| \|\Phi'((1-\ell)x + \ell g)\|_{[0,1],1} \right\|_{\Omega,\infty}; \end{cases}
\end{aligned}$$

for any $x \in [a,b]$.

PROOF. Taking the modulus in the equality (1.10), we have

$$\begin{aligned}
(1.31)\qquad & \left|\int_\Omega \Phi\circ g d\mu - \Phi(x) - \lambda\left(\int_\Omega g d\mu - x\right)\right| \\
&\le \int_\Omega \left|(g-x)\int_0^1 \left(\Phi'((1-s)x+sg)-\lambda\right)ds\right| d\mu \\
&\le \int_\Omega |g-x| \int_0^1 \left|\Phi'((1-s)x+sg)-\lambda\right| ds d\mu \\
&= \int_\Omega |g-x| \left\|\Phi'((1-\ell)x+\ell g)-\lambda\right\|_{[0,1],1} d\mu
\end{aligned}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$.

Utilising Hölder's inequality for the μ-measurable functions $F, G : \Omega \to \mathbb{C}$,

$$\left|\int_\Omega FG d\mu\right| \le \left(\int_\Omega |F|^p d\mu\right)^{1/p}\left(\int_\Omega |G|^q d\mu\right)^{1/q},\ p>1, \frac{1}{p}+\frac{1}{q}=1$$

and

$$\left|\int_\Omega FG d\mu\right| \le ess\sup_{t\in\Omega}|F(t)|\int_\Omega |G| d\mu$$

we get from (1.31) the desired result (1.29). ∎

REMARK 1.5. If we take $x = \frac{a+b}{2}$ in (1.29), then we get

$$\begin{aligned}
(1.32)\qquad & \left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \lambda\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \\
&\le \int_\Omega \left|g-\frac{a+b}{2}\right| \left\|\Phi'\left((1-\ell)\frac{a+b}{2}+\ell g\right)-\lambda\right\|_{[0,1],1} d\mu \\
&\le \begin{cases}
\left\|g-\frac{a+b}{2}\right\|_{\Omega,\infty}\left\|\left\|\Phi'\left((1-\ell)\frac{a+b}{2}+\ell g\right)-\lambda\right\|_{[0,1],1}\right\|_{\Omega,1}; \\[2ex]
\left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left\|\left\|\Phi'\left((1-\ell)\frac{a+b}{2}+\ell g\right)-\lambda\right\|_{[0,1],1}\right\|_{\Omega,q} \\
p>1, \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\left\|g-\frac{a+b}{2}\right\|_{\Omega,1}\left\|\left\|\Phi'\left((1-\ell)\frac{a+b}{2}+\ell g\right)-\lambda\right\|_{[0,1],1}\right\|_{\Omega,\infty};
\end{cases}
\end{aligned}$$

for any $\lambda \in \mathbb{C}$ and, in particular, for $\lambda = 0$ we have

$$\begin{aligned} &\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right)\right| \\ &\leq \int_\Omega \left|g - \frac{a+b}{2}\right| \left\|\Phi'\left((1-\ell)\frac{a+b}{2} + \ell g\right)\right\|_{[0,1],1} d\mu \\ &\leq \begin{cases} \left\|g - \frac{a+b}{2}\right\|_{\Omega,\infty} \left\|\left\|\Phi'\left((1-\ell)\frac{a+b}{2} + \ell g\right)\right\|_{[0,1],1}\right\|_{\Omega,1}; \\ \\ \left\|g - \frac{a+b}{2}\right\|_{\Omega,p} \left\|\left\|\Phi'\left((1-\ell)\frac{a+b}{2} + \ell g\right)\right\|_{[0,1],1}\right\|_{\Omega,q} \\ p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left\|g - \frac{a+b}{2}\right\|_{\Omega,1} \left\|\left\|\Phi'\left((1-\ell)\frac{a+b}{2} + \ell g\right)\right\|_{[0,1],1}\right\|_{\Omega,\infty}; \end{cases} \end{aligned} \tag{1.33}$$

If we take $x = \int_\Omega g d\mu$ in (1.29), then we get

$$\begin{aligned} &\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \\ &\leq \int_\Omega \left|g - \int_\Omega g d\mu\right| \left\|\Phi'\left((1-\ell)\int_\Omega g d\mu + \ell g\right) - \lambda\right\|_{[0,1],1} d\mu \\ &\leq \begin{cases} \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty} \left\|\left\|\Phi'\left((1-\ell)\int_\Omega g d\mu + \ell g\right) - \lambda\right\|_{[0,1],1}\right\|_{\Omega,1}; \\ \\ \left\|g - \int_\Omega g d\mu\right\|_{\Omega,p} \left\|\left\|\Phi'\left((1-\ell)\int_\Omega g d\mu + \ell g\right) - \lambda\right\|_{[0,1],1}\right\|_{\Omega,q} \\ p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left\|g - \int_\Omega g d\mu\right\|_{\Omega,1} \left\|\left\|\Phi'\left((1-\ell)\int_\Omega g d\mu + \ell g\right) - \lambda\right\|_{[0,1],1}\right\|_{\Omega,\infty}; \end{cases} \end{aligned} \tag{1.34}$$

for any $\lambda \in \mathbb{C}$ and, in particular, for $\lambda = 0$ we have

$$\begin{aligned} &\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \\ &\leq \int_\Omega \left|g - \int_\Omega g d\mu\right| \left\|\Phi'\left((1-\ell)\int_\Omega g d\mu + \ell g\right)\right\|_{[0,1],1} d\mu \\ &\leq \begin{cases} \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty} \left\|\left\|\Phi'\left((1-\ell)\int_\Omega g d\mu + \ell g\right)\right\|_{[0,1],1}\right\|_{\Omega,1}; \\ \\ \left\|g - \int_\Omega g d\mu\right\|_{\Omega,p} \left\|\left\|\Phi'\left((1-\ell)\int_\Omega g d\mu + \ell g\right)\right\|_{[0,1],1}\right\|_{\Omega,q} \\ p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left\|g - \int_\Omega g d\mu\right\|_{\Omega,1} \left\|\left\|\Phi'\left((1-\ell)\int_\Omega g d\mu + \ell g\right)\right\|_{[0,1],1}\right\|_{\Omega,\infty}. \end{cases} \end{aligned} \tag{1.35}$$

COROLLARY 1.10 (Dragomir, 2014 [**4**]). *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and* *such that* $\Phi \circ g,\, g \in L(\Omega,\mu)$ *, then*

$$\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) \right| \leq \|\Phi'\|_{[a,b],\infty} \int_{\Omega} |g - x| \, d\mu \tag{1.36}$$

for any $x \in [a,b]$ *.*

In particular, we have

$$\left| \int_{\Omega} \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) \right| \leq \|\Phi'\|_{[a,b],\infty} \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu \tag{1.37}$$

and

$$\left| \int_{\Omega} \Phi \circ g d\mu - \Phi\left(\int_{\Omega} g d\mu\right) \right| \leq \|\Phi'\|_{[a,b],\infty} \int_{\Omega} \left| g - \int_{\Omega} g d\mu \right| d\mu. \tag{1.38}$$

PROOF. We have from (1.29) that

$$\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) \right| \leq \int_{\Omega} |g - x| \left(\int_0^1 |\Phi'((1-s)x + sg)| \, ds \right) d\mu \tag{1.39}$$

for any $x \in [a,b]$.

However, for any $t \in \Omega$ and almost every $s \in [0,1]$ we have

$$|\Phi'((1-s)x + sg(t))| \leq ess \sup_{u \in [a,b]} |\Phi'(u)| = \|\Phi'\|_{[a,b],\infty},$$

for any $x \in [a,b]$.

Making use of (1.39) we get (1.36). ∎

REMARK 1.6. We remark that, the quantity from Corollary 1.10

$$\delta_{\mu}(g,x) := \int_{\Omega} |g - x| \, d\mu$$

cannot be computed in general.

However, in the case when $\Omega = [a,b]$, $g : [a,b] \to [a,b]$, $g(t) = t$ and $\mu(t) = \frac{1}{b-a}dt$, we have

$$\begin{aligned} \delta_{\mu}(g,x) &:= \frac{1}{b-a} \int_a^b |t - x| \, dt = \frac{1}{b-a} \left[\int_a^x (x-t) \, dt + \int_x^b (t-x) \, dt \right] \\ &= \frac{1}{b-a} \left[(x-a)^2 + (b-x)^2 \right] \\ &= \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a), \end{aligned}$$

where $x \in [a,b]$.

Utilising the inequality (1.36) we get Ostrowski's inequality

$$\left| \frac{1}{b-a} \int_a^b \Phi(t) \, dt - \Phi(x) \right| \leq \|\Phi'\|_{[a,b],\infty} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \tag{1.40}$$

for any $x \in [a,b]$.

From the inequalities (1.37) and (1.38) we get the midpoint inequality

$$\left| \frac{1}{b-a} \int_a^b \Phi(t) \, dt - \Phi\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{4} \|\Phi'\|_{[a,b],\infty} (b-a). \tag{1.41}$$

REMARK 1.7. If we consider the dispersion or the standard variation

$$\sigma_\mu(g) := \left(\int_\Omega \left(g - \int_\Omega g d\mu\right)^2 d\mu\right)^{1/2} = \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right)^{1/2},$$

then by (1.38) we have the inequalities

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \le \|\Phi'\|_{[a,b],\infty} \delta_\mu\left(g, \int_\Omega g d\mu\right) \le \|\Phi'\|_{[a,b],\infty} \sigma_\mu(g). \tag{1.42}$$

In general, we have by Cauchy-Bunyakovsky-Schwarz's inequality that

$$\delta_\mu(g,x) := \int_\Omega |g - x| \, d\mu \le \left(\int_\Omega (g-x)^2 d\mu\right)^{1/2}. \tag{1.43}$$

Since

$$\begin{aligned}
&\int_\Omega (g-x)^2 d\mu \\
&= \int_\Omega \left(g - \int_\Omega g d\mu + \int_\Omega g d\mu - x\right)^2 d\mu \\
&= \int_\Omega \left(g - \int_\Omega g d\mu\right)^2 d\mu + 2\int_\Omega \left(g - \int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right) d\mu \\
&+ \int_\Omega \left(\int_\Omega g d\mu - x\right)^2 d\mu \\
&= \int_\Omega \left(g - \int_\Omega g d\mu\right)^2 d\mu + \left(\int_\Omega g d\mu - x\right)^2
\end{aligned}$$

for any $x \in [a,b]$, then by (1.36) and (1.43) we get the inequalities

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi(x)\right| \le \|\Phi'\|_{[a,b],\infty} \delta_\mu(g,x) \le \|\Phi'\|_{[a,b],\infty} \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x\right)^2\right]^{1/2} \tag{1.44}$$

for any $x \in [a,b]$.

If we use the discrete measure, then from (1.44) we have

$$\begin{aligned}
&\left|\sum_{j=1}^n p_j \Phi(x_j) - \Phi(x)\right| \\
&\le \|\Phi'\|_{[a,b],\infty} \sum_{j=1}^n p_j |x_j - x| \\
&\le \|\Phi'\|_{[a,b],\infty} \left[\sum_{j=1}^n p_j x_j^2 - \left(\sum_{j=1}^n p_j x_j\right)^2 + \left(\sum_{j=1}^n p_j x_j - x\right)^2\right]^{1/2},
\end{aligned} \tag{1.45}$$

for any $x \in [a,b]$, where $x_j \in [a,b]$ and $p_j \ge 0$ with $\sum_{j=1}^n p_j = 1$.

In particular, we have

$$\left|\sum_{j=1}^{n} p_j \Phi(x_j) - \Phi\left(\frac{a+b}{2}\right)\right| \le \|\Phi'\|_{[a,b],\infty} \sum_{j=1}^{n} p_j \left|x_j - \frac{a+b}{2}\right| \le \frac{1}{2}(b-a)\|\Phi'\|_{[a,b],\infty} \tag{1.46}$$

and

$$\begin{aligned}\left|\sum_{j=1}^{n} p_j \Phi(x_j) - \Phi\left(\sum_{k=1}^{n} p_k x_k\right)\right| &\le \|\Phi'\|_{[a,b],\infty} \sum_{j=1}^{n} p_j \left|x_j - \sum_{k=1}^{n} p_k x_k\right| \\ &\le \|\Phi'\|_{[a,b],\infty}\left[\sum_{j=1}^{n} p_j x_j^2 - \left(\sum_{j=1}^{n} p_j x_j\right)^2\right]^{1/2} \le \frac{1}{2}(b-a)\|\Phi'\|_{[a,b],\infty}.\end{aligned} \tag{1.47}$$

1.4. Inequalities for Bounded Derivatives. Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\begin{aligned}&\bar{U}_{[a,b]}(\gamma,\Gamma) \\ &:= \left\{ f : [a,b] \to \mathbb{C} \mid \operatorname{Re}\left[(\Gamma - f(t))\left(\overline{f(t)} - \overline{\gamma}\right)\right] \ge 0 \text{ for almost every } t \in [a,b]\right\}\end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{ f : [a,b] \to \mathbb{C} \mid \left|f(t) - \frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}|\Gamma - \gamma| \text{ for a.e. } t \in [a,b]\right\}.$$

The following representation result may be stated.

PROPOSITION 1.11. *For any* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \ne \Gamma$, *we have that* $\bar{U}_{[a,b]}(\gamma,\Gamma)$ *and* $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ *are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{1.48}$$

PROOF. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left|z - \frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}|\Gamma - \gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma})\right] \ge 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma - \gamma|^2 - \left|z - \frac{\gamma+\Gamma}{2}\right|^2 = \operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma})\right]$$

that holds for any $z \in \mathbb{C}$.

The equality (1.29) is thus a simple consequence of this fact. ∎

On making use of the complex numbers field properties we can also state that:

COROLLARY 1.12. *For any* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \ne \Gamma$,*we have that*

$$\begin{aligned}\bar{U}_{[a,b]}(\gamma,\Gamma) = \{ f : [a,b] \to \mathbb{C} \mid\ &(\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) \\ &+ (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \ge 0 \textit{ for a.e. } t \in [a,b]\}.\end{aligned} \tag{1.49}$$

Now, if we assume that $\mathrm{Re}(\Gamma) \geq \mathrm{Re}(\gamma)$ and $\mathrm{Im}(\Gamma) \geq \mathrm{Im}(\gamma)$, then we can define the following set of functions as well:

$$\bar{S}_{[a,b]}(\gamma,\Gamma) := \{f : [a,b] \to \mathbb{C} \mid \mathrm{Re}(\Gamma) \geq \mathrm{Re} f(t) \geq \mathrm{Re}(\gamma) \text{ and } \mathrm{Im}(\Gamma) \geq \mathrm{Im} f(t) \geq \mathrm{Im}(\gamma) \text{ for a.e. } t \in [a,b]\}. \tag{1.50}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma,\Gamma) \subseteq \bar{U}_{[a,b]}(\gamma,\Gamma). \tag{1.51}$$

The following result holds:

THEOREM 1.13 (Dragomir, 2014 [**4**]). *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I. For some $\gamma,\Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, assume that $\Phi' \in \bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g, g \in L(\Omega,\mu)$, then we have the inequality*

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - x\right)\right| \leq \frac{1}{2}|\Gamma-\gamma|\int_\Omega |g-x|\,d\mu \tag{1.52}$$

for any $x \in [a,b]$.

In particular, we have

$$\begin{aligned}\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \\ \leq \frac{1}{2}|\Gamma-\gamma|\int_\Omega \left|g - \frac{a+b}{2}\right| d\mu \leq \frac{1}{4}(b-a)|\Gamma-\gamma|\end{aligned} \tag{1.53}$$

and

$$\begin{aligned}\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| &\leq \frac{1}{2}|\Gamma-\gamma|\int_\Omega \left|g - \int_\Omega g d\mu\right| d\mu \\ &\leq \frac{1}{2}|\Gamma-\gamma|\left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right)^{1/2} \\ &\leq \frac{1}{4}(b-a)|\Gamma-\gamma|.\end{aligned} \tag{1.54}$$

PROOF. By the equality (1.10) for $\lambda = \frac{\gamma+\Gamma}{2}$ we have

$$\begin{aligned}&\int_\Omega \Phi \circ g d\mu - \Phi(x) - \frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - x\right) \\ &= \int_\Omega \left[(g-x)\int_0^1 \left(\Phi'((1-s)x+sg) - \frac{\gamma+\Gamma}{2}\right) ds\right] d\mu.\end{aligned} \tag{1.55}$$

Since $\Phi' \in \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ we have

$$\left|\Phi'((1-s)x+sg(t)) - \frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}|\Gamma-\gamma| \tag{1.56}$$

for a.e. $s \in [0,1]$ and for any $x \in [a,b]$ and any $t \in \Omega$.

Integrating (1.56) over s on $[0,1]$ we get

$$\int_0^1 \left|\Phi'((1-s)x+sg(t)) - \frac{\gamma+\Gamma}{2}\right| ds \leq \frac{1}{2}|\Gamma-\gamma| \tag{1.57}$$

for any $x \in [a,b]$ and any $t \in \Omega$.

Taking the modulus in (1.55) we get via (1.57) that

$$\begin{aligned}(1.58)\qquad &\left|\int_\Omega \Phi\circ g d\mu-\Phi(x)-\frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - x\right)\right| \\ &\le \int_\Omega\left[|g-x|\left|\int_0^1\left(\Phi'((1-s)x+sg)-\frac{\gamma+\Gamma}{2}\right)ds\right|\right]d\mu \\ &\le \int_\Omega\left[|g-x|\int_0^1\left|\Phi'((1-s)x+sg(t))-\frac{\gamma+\Gamma}{2}\right|ds\right]d\mu \\ &\le \frac{1}{2}|\Gamma-\gamma|\int_\Omega |g-x|\,d\mu\end{aligned}$$

and the proof of (1.52) is completed. ∎

COROLLARY 1.14 (Dragomir, 2014 [**4**]). *Let $\Phi: I\to\mathbb{R}$ be a convex function on $[a,b]\subset \mathring{I}$, the interior of I. If $g:\Omega\to[a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi\circ g,\ g\in L(\Omega,\mu)$, then we have the inequality*

$$\begin{aligned}(1.59)\qquad &\left|\int_\Omega \Phi\circ g d\mu-\Phi(x)-\frac{\Phi'_+(a)+\Phi'_-(b)}{2}\left(\int_\Omega g d\mu - x\right)\right| \\ &\le \frac{1}{2}\left[\Phi'_-(b)-\Phi'_+(a)\right]\int_\Omega |g-x|\,d\mu\end{aligned}$$

for any $x\in[a,b]$.

In particular, we have

$$\begin{aligned}(1.60)\qquad &\left|\int_\Omega \Phi\circ g d\mu-\Phi\left(\frac{a+b}{2}\right)-\frac{\Phi'_+(a)+\Phi'_-(b)}{2}\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \\ &\le \frac{1}{2}\left[\Phi'_-(b)-\Phi'_+(a)\right]\int_\Omega \left|g-\frac{a+b}{2}\right| d\mu \le \frac{1}{2}(b-a)\left[\Phi'_-(b)-\Phi'_+(a)\right]\end{aligned}$$

and

$$(1.61)\qquad 0\le \int_\Omega \Phi\circ g d\mu-\Phi\left(\int_\Omega g d\mu\right)\le \frac{1}{2}\left[\Phi'_-(b)-\Phi'_+(a)\right]\int_\Omega\left|g-\int_\Omega g d\mu\right|d\mu.$$

The inequality (1.61) is not as good as (1.1).

The discrete case is as follows:

REMARK 1.8. Let $\Phi: I\to\mathbb{C}$ be an absolutely continuous functions on $[a,b]\subset\mathring{I}$, the interior of I. For some $\gamma,\Gamma\in\mathbb{C}$, $\gamma\ne\Gamma$, assume that $\Phi'\in\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$. If $x_j\in[a,b]$ and $p_j\ge 0$ with $\sum_{j=1}^n p_j=1$ then we have the inequality

$$(1.62)\qquad \left|\sum_{j=1}^n p_j\Phi(x_j)-\Phi(x)-\frac{\gamma+\Gamma}{2}\left(\sum_{k=1}^n p_k x_k - x\right)\right|\le\frac{1}{2}|\Gamma-\gamma|\sum_{j=1}^n p_j|x_j-x|$$

for any $x\in[a,b]$.

In particular, we have

$$\begin{aligned}(1.63)\qquad &\left|\sum_{j=1}^n p_j\Phi(x_j)-\Phi\left(\frac{a+b}{2}\right)-\frac{\gamma+\Gamma}{2}\left(\sum_{k=1}^n p_k x_k-\frac{a+b}{2}\right)\right| \\ &\le\frac{1}{2}|\Gamma-\gamma|\sum_{j=1}^n p_j\left|x_j-\frac{a+b}{2}\right|\le\frac{1}{4}(b-a)|\Gamma-\gamma|\end{aligned}$$

and

$$\begin{aligned}
&\left|\sum_{j=1}^{n} p_j \Phi\left(x_j\right)-\Phi\left(\sum_{k=1}^{n} p_k x_k\right)\right| \\
&\leq \frac{1}{2}|\Gamma-\gamma| \sum_{j=1}^{n} p_j\left|x_j-\sum_{k=1}^{n} p_k x_k\right| \\
&\leq \frac{1}{2}|\Gamma-\gamma|\left(\sum_{j=1}^{n} p_j x_j^2-\left(\sum_{k=1}^{n} p_k x_k\right)^2\right)^{1/2} \leq \frac{1}{4}(b-a)|\Gamma-\gamma| .
\end{aligned} \tag{1.64}$$

If $\Phi: I \rightarrow \mathbb{R}$ is a convex function on $[a, b]$, then we have

$$\begin{aligned}
&\left|\sum_{j=1}^{n} p_j \Phi\left(x_j\right)-\Phi(x)-\frac{\Phi_{+}^{\prime}(a)+\Phi_{-}^{\prime}(b)}{2}\left(\sum_{k=1}^{n} p_k x_k-x\right)\right| \\
&\leq \frac{1}{2}\left[\Phi_{-}^{\prime}(b)-\Phi_{+}^{\prime}(a)\right] \sum_{j=1}^{n} p_j\left|x_j-x\right|
\end{aligned} \tag{1.65}$$

for any $x \in [a, b]$.

In particular, we have

$$\begin{aligned}
&\left|\sum_{j=1}^{n} p_j \Phi\left(x_j\right)-\Phi\left(\frac{a+b}{2}\right)-\frac{\Phi_{+}^{\prime}(a)+\Phi_{-}^{\prime}(b)}{2}\left(\sum_{k=1}^{n} p_k x_k-\frac{a+b}{2}\right)\right| \\
&\leq \frac{1}{2}\left[\Phi_{-}^{\prime}(b)-\Phi_{+}^{\prime}(a)\right] \sum_{j=1}^{n} p_j\left|x_j-\frac{a+b}{2}\right| \leq \frac{1}{4}(b-a)\left[\Phi_{-}^{\prime}(b)-\Phi_{+}^{\prime}(a)\right] .
\end{aligned} \tag{1.66}$$

2. OTHER OSTROWSKI-JENSEN TYPE INEQUALITIES

2.1. Some Identities. The following result holds [**4**]:

LEMMA 2.1 (Dragomir, 2014 [**4**]). *Let $\Phi: I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, the interior of I. If $g: \Omega \rightarrow [a, b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega, \mu)$, then we have the equality*

$$\begin{aligned}
&\int_{\Omega} \Phi \circ g d\mu-\Phi(x)-\lambda\left(\int_{\Omega} g d\mu-x\right) \\
&=\int_{\Omega}\left[(g-x) \int_0^1\left(\Phi^{\prime}((1-s) x+s g)-\lambda\right) d s\right] d\mu
\end{aligned} \tag{2.1}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$.

In particular, we have

$$\int_{\Omega} \Phi \circ g d\mu-\Phi(x)=\int_{\Omega}\left[(g-x) \int_0^1 \Phi^{\prime}((1-s) x+s g) d s\right] d\mu, \tag{2.2}$$

for any $x \in [a, b]$.

REMARK 2.1. With the assumptions of Lemma 2.1 we have

$$\begin{aligned}\int_\Omega \Phi\circ g d\mu-\Phi\left(\frac{a+b}{2}\right) \\ =\int_\Omega\left[\left(g-\frac{a+b}{2}\right)\int_0^1\Phi'\left((1-s)\frac{a+b}{2}+sg\right)ds\right]d\mu.\end{aligned}\tag{2.3}$$

COROLLARY 2.2 (Dragomir, 2014 [**4**]). *With the assumptions of Lemma 2.1 we have*

$$\begin{aligned}\int_\Omega \Phi\circ g d\mu-\Phi\left(\int_\Omega g d\mu\right) \\ =\int_\Omega\left[\left(g-\int_\Omega g d\mu\right)\int_0^1\Phi'\left((1-s)\int_\Omega g d\mu+sg\right)ds\right]d\mu.\end{aligned}\tag{2.4}$$

PROOF. We observe that since $g:\Omega\to[a,b]$ and $\int_\Omega d\mu=1$ then $\int_\Omega g d\mu\in[a,b]$ and by taking $x=\int_\Omega g d\mu$ in (2.2) we get (2.4). ∎

COROLLARY 2.3 (Dragomir, 2014 [**4**]). *With the assumptions of Lemma 2.1 we have*

$$\begin{aligned}\int_\Omega \Phi\circ g d\mu-\frac{1}{b-a}\int_a^b\Phi(x)\,dx-\lambda\left(\int_\Omega g d\mu-\frac{a+b}{2}\right) \\ =\int_\Omega\left\{\frac{1}{b-a}\int_a^b\left[(g-x)\int_0^1\left(\Phi'\left((1-s)x+sg\right)-\lambda\right)ds\right]dx\right\}d\mu.\end{aligned}\tag{2.5}$$

PROOF. Follows by integrating the identity (2.1) over $x\in[a,b]$, dividing by $b-a>0$ and using Fubini's theorem. ∎

COROLLARY 2.4 (Dragomir, 2014 [**4**]). *Let $\Phi:I\to\mathbb{C}$ be an absolutely continuous functions on $[a,b]\subset\mathring{I}$, the interior of I. If $g,h:\Omega\to[a,b]$ are Lebesgue μ-measurable on Ω and such that $\Phi\circ g,\ \Phi\circ h,\ g,\ h\in L(\Omega,\mu)$, then we have the equality*

$$\begin{aligned}\int_\Omega \Phi\circ g d\mu-\int_\Omega\Phi\circ h d\mu-\lambda\left(\int_\Omega g d\mu-\int_\Omega h d\mu\right) \\ =\int_\Omega\int_\Omega\left[(g(t)-h(\tau))\int_0^1\left(\Phi'\left((1-s)h(\tau)+sg(t)\right)-\lambda\right)ds\right] \\ \times d\mu(t)\,d\mu(\tau)\end{aligned}\tag{2.6}$$

for any $\lambda\in\mathbb{C}$ and $x\in[a,b]$.

In particular, we have

$$\begin{aligned}\int_\Omega \Phi\circ g d\mu-\int_\Omega\Phi\circ h d\mu \\ =\int_\Omega\int_\Omega\left[(g(t)-h(\tau))\int_0^1\Phi'\left((1-s)h(\tau)+sg(t)\right)ds\right]d\mu(t)\,d\mu(\tau),\end{aligned}\tag{2.7}$$

for any $x\in[a,b]$.

REMARK 2.2. The above inequality (2.6) can be extended for two measures as follows

$$\begin{aligned}\int_{\Omega_1} \Phi\circ g d\mu_1-\int_{\Omega_2}\Phi\circ h d\mu_2-\lambda\left(\int_{\Omega_1} g d\mu_1-\int_{\Omega_2} h d\mu_2\right) \\ =\int_{\Omega_1}\int_{\Omega_2}\left[(g(t)-h(\tau))\int_0^1\left(\Phi'\left((1-s)h(\tau)+sg(t)\right)-\lambda\right)ds\right] \\ \times d\mu_1(t)\,d\mu_2(\tau),\end{aligned}\tag{2.8}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$ and provided that $\Phi \circ g,\, g \in L(\Omega_1, \mu_1)$ while $\Phi \circ h, h \in L(\Omega_2, \mu_2)$.

REMARK 2.3. If $w \geq 0$ μ-almost everywhere (μ-a.e.) on Ω with $\int_\Omega w d\mu > 0$, then by replacing $d\mu$ with $\frac{wd\mu}{\int_\Omega wd\mu}$ in (2.1) we have the weighted equality

$$\frac{1}{\int_\Omega wd\mu}\int_\Omega w(\Phi \circ g)\,d\mu - \Phi(x) - \lambda\left(\frac{1}{\int_\Omega wd\mu}\int_\Omega wgd\mu - x\right) = \frac{1}{\int_\Omega wd\mu}\int_\Omega w\cdot\left[(g-x)\int_0^1 (\Phi'((1-s)x+sg)-\lambda)\,ds\right]d\mu \tag{2.9}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$, provided $\Phi \circ g,\, g \in L_w(\Omega, \mu)$ where

$$L_w(\Omega,\mu) := \left\{g \mid \int_\Omega w\,|g|\,d\mu < \infty\right\}.$$

The other equalities have similar weighted versions. However the details are omitted.

2.2. Inequalities for Derivatives of Bounded Variation. The following result holds:

THEOREM 2.5 (Dragomir, 2014 [**5**]). *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I and with the property that the derivative Φ' is of bounded variation on $[a,b]$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g,\, g \in L(\Omega,\mu)$, then we have*

$$\left|\int_\Omega \Phi \circ gd\mu - \Phi(x) - \frac{\Phi'(a)+\Phi'(b)}{2}\left(\int_\Omega gd\mu - x\right)\right| \leq \frac{1}{2}\bigvee_a^b(\Phi')\int_\Omega |g-x|\,d\mu \tag{2.10}$$

for any $x \in [a,b]$.

In particular, we have

$$\left|\int_\Omega \Phi \circ gd\mu - \Phi\left(\frac{a+b}{2}\right) - \frac{\Phi'(a)+\Phi'(b)}{2}\left(\int_\Omega gd\mu - \frac{a+b}{2}\right)\right| \leq \frac{1}{2}\bigvee_a^b(\Phi')\int_\Omega\left|g-\frac{a+b}{2}\right|d\mu \leq \frac{1}{2}(b-a)\bigvee_a^b(\Phi') \tag{2.11}$$

and

$$\begin{aligned}\left|\int_\Omega \Phi \circ gd\mu - \Phi\left(\int_\Omega gd\mu\right)\right| &\leq \frac{1}{2}\bigvee_a^b(\Phi')\int_\Omega\left|g - \int_\Omega gd\mu\right|d\mu \\ &\leq \frac{1}{2}\bigvee_a^b(\Phi')\left(\int_\Omega g^2d\mu - \left(\int_\Omega gd\mu\right)^2\right)^{1/2} \\ &\leq \frac{1}{4}(b-a)\bigvee_a^b(\Phi').\end{aligned} \tag{2.12}$$

PROOF. From the identity (2.1) we have

$$\int_\Omega \Phi \circ gd\mu - \Phi(x) - \frac{\Phi'(a)+\Phi'(b)}{2}\left(\int_\Omega gd\mu - x\right) = \int_\Omega\left[(g-x)\int_0^1\left(\Phi'((1-s)x+sg) - \frac{\Phi'(a)+\Phi'(b)}{2}\right)ds\right]d\mu \tag{2.13}$$

for any $x \in [a, b]$.

Taking the modulus in (2.13) we get

$$\begin{aligned}
(2.14) \quad & \left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left(\int_\Omega g d\mu - x \right) \right| \\
& \le \int_\Omega \left| (g - x) \int_0^1 \left(\Phi'((1-s)x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right) \right| ds d\mu \\
& \le \int_\Omega |g - x| \int_0^1 \left| \Phi'((1-s)x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right| ds d\mu
\end{aligned}$$

for any $x \in [a, b]$.

Since Φ' is of bounded variation on $[a, b]$, then for any $s \in [0, 1]$, $x \in [a, b]$ and $t \in \Omega$ we have

$$\begin{aligned}
& \left| \Phi'((1-s)x + sg(t)) - \frac{\Phi'(a) + \Phi'(b)}{2} \right| \\
& = \frac{1}{2} |\Phi'((1-s)x + sg(t)) - \Phi'(a) + \Phi'((1-s)x + sg(t)) - \Phi'(b)| \\
& \le \frac{1}{2} [|\Phi'((1-s)x + sg(t)) - \Phi'(a)| + |\Phi'(b) - \Phi'((1-s)x + sg(t))|] \\
& \le \frac{1}{2} \bigvee_a^b (\Phi').
\end{aligned}$$

Then we have

$$\begin{aligned}
(2.15) \quad & \int_\Omega |g - x| \int_0^1 \left| \Phi'((1-s)x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right| ds d\mu \\
& \le \frac{1}{2} \bigvee_a^b (\Phi') \int_\Omega |g - x| \, d\mu
\end{aligned}$$

for any $x \in [a, b]$.

Making use of (2.14) and (2.15) we deduce the desired result (2.10). ∎

REMARK 2.4. Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, the interior of I and with the property that the derivative Φ' is of bounded variation on $[a, b]$. If $x_i \in [m, M]$ and $w_i \ge 0 \ (i = 1, \dots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has the weighted discrete inequality:

$$\begin{aligned}
(2.16) \quad & \left| \sum_{i=1}^n w_i \Phi(x_i) - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left(\sum_{i=1}^n w_i x_i - x \right) \right| \\
& \le \frac{1}{2} \bigvee_a^b (\Phi') \sum_{i=1}^n w_i |x_i - x|
\end{aligned}$$

for any $x \in [a, b]$.

In particular, we have

$$\left|\sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\frac{a+b}{2}\right) - \frac{\Phi'(a)+\Phi'(b)}{2}\left(\sum_{i=1}^{n} w_i x_i - \frac{a+b}{2}\right)\right| \leq \frac{1}{2}\bigvee_a^b(\Phi')\sum_{i=1}^{n} w_i\left|x_i - \frac{a+b}{2}\right| \leq \frac{1}{4}(b-a)\bigvee_a^b(\Phi') \tag{2.17}$$

and

$$\begin{aligned}\left|\sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right)\right| &\leq \frac{1}{2}\bigvee_a^b(\Phi')\sum_{i=1}^{n} w_i\left|x_i - \sum_{i=1}^{n} w_i x_i\right| \\ &\leq \frac{1}{2}\bigvee_a^b(\Phi')\left(\sum_{j=1}^{n} w_i x_j^2 - \left(\sum_{k=1}^{n} w_k x_k\right)^2\right)^{1/2} \\ &\leq \frac{1}{4}(b-a)\bigvee_a^b(\Phi').\end{aligned} \tag{2.18}$$

2.3. Inequalities for Lipschitzian Derivatives. The following result holds:

THEOREM 2.6 (Dragomir, 2014 [**5**]). *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I and with the property that the derivative Φ' is Lipschitzian with the constant $K > 0$ on $[a,b]$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g,\ g \in L(\Omega,\mu)$, then we have*

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right)\right| \leq \frac{1}{2}K\left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x\right)^2\right] \tag{2.19}$$

for any $x \in [a,b]$, where $\sigma_\mu(g)$ is the dispersion or the standard variation, namely

$$\sigma_\mu(g) := \left(\int_\Omega \left(g - \int_\Omega g d\mu\right)^2 d\mu\right)^{1/2} = \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right)^{1/2}.$$

In particular, we have

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\frac{a+b}{2}\right)\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \leq \frac{1}{2}K\left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - \frac{a+b}{2}\right)^2\right] \tag{2.20}$$

and

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \leq \frac{1}{2}K\sigma_\mu^2(g) \leq \frac{1}{8}K(b-a)^2. \tag{2.21}$$

PROOF. From the identity (2.1) we have for $\lambda = \Phi'(x)$ that

$$\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right) = \int_\Omega \left[(g-x)\int_0^1 (\Phi'((1-s)x+sg) - \Phi'(x))\,ds\right] d\mu \tag{2.22}$$

for any $x \in [a,b]$.

Taking the modulus in (2.22) we get

$$\begin{aligned} &\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right)\right| \\ &\le \int_\Omega |g-x| \left|\int_0^1 (\Phi'((1-s)x+sg) - \Phi'(x))\,ds\right| d\mu \\ &\le \int_\Omega \left[|g-x|\int_0^1 |(\Phi'((1-s)x+sg) - \Phi'(x))|\,ds\right] d\mu \\ &\le K\int_\Omega \left[|g-x|\int_0^1 s|g-x|\,ds\right] d\mu = \frac{1}{2}K\int_\Omega (g-x)^2 d\mu \end{aligned} \tag{2.23}$$

for any $x \in [a,b]$.

However,

$$\begin{aligned} &\int_\Omega (g-x)^2 d\mu \\ &= \int_\Omega \left(g - \int_\Omega g d\mu + \int_\Omega g d\mu - x\right)^2 d\mu \\ &= \int_\Omega \left(g - \int_\Omega g d\mu\right)^2 d\mu + 2\int_\Omega \left(g - \int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right) d\mu \\ &+ \int_\Omega \left(\int_\Omega g d\mu - x\right)^2 d\mu \\ &= \int_\Omega \left(g - \int_\Omega g d\mu\right)^2 d\mu + \left(\int_\Omega g d\mu - x\right)^2 \end{aligned}$$

for any $x \in [a,b]$, and by (2.23) we get the desired result (2.19). ∎

COROLLARY 2.7. *Let $\Phi : I \to \mathbb{C}$ be a twice differentiable functions on $[a,b] \subset \mathring{I}$ with $\|\Phi''\|_{[a,b],\infty} := ess\sup_{t\in[a,b]} |\Phi''(t)| < \infty$. Then the inequalities (2.19)-(2.21) hold for $K = \|\Phi''\|_{[a,b],\infty}$.*

REMARK 2.5. Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$ and with the property that the derivative Φ' is Lipschitzian with the constant $K > 0$ on $[a,b]$. If $x_i \in [m,M]$ and $w_i \ge 0$ $(i = 1,\dots,n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has the weighted discrete inequality:

$$\left|\sum_{i=1}^n w_i\Phi(x_i) - \Phi(x) - \Phi'(x)\left(\sum_{i=1}^n w_i x_i - x\right)\right| \le \frac{1}{2}K\left[\sigma_w^2(\mathbf{x}) + \left(\sum_{i=1}^n w_i x_i - x\right)^2\right] \tag{2.24}$$

for any $x \in [a,b]$, where

$$\sigma_w(\mathbf{x}) := \left(\sum_{i=1}^{n} w_i \left(x_i - \sum_{k=1}^{n} w_k x_k\right)^2\right)^{1/2} = \left(\sum_{i=1}^{n} w_i x_i^2 - \left(\sum_{k=1}^{n} w_k x_k\right)^2\right)^{1/2}.$$

The following lemma may be stated:

LEMMA 2.8. *Let $u : [a,b] \to \mathbb{R}$ and $l, L \in \mathbb{R}$ with $L > l$. The following statements are equivalent:*

(i) *The function $u - \frac{l+L}{2} \cdot e$, where $e(t) = t$, $t \in [a,b]$ is $\frac{1}{2}(L-l)$ $-$Lipschitzian;*

(ii) *We have the inequalities*

$$l \le \frac{u(t) - u(s)}{t - s} \le L \quad \text{for each } t, s \in [a,b] \quad \text{with } t \ne s; \tag{2.25}$$

(iii) *We have the inequalities*

$$l(t-s) \le u(t) - u(s) \le L(t-s) \quad \text{for each } t, s \in [a,b] \quad \text{with } t > s. \tag{2.26}$$

DEFINITION 2.1. The function $u : [a,b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) from Lemma 2.8 is said to be (l, L)-Lipschitzian on $[a,b]$.

If $L > 0$ and $l = -L$, then $(-L, L)$ $-$Lipschitzian means L-Lipschitzian in the classical sense.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides examples of (l, L)-Lipschitzian functions.

PROPOSITION 2.9. *Let $u : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $-\infty < l = \inf_{t\in[a,b]} u'(t)$ and $\sup_{t\in[a,b]} u'(t) = L < \infty$, then u is (l, L)-Lipschitzian on $[a,b]$.*

The following result holds.

COROLLARY 2.10 (Dragomir, 2014 [**5**]). *Let $\Phi : I \to \mathbb{R}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, with the property that the derivative Φ' is (l, L)-Lipschitzian on $[a,b]$, where $l, L \in \mathbb{R}$ with $L > l$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega, \mu)$, then we have*

$$\begin{aligned} &\left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x) \left(\int_\Omega g d\mu - x \right) \right. \\ &\left. - \frac{1}{4}(L+l) \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x \right)^2 \right] \right| \\ &\le \frac{1}{4}(L-l) \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x \right)^2 \right] \end{aligned} \tag{2.27}$$

for any $x \in [a,b]$.

In particular, we have

$$\begin{aligned} &\left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\frac{a+b}{2}\right) \left(\int_\Omega g d\mu - \frac{a+b}{2} \right) \right. \\ &\left. - \frac{1}{4}(L+l) \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - \frac{a+b}{2} \right)^2 \right] \right| \\ &\le \frac{1}{4}(L-l) \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - \frac{a+b}{2} \right)^2 \right] \end{aligned} \tag{2.28}$$

and

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right) - \frac{1}{4}(L+l)\sigma_\mu^2(g)\right| \leq \frac{1}{4}(L-l)\sigma_\mu^2(g) \leq \frac{1}{16}(L-l)(b-a)^2. \tag{2.29}$$

PROOF. Consider the auxiliary function $\Psi : [a,b] \to \mathbb{R}$ given by

$$\Psi(x) = \Phi(x) - \frac{1}{4}(L+l)x^2.$$

We observe that Ψ is differentiable and

$$\Psi'(x) = \Phi'(x) - \frac{1}{2}(L+l)x.$$

Since Φ' is (l,L)-Lipschitzian on $[a,b]$ it follows that Ψ' is Lipschitzian with the constant $\frac{1}{2}(L-l)$, so we can apply Theorem 2.6 for Ψ, i.e. we have the inequality

$$\left|\int_\Omega \Psi \circ g d\mu - \Psi(x) - \Psi'(x)\left(\int_\Omega g d\mu - x\right)\right| \leq \frac{1}{4}(L-l)\left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x\right)^2\right]. \tag{2.30}$$

However

$$\begin{aligned}
&\int_\Omega \Psi \circ g d\mu - \Psi(x) - \Psi'(x)\left(\int_\Omega g d\mu - x\right) \\
&= \int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right) \\
&- \frac{1}{4}(L+l)\left[\int_\Omega g^2 d\mu - x^2 - 2x\left(\int_\Omega g d\mu - x\right)\right] \\
&= \int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right) \\
&- \frac{1}{4}(L+l)\left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x\right)^2\right]
\end{aligned}$$

and by (2.30) we get the desired result (2.27). ∎

REMARK 2.6. We observe that if the function Φ is twice differentiable on $\mathring{I}$ and for $[a,b] \subset \mathring{I}$ we have

$$-\infty < l \leq \Phi''(x) \leq L < \infty \text{ for any } x \in [a,b],$$

then Φ' is (l,L)-Lipschitzian on $[a,b]$ and the inequalities (2.27)-(2.29) hold true.

The following result also holds:

THEOREM 2.11 (Dragomir, 2014 [**5**]). *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I and with the property that the derivative Φ' is Lipschitzian with the constant $K > 0$ on $[a,b]$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that*

$\Phi \circ g$, $g \in L(\Omega, \mu)$, *then we have*

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right)\right| \tag{2.31}$$
$$\leq \frac{1}{2}K\left[\left|x - \int_\Omega g d\mu\right| \int_\Omega |g-x| d\mu + \int_\Omega |g-x| \left|g - \int_\Omega g d\mu\right| d\mu\right]$$
$$\leq \frac{1}{2}K\left[\left|x - \int_\Omega g d\mu\right| + \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\right] \int_\Omega |g-x| d\mu$$

for any $x \in [a,b]$*, where*

$$\left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty} := ess\sup_{t\in\Omega}\left|g(t) - \int_\Omega g d\mu\right| < \infty.$$

In particular, we have

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \tag{2.32}$$
$$\leq \frac{1}{2}K\left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right| \int_\Omega \left|g - \frac{a+b}{2}\right| d\mu\right.$$
$$\left. + \int_\Omega \left|g - \frac{a+b}{2}\right| \left|g - \int_\Omega g d\mu\right| d\mu\right]$$
$$\leq \frac{1}{2}K\left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right| + \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\right] \int_\Omega \left|g - \frac{a+b}{2}\right| d\mu.$$

PROOF. From the identity (2.1) we have for $\lambda = \Phi'\left(\int_\Omega g d\mu\right)$ that

$$\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right) \tag{2.33}$$
$$= \int_\Omega \left[(g-x)\int_0^1 \left(\Phi'((1-s)x + sg) - \Phi'\left(\int_\Omega g d\mu\right)\right) ds\right] d\mu$$

for any $x \in [a,b]$.

Taking the modulus in (2.33) we get

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right)\right| \tag{2.34}$$
$$\leq \int_\Omega |g-x| \left|\int_0^1 \left(\Phi'((1-s)x + sg) - \Phi'\left(\int_\Omega g d\mu\right)\right) ds\right| d\mu$$
$$\leq \int_\Omega \left[|g-x| \int_0^1 \left|\left(\Phi'((1-s)x + sg) - \Phi'\left(\int_\Omega g d\mu\right)\right)\right| ds\right] d\mu$$
$$\leq K \int_\Omega \left[|g-x| \int_0^1 \left|(1-s)x + sg - \int_\Omega g d\mu\right| ds\right] d\mu$$
$$= K \int_\Omega \left[|g-x| \int_0^1 \left|(1-s)x + sg - (1-s)\int_\Omega g d\mu - s\int_\Omega g d\mu\right| ds\right] d\mu$$
$$:= B.$$

Using the triangle inequality we have for any $t \in \Omega$

$$\begin{aligned}
&\int_0^1 \left|(1-s)\,x + sg(t) - (1-s)\int_\Omega g d\mu - s\int_\Omega g d\mu\right| ds \\
&\le \int_0^1 (1-s)\left|x - \int_\Omega g d\mu\right| ds + \int_0^1 s\left|g(t) - \int_\Omega g d\mu\right| ds \\
&= \frac{1}{2}\left[\left|x - \int_\Omega g d\mu\right| + \left|g(t) - \int_\Omega g d\mu\right|\right]
\end{aligned}$$

and then

$$\begin{aligned}
B &\le \frac{1}{2}K\int_\Omega |g-x|\left[\left|x - \int_\Omega g d\mu\right| + \left|g(t) - \int_\Omega g d\mu\right|\right] d\mu \\
&= \frac{1}{2}K\left[\left|x - \int_\Omega g d\mu\right|\int_\Omega |g-x|\, d\mu + \int_\Omega |g-x|\left|g - \int_\Omega g d\mu\right| d\mu\right].
\end{aligned} \tag{2.35}$$

Making use of (2.34) and (2.35) we deduce the desired result (2.31). ∎

COROLLARY 2.12 (Dragomir, 2014 [**5**]). *Let $\Phi : I \to \mathbb{R}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, with the property that the derivative Φ' is (l,L)-Lipschitzian on $[a,b]$, where $l, L \in \mathbb{R}$ with $L > l$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g,\ g \in L(\Omega,\mu)$, then we have*

$$\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right)\right. \\
&\left. -\frac{1}{4}(L+l)\left[\sigma_\mu^2(g) - \left(x - \int_\Omega g d\mu\right)^2\right]\right| \\
&\le \frac{1}{4}(L-l)\left[\left|x - \int_\Omega g d\mu\right|\int_\Omega |g-x|\, d\mu + \int_\Omega |g-x|\left|g - \int_\Omega g d\mu\right| d\mu\right] \\
&\le \frac{1}{4}(L-l)\left[\left|x - \int_\Omega g d\mu\right| + \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\right]\int_\Omega |g-x|\, d\mu
\end{aligned} \tag{2.36}$$

for any $x \in [a,b]$.

In particular, we have

$$\begin{aligned}
&\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right. \\
&\left. -\frac{1}{4}(L+l)\left[\sigma_\mu^2(g) - \left(\frac{a+b}{2} - \int_\Omega g d\mu\right)^2\right]\right| \\
&\le \frac{1}{4}(L-l)\left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right|\int_\Omega \left|g - \frac{a+b}{2}\right| d\mu\right. \\
&\left. + \int_\Omega \left|g - \frac{a+b}{2}\right|\left|g - \int_\Omega g d\mu\right| d\mu\right] \\
&\le \frac{1}{4}(L-l)\left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right| + \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\right]\int_\Omega \left|g - \frac{a+b}{2}\right| d\mu.
\end{aligned} \tag{2.37}$$

3. INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES IN ABSOLUTE VALUE ARE h-CONVEX

3.1. Inequalities for $|\Phi'|$ is h-Convex, Quasi-convex or Log-convex. We use the notations

$$\|k\|_{\Omega,p} := \begin{cases} \left(\displaystyle\int_{\Omega} |k(t)|^p \, d\mu(t)\right)^{1/p} < \infty, \ p \geq 1, \ k \in L_p(\Omega, \mu); \\ ess\sup_{t\in\Omega} |k(t)| < \infty, \ p = \infty, \ k \in L_\infty(\Omega, \mu); \end{cases}$$

and

$$\|\Phi\|_{[0,1],p} := \begin{cases} \left(\int_0^1 |\Phi(s)|^p \, ds\right)^{1/p} < \infty, \ p \geq 1, \ \Phi \in L_p(0,1); \\ ess\sup_{s\in[0,1]} |\Phi(s)| < \infty, \ p = \infty, \ \Phi \in L_\infty(0,1). \end{cases}$$

The following result holds:

THEOREM 3.1 (Dragomir, 2014 [**5**]). *Let $\Phi : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of I and such that $|\Phi'|$ is h-convex on the interval $[a,b] \subset \mathring{I}$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega, \mu)$, then we have the inequality*

$$\left|\int_{\Omega} \Phi \circ g d\mu - \Phi(x)\right| \leq \int_0^1 h(s)\,ds \begin{cases} \|g - x\|_{\Omega,\infty} \left[|\Phi'(x)| + \|\Phi' \circ g\|_{\Omega,1}\right] \\ \text{if } \Phi' \circ g \in L(\Omega,\mu); \\ \\ \|g - x\|_{\Omega,p} \left\||\Phi'(x)| + |\Phi' \circ g|\right\|_{\Omega,q} \\ \text{if } \Phi' \circ g \in L_q(\Omega,\mu), \ p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \|g - x\|_{\Omega,1} \left[|\Phi'(x)| + \|\Phi' \circ g\|_{\Omega,\infty}\right] \\ \text{if } \Phi' \circ g \in L_\infty(\Omega,\mu) \end{cases} \tag{3.1}$$

for any $x \in [a,b]$.

In particular, we have

$$\left|\int_{\Omega} \Phi \circ g d\mu - \Phi\left(\int_{\Omega} g d\mu\right)\right| \leq \int_0^1 h(s)\,ds \begin{cases} \left\|g - \int_{\Omega} g d\mu\right\|_{\Omega,\infty} \left[\left|\Phi'\left(\int_{\Omega} g d\mu\right)\right| + \|\Phi' \circ g\|_{\Omega,1}\right], \\ \text{if } \Phi' \circ g \in L(\Omega,\mu); \\ \\ \left\|g - \int_{\Omega} g d\mu\right\|_{\Omega,p} \left\|\left|\Phi'\left(\int_{\Omega} g d\mu\right)\right| + |\Phi' \circ g|\right\|_{\Omega,q} \\ \text{if } \Phi' \circ g \in L_q(\Omega,\mu), \ p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left\|g - \int_{\Omega} g d\mu\right\|_{\Omega,1} \left[\left|\Phi'\left(\int_{\Omega} g d\mu\right)\right| + \|\Phi' \circ g\|_{\Omega,\infty}\right] \\ \text{if } \Phi' \circ g \in L_\infty(\Omega,\mu) \end{cases} \tag{3.2}$$

and

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi\left(\frac{a+b}{2}\right)\right| \tag{3.3}$$

$$\leq\int_{0}^{1}h(s)\,ds\begin{cases}\left\|g-\frac{a+b}{2}\right\|_{\Omega,\infty}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right|+\left\|\Phi'\circ g\right\|_{\Omega,1}\right]\\ \textit{if } \Phi'\circ g\in L(\Omega,\mu)\,;\,;\\ \\ \left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left\|\left|\Phi'\left(\frac{a+b}{2}\right)\right|+\left|\Phi'\circ g\right|\right\|_{\Omega,q}\\ \textit{if } \Phi'\circ g\in L_{q}(\Omega,\mu)\,,\ p>1,\frac{1}{p}+\frac{1}{q}=1;\\ \\ \left\|g-\frac{a+b}{2}\right\|_{\Omega,1}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right|+\left\|\Phi'\circ g\right\|_{\Omega,\infty}\right]\\ \textit{if } \Phi'\circ g\in L_{\infty}(\Omega,\mu)\end{cases}$$

$$\leq\frac{1}{2}(b-a)\int_{0}^{1}h(s)\,ds\begin{cases}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right|+\left\|\Phi'\circ g\right\|_{\Omega,1}\right];\\ \\ \left\|\left|\Phi'\left(\frac{a+b}{2}\right)\right|+\left|\Phi'\circ g\right|\right\|_{\Omega,q}\\ p>1,\frac{1}{p}+\frac{1}{q}=1;\\ \\ \left[\left|\Phi'\left(\frac{a+b}{2}\right)\right|+\left\|\Phi'\circ g\right\|_{\Omega,\infty}\right].\end{cases}$$

PROOF. We have

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi(x)\right|\leq\int_{\Omega}|g-x|\left|\int_{0}^{1}\Phi'((1-s)x+sg)\,ds\right|d\mu, \tag{3.4}$$

for any $x\in[a,b]$.

Utilising Hölder's inequality for the μ-measurable functions $F,G:\Omega\to\mathbb{C}$,

$$\left|\int_{\Omega}FGd\mu\right|\leq\left(\int_{\Omega}|F|^{p}\,d\mu\right)^{1/p}\left(\int_{\Omega}|G|^{q}\,d\mu\right)^{1/q},\ p>1,\frac{1}{p}+\frac{1}{q}=1$$

and

$$\left|\int_{\Omega}FGd\mu\right|\leq ess\sup_{t\in\Omega}|F(t)|\int_{\Omega}|G|\,d\mu,$$

we have

$$B:=\int_{\Omega}|g-x|\left|\int_{0}^{1}\Phi'((1-s)x+sg)\,ds\right|d\mu \tag{3.5}$$

$$\leq\begin{cases}ess\sup_{t\in\Omega}|g(t)-x|\int_{\Omega}\left|\int_{0}^{1}\Phi'((1-s)x+sg)\,ds\right|d\mu;\\ \\ \left(\int_{\Omega}|g-x|^{p}\,d\mu\right)^{1/p}\left(\int_{\Omega}\left|\int_{0}^{1}\Phi'((1-s)x+sg)\,ds\right|^{q}d\mu\right)^{1/q}\\ p>1,\frac{1}{p}+\frac{1}{q}=1;\\ \\ \int_{\Omega}|g-x|\,d\mu\,ess\sup_{t\in\Omega}\left|\int_{0}^{1}\Phi'((1-s)x+sg)\,ds\right|,\end{cases}$$

for any $x\in[a,b]$.

Since $|\Phi'|$ is h-convex on the interval $[a,b]$, then we have for any $t \in \Omega$ that

$$\begin{aligned}\left|\int_0^1 \Phi'((1-s)x+sg(t))\,ds\right| &\leq \int_0^1 |\Phi'((1-s)x+sg(t))|\,ds \\ &\leq |\Phi'(x)|\int_0^1 h(1-s)\,ds + |\Phi'(g(t))|\int_0^1 h(s)\,ds \\ &= [|\Phi'(x)| + |\Phi'(g(t))|]\int_0^1 h(s)\,ds,\end{aligned}$$

for any $x \in [a,b]$.

This implies that

$$\int_\Omega \left|\int_0^1 \Phi'((1-s)x+sg)\,ds\right| d\mu \leq \int_0^1 h(s)\,ds\left[|\Phi'(x)| + \int_\Omega |\Phi'\circ g|\,d\mu\right] \tag{3.6}$$

for any $x \in [a,b]$.

We have for any $t \in \Omega$ that

$$\begin{aligned}\left|\int_0^1 \Phi'((1-s)x+sg(t))\,ds\right|^q &\leq \left[\int_0^1 |\Phi'((1-s)x+sg(t))|\,ds\right]^q \\ &\leq \left[[|\Phi'(x)| + |\Phi'(g(t))|]\int_0^1 h(s)\,ds\right]^q \\ &= \left[\int_0^1 h(s)\,ds\right]^q [|\Phi'(x)| + |\Phi'(g(t))|]^q\end{aligned}$$

for any $x \in [a,b]$.

This implies

$$\begin{aligned}&\left(\int_\Omega \left|\int_0^1 \Phi'((1-s)x+sg)\,ds\right|^q d\mu\right)^{1/q} \\ &\leq \int_0^1 h(s)\,ds\left[\int_\Omega [|\Phi'(x)| + |\Phi'(g(t))|]^q\,d\mu\right]^{1/q} \\ &= \int_0^1 h(s)\,ds\left[\int_\Omega [|\Phi'(x)| + |\Phi'\circ g|]^q\,d\mu\right]^{1/q}.\end{aligned} \tag{3.7}$$

Also

$$\begin{aligned}&ess\sup_{t\in\Omega}\left|\int_0^1 \Phi'((1-s)x+sg)\,ds\right| \\ &\leq \left[|\Phi'(x)| + ess\sup_{t\in\Omega}|\Phi'(g(t))|\right]\int_0^1 h(s)\,ds \\ &= \left[|\Phi'(x)| + ess\sup_{t\in\Omega}|\Phi'\circ g|\right]\int_0^1 h(s)\,ds\end{aligned} \tag{3.8}$$

for any $x \in [a,b]$.

Making use of (3.6)-(3.8) we get the desired result (3.1). ∎

REMARK 3.1. With the assumptions of Theorem 3.1 and if $|\Phi'|$ is convex on the interval $[a,b]$, then $\int_0^1 h(s)\,ds = \frac{1}{2}$ and the inequalities (3.1)-(3.3) hold with $\frac{1}{2}$ instead of $\int_0^1 h(s)\,ds$. If $|\Phi'|$ is of s-Godunova-Levin type, with $s \in [0,1)$ on the interval $[a,b]$, then $\int_0^1 \frac{1}{t^s}dt = \frac{1}{1-s}$ and the inequalities (3.1)-(3.3) hold with $\frac{1}{1-s}$ instead of $\int_0^1 h(s)\,ds$.

Firstly, let us recall the definition of quasi-convex functions.

DEFINITION 3.1. The function $h:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is said to be *quasi-convex* (QC) on the interval I if

$$h(\lambda x+(1-\lambda)y)\leq\max\{h(x),h(y)\} \tag{3.9}$$

for any $x,y\in I$ and $\lambda\in[0,1]$.

Following [**7**], we say that for an interval $I\subseteq\mathbb{R}$, the mapping $h:I\to\mathbb{R}$ is *quasi-monotone* on I if it is either monotone on $I=[c,d]$ or monotone nonincreasing on a proper subinterval $[c,c']\subset I$ and monotone nondecreasing on $[c',d]$.

The class $QM(I)$ of quasi-monotone functions on I provides an immediate characterization of quasi-convex functions [**7**].

PROPOSITION 3.2. *Suppose $I\subseteq\mathbb{R}$. Then the following statements are equivalent for a function $h:I\to\mathbb{R}$:*

(a) $h\in QM(I)$;
(b) *On any subinterval of I, h achieves its supremum at an end point;*
(c) $h\in QC(I)$.

As examples of quasi-convex functions we may consider the class of monotonic functions on an interval I for the class of convex functions on that interval.

THEOREM 3.3 (Dragomir, 2014 [**5**]). *Let $\Phi:I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of I and such that $|\Phi'|$ is quasi-convex on the interval $[a,b]\subset\mathring{I}$. If $g:\Omega\to[a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi\circ g$, $g\in L(\Omega,\mu)$ and $\Phi'\circ g\in L_\infty(\Omega,\mu)$, then we have the inequality*

$$\begin{aligned}\left|\int_\Omega\Phi\circ g d\mu-\Phi(x)\right| &\leq \int_\Omega|g-x|\max\{|\Phi'(x)|,|\Phi'\circ g|\}d\mu\\ &\leq \max\left\{|\Phi'(x)|,\|\Phi'\circ g\|_{\Omega,\infty}\right\}\|g-x\|_{\Omega,1}\end{aligned} \tag{3.10}$$

for any $x\in[a,b]$.

In particular, we have

$$\begin{aligned}&\left|\int_\Omega\Phi\circ g d\mu-\Phi\left(\int_\Omega g d\mu\right)\right|\\ &\leq\int_\Omega\left|g-\int_\Omega g d\mu\right|\max\left\{\left|\Phi'\left(\int_\Omega g d\mu\right)\right|,|\Phi'\circ g|\right\}d\mu\\ &\leq\max\left\{|\Phi'(x)|,\|\Phi'\circ g\|_{\Omega,\infty}\right\}\left\|g-\int_\Omega g d\mu\right\|_{\Omega,1}\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}&\left|\int_\Omega\Phi\circ g d\mu-\Phi\left(\frac{a+b}{2}\right)\right|\\ &\leq\int_\Omega\left|g-\frac{a+b}{2}\right|\max\left\{\left|\Phi'\left(\frac{a+b}{2}\right)\right|,|\Phi'\circ g|\right\}d\mu\\ &\leq\max\left\{\left|\Phi'\left(\frac{a+b}{2}\right)\right|,\|\Phi'\circ g\|_{\Omega,\infty}\right\}\left\|g-\frac{a+b}{2}\right\|_{\Omega,1}.\end{aligned} \tag{3.12}$$

PROOF. From (3.4) we have

$$\left|\int_\Omega \Phi\circ g d\mu-\Phi(x)\right|\le\int_\Omega |g-x|\left(\int_0^1|\Phi'((1-s)x+sg)|ds\right)d\mu \tag{3.13}$$
$$\le\int_\Omega |g-x|\max\{|\Phi'(x)|,|\Phi'\circ g|\}d\mu,$$

for any $x\in[a,b]$.

Observe that

$$|(\Phi'\circ g)(t)|\le\|\Phi'\circ g\|_{\Omega,\infty}\text{ for almost every }t\in\Omega$$

and then

$$\int_\Omega |g-x|\max\{|\Phi'(x)|,|\Phi'\circ g|\}d\mu \tag{3.14}$$
$$\le\int_\Omega |g-x|\max\left\{|\Phi'(x)|,\|\Phi'\circ g\|_{\Omega,\infty}\right\}d\mu$$
$$=\max\left\{|\Phi'(x)|,\|\Phi'\circ g\|_{\Omega,\infty}\right\}\int_\Omega|g-x|d\mu,$$

for any $x\in[a,b]$.

Using (3.13) and (3.14) we get the desired result (3.10). ∎

In what follows, I will denote an interval of real numbers. A function $f:I\to(0,\infty)$ is said to be *log-convex* or *multiplicatively convex* if $\log f$ is convex, or, equivalently, if for any $x,y\in I$ and $t\in[0,1]$ one has the inequality

$$f(tx+(1-t)y)\le[f(x)]^t[f(y)]^{1-t}. \tag{3.15}$$

We note that if f and g are convex and g is increasing, then $g\circ f$ is convex, moreover, since $f=\exp[\log f]$, it follows that a log-convex function is convex, but the converse may not necessarily be true. This follows directly from (3.15) since, by the arithmetic-geometric mean inequality we have

$$[f(x)]^t[f(y)]^{1-t}\le tf(x)+(1-t)f(y) \tag{3.16}$$

for all $x,y\in I$ and $t\in[0,1]$.

THEOREM 3.4 (Dragomir, 2014 [**5**]). *Let $\Phi:I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of I and such that $|\Phi'|$ is log-convex on the interval $[a,b]\subset\mathring{I}$. If $g:\Omega\to[a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi\circ g$, $\Phi'\circ g$, $g\in L(\Omega,\mu)$ then we have the inequality*

$$\left|\int_\Omega \Phi\circ g d\mu-\Phi(x)\right| \tag{3.17}$$
$$\le\int_\Omega|g-x|L(|\Phi'\circ g|,|\Phi'(x)|)d\mu$$
$$\le\frac12\left[|\Phi'(x)|\int_\Omega|g-x|d\mu+\int_\Omega|g-x||\Phi'\circ g|d\mu\right]$$
$$\left(\le\frac12\left[|\Phi'(x)|+\|\Phi'\circ g\|_{\Omega,\infty}\right]\|g-x\|_{\Omega,1}\ \textit{if}\ \Phi'\circ g\in L_\infty(\Omega,\mu)\right)$$

for any $x \in [a,b]$, where $L(\cdot,\cdot)$ is the logarithmic mean, namely for $\alpha,\beta>0$

$$L(\alpha,\beta) := \begin{cases} \frac{\alpha-\beta}{\ln\alpha-\ln\beta}, \ \alpha\neq\beta \\ \\ \alpha, \ \alpha=\beta. \end{cases}$$

In particular, we have

$$\begin{aligned}(3.18)\qquad & \left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \\ & \le \int_\Omega \left|g-\int_\Omega g d\mu\right| L\left(\left|\Phi'\circ g\right|, \left|\Phi'\left(\int_\Omega g d\mu\right)\right|\right) d\mu \\ & \le \frac{1}{2}\left[\left|\Phi'\left(\int_\Omega g d\mu\right)\right| \int_\Omega \left|g-\int_\Omega g d\mu\right| d\mu + \int_\Omega \left|g-\int_\Omega g d\mu\right| \left|\Phi'\circ g\right| d\mu\right] \\ & \left(\le \frac{1}{2}\left[\left|\Phi'\left(\int_\Omega g d\mu\right)\right| + \left\|\Phi'\circ g\right\|_{\Omega,\infty}\right] \left\|g-\int_\Omega g d\mu\right\|_{\Omega,1} \right. \\ & \left. \textit{if } \Phi'\circ g \in L_\infty(\Omega,\mu)\right)\end{aligned}$$

and

$$\begin{aligned}(3.19)\qquad & \left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\frac{a+b}{2}\right)\right| \\ & \le \int_\Omega \left|g-\frac{a+b}{2}\right| L\left(\left|\Phi'\circ g\right|, \left|\Phi'\left(\frac{a+b}{2}\right)\right|\right) d\mu \\ & \le \frac{1}{2}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right| \int_\Omega \left|g-\frac{a+b}{2}\right| d\mu + \int_\Omega \left|g-\frac{a+b}{2}\right| \left|\Phi'\circ g\right| d\mu\right] \\ & \left(\le \frac{1}{2}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right| + \left\|\Phi'\circ g\right\|_{\Omega,\infty}\right] \left\|g-\frac{a+b}{2}\right\|_{\Omega,1} \right. \\ & \left. \textit{if } \Phi'\circ g \in L_\infty(\Omega,\mu)\right).\end{aligned}$$

PROOF. From (3.4) we have

$$\begin{aligned}(3.20)\qquad \left|\int_\Omega \Phi\circ g d\mu - \Phi(x)\right| & \le \int_\Omega |g-x| \left(\int_0^1 \left|\Phi'\left((1-s)x+sg\right)\right| ds\right) d\mu \\ & \le \int_\Omega |g-x| \left(\int_0^1 \left|\Phi'(x)\right|^{1-s} \left|\Phi'\circ g\right|^s ds\right) d\mu,\end{aligned}$$

for any $x\in[a,b]$.

Since, for any $C>0$, one has

$$\int_0^1 C^\lambda d\lambda = \frac{C-1}{\ln C},$$

then for any $t \in \Omega$ we have

$$\begin{aligned}
(3.21)\qquad \int_0^1 |\Phi'(x)|^{1-s} |\Phi'(g(t))|^s ds &= |\Phi'(x)| \int_0^1 \left| \frac{\Phi'(g(t))}{\Phi'(x)} \right|^s ds \\
&= |\Phi'(x)| \frac{\left| \frac{\Phi'(g(t))}{\Phi'(x)} \right| - 1}{\ln \left| \frac{\Phi'(g(t))}{\Phi'(x)} \right|} \\
&= \frac{|\Phi'(g(t))| - |\Phi'(x)|}{\ln |\Phi'(g(t))| - \ln |\Phi'(x)|} \\
&= L(|\Phi'(g(t))|, |\Phi'(x)|),
\end{aligned}$$

for any $x \in [a, b]$.

Making use of (3.20) and (3.21) we get the first inequality in (3.17).

The second inequality in (3.17) follows by the fact that

$$L(\alpha, \beta) \leq \frac{\alpha + \beta}{2} \text{ for any } \alpha, \beta > 0.$$

The last inequality in (3.17) is obvious. ∎

3.2. Inequalities for $|\Phi'|^q$ is h-Convex or Log-convex.

We have:

THEOREM 3.5 (Dragomir, 2014 [**5**]). *Let $\Phi : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of I and such that for $p > 1, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $|\Phi'|^q$ is h-convex on the interval $[a, b] \subset \mathring{I}$. If $g : \Omega \to [a, b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega, \mu)$ and $\Phi' \circ g \in L_q(\Omega, \mu)$ then we have the inequality*

$$\begin{aligned}
(3.22)\qquad & \left| \int_\Omega \Phi \circ g d\mu - \Phi(x) \right| \\
& \leq \left(\int_0^1 h(s) ds \right)^{1/q} \|g - x\|_{\Omega,p} \left(|\Phi'(x)|^q + \int_\Omega |\Phi' \circ g|^q d\mu \right)^{1/q} \\
& \leq \left(\int_0^1 h(s) ds \right)^{1/q} \|g - x\|_{\Omega,p} \left(|\Phi'(x)| + \|\Phi' \circ g\|_{\Omega,q} \right)
\end{aligned}$$

for any $x \in [a, b]$.

In particular, we have

$$\begin{aligned}
(3.23)\qquad & \left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu \right) \right| \\
& \leq \left(\int_0^1 h(s) ds \right)^{1/q} \\
& \times \left\| g - \int_\Omega g d\mu \right\|_{\Omega,p} \left(\left| \Phi'\left(\int_\Omega g d\mu \right) \right|^q + \int_\Omega |\Phi' \circ g|^q d\mu \right)^{1/q} \\
& \leq \left(\int_0^1 h(s) ds \right)^{1/q} \\
& \times \left\| g - \int_\Omega g d\mu \right\|_{\Omega,p} \left(\left| \Phi'\left(\int_\Omega g d\mu \right) \right| + \|\Phi' \circ g\|_{\Omega,q} \right)
\end{aligned}$$

and

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi\left(\frac{a+b}{2}\right)\right| \tag{3.24}$$
$$\leq\left(\int_{0}^{1}h\left(s\right)ds\right)^{1/q}$$
$$\times\left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left(\left|\Phi'\left(\frac{a+b}{2}\right)\right|^{q}+\int_{\Omega}\left|\Phi'\circ g\right|^{q}d\mu\right)^{1/q}$$
$$\leq\left(\int_{0}^{1}h\left(s\right)ds\right)^{1/q}$$
$$\times\left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left(\left|\Phi'\left(\frac{a+b}{2}\right)\right|+\left\|\Phi'\circ g\right\|_{\Omega,q}\right).$$

PROOF. From the proof of Theorem 3.1 we have

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi\left(x\right)\right| \tag{3.25}$$
$$\leq\int_{\Omega}\left|g-x\right|\left|\int_{0}^{1}\Phi'\left(\left(1-s\right)x+sg\right)ds\right|d\mu$$
$$\leq\left(\int_{\Omega}\left|g-x\right|^{p}d\mu\right)^{1/p}\left(\int_{\Omega}\left|\int_{0}^{1}\Phi'\left(\left(1-s\right)x+sg\right)ds\right|^{q}d\mu\right)^{1/q}$$
$$\leq\left(\int_{\Omega}\left|g-x\right|^{p}d\mu\right)^{1/p}\left(\int_{\Omega}\left(\int_{0}^{1}\left|\Phi'\left(\left(1-s\right)x+sg\right)\right|^{q}ds\right)d\mu\right)^{1/q}$$

for $p>1,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $x\in[a,b]$.

Since $|\Phi'|^{q}$ is h-convex on the interval $[a,b]$, then

$$\int_{0}^{1}\left|\Phi'\left(\left(1-s\right)x+sg\left(t\right)\right)\right|^{q}ds\leq\left|\Phi'\left(x\right)\right|^{q}\int_{0}^{1}h\left(1-s\right)ds+\left|\Phi'\left(g\left(t\right)\right)\right|^{q}\int_{0}^{1}h\left(s\right)ds$$
$$=\left[\left|\Phi'\left(x\right)\right|^{q}+\left|\Phi'\left(g\left(t\right)\right)\right|^{q}\right]\int_{0}^{1}h\left(s\right)ds$$

for any $x\in[a,b]$ and $t\in\Omega$.

Therefore

$$\left(\int_{\Omega}\left(\int_{0}^{1}\left|\Phi'\left(\left(1-s\right)x+sg\right)\right|^{q}ds\right)d\mu\right)^{1/q} \tag{3.26}$$
$$\leq\left(\int_{\Omega}\left(\left[\left|\Phi'\left(x\right)\right|^{q}+\left|\Phi'\left(g\left(t\right)\right)\right|^{q}\right]\int_{0}^{1}h\left(s\right)ds\right)d\mu\right)^{1/q}$$
$$=\left(\int_{0}^{1}h\left(s\right)ds\right)^{1/q}\left(\left|\Phi'\left(x\right)\right|^{q}+\int_{\Omega}\left|\Phi'\circ g\right|^{q}d\mu\right)^{1/q}$$

for any $x\in[a,b]$.

This proves the first inequality in (3.22).

Now, we observe that the following elementary inequality holds:

$$\left(\alpha+\beta\right)^{r}\geq\left(\leq\right)\alpha^{r}+\beta^{r} \tag{3.27}$$

for any $\alpha,\beta\geq0$ and $r\geq1$ $\left(0<r<1\right)$.

Indeed, if we consider the function $f_r : [0,\infty) \to \mathbb{R}$, $f_r(t) = (t+1)^r - t^r$ we have $f_r'(t) = r\left[(t+1)^{r-1} - t^{r-1}\right]$. Observe that for $r > 1$ and $t > 0$ we have that $f_r'(t) > 0$ showing that f_r is strictly increasing on the interval $[0,\infty)$. Now for $t = \frac{\alpha}{\beta}$ $(\beta > 0, \alpha \geq 0)$ we have $f_r(t) > f_r(0)$ giving that $\left(\frac{\alpha}{\beta}+1\right)^r - \left(\frac{\alpha}{\beta}\right)^r > 1$, i.e., the desired inequality (3.27).

For $r \in (0,1)$ we have that f_r is strictly decreasing on $[0,\infty)$ which proves the second case in (3.27).

Making use of (3.27) for $r = 1/q \in (0,1)$ we have

$$\left(|\Phi'(x)|^q + \int_\Omega |\Phi' \circ g|^q \, d\mu\right)^{1/q} \leq |\Phi'(x)| + \left(\int_\Omega |\Phi' \circ g|^q \, d\mu\right)^{1/q}$$

and then we get the second part of (3.22). ∎

Finally, we have:

THEOREM 3.6 (Dragomir, 2014 [**5**]). *Let $\Phi : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of I and such that for $p > 1, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $|\Phi'|^q$ is log-convex on the interval $[a,b] \subset \mathring{I}$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega,\mu)$ and $\Phi' \circ g \in L_q(\Omega,\mu)$ then we have the inequality*

$$\begin{aligned} (3.28)\qquad & \left|\int_\Omega \Phi \circ g d\mu - \Phi(x)\right| \\ & \leq \|g - x\|_{\Omega,p} \left(\int_\Omega L\left(|\Phi' \circ g|^q, |\Phi'(x)|^q\right) d\mu\right)^{1/q} \\ & \leq \frac{1}{2^{1/q}} \|g - x\|_{\Omega,p} \left[|\Phi'(x)|^q + \int_\Omega |\Phi' \circ g|^q \, d\mu\right]^{1/q} \\ & \leq \frac{1}{2^{1/q}} \|g - x\|_{\Omega,p} \left[|\Phi'(x)| + \|\Phi' \circ g\|_{\Omega,q}\right] \end{aligned}$$

for any $x \in [a,b]$.

In particular, we have

$$\begin{aligned} (3.29)\qquad & \left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \\ & \leq \left\|g - \int_\Omega g d\mu\right\|_{\Omega,p} \left(\int_\Omega L\left(|\Phi' \circ g|^q, \left|\Phi'\left(\int_\Omega g d\mu\right)\right|^q\right) d\mu\right)^{1/q} \\ & \leq \frac{1}{2^{1/q}} \left\|g - \int_\Omega g d\mu\right\|_{\Omega,p} \left[\left|\Phi'\left(\int_\Omega g d\mu\right)\right|^q + \int_\Omega |\Phi' \circ g|^q \, d\mu\right]^{1/q} \\ & \leq \frac{1}{2^{1/q}} \left\|g - \int_\Omega g d\mu\right\|_{\Omega,p} \left[\left|\Phi'\left(\int_\Omega g d\mu\right)\right| + \|\Phi' \circ g\|_{\Omega,q}\right] \end{aligned}$$

and

$$\left|\int_\Omega \Phi\circ g d\mu-\Phi\left(\frac{a+b}{2}\right)\right| \tag{3.30}$$
$$\le\left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left(\int_\Omega L\left(\left|\Phi'\circ g\right|^q,\left|\Phi'\left(\frac{a+b}{2}\right)\right|^q\right)d\mu\right)^{1/q}$$
$$\le\frac{1}{2^{1/q}}\left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right|^q+\int_\Omega\left|\Phi'\circ g\right|^q d\mu\right]^{1/q}$$
$$\le\frac{1}{2^{1/q}}\left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right|+\left\|\Phi'\circ g\right\|_{\Omega,q}\right].$$

PROOF. Since $|\Phi'|^q$ is log-convex on the interval $[a,b]$, then

$$\begin{aligned}\int_0^1\left|\Phi'\left((1-s)x+sg(t)\right)\right|^q ds&\le\int_0^1\left|\Phi'(x)\right|^{q(1-s)}\left|g(t)\right|^{sq}ds\\&=\left|\Phi'(x)\right|^q\int_0^1\left|\frac{g(t)}{\Phi'(x)}\right|^{sq}ds\\&=L\left(\left|\Phi'(g(t))\right|^q,\left|\Phi'(x)\right|^q\right)\end{aligned}$$

for any $x\in[a,b]$ and $t\in\Omega$.

Then

$$\left(\int_\Omega\left(\int_0^1\left|\Phi'\left((1-s)x+sg\right)\right|^q ds\right)d\mu\right)^{1/q}\le\left(\int_\Omega L\left(\left|\Phi'\circ g\right|^q,\left|\Phi'(x)\right|^q\right)d\mu\right)^{1/q}$$

and by (3.25) we get the first inequality in (3.28).

Since, in general

$$L(\alpha,\beta)\le\frac{\alpha+\beta}{2}\text{ for any }\alpha,\beta>0,$$

then

$$\begin{aligned}\int_\Omega L\left(\left|\Phi'\circ g\right|^q,\left|\Phi'(x)\right|^q\right)d\mu&\le\frac{1}{2}\int_\Omega\left[\left|\Phi'\circ g\right|^q+\left|\Phi'(x)\right|^q\right]d\mu\\&=\frac{1}{2}\left[\left|\Phi'(x)\right|^q+\int_\Omega\left|\Phi'\circ g\right|^q d\mu\right]\end{aligned}$$

and we get the second inequality in (3.28).

The last part is obvious. ∎

REFERENCES

[1] S. S. Dragomir, A Grüss type inequality for isotonic linear functionals and applications. *Demonstratio Math.* **36** (2003), no. 3, 551–562. Preprint RGMIA *Res. Rep. Coll.* **5**(2002), Suplement, Art. 12. [Online http://rgmia.org/v5(E).php].

[2] S. S. Dragomir, Reverses of the Jensen inequality in terms of first derivative and applications, *Acta Math. Vietnam.* **38** (2013), no. 3, 429–446. Preprint *RGMIA Res. Rep. Coll.* 14 (2011), Art. 71. [http://rgmia.org/papers/v14/v14a71.pdf].

[3] S. S. Dragomir. General Lebesgue integral inequalities of Jensen and Ostrowski type for differentiable functions whose derivatives in absolute value are h-convex and applications, *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **69** (2015), no. 2, 17–45. Preprint *RGMIA Res. Rep. Coll.* **17** (2014), Art. 38. [http://rgmia.org/papers/v17/v17a38.pdf].

[4] S. S. Dragomir, Jensen and Ostrowski type inequalities for general Lebesgue integral with applications, *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **70** (2016) , No.2, 29–49. Preprint *RGMIA Res. Rep. Coll.* **17** (2014), Art. 25. [`http://rgmia.org/papers/v17/v17a25.pdf`].

[5] S. S. Dragomir, New Jensen and Ostrowski type inequalities for general Lebesgue integral with applications, *Iranian Journal of Mathematical Sciences and Informatics*, Vol. **11**, No. 2 (2016), pp. 1-22. DOI: 10.7508/ijmsi.2016.02.001. Preprint *RGMIA Res. Rep. Coll.* **17** (2014), Art. 27. [`http://rgmia.org/papers/v17/v17a27.pdf`].

[6] S. S. Dragomir and N. M. Ionescu, Some converse of Jensen's inequality and applications. *Rev. Anal. Numér. Théor. Approx.* **23** (1994), no. 1, 71–78.

[7] S. S. Dragomir and C. E. M. Pearce, Quasi-convex functions and Hadamard's inequality, *Bull. Australian Math. Soc.*, **57**(1998), 377-385.

[8] A. Ostrowski, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel.*, **10** (1938), 226-227.

CHAPTER 10

List of Papers Related to Ostrowski's Inequality

1. Before 2000

1. Milovanović, Gradimir V.; Pečarić, Josip E. On generalization of the inequality of A. Ostrowski and some related applications. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 544-576 (1976), 155–158.

2. Fink, A. M. Bounds on the deviation of a function from its averages. Czechoslovak Math. J. 42(117) (1992), no. 2, 289–310.

3. Anastassiou, George A. Ostrowski type inequalities. Proc. Amer. Math. Soc. 123 (1995), no. 12, 3775–3781.

4. Dragomir, S. S.; Wang, S. An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules. Comput. Math. Appl. 33 (1997), no. 11, 15–20.

5. Anastassiou, G. A. Multivariate Ostrowski type inequalities. Acta Math. Hungar. 76 (1997), no. 4, 267–278.

6. Dragomir, Sever S.; Wang, Song A new inequality of Ostrowski's type in L_1 norm and applications to some special means and to some numerical quadrature rules. Tamkang J. Math. 28 (1997), no. 3, 239–244.

7. Dragomir, S. S.; Wang, S. Applications of Ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rules. Appl. Math. Lett. 11 (1998), no. 1, 105–109.

8. Dragomir, Sever S.; Wang, Song A new inequality of Ostrowski's type in L_p-norm. Indian J. Math. 40 (1998), no. 3, 299–304.

9. Cerone, P.; Dragomir, S. S.; Roumeliotis, J. An inequality of Ostrowski type for mappings whose second derivatives belong to $L_1(a,b)$ and applications. Honam Math. J. 21 (1999), no. 1, 127–137.

10. Merkle, Milan Remarks on Ostrowski's and Hadamard's inequality. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. 10 (1999), 113–117.

11. Peachey, T. C.; Mcandrew, A.; Dragomir, S. S. The best constant in an inequality of Ostrowski type. Tamkang J. Math. 30 (1999), no. 3, 219–222.

12. Fedotov, I.; Dragomir, S. S. An inequality of Ostrowski type and its applications for Simpson's rule and special means. Math. Inequal. Appl. 2 (1999), no. 4, 491–499.

13. Dragomir, S. S.; Barnett, N. S.; Wang, S. An Ostrowski type inequality for a random variable whose probability density function belongs to $L_p[a,b]$, $p>1$. Math. Inequal. Appl. 2 (1999), no. 4, 501–508.

14. Dragomir, S. S. The Ostrowski integral inequality for mappings of bounded variation. Bull. Austral. Math. Soc. 60 (1999), no. 3, 495–508.

15. Barnett, N. S.; Dragomir, S. S. An inequality of Ostrowski's type for cumulative distribution functions. Kyungpook Math. J. 39 (1999), no. 2, 303–311.

16. Cerone, P.; Dragomir, S. S.; Roumeliotis, J. An inequality of Ostrowski-Grüss type for twice differentiable mappings and applications in numerical integration. Kyungpook Math. J. 39 (1999), no. 2, 333–341.

17. Dragomir, S. S. The Ostrowski's integral inequality for Lipschitzian mappings and applications. Comput. Math. Appl. 38 (1999), no. 11-12, 33–37.

18. Cerone, P.; Dragomir, S. S.; Roumeliotis, J. Some Ostrowski type inequalities for n-time differentiable mappings and applications. Demonstratio Math. 32 (1999), no. 4, 697–712.

19. Dragomir, S. S.; Barnett, N. S. An Ostrowski type inequality for mappings whose second derivatives are bounded and applications. J. Indian Math. Soc. (N.S.) 66 (1999), no. 1-4, 237–245.

20. Dragomir, S. S.; Cerone, P.; Roumeliotis, J.; Wang, S. A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis. Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 42(90) (1999), no. 4, 301–314.

2. PERIOD 2000-2010

21. Dedić, Lj.; Matić, M.; Pečarić, J. On some generalizations of Ostrowski inequality for Lipschitz functions and functions of bounded variation. Math. Inequal. Appl. 3 (2000), no. 1, 1–14.

22. Pearce, C. E. M.; Pečarić, J.; Ujević, N.; Varošanec, S. Generalizations of some inequalities of Ostrowski-Grüss type. Math. Inequal. Appl. 3 (2000), no. 1, 25–34.

23. Brnetić, I.; Pečarić, J. On an Ostrowski type inequality for a random variable. Math. Inequal. Appl. 3 (2000), no. 1, 143–145.

24. Matić, M.; Pečarić, J.; Ujević, N. Improvement and further generalization of inequalities of Ostrowski-Grüss type. Comput. Math. Appl. 39 (2000), no. 3-4, 161–175.

25. Dragomir, S. S.; Cerone, P.; Roumeliotis, J. A new generalization of Ostrowski's integral inequality for mappings whose derivatives are bounded and applications in numerical integration and for special means. Appl. Math. Lett. 13 (2000), no. 1, 19–25.

26. Dragomir, S. S.; Pečarić, J.; Wang, S. The unified treatment of trapezoid, Simpson, and Ostrowski type inequality for monotonic mappings and applications. Math. Comput. Modelling 31 (2000), no. 6-7, 61–70.

27. Dragomir, S. S.; Barnett, N. S.; Cerone, P. An n-dimensional version of Ostrowski's inequality for mappings of the Hölder type. Kyungpook Math. J. 40 (2000), no. 1, 65–75.

28. Dedić, Lj.; Matić, M.; Pečarić, J. On generalizations of Ostrowski inequality via some Euler-type identities. Math. Inequal. Appl. 3 (2000), no. 3, 337–353.

29. Dragomir, S. S.; Cerone, P.; Barnett, N. S.; Roumeliotis, J. An inequality of the Ostrowski type for double integrals and applications for cubature formulae. Tamsui Oxf. J. Math. Sci. 16 (2000), no. 1, 1–16.

30. Dragomir, S. S. On the Ostrowski's inequality for Riemann-Stieltjes integral and applications. Korean J. Comput. Appl. Math. 7 (2000), no. 3, 611–627.

31. Pearce, C. E. M.; Pečarić, J. On Anastassiou's generalizations of the Ostrowski inequality and related results. J. Comput. Anal. Appl. 2 (2000), no. 3, 215–235.

32. Pachpatte, B. G. On an inequality of Ostrowski type in three independent variables. J. Math. Anal. Appl. 249 (2000), no. 2, 583–591.

33. Gavrea, B.; Gavrea, I. Ostrowski type inequalities from a linear functional point of view. JIPAM. J. Inequal. Pure Appl. Math. 1 (2000), no. 2, Article 11, 1–9.

34. Barnett, N. S.; Dragomir, S. S. An Ostrowski's type inequality for a random variable whose probability density function belongs to $L_\infty[a,b]$. Nonlinear Anal. Forum 5 (2000), 125–135.

35. Čuljak, V.; Pearce, C. E. M.; Pečarić, J. The unified treatment of some inequalities of Ostrowski and Simpson types. Soochow J. Math. 26 (2000), no. 4, 377–390.

36. Matić, M.; Pečarić, J.; Ujević, N. Generalizations of weighted version of Ostrowski's inequality and some related results. J. Inequal. Appl. 5 (2000), no. 6, 639–666.

37. Hanna, G.; Cerone, P.; Roumeliotis, J. An Ostrowski type inequality in two dimensions using the three point rule. Proceedings of the 1999 International Conference on Computational Techniques and Applications (Canberra). ANZIAM J. 42 (2000), (C), C671–C689.

38. Cerone, P.; Dragomir, S. S.; Roumeliotis, J. An Ostrowski type inequality for mappings whose second derivatives belong to $L_p(a, b)$and applications. J. Indian Math. Soc. (N.S.) 67 (2000), no. 1-4, 59–67.

39. Dragomir, S. S. On the Ostrowski's integral inequality for mappings with bounded variation and applications. Math. Inequal. Appl. 4 (2001), no. 1, 59–66.

40. Pachpatte, B. G. On a new Ostrowski type inequality in two independent variables. Tamkang J. Math. 32 (2001), no. 1, 45–49.

41. Dragomir, S. S. A generalization of the Ostrowski integral inequality for mappings whose derivatives belong to $L_p[a, b]$ and applications in numerical integration. J. Math. Anal. Appl. 255 (2001), no. 2, 605–626.

42. Barnett, N. S.; Dragomir, S. S. An Ostrowski type inequality for double integrals and applications for cubature formulae. Soochow J. Math. 27 (2001), no. 1, 1–10.

43. Sofo, A.; Dragomir, S. S. An inequality of Ostrowski type for twice differentiable mappings in terms of the L_p norm and applications. Soochow J. Math. 27 (2001), no. 1, 97–111.

44. Matić, M.; Pečarić, J. Two-point Ostrowski inequality. Math. Inequal. Appl. 4 (2001), no. 2, 215–221.

45. Cheng, Xiao-Liang Improvement of some OstrowskiGrüss type inequalities. Comput. Math. Appl. 42 (2001), no. 1-2, 109–114.

46. Matić, M.; Pečarić, J.; Ujević, N. Weighted version of multivariate Ostrowski type inequalities. Rocky Mountain J. Math. 31 (2001), no. 2, 511–538.

47. Dragomir, S. S. Better bounds in some OstrowskiGrüss type inequalities. Tamkang J. Math. 32 (2001), no. 3, 211–216.

48. Barnett, N. S.; Dragomir, S. S.; Sofo, A. Better bounds for an inequality of the Ostrowski type with applications. Demonstratio Math. 34 (2001), no. 3, 533–542.

49. Dragomir, S. S. A generalization of Ostrowski integral inequality for mappings whose derivatives belong to $L_1[a, b]$ and applications in numerical integration. J. Comput. Anal. Appl. 3 (2001), no. 4, 343–360.

50. Sofo, A.; Dragomir, S. S. A perturbed version of the Ostrowski inequality for twice differentiable mappings. Turkish J. Math. 25 (2001), no. 3, 379–412.

51. Cerone, P.; Dragomir, S. S. Some bounds in terms of Δ-seminorms for OstrowskiGrüss type inequalities. Soochow J. Math. 27 (2001), no. 4, 423–434.

52. Dragomir, S. S.; Fang, M. L. Improvement of an Ostrowski type inequality for monotonic mappings and its application for some special means. JIPAM. J. Inequal. Pure Appl. Math. 2 (2001), no. 3, Article 31, 6 pp.

53. Dragomir, S. S. On the Ostrowski's integral inequality for Lipschitzian mappings and applications. Studia Univ. Babeş-Bolyai Math. 46 (2001), no. 1, 33–40.

54. Sofo, A. A perturbed version of the Ostrowski integral inequality. Proceedings of the Third World Congress of Nonlinear Analysts, Part 4 (Catania, 2000). Nonlinear Anal. 47 (2001), no. 4, 2353–2364.

55. Pečarić, Josip; Varošanec, Sanja Harmonic polynomials and generalization of Ostrowski inequality with applications in numerical integration. Proceedings of the Third World Congress of Nonlinear Analysts, Part 4 (Catania, 2000). Nonlinear Anal. 47 (2001), no. 4, 2365–2374.

56. Dragomir, S. S.; Gluščević, V.; Pearce, C. E. M. Csiszár f-divergence, Ostrowski's inequality and mutual information. Proceedings of the Third World Congress of Nonlinear Analysts, Part 4 (Catania, 2000). Nonlinear Anal. 47 (2001), no. 4, 2375–2386.

57. Pachpatte, B. G. An inequality of Ostrowski type in n independent variables. Facta Univ. Ser. Math. Inform. No. 16 (2001), 21–24.

58. Barnett, N. S.; Cerone, P.; Dragomir, S. S.; Roumeliotis, J.; Sofo, A. A survey on Ostrowski type inequalities for twice differentiable mappings and applications. Inequality theory and applications. Vol. I, 33–85, Nova Sci. Publ., Huntington, NY, 2001.

59. Sofo, A.; Dragomir, S. S. An OstrowskiGrüss type inequality for twice differentiable mappings using a multi-branch Peano kernel. Inequality theory and applications. Vol. I, 281–294, Nova Sci. Publ., Huntington, NY, 2001.

60. Pachpatte, B. G. A note on Ostrowski type inequalities. Demonstratio Math. 35 (2002), no. 1, 27–30.

61. Dragomir, S. S. The discrete version of Ostrowski's inequality in normed linear spaces. JIPAM. J. Inequal. Pure Appl. Math. 3 (2002), no. 1, Article 2, 16 pp.

62. Barnett, N. S.; Buşe, C.; Cerone, P.; Dragomir, S. S. On weighted Ostrowski type inequalities for operators and vector-valued functions. JIPAM. J. Inequal. Pure Appl. Math. 3 (2002), no. 1, Article 12, 21 pp.

63. Cerone, P. A new Ostrowski type inequality involving integral means over end intervals. Tamkang J. Math. 33 (2002), no. 2, 109–118.

64. Anastassiou, George A. Univariate Ostrowski inequalities, revisited. Monatsh. Math. 135 (2002), no. 3, 175–189.

65. Sofo, A. Integral inequalities of the Ostrowski type. JIPAM. J. Inequal. Pure Appl. Math. 3 (2002), no. 2, Article 21, 39 pp.

66. Anastassiou, George A. Multivariate Montgomery identities and Ostrowski inequalities. Numer. Funct. Anal. Optim. 23 (2002), no. 3-4, 247–263.

67. Dragomir, S. S. Approximating the finite Hilbert transform via an Ostrowski type inequality for functions of bounded variation. JIPAM. J. Inequal. Pure Appl. Math. 3 (2002), no. 4, Article 51, 19 pp.

68. Pachpatte, B. G. On multivariate Ostrowski type inequalities. JIPAM. J. Inequal. Pure Appl. Math. 3 (2002), no. 4, Article 58, 5 pp.

69. Barnett, N. S.; Buşe, C.; Cerone, P.; Dragomir, S. S. Ostrowski's inequality for vector-valued functions and applications. Comput. Math. Appl. 44 (2002), no. 5-6, 559–572.

70. Cerone, P. On relationships between Ostrowski, trapezoidal and Chebychev identities and inequalities. Soochow J. Math. 28 (2002), no. 3, 311–328.

71. Dragomir, S. S. A refinement of Ostrowski's inequality for absolutely continuous functions whose derivatives belong to L_∞ and applications. Libertas Math. 22 (2002), 49–63.

72. Lü, Zhong Xue On a generalized Ostrowski integral inequality. (Chinese) J. Xuzhou Norm. Univ. Nat. Sci. Ed. 20 (2002), no. 3, 19–22.

73. Dragomir, Sever S.; Rassias, Themistocles M. Generalisations of the Ostrowski inequality and applications. Ostrowski type inequalities and applications in numerical integration, 1–63, Kluwer Acad. Publ., Dordrecht, 2002.

74. Barnett, Neil S.; Cerone, Pietro; Dragomir, Sever S. Ostrowski type inequalities for multiple integrals. Ostrowski type inequalities and applications in numerical integration, 285–329, Kluwer Acad. Publ., Dordrecht, 2002.

75. Hanna, George Some results for double integrals based on an Ostrowski type inequality. Ostrowski type inequalities and applications in numerical integration, 331–371, Kluwer Acad. Publ., Dordrecht, 2002.

76. Dedić, Lj.; Matic, M.; Pečarić, J. Some further generalizations of Ostrowski inequality for Hölder functions and functions with bounded derivatives. J. Comput. Anal. Appl. 4 (2002), no. 4, 313–337.

77. Hanna, G.; Dragomir, S. S.; Cerone, P. A general Ostrowski type inequality for double integrals. Tamkang J. Math. 33 (2002), no. 4, 319–333.

78. Zhongxue, Lü; Hongzheng, Xie On some new generalizations of Ostrowski integral inequality. Int. J. Math. Math. Sci. 32 (2002), no. 2, 93–101.

79. Dragomir, S. S. Approximating the finite Hilbert transform via Ostrowski type inequalities for absolutely continuous functions. Bull. Korean Math. Soc. 39 (2002), no. 4, 543–559.

80. Dragomir, S. S. A refinement of Ostrowski's inequality for absolutely continuous functions and applications. Acta Math. Vietnam. 27 (2002), no. 2, 203–217.

81. Anastassiou, G. A. Multidimensional Ostrowski inequalities, revisited. Acta Math. Hungar. 97 (2002), no. 4, 339–353.

82. Matić, M.; Pečarić, J.; Ujević, N. Generalization of an inequality of Ostrowski type and some related results. Indian J. Math. 44 (2002), no. 2, 189–209.

83. Anastassiou, George A. Probabilistic Ostrowski type inequalities. Stochastic Anal. Appl. 20 (2002), no. 6, 1177–1189.

84. Ujević, Nenad Inequalities of Ostrowski-Grüss type and applications. Appl. Math. (Warsaw) 29 (2002), no. 4, 465–479.

85. Dedić, L. J.; Matić, M.; Pečarić, J.; Vukelić, A. On generalizations of Ostrowski inequality via Euler harmonic identities. J. Inequal. Appl. 7 (2002), no. 6, 787–805.

86. Buşe, C.; Dragomir, S. S.; Sofo, A. Ostrowski's inequality for vector-valued functions of bounded semivariation and applications. New Zealand J. Math. 31 (2002), no. 2, 137–152.

87. Ujević, Nenad Perturbations of an Ostrowski type inequality and applications. Int. J. Math. Math. Sci. 32 (2002), no. 8, 491–500.

88. Dragomir, S. S. Ostrowski type inequalities for isotonic linear functionals. JIPAM. J. Inequal. Pure Appl. Math. 3 (2002), no. 5, Article 68, 13 pp.

89. Yang, Xiaojing Refinement of Hölder inequality and application to Ostrowski inequality. Appl. Math. Comput. 138 (2003), no. 2-3, 455–461.

90. Dragomir, S. S. A weighted Ostrowski type inequality for functions with values in Hilbert spaces and applications. J. Korean Math. Soc. 40 (2003), no. 2, 207–224.

91. Dedić, Lj.; Pečarić, J.; Ujević, N. On generalizations of Ostrowski inequality and some related results. Czechoslovak Math. J. 53(128) (2003), no. 1, 173–189

92. Pachpatte, B. G. On multidimensional Ostrowski and Grüss type finite difference inequalities. JIPAM. J. Inequal. Pure Appl. Math. 4 (2003), no. 1, Article 7, 8 pp.

93. Ujević, Nenad Inequalities of Ostrowski type and applications in numerical integration. Appl. Math. E-Notes 3 (2003), 71–79

94. Dragomir, S. S. Some new inequalities of Ostrowski type. Austral. Math. Soc. Gaz. 30 (2003), no. 1, 25–30

95. Anastassiou, George A. Fractional Ostrowski type inequalities. Commun. Appl. Anal. 7 (2003), no. 2-3, 203–208.

96. Dragomir, S. S. Improvements of Ostrowski and generalised trapezoid inequality in terms of the upper and lower bounds of the first derivative. Tamkang J. Math. 34 (2003), no. 3, 213–222.

97. Ujević, N. New bounds for the first inequality of Ostrowski-Grüss type and applications. Comput. Math. Appl. 46 (2003), no. 2-3, 421–427.

98. Anastassiou, George A. Fuzzy Ostrowski type inequalities. Comput. Appl. Math. 22 (2003), no. 2, 279–292.

99. Dragomir, Sever S. An Ostrowski like inequality for convex functions and applications. Rev. Mat. Complut. 16 (2003), no. 2, 373–382.

100. Barnett, N. S.; Cerone, P.; Dragomir, S. S.; Pinheiro, M. R.; Sofo, A. Ostrowski type inequalities for functions whose modulus of the derivatives are convex and applications. Inequality theory and applications, Vol. 2 (Chinju/Masan, 2001), 19–32, Nova Sci. Publ., Hauppauge, NY, 2003

101. Dragomir, S. S.; Sofo, A. An integral inequality related to the Ostrowski result and applications. Inequality theory and applications, Vol. 2 (Chinju/Masan, 2001), 91–101, Nova Sci. Publ., Hauppauge, NY, 2003.

102. Hanna, G.; Dragomir, S. S. Some Ostrowski type inequalities for double integrals of functions whose partial derivatives satisfy certain convexity properties. Inequality theory and applications, Vol. 2 (Chinju/Masan, 2001), 113–126, Nova Sci. Publ., Hauppauge, NY, 2003.

103. Ujević, Nenad Ostrowski-Grüss type inequalities in two dimensions. JIPAM. J. Inequal. Pure Appl. Math. 4 (2003), no. 5, Article 101, 12 pp.

104. Lü, Zhongxue On Ostrowski integral inequality in three independent variables and applications. Int. J. Math. Sci. 2 (2003), no. 1, 185–203.

105. Ujević, Nenad Two sharp Ostrowskilike inequalities and applications. Methods Appl. Anal. 10 (2003), no. 3, 477–486.

106. Roumeliotis, John Improved weighted OstrowskiGrüss type inequalities. Inequality theory and applications. Vol. 3, 153–160, Nova Sci. Publ., Hauppauge, NY, 2003.

107. Ujević, Nenad Generalized perturbed inequalities of Ostrowski type and applications. Inequality theory and applications. Vol. 3, 197–206, Nova Sci. Publ., Hauppauge, NY, 2003.

108. Ujević, N. A generalization of Ostrowski's inequality and applications in numerical integration. Appl. Math. Lett. 17 (2004), no. 2, 133–137.

109. Pachpatte, B. G. A note on Ostrowski and Grüss type discrete inequalities. Tamkang J. Math. 35 (2004), no. 1, 61–65.

110. Hwang, Dah-Yan; Yang, Gou-Sheng On Ostrowski and Grüss type discrete inequalities for second forward differences. Tamkang J. Math. 35 (2004), no. 1, 87–93.

111. Pachpatte, B. G. New Ostrowski type inequalities for mappings whose derivatives belong to Lp spaces. Demonstratio Math. 37 (2004), no. 2, 293–298.

112. Cerone, Pietro; Dragomir, Sever S. Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions. Demonstratio Math. 37 (2004), no. 2, 299–308.

113. Pachpatte, B. G. New inequalities of Ostrowski type for twice differentiable mappings. Tamkang J. Math. 35 (2004), no. 3, 219–226.

114. Pachpatte, B. G. On a new generalisation of Ostrowski's inequality. JIPAM. J. Inequal. Pure Appl. Math. 5 (2004), no. 2, Article 36, 4 pp.

115. Ujević, N. Sharp inequalities of Simpson type and Ostrowski type. Comput. Math. Appl. 48 (2004), no. 1-2, 145–151.

116. Pachpatte, B. G. New inequalities of Ostrowski and trapezoid type for n-time differentiable functions. Bull. Korean Math. Soc. 41 (2004), no. 4, 633–639.

117. Aljinović, A. A.; Pečarić, J. Discrete weighted Montgomery identity and discrete Ostrowski type inequalities. Comput. Math. Appl. 48 (2004), no. 5-6, 731–745.

118. Dragomir, S. S. Some companions of Ostrowski's inequality for absolutely continuous functions and applications. Facta Univ. Ser. Math. Inform. No. 19 (2004), 1–16.

119. Ujević, Nenad Inequalities of Ostrowski type in two dimensions. Rocky Mountain J. Math. 35 (2005), no. 1, 331–347.

120. Wu, Qingbiao; Yang, Shijun A note to Ujević's generalization of Ostrowski's inequality. Appl. Math. Lett. 18 (2005), no. 6, 657–665.

121. de la Cal, J.; Cárcamo, J. A general Ostrowskitype inequality. Statist. Probab. Lett. 72 (2005), no. 2, 145–152.

122. Dragomir, S. S. Some Ostrowski type inequalities via Cauchy's mean value theorem. New Zealand J. Math. 34 (2005), no. 1, 31–42.

123. Dragomir, S. S. Some companions of Ostrowski's inequality for absolutely continuous functions and applications. Bull. Korean Math. Soc. 42 (2005), no. 2, 213–230.

124. Kumar, P. The Ostrowski type moment integral inequalities and moment-bounds for continuous random variables. Comput. Math. Appl. 49 (2005), no. 11-12, 1929–1940.

125. Dragomir, S. S. An Ostrowski type inequality for convex functions. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. 16 (2005), 12–25.

126. Dragomir, S. S. An inequality of Ostrowski type via Pompeiu's mean value theorem. JIPAM. J. Inequal. Pure Appl. Math. 6 (2005), no. 3, Article 83, 9 pp.

127. Pachpatte, B. G. A note on Ostrowski like inequalities. JIPAM. J. Inequal. Pure Appl. Math. 6 (2005), no. 4, Article 114, 4 pp.

128. Pachpatte, B. G. On OstrowskiGrüss-Čebyšev type inequalities for functions whose modulus of derivatives are convex. JIPAM. J. Inequal. Pure Appl. Math. 6 (2005), no. 4, Article 128, 15 pp.

129. Aglić Aljinović, A.; Pečarić, J. On some Ostrowski type inequalities via Montgomery identity and Taylor's formula. Tamkang J. Math. 36 (2005), no. 3, 199–218.

130. Čuljak, V.; Pečarić, J.; Persson, L. E. A generalized Simpson, trapezoid and Ostrowski type inequalities for convex functions. Soochow J. Math. 31 (2005), no. 4, 617–627.

131. Aglić Aljinović, A.; Marangunić, Lj.; Pečarić, J. On Landau type inequalities via Ostrowski inequalities. Nonlinear Funct. Anal. Appl. 10 (2005), no. 4, 565–579.

132. Aglić Aljinović, A.; Pečarić, J.; Vukelić, A. On some Ostrowski type inequalities via Montgomery identity and Taylor's formula. II. Tamkang J. Math. 36 (2005), no. 4, 279–301.

133. Florea, Aurelia; Niculescu, Constantin P. A note on Ostrowski's inequality. J. Inequal. Appl. 2005, no. 5, 459–468.

134. Dragomir, S. S. Ostrowski type inequalities for functions defined on linear spaces and applications for semi-inner products. J. Concr. Appl. Math. 3 (2005), no. 1, 91–103

135. Chen, Ying Shu; Yu, Yuan Hong An inequality of Ostrowski type for twice differentiable functions. (Chinese) Math. Practice Theory 35 (2005), no. 12, 188–192.

136. Aglić Aljinović, A.; Matić, M.; Pečarić, J. Improvements of some Ostrowski type inequalities. J. Comput. Anal. Appl. 7 (2005), no. 3, 289–304.

137. Pachpatte, B. G. New Ostrowski and Grüss type inequalities. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) 51 (2005), no. 2, 377–386 (2006).

138. Gavrea, Ioan On an Ostrowski type inequality. Automat. Comput. Appl. Math. 14 (2005), no. 1, 57–61.

139. Dragomir, S. S.; Sofo, A. An inequality for monotonic functions generalizing Ostrowski and related results. Comput. Math. Appl. 51 (2006), no. 3-4, 497–506.

140. Anastassiou, George A. Csiszar and Ostrowski type inequalities via Euler-type and Fink identities. Panamer. Math. J. 16 (2006), no. 2, 77–91

141. Anastasiou, K. S.; Kechriniotis, Aristides I.; Kotsos, B. A. Generalizations of the Ostrowski's inequality. J. Interdiscip. Math. 9 (2006), no. 1, 49–60.

142. Yang, Shijun A unified approach to some inequalities of Ostrowski-Grüss type. Comput. Math. Appl. 51 (2006), no. 6-7, 1047–1056.

143. Liu, Zheng A sharp integral inequality of Ostrowski-Grüss type. Soochow J. Math. 32 (2006), no. 2, 223–231.

144. Pachpatte, B. G. A new Ostrowski type inequality for double integrals. Soochow J. Math. 32 (2006), no. 2, 317–322.

145. Mir, Nazir Ahmad; Rafiq, Arif A note on Ostrowski like inequalities in $L_1(a,b)$ spaces. Gen. Math. 14 (2006), no. 1, 23–30.

146. Barnett, N. S.; Dragomir, S. S. Applications of Ostrowski's version of the Grüss inequality for trapezoid type rules. Tamkang J. Math. 37 (2006), no. 2, 163–173.

147. Li, Xiu Qin; Song, Guo Hua; Yu, Yuan Hong Some generalizations of Ostrowski's inequality. (Chinese) Heilongjiang Daxue Ziran Kexue Xuebao 23 (2006), no. 4, 483–485.

148. Pachpatte, B. G. New Ostrowski type inequalities involving the product of two functions. JIPAM. J. Inequal. Pure Appl. Math. 7 (2006), no. 3, Article 104, 5 pp.

149. Rafiq, A.; Mir, Nazir Ahmad An Ostrowski type inequality for p-norms. JIPAM. J. Inequal. Pure Appl. Math. 7 (2006), no. 3, Article 112, 7 pp.

150. Rafiq, A.; Mir, N. A.; Zafar, Fiza A generalized OstrowskiGrüss type inequality for twice differentiable mappings and applications. JIPAM. J. Inequal. Pure Appl. Math. 7 (2006), no. 4, Article 124, 7 pp.

151. Pachpatte, B. G. New Ostrowski type inequalities via mean value theorems. JIPAM. J. Inequal. Pure Appl. Math. 7 (2006), no. 4, Article 137, 5 pp.

152. Pečarić, Josip; Ungar, Šime On an inequality of Ostrowski type. JIPAM. J. Inequal. Pure Appl. Math. 7 (2006), no. 4, Article 151, 5 pp.

153. Jia, Baoguo A generalization for Ostrowski's inequality in R2. JIPAM. J. Inequal. Pure Appl. Math. 7 (2006), no. 5, Article 182, 4 pp. (electronic).

154. Rafiq, A.; Barnett, N. S. A note on a weighted Ostrowski type inequality for cumulative distribution functions. JIPAM. J. Inequal. Pure Appl. Math. 7 (2006), no. 5, Article 188, 5 pp.

155. Liu, Zheng A sharp inequality of Ostrowski-Grüss type. JIPAM. J. Inequal. Pure Appl. Math. 7 (2006), no. 5, Article 192, 10 pp.

156. Pachpatte, B. G. New Ostrowski and Grüss type inequalities. Tamsui Oxf. J. Math. Sci. 22 (2006), no. 2, 159–166.

157. Wang, Chung-Shin; Ho, Ming-In; Hwang, Shiow-Ru Ostrowski and Grüss type inequalities involving several functions. Tamsui Oxf. J. Math. Sci. 22 (2006), no. 2, 231–241.

158. Anastassiou, George A. Multivariate Euler type identity and Ostrowski type inequalities. Numerical analysis and approximation theory, 27–54, Casa Cărţii de ştiinţă, Cluj-Napoca, 2006.

159. Pachpatte, B. G. New Ostrowski and Grüss type inequalities for double integrals. Demonstratio Math. 39 (2006), no. 4, 783–791

160. Rafiq, Arif; Mir, Nazir Ahmad; Ahmad, Farooq Weighted Ostrowski type inequality for differentiable mappings whose first derivatives belong to Lp(a,b). Gen. Math. 14 (2006), no. 3, 91–102.

161. Ahmad, Farooq; Rafiq, Arif; Mir, Nazir Ahmad Weighted OstrowskiGrüss type inequality for differentiable mappings. Glob. J. Pure Appl. Math. 2 (2006), no. 2, 147–154.

162. Pachpatte, B. G. On generalizations of Ostrowski type inequalities and some related results. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) 52 (2006), no. 2, 365–375.

163. Liu, Zheng Some inequalities of Ostrowski type and applications. Appl. Math. E-Notes 7 (2007), 93–101

164. Cheung, W.-S.; Dragomir, S. S. Two Ostrowski type inequalities for the Stieltjes integral of monotonic functions. Bull. Austral. Math. Soc. 75 (2007), no. 2, 299–311.

165. Liu, Zheng Some sharp OstrowskiGrüss type inequalities. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. 18 (2007), 14–21.

166. Anastassiou, George A. High order Ostrowski type inequalities. Appl. Math. Lett. 20 (2007), no. 6, 616–621.

167. Liu, Zheng Note on a paper by N. Ujević: "Sharp inequalities of Simpson type and Ostrowski type” [Comput. Math. Appl. 48 (2004), no. 1-2, 145–151; MR2086792]. Appl. Math. Lett. 20 (2007), no. 6, 659–663.

168. Hwang, Shiow-Ru Ostrowski-Grüss-Čebyšev type inequalities involving several functions. Tamsui Oxf. J. Math. Sci. 23 (2007), no. 1, 105–125.

169. Liu, Zheng Some OstrowskiGrüss type inequalities and applications. Comput. Math. Appl. 53 (2007), no. 1, 73–79.

170. Aglić Aljinović, A.; Hoxha, R.; Pečarić, J. On some Ostrowski type integral inequalities. Sarajevo J. Math. 3(15) (2007), no. 1, 29–40.

171. Rafiq, Arif; Ahmad, Farooq; Ahmad Mir, Nazir New OstrowskiGrüss type inequalities. Int. J. Appl. Math. 20 (2007), no. 4, 483–493.

172. Anastassiou, George A. Multivariate Fink type identity and multivariate Ostrowski, comparison of means and Grüss type inequalities. Math. Comput. Modelling 46 (2007), no. 3-4, 351–374.

173. Anastassiou, George A. Optimal multivariate Ostrowski Euler type inequalities. Stud. Univ. Babeş-Bolyai Math. 52 (2007), no. 1, 25–61.

174. Anastassiou, George A.; Goldstein, Jerome A. Ostrowski type inequalities over Euclidean domains. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 18 (2007), no. 3, 305–310.

175. Anastassiou, George A. Taylor-Widder representation formulae and Ostrowski, Grüss, integral means and Csiszar type inequalities. Comput. Math. Appl. 54 (2007), no. 1, 9–23.

176. Cerone, P.; Cheung, W. S.; Dragomir, S. S. On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrators of bounded variation. Comput. Math. Appl. 54 (2007), no. 2, 183–191.

177. Anastassiou, George Multivariate Euler type identity and optimal multivariate Ostrowski type inequalities. Adv. Nonlinear Var. Inequal. 10 (2007), no. 2, 51–104.

178. Anastassiou, George A. Multivariate fractional Ostrowski type inequalities. Comput. Math. Appl. 54 (2007), no. 3, 434–447.

179. Rafiq, Arif; Mir, Nazir Ahmad; Ahmad, Farooq Weighted Čebyšev-Ostrowski type inequalities. Appl. Math. Mech. (English Ed.) 28 (2007), no. 7, 901–906.

180. Aljinović, A. Aglić; Pečarić, J. Generalizations of weighted Euler identity and Ostrowski type inequalities. Adv. Stud. Contemp. Math. (Kyungshang) 14 (2007), no. 1, 141–151.

181. Pachpatte, B. G. Some new Ostrowski and Grüss type inequalities. Tamkang J. Math. 38 (2007), no. 2, 111–120.

182. Čivljak, A.; Dedić, Lj.; Matić, M. On Ostrowski and Euler-Grüss type inequalities involving measures. J. Math. Inequal. 1 (2007), no. 1, 65–81.

183. Rafiq, Arif; Ahmad, Farooq Ostrowski, Grüss, Čebyšev type inequalities for functions whose second derivatives belong to Lp(a,b) and whose modulus of second derivatives are convex. Rev. Colombiana Mat. 41 (2007), no. 1, 1–13.

184. Pachpatte, B. G. New Ostrowski type inequality for triple integrals. Tamkang J. Math. 38 (2007), no. 3, 253–259.

185. Pachpatte, B. G. New inequalities of Ostrowski-Grüss type. Fasc. Math. No. 38 (2007), 97–104.

186. Pachpatte, B. G. New discrete OstrowskiGrüss like inequalities. Facta Univ. Ser. Math. Inform. 22 (2007), no. 1, 15–20.

187. Popa, Emil C. An inequality of Ostrowski type via a mean value theorem. Gen. Math. 15 (2007), no. 1, 93–100.

188. Barnett, N. S.; Dragomir, S. S. On the weighted Ostrowski inequality. JIPAM. J. Inequal. Pure Appl. Math. 8 (2007), no. 4, Article 96, 10 pp.

189. Anastassiou, George A. Ostrowski type inequalities over balls and shells via a Taylor-Widder formula. JIPAM. J. Inequal. Pure Appl. Math. 8 (2007), no. 4, Article 106, 13 pp.

190. Liu, Wen-jun; Xue, Qiao-ling; Wang, Shun-Feng Several new perturbed Ostrowskilike type inequalities. JIPAM. J. Inequal. Pure Appl. Math. 8 (2007), no. 4, Article 110, 6 pp.

191. Pachpatte, B. G. On a new generalization of Ostrowski type inequality. Tamkang J. Math. 38 (2007), no. 4, 335–339.

192. Rafiq, Arif; Zafar, Fiza New bounds for the first inequality of Ostrowski-Grüss type and applications in numerical integration. Nonlinear Funct. Anal. Appl. 12 (2007), no. 1, 75–85.

193. Pachpatte, B. G. New generalization of certain Ostrowski type inequalities. Tamsui Oxf. J. Math. Sci. 23 (2007), no. 4, 393–403.

194. Dinu, Cristian Ostrowski type inequalities on time scales. An. Univ. Craiova Ser. Mat. Inform. 34 (2007), 43–58.

195. Kechriniotis, Aristides I. Complements of Ostrowski type inequalities. Appl. Math. E-Notes 8 (2008), 89–97.

196. Anastassiou, George A.; Goldstein, Jerome A. Higher order Ostrowski type inequalities over Euclidean domains. J. Math. Anal. Appl. 337 (2008), no. 2, 962–968.

197. Bohner, Martin; Matthews, Thomas Ostrowski inequalities on time scales. JIPAM. J. Inequal. Pure Appl. Math. 9 (2008), no. 1, Article 6, 8 pp.

198. Tseng, Kuei-Lin; Hwang, Shiow Ru; Dragomir, S. S. Generalizations of weighted Ostrowski type inequalities for mappings of bounded variation and their applications. Comput. Math. Appl. 55 (2008), no. 8, 1785–1793.

199. Yeh, Cheh-Chih Ostrowski's inequality on time scales. Appl. Math. Lett. 21 (2008), no. 4, 404–409.

200. Liu, Wenjun; Dong, Jianwei On new Ostrowski type inequalities. Demonstratio Math. 41 (2008), no. 2, 317–322.

201. Liu, Zheng A note on an Ostrowski type inequality. Tamsui Oxf. J. Math. Sci. 24 (2008), no. 1, 7–10.

202. Hwang, Shiow-Ru; Cheng, Kuang-Tsan; Wang, Chung-Shin A note on multivariate Ostrowski type inequalities. Tamsui Oxf. J. Math. Sci. 24 (2008), no. 1, 97–108.

203. Liu, Zheng A sharp generalized OstrowskiGrüss inequality. Tamsui Oxf. J. Math. Sci. 24 (2008), no. 2, 175–184.

204. Rafiq, Arif; Mir, Nazir Ahmad; Zafar, Fiza A generalized Ostrowski type inequality for a random variable whose probability density function belongs to $L_\infty[a,b]$. Demonstratio Math. 41 (2008), no. 3, 723–732.

205. Liu, Wen-Jun; Huang, Yu; Pan, Xing-xia New weighted OstrowskiGrüss-Čebyšev type inequalities. Bull. Korean Math. Soc. 45 (2008), no. 3, 477–483.

206. Liu, Zheng A sharp L_2 inequality of Ostrowski type. ANZIAM J. 49 (2008), no. 3, 423–429.

207. Rafiq, Arif; Mir, Nazir Ahmad; Zafar, Fiza A generalized Ostrowski type inequality for a random variable whose probability density function belongs to Lp[a,b]. Appl. Math. E-Notes 8 (2008), 246–253.

208. Liu, Zheng Some Ostrowski type inequalities. Math. Comput. Modelling 48 (2008), no. 5-6, 949–960.

209. Anastassiou, George A. Ostrowski type inequalities over spherical shells. Serdica Math. J. 34 (2008), no. 3, 629–650.

210. Liu, Wenjun; Ngô, Quõc-Anh A generalization of Ostrowski inequality on time scales for k points. Appl. Math. Comput. 203 (2008), no. 2, 754–760.

211. Dragomir, Sever S. Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation. Arch. Math. (Basel) 91 (2008), no. 5, 450–460.

212. Lü, Zhongxue On sharp inequalities of Simpson type and Ostrowski type in two independent variables. Comput. Math. Appl. 56 (2008), no. 8, 2043–2047.

213. Miao, Yu; Shen, Luming Some general OstrowskiGrüss type inequalities. Acta Math. Acad. Paedagog. Nyházi. (N.S.) 24 (2008), no. 2, 243–248.

214. Kikianty, Eder; Dragomir, S. S.; Cerone, P. Sharp inequalities of Ostrowski type for convex functions defined on linear spaces and application. Comput. Math. Appl. 56 (2008), no. 9, 2235–2246.

215. Qayyum, A. A weighted OstrowskiGrüss type inequality for twice differentiable mappings and applications. Int. J. Math. Comput. 1 (2008), N08, 63–71.

216. Kikianty, Eder; Dragomir, Sever S.; Cerone, Pietro Ostrowski type inequality for absolutely continuous functions on segments in linear spaces. Bull. Korean Math. Soc. 45 (2008), no. 4, 763–780.

217. Liu, Wenjun; Quoc Anh Ngô; Chen, Wenbing A perturbed Ostrowskitype inequality on time scales for k points for functions whose second derivatives are bounded. J. Inequal. Appl. 2008, Art. ID 597241, 12 pp.

218. Karpuz, Başak; Özkan, Umut Mutlu Generalized Ostrowski's inequality on time scales. JIPAM. J. Inequal. Pure Appl. Math. 9 (2008), no. 4, Article 112, 7 pp.

219. Rafiq, Arif; Mir, Nazir Ahmad; Zafar, Fiza A generalization of OstrowskiGrüss type inequality for twice differentiable mappings in Euclidean norm. Gen. Math. 16 (2008), no. 3, 51–72.

220. Mir, Nazir Ahmad; Rafiq, Arif; Rizwan, Muhammad Ostrowski Grüss Čebyšev type inequalities for functions whose modulus of second derivatives are convex. Gen. Math. 16 (2008), no. 3, 73–97.

221. Rafiq, Arif; Ahmad, Farooq Another weighted OstrowskiGrüss type inequality for twice differentiable mappings. Kragujevac J. Math. 31 (2008), 43–51.

222. Tseng, Kuei-Lin Improvements of some inequalities of Ostrowski type and their applications. Taiwanese J. Math. 12 (2008), no. 9, 2427–2441.

223. Tong, Fengru; Guan, Linlin A simple proof of the generalized OstrowskiGrüss type integral inequality. Int. J. Math. Anal. (Ruse) 2 (2008), no. 17-20, 889–892.

224. Liu, Zheng Another sharp L2 inequality of Ostrowski type. ANZIAM J. 50 (2008), no. 1, 129–136.

225. Ngô, Quoc Anh; Liu, Wenjun A sharp Grüss type inequality on time scales and application to the sharp OstrowskiGrüss inequality. Commun. Math. Anal. 6 (2009), no. 2, 33–41.

226. Barnett, N. S.; Cheung, W.-S.; Dragomir, S. S.; Sofo, A. Ostrowski and trapezoid type inequalities for the Stieltjes integral with Lipschitzian integrands or integrators. Comput. Math. Appl. 57 (2009), no. 2, 195–201.

227. Miao, Yu Note on the discrete OstrowskiGrüss type inequality. J. Math. Inequal. 3 (2009), no. 1, 93–98.

228. Liu, Zheng Some companions of an Ostrowski type inequality and applications. JIPAM. J. Inequal. Pure Appl. Math. 10 (2009), no. 2, Article 52, 12 pp.

229. Huy, Vu Nhat; Ngô, Quoc-Anh New inequalities of Ostrowski-like type involving n knots and the L_p-norm of the m-th derivative. Appl. Math. Lett. 22 (2009), no. 9, 1345–1350.

230. Acu, Ana Maria; Gonska, Heiner Ostrowski inequalities and moduli of smoothness. Results Math. 53 (2009), no. 3-4, 217–228.

231. Liu, Zheng Some Ostrowski type inequalities and applications. Vietnam J. Math. 37 (2009), no. 1, 15–22.

232. Barnett, N. S.; Dragomir, S. S.; Gomm, I. A companion for the Ostrowski and the generalised trapezoid inequalities. Math. Comput. Modelling 50 (2009), no. 1-2, 179–187.

233. Niezgoda, Marek A new inequality of Ostrowski-Grüss type and applications to some numerical quadrature rules. Comput. Math. Appl. 58 (2009), no. 3, 589–596.

234. Wang, Mingjin; Zhao, Xilai Ostrowski type inequalities for higher-order derivatives. J. Inequal. Appl. 2009, Art. ID 162689, 8 pp.

235. Pachpatte, B. G. New inequalities of Ostrowski and Grüss type for triple integrals. Tamkang J. Math. 40 (2009), no. 2, 117–127.

236. Liu, Wenjun; Ngô, Quoc-Anh An OstrowskiGrüss type inequality on time scales. Comput. Math. Appl. 58 (2009), no. 6, 1207–1210.

237. Anastassiou, George A. Caputo fractional Ostrowski type inequalities. J. Appl. Funct. Anal. 4 (2009), no. 2, 218–236.

238. Liu, Wenjun J.; Quoc-Anh Ngô; Chen, Wenbing B. A new generalization of Ostrowski type inequality on time scales. An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 17 (2009), no. 2, 101–114.

239. Aglić Aljinović, A. A note on generalization of weightednn Čebyšev and Ostrowski inequalities. J. Math. Inequal. 3 (2009), no. 3, 409–416.

240. Matić, Marko; Ungar, Šime More on the two-point Ostrowski inequality. J. Math. Inequal. 3 (2009), no. 3, 417–426.

241. Ahmad, Farooq; Barnett, N. S.; Dragomir, S. S. New weighted Ostrowski and Čebyšev type inequalities. Nonlinear Anal. 71 (2009), no. 12, e1408–e1412.

242. Rafiq, Arif; Ahmad, Farooq On weighted Ostrowski type inequalities in $L_1(a,b)$ spaces. Math. Commun. 14 (2009), no. 1, 27–33.

3. After 2010

243. Liu, Wen-Jun; Xue, Qiao-Ling; Wang, Shun-Feng New generalization of perturbed Ostrowski type inequalities and applications. J. Appl. Math. Comput. 32 (2010), no. 1, 157–169.

244. Lian, Bao-Sheng; Yang, Qiao-Hua Ostrowski type inequalities on H-type groups. J. Math. Anal. Appl. 365 (2010), no. 1, 158–166.

245. Zhao, Changjian; Cheung, Wing-Sum On Ostrowskitype inequalities for higher-order partial derivatives. J. Inequal. Appl. 2010, Art. ID 960672, 8 pp.

246. Özkan, Umut Mutlu; Yildirim, Hüseyin Ostrowski type inequality for double integrals on time scales. Acta Appl. Math. 110 (2010), no. 1, 283–288.

247. Liu, Wenjun; Ngô, Quoc Anh; Chen, Wenbing Ostrowski type inequalities on time scales for double integrals. Acta Appl. Math. 110 (2010), no. 1, 477–497.

248. Liu, Heng-Xing; Luan, Jing-Wen Ostrowski type inequalities in the Grushin plane. J. Inequal. Appl. 2010, Art. ID 987484, 9 pp.

249. Huy, Vu Nhat; Ngô, Quoc-Anh A new way to think about Ostrowskilike type inequalities. Comput. Math. Appl. 59 (2010), no. 9, 3045–3052.

250. Čivljak, Ambroz; Dedić, Ljuban Generalizations of Ostrowski inequality via biparametric Euler harmonic identities for measures. Banach J. Math. Anal. 4 (2010), no. 1, 170–184.

251. Liu, Zheng A sharp OstrowskiGrüss type inequality. Math. Inequal. Appl. 13 (2010), no. 2, 243–253.

252. Pečarić, Josip; Ungar, Šime On the two-point Ostrowski inequality. Math. Inequal. Appl. 13 (2010), no. 2, 339–347

253. Liu, Wenjun; Ngô, Quoc Anh; Chen, Wenbing On new Ostrowski type inequalities for double integrals on time scales. Dynam. Systems Appl. 19 (2010), no. 1, 189–198.

254. Qayyum, Ather On the weighted Ostrowski type inequality for $L_p(a, b)$ and application. Int. J. Pure Appl. Math. 61 (2010), no. 1, 27–35.

255. Liu, Zheng New sharp bound for a general Ostrowski type inequality. Tamsui Oxf. J. Math. Sci. 26 (2010), no. 1, 53–59.

256. Zafar, Fiza; Mir, Nazir Ahmad A generalization of OstrowskiGrüss type inequality for first differentiable mappings. Tamsui Oxf. J. Math. Sci. 26 (2010), no. 1, 61–76.

257. Sarikaya, M. Z. On the Ostrowski type integral inequality. Acta Math. Univ. Comenian. (N.S.) 79 (2010), no. 1, 129–134.

258. Tuna, Adnan; Daghan, Durmus Generalization of Ostrowski and OstrowskiGrüss type inequalities on time scales. Comput. Math. Appl. 60 (2010), no. 3, 803–811.

259. Sarikaya, Mehmet Zeki New weighted Ostrowski and Čebyšev type inequalities on time scales. Comput. Math. Appl. 60 (2010), no. 5, 1510–1514.

260. Alomari, M.; Darus, M.; Dragomir, S. S.; Cerone, P. Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense. Appl. Math. Lett. 23 (2010), no. 9, 1071–1076.

261. Xue, Qiaoling; Zhu, Jian; Liu, Wenjun A new generalization of Ostrowskitype inequality involving functions of two independent variables. Comput. Math. Appl. 60 (2010), no. 8, 2219–2224.

262. Tseng, Kuei-Lin; Hwang, Shiow-Ru; Yang, Gou-Sheng; Chou, Yi-Ming Improvements of the Ostrowski integral inequality for mappings of bounded variation I. Appl. Math. Comput. 217 (2010), no. 6, 2348–2355.

263. Gavrea, Bogdan; Gavrea, Ioan On some Ostrowski type inequalities. Gen. Math. 18 (2010), no. 1, 33–44.

264. Özdemir, Muhamet Emin; Kavurmaci, Havva; Set, Erhan Ostrowski's type inequalities for (α, m)-convex functions. Kyungpook Math. J. 50 (2010), no. 3, 371–378.

265. Yang, Wengui Some new OstrowskiGrüss type inequalities with two mappings and applications. Tamsui Oxf. J. Math. Sci. 26 (2010), no. 4, 397–409.

266. Wang, Mingjin A probabilistic version of Ostrowski inequality. Appl. Math. Lett. 24 (2011), no. 3, 270–274.

267. Vu Nhat Huy; Ngô, Quoc-Anh New bounds for the Ostrowskilike type inequalities. Bull. Korean Math. Soc. 48 (2011), no. 1, 95–104.

268. Liu, Zheng A sharp general Ostrowski type inequality. Bull. Aust. Math. Soc. 83 (2011), no. 2, 189–209.

269. Park, Jaekeun Ostrowski type inequalities for mappings whose derivatives are (s, m)-convex in the second sense. Far East J. Math. Sci. (FJMS) 49 (2011), no. 2, 181–195.

270. Park, Jaekeun Generalization of Ostrowskitype inequalities for differentiable real (s, m)-convex mappings. Far East J. Math. Sci. (FJMS) 49 (2011), no. 2, 157–171.

271. Tseng, Kuei-Lin; Hwang, Shiow-Ru; Yang, Gou-Sheng; Chou, Yi-Ming Weighted Ostrowski integral inequality for mappings of bounded variation. Taiwanese J. Math. 15 (2011), no. 2, 573–585.

272. Hussain, Sabir; Latif, Muhammad Amer; Alomari, Mohammad Generalized double-integral Ostrowski type inequalities on time scales. Appl. Math. Lett. 24 (2011), no. 8, 1461–1467.

273. Dragomir, S. S. Ostrowski's type inequalities for continuous functions of selfadjoint operators on Hilbert spaces: a survey of recent results. Ann. Funct. Anal. 2 (2011), no. 1, 139–205.

274. Niezgoda, Marek Grüss and Ostrowski type inequalities. Appl. Math. Comput. 217 (2011), no. 23, 9779–9789.

275. Anastassiou, George A. Shell mixed fractional Ostrowski inequalities. Commun. Appl. Anal. 15 (2011), no. 1, 101–111.

276. Kavurmacı, Havva; Özdemir, M. Emin; Avcı, Merve New Ostrowski type inequalities for m-convex functions and applications. Hacet. J. Math. Stat. 40 (2011), no. 2, 135–145.

277. Set, Erhan; Sarıkaya, Mehmet Zeki On the generalization of Ostrowski and Grüss type discrete inequalities. Comput. Math. Appl. 62 (2011), no. 1, 455–461.

278. Vong, Seak Weng A note on some Ostrowskilike type inequalities. Comput. Math. Appl. 62 (2011), no. 1, 532–535.

279. Bohner, Martin; Matthews, Thomas; Tuna, Adnan Weighted OstrowskiGrüss inequalities on time scales. Afr. Diaspora J. Math. 12 (2011), no. 1, 89–99.

280. Alomari, Mohammad Wajeeh A companion of Ostrowski's inequality with applications. Transylv. J. Math. Mech. 3 (2011), no. 1, 9–14.

281. Özdemir, M. Emin; Akdemir, Ahmet Ocak; Set, Erhan A new Ostrowskitype inequality for double integrals. J. Inequal. Spec. Funct. 2 (2011), no. 1, 27–34.

282. Anastassiou, George A. Fuzzy fractional calculus and Ostrowski inequality. J. Fuzzy Math. 19 (2011), no. 3, 577–590

283. Sarikaya, Mehmet Zeki; Ogunmez, Hasan On the weighted Ostrowskitype integral inequality for double integrals. Arab. J. Sci. Eng. 36 (2011), no. 6, 1153–1160.

284. Anastassiou, George A. Representations and Ostrowski type inequalities on time scales. Comput. Math. Appl. 62 (2011), no. 10, 3933–3958.

285. Dragomir, S. S. Ostrowski's type inequalities for some classes of continuous functions of selfadjoint operators in Hilbert spaces. Comput. Math. Appl. 62 (2011), no. 12, 4439–4448.

286. Anastassiou, George A. Multivariate radial mixed Caputo fractional Ostrowski inequalities. Indian J. Math. 53 (2011), no. 2, 393–405.

287. Mir, Nazir Ahmad; Ullah, Roman Some inequalities of Ostrowski and Grüss type for triple integrals on time scales. Tamkang J. Math. 42 (2011), no. 4, 415–426.

288. Masjed-Jamei, Mohammad A linear constructive approximation for integrable functions and a parametric quadrature model based on a generalization of OstrowskiGrüss type inequalities. Electron. Trans. Numer. Anal. 38 (2011), 218–232.

289. Xue, Qiaoling; Wang, Shunfeng; Liu, Wenjun A new generalization of OstrowskiGrüss type inequalities involving functions of two independent variables. Miskolc Math. Notes 12 (2011), no. 2, 265–272

290. Jawarneh, Y.; Noorani, M. S. M. Inequalities of Ostrowski and Simpson type for mappings of two variables with bounded variation and applications. Transylv. J. Math. Mech. 3 (2011), no. 2, 81–94.

291. Sulaiman, W. T. Inequalities concerning some bounds for Čebyšev and Ostrowski type. J. Contemp. Appl. Math. 1 (2011), no. 1, 71–80.

292. Masjed-Jamei, Mohammad; Dragomir, Sever S. A new generalization of the Ostrowski inequality and applications. Filomat 25 (2011), no. 1, 115–123.

293. Acu, Ana Maria; Sofonea, Daniel Florin On an inequality of Ostrowski type. J. Sci. Arts 2011, no. 3(16), 281–287.

294. Acu, Ana Maria; Baboş, Alina; Sofonea, Florin The mean value theorems and inequalities of Ostrowski type. Sci. Stud. Res. Ser. Math. Inform. 21 (2011), no. 1, 5–16.

295. Dragomir, Silvestru Sever Operator inequalities of Ostrowski and trapezoidal type. Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1.

296. Liu, Zheng Another generalization of weighted Ostrowski type inequality for mappings of bounded variation. Appl. Math. Lett. 25 (2012), no. 3, 393–397.

297. Feng, Qinghua; Meng, Fanwei Some generalized OstrowskiGrüss type integral inequalities. Comput. Math. Appl. 63 (2012), no. 3, 652–659.

298. Tseng, Kuei-Lin Improvements of the Ostrowski integral inequality for mappings of bounded variation II. Appl. Math. Comput. 218 (2012), no. 10, 5841–5847.

299. Sulaiman, W. T. Some new Ostrowski type inequalities. J. Appl. Funct. Anal. 7 (2012), no. 1-2, 102–107.

300. Gonska, Heiner; Raşa, Ioan; Rusu, Maria-Daniela Applications of an Ostrowskitype inequality. J. Comput. Anal. Appl. 14 (2012), no. 1, 19–31.

301. Anastassiou, George A. Univariate mixed Caputo fractional Ostrowski inequalities. J. Comput. Anal. Appl. 14 (2012), no. 4, 706–713.

302. Set, Erhan New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. Comput. Math. Appl. 63 (2012), no. 7, 1147–1154.

303. Park, Jaekeun Some Ostrowskilike type inequalities for differentiable real (α, m)-convex mappings. Far East J. Math. Sci. (FJMS) 61 (2012), no. 1, 75–91.

304. Feng, Qinghua; Meng, Fanwei Generalized Ostrowski type inequalities for multiple points on time scales involving functions of two independent variables. J. Inequal. Appl. 2012, 2012:74, 11 pp.

305. Anastassiou, George A. Ostrowski and Landau inequalities for Banach space valued functions. Math. Comput. Modelling 55 (2012), no. 3-4, 312–329.

306. Akın-Bohner, Elvan; Bohner, Martin; Matthews, Thomas Time scales Ostrowski and Grüss type inequalities involving three functions. Nonlinear Dyn. Syst. Theory 12 (2012), no. 2, 119–135.

307. Anastassiou, George A. Univariate right fractional Ostrowski inequalities. Cubo 14 (2012), no. 1, 1–7.

308. Latif, Muhammad Amer; Dragomir, Sever S. New Ostrowski type inequalities for co-ordinated s-convex functions in the second sense. Matematiche (Catania) 67 (2012), no. 1, 57–72.

309. Allahviranloo, T.; Salahshour, S.; Avazpour, L. On the fractional Ostrowski inequality with uncertainty. J. Math. Anal. Appl. 395 (2012), no. 1, 191–201.

310. Set, Erhan; Özdemir, M. Emin; Sarıkaya, Mehmet Zeki New inequalities of Ostrowski's type for s-convex functions in the second sense with applications. Facta Univ. Ser. Math. Inform. 27 (2012), no. 1, 67–82.

311. Tuna, Adnan; Jiang, Yong; Liu, Wenjun Weighted Ostrowski, OstrowskiGrüss and OstrowskiČebyšev type inequalities on time scales. Publ. Math. Debrecen 81 (2012), no. 1-2, 81–102.

312. Anastassiou, George A. Multivariate right fractional Ostrowski inequalities. J. Appl. Math. Inform. 30 (2012), no. 3-4, 445–454.

313. Qayyum, Ather; Hussain, Sabir A new generalized Ostrowski Grüss type inequality and applications. Appl. Math. Lett. 25 (2012), no. 11, 1875–1880.

314. Anastassiou, George A. Ostrowski inequalities for cosine and sine operator functions. Mat. Vesnik 64 (2012), no. 4, 336–346.

315. Liu, Zheng A note on Ostrowski type inequalities related to some s-convex functions in the second sense. Bull. Korean Math. Soc. 49 (2012), no. 4, 775–785.

316. Xiao, Ying-Xiong Remarks on Ostrowskilike inequalities. Appl. Math. Comput. 219 (2012), no. 3, 1158–1162.

317. Sarikaya, Mehmet Zeki On the Ostrowski type integral inequality for double integrals. Demonstratio Math. 45 (2012), no. 3, 533–540.

318. Liu, Wenjun; Tuna, Adnan Weighted Ostrowski, trapezoid and Grüss type inequalities on time scales. J. Math. Inequal. 6 (2012), no. 3, 381–399.

319. Gonska, Heiner; Raşa, Ioan; Rusu, Maria-Daniela Generalized Ostrowski Grüss-type inequalities. Results Math. 62 (2012), no. 3-4, 311–318.

320. Sarikaya, Mehmet Zeki Ostrowski type inequalities involving the right Caputo fractional derivatives belong to L_p. Facta Univ. Ser. Math. Inform. 27 (2012), no. 2, 191–197.

321. Park, Jaekeun On the Ostrowskilike type integral inequalities for mappings whose second derivatives are s^*-convex. Far East J. Math. Sci. (FJMS) 67 (2012), no. 1, 21–35.

322. Alomari, Mohammad Wajeeh A companion of Ostrowski's inequality for mappings whose first derivatives are bounded and applications in numerical integration. Kragujevac J. Math. 36 (2012), no. 1, 77–82.

323. Alomari, Mohammad W.; Özdemir, M. Emin; Kavurmac, Havva On companion of Ostrowski inequality for mappings whose first derivatives absolute value are convex with applications. Miskolc Math. Notes 13 (2012), no. 2, 233–248.

324. Latif, M. A.; Alomari, M. W.; Hussain, S. On Ostrowskitype inequalities for functions whose derivatives are m-convex and (α, m)-convex functions with applications. Tamkang J. Math. 43 (2012), no. 4, 521–532.

325. Chalco-Cano, Y.; Flores-Franulič, A.; Román-Flores, H. Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative. Comput. Appl. Math. 31 (2012), no. 3, 457–472.

326. Anastassiou, George A. Ostrowski inequalities for semigroups. Commun. Appl. Anal. 16 (2012), no. 4, 565–577.

327. Park, Jaekeun On the Ostrowskilike type inequalities for convex mappings of 2-parameters on the co-ordinates. Far East J. Math. Sci. (FJMS) 69 (2012), no. 2, 155–172.

328. Park, Jaekeun On the Ostrowskilike type inequalities for convex mappings of 2-parameters on the co-ordinates. Far East J. Math. Sci. (FJMS) 69 (2012), no. 2, 155–172.

329. Özdemir, M. Emin; Akdemir, Ahmet Ocak; Set, Erhan On the OstrowskiGrüss type inequality for twice differentiable functions. Hacet. J. Math. Stat. 41 (2012), no. 5, 651–655.

330. Alomari, Mohammad W. A generalization of companion inequality of Ostrowski's type for mappings whose first derivatives are bounded and applications in numerical integration. Transylv. J. Math. Mech. 4 (2012), no. 2, 103–109.

331. Latif, Muhammad Amer; Hussain, Sabir; Dragomir, Sever S. New Ostrowski type inequalities for co-ordinated convex functions. Transylv. J. Math. Mech. 4 (2012), no. 2, 125–136.

332. Liu, Zheng A sharp general Ostrowski type inequality for double integrals. Tamsui Oxf. J. Inf. Math. Sci. 28 (2012), no. 2, 217–226.

333. Zhao, Chang-Jian; Cheung, Wing-Sum On multivariate Ostrowski type inequalities. Functional equations in mathematical analysis, 361–369, Springer Optim. Appl., 52, Springer, New York, 2012.

334. Alomari, M. W.; A companion of Dragomir's generalization of the Ostrowski inequality and applications to numerical integration. Ukrainian Math. J. 64 (2012), no. 4, 491–510.

335. Wang, Zhibo; Vong, Seakweng On some Ostrowskilike type inequalities involving n knots. Appl. Math. Lett. 26 (2013), no. 2, 296–300.

336. Alomari, Mohammad Wajeeh A companion of Ostrowski's inequality for the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$, where f is of bounded variation and u is of r-H-Hölder type and applications. Appl. Math. Comput. 219 (2013), no. 9, 4792–4799.

337. Liu, Wenjun; Park, Jaekeun A generalization of the companion of Ostrowskilike inequality and applications. Appl. Math. Inf. Sci. 7 (2013), no. 1, 273–278.

338. Hussain, Sabir; Qayyum, Ather A generalized OstrowskiGrüss type inequality for bounded differentiable mappings and its applications. J. Inequal. Appl. 2013, 2013:1, 7 pp.

339. Wu, Ying; Wang, Yapeng On the optimal constants of Ostrowskilike inequalities involving n knots. Appl. Math. Comput. 219 (2013), no. 14, 7789–7794.

340. Hwang, Shiow-Ru; Hsu, Kai-Chen; Tseng, Kuei-Lin Weighted Ostrowski integral inequalities for mappings whose derivatives belong to LP(a,b). Appl. Math. Comput. 219 (2013), no. 17, 9516–9523.

341. Liu, Wenjun; Zhu, Yiting Some companions of perturbed Ostrowski type inequalities for functions whose second derivatives are bounded and applications in numerical integration. Int. J. Appl. Math. Stat. 34 (2013), no. 4, 52–59.

342. Park, Jaekeun Some companions of perturbed Ostrowski type inequalities for functions whose second derivatives are bounded. Int. J. Appl. Math. Stat. 36 (2013), no. 6, 95–103.

343. Park, Jaekeun Generalizations of OstrowskiGrüss type integral inequalities for twice differentiable mappings. Int. J. Appl. Math. Stat. 39 (2013), no. 9, 10–23.

344. Liu, Wenjun; Zhu, Yiting; Park, Jaekeun Some companions of perturbed Ostrowski-type inequalities based on the quadratic kernel function with three sections and applications. J. Inequal. Appl. 2013, 2013:226, 14 pp.

345. Hussain, Sabir Generalization of Ostrowski and Čebyšev type inequalities involving many functions. Aequationes Math. 85 (2013), no. 3, 409–419.

346. Jawarneh, Y.; Noorani, M. S. M. Inequalities of Ostrowski and Simpson type for mappings of two variables with bounded variation and applications. Int. J. Math. Anal. (Ruse) 7 (2013), no. 33-36, 1609–1622.

347. Aglić Aljinović, A.; Pečarić, J. A discrete weighted Montgomery identity and Ostrowski type inequalities for functions of two variables. Sarajevo J. Math. 9(21) (2013), no. 1, 47–56.

348. Wang, Shunfeng; Xue, Qiaoling; Liu, Wenjun Some new perturbed generalizations of OstrowskiGrüss type inequalities for bounded differentiable mappings and applications. Appl. Math. Inf. Sci. 7 (2013), no. 5, 2077–2081.

349. Zheng, Bin Some new generalized 2D OstrowskiGrüss type inequalities on time scales. Arab J. Math. Sci. 19 (2013), no. 2, 159–172.

350. Pachpatte, Deepak B. On right Caputo fractional Ostrowski inequalities involving three functions. Int. J. Anal. 2013, Art. ID 127061, 5 pp.

351. Yue, Hu Ostrowski inequality for fractional integrals and related fractional inequalities. Transylv. J. Math. Mech. 5 (2013), no. 1, 85–89.

352. Acu, Ana Maria; Sofonea, Florin; Muraru, Carmen Violeta Grüss and Ostrowski type inequalities and their applications. Sci. Stud. Res. Ser. Math. Inform. 23 (2013), no. 1, 5–14.

353. Tunç, Mevlüt Ostrowski-type inequalities via h-convex functions with applications to special means. J. Inequal. Appl. 2013, 2013:326, 10 pp.

354. Dou, Jingbo; Han, Yazhou Ostrowski type inequalities related to the generalized Baouendi-Grushin vector fields. Rend. Semin. Mat. Univ. Padova 129 (2013), 225–244.

355. Anastassiou, George A. General Grüss and Ostrowski type inequalities involving S-convexity. Bull. Allahabad Math. Soc. 28 (2013), no. 1, 101–129.

356. Park, Jaekeun On the Ostrowskilike type integral inequality based on the quadratic kernel function with three sections. Int. J. Appl. Math. Stat. 42 (2013), no. 12, 10–17.

357. Park, Jaekeun Some integral inequalities of Simpson-like type, Ostrowskilike type and Ostrowski-Grüss-like type. Int. J. Appl. Math. Stat. 42 (2013), no. 12, 241–248.

358. Alomari, M. W.; New Sharp Ostrowskitype Inequalities and Generalized Trapezoid-type Inequalities for Riemann–Stieltjes Integrals and their Applications. Ukrainian Math. J. 65 (2013), no. 7, 995–1018.

359. Sarikaya, Mehmet Zeki; Yaldiz, Hatice; New generalization fractional inequalities of Ostrowski-Grüss type. Lobachevskii J. Math. 34 (2013), no. 4, 326–331.

360. Wang, Zhibo; Vong, Seakweng On some generalizations of an Ostrowski-Grüss type integral inequality. Appl. Math. Comput. 229 (2014), 239–244.

361. Kovač, S.; Pečarić, J.; Tipurić-Spužević, S. Weighted Ostrowski type inequalities with application to onepoint integral formula. Mediterr. J. Math. 11 (2014), no. 1, 13–30.

362. Akdemir, Ahmet Ocak Inequalities of Ostrowski's type for m- and (α, m)-logarithmically convex functions via Riemann-Liouville fractional integrals. J. Comput. Anal. Appl. 16 (2014), no. 2, 375–383.

363. Kavurmaci, Havva Ostrowski type inequalities for m- and (α, m) -logaritmically convex functions. J. Comput. Anal. Appl. 16 (2014), no. 5, 820–826.

364. Liu, Wenjun Some Ostrowski type inequalities via Riemann-Liouville fractional integrals for h-convex functions. J. Comput. Anal. Appl. 16 (2014), no. 5, 998–1004.

365. Acu, Ana Maria; Rusu, Maria-Daniela Improvement and generalization of some Ostrowski-type inequalities. J. Appl. Funct. Anal. 9 (2014), no. 3-4, 216–229.

366. Anastassiou, George A. Further interpretation of some fractional Ostrowski and Grüss type inequalities. J. Appl. Funct. Anal. 9 (2014), no. 3-4, 392–403.

367. Liu, Wenjun; Tuna, Adnan; Jiang, Yong On weighted Ostrowski type, trapezoid type, Grüss type and Ostrowski-Grüss like inequalities on time scales. Appl. Anal. 93 (2014), no. 3, 551–571.

368. Tunç, Mevlüt Ostrowski type inequalities for functions whose derivatives are MT-convex. J. Comput. Anal. Appl. 17 (2014), no. 4, 691–696.

369. Huy, Vu Nhat; Ngô, Quoc-Anh A new Ostrowski-Grüss inequality involving $3n$ knots. Appl. Math. Comput. 235 (2014), 272–282.

370. Masjed-Jamei, Mohammad; Dragomir, Sever S. A generalization of the Ostrowski-Gruss inequality. Anal. Appl. (Singap.) 12 (2014), no. 2, 117–130.

371. Sarikaya, Mehmet Zeki; Set, Erhan On new Ostrowski type integral inequalities. Thai J. Math. 12 (2014), no. 1, 145–154.

372. Zheng, Bin; Feng, Qinghua Generalized n dimensional Ostrowski type and Grüss type inequalities on time scales. J. Appl. Math. 2014, Art. ID 434958, 11 pp.

373. Set, Erhan; Sarikaya, Mehmet Zeki; Ozdemir, M. Emin Some Ostrowski's type inequalities for functions whose second derivatives are s-convex in the second sense. Demonstr. Math. 47 (2014), no. 1, 37–47.

374. Işcan, I. Ostrowski type inequalities for functions whose derivatives are preinvex. Bull. Iranian Math. Soc. 40 (2014), no. 2, 373–386.

375. Anastassiou, George A. Multivariate weighted fractional representation formulae and Ostrowski type inequalities. Stud. Univ. Babeş-Bolyai Math. 59 (2014), no. 1, 3–10.

376. Qayyum, Ather; Shoaib, Muhammad; Latif, Muhammad Amer A generalized inequality of Ostrowski type for twice differentiable bounded mappings and applications. Appl. Math. Sci. (Ruse) 8 (2014), no. 37-40, 1889–1901

377. Dragomir, Sever S. Refinements of the Ostrowski inequality in terms of the cumulative variation and applications. Analysis (Berlin) 34 (2014), no. 2, 223–240.

378. Qayyum, A.; Shoaib, M.; Matouk, A. E.; Latif, M. A. On new generalized Ostrowski type integral inequalities. Abstr. Appl. Anal. 2014, Art. ID 275806, 8 pp.

379. Park, Jaekeun Some companions of an Ostrowski-like type inequality for twice differentiable functions. Appl. Math. Sci. (Ruse) 8 (2014), no. 45-48, 2339–2351.

380. Anastassiou, George A. Multivariate fractional representation formula and Ostrowski type inequality. Sarajevo J. Math. 10(22) (2014), no. 1, 27–35.

381. Sarikaya, M. Zeki; Yaldiz, H.; Basak, N. New fractional inequalities of Ostrowski-Grüss type. Matematiche (Catania) 69 (2014), no. 1, 227–235.

382. Dragomir, S. S. Power Pompeiu's type inequalities for absolutely continuous functions with applications to Ostrowski's inequality. Stud. Univ. Babeş-Bolyai Math. 59 (2014), no. 2, 131–140.

383. Liu, Wenjun; Tuna, Adnan; Jiang, Yong New weighted Ostrowski and Ostrowski-Grüss type inequalities on time scales. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) 60 (2014), no. 1, 57–76.

384. Liu, Zheng Some weighted Ostrowski type inequalities. Vietnam J. Math. 42 (2014), no. 2, 141–151.

385. Sarikaya, Mehmet Zeki; Filiz, Hatice Note on the Ostrowski type inequalities for fractional integrals. Vietnam J. Math. 42 (2014), no. 2, 187–190.

386. Aglić Aljinović, A.; Pečarić, J.; Tipurić-Spužević, S. Weighted Ostrowski type inequalities for functions with one point of nondifferentiability. Arab J. Math. Sci. 20 (2014), no. 2, 177–190.

387. Aglić Aljinović, Andrea Montgomery identity and Ostrowski type inequalities for Riemann-Liouville fractional integral. J. Math. 2014, Art. ID 503195, 6 pp.

388. Franjić, Iva; Pečarić, Josip; Tipurić-Spužević, Sanja Ostrowski type inequalities for functions whose higher order derivatives have a single point of non-differentiability. Appl. Math. Comput. 245 (2014), 557–565.

389. Liu, Wenjun; Deng, Zihao Some refinements of the weighted Ostrowski's inequality for functions of bounded variation and applications. Math. Theory Appl. (Changsha) 34 (2014), no. 2, 49–54.

390. Liu, Wenjun; Gao, Xingyue Approximating the finite Hilbert transform via a companion of Ostrowski's inequality for function of bounded variation and applications. Appl. Math. Comput. 247 (2014), 373–385.

391. Jiang, Yong; Rüzgar, Hüseyin; Liu, Wenjun; Tuna, Adnan Some new generalizations of Ostrowski type inequalities on time scales involving combination of Δ-integral means. J. Nonlinear Sci. Appl. 7 (2014), no. 5, 311–324.

392. Anastassiou, George A. Fractional Ostrowski and Grüss type inequalities involving several functions. Panamer. Math. J. 24 (2014), no. 3, 1–14.

393. Jiang, Yong; Rüzgar, Hüseyin; Liu, Wenjun; Tuna, Adnan Some new generalizations of Ostrowski type inequalities on time scales involving combination of Δ-integral means. J. Nonlinear Sci. Appl. 7 (2014), no. 5, 311–324.

394. Anastassiou, George A. Fractional Ostrowski and Grüss type inequalities involving several functions. Panamer. Math. J. 24 (2014), no. 3, 1–14.

395. Özdemir, Muhamet Emin; Yildiz, Çetin An Ostrowski type inequality for derivatives of q-th power of s-convex functions via fractional integrals. Georgian Math. J. 21 (2014), no. 4, 491–498.

396. Dragomir, S. S. Inequalities of Pompeiu's type for absolutely continuous functions with applications to Ostrowski's inequality. Acta Math. Acad. Paedagog. Nyházi. (N.S.) 30 (2014), no. 1, 43–52.

397. Liu, Zheng Some general Ostrowski type inequalities. Acta Math. Acad. Paedagog. Nyházi. (N.S.) 30 (2014), no. 1, 53–65.

398. Anastassiou, George A. The reduction method in fractional calculus and fractional Ostrowski type inequalities. Indian J. Math. 56 (2014), no. 3, 333–357.

399. Wang, JinRong; Deng, JianHua; Fečkan, Michal Exploring $s-e$-condition and applications to some Ostrowski type inequalities via Hadamard fractional integrals. Math. Slovaca 64 (2014), no. 6, 1381–1396.

400. Set, Erhan; Sarikaya, Mehmet Zeki On a new Ostrowski-type inequality and related results. Kyungpook Math. J. 54 (2014), no. 4, 545–554.

401. Özdemir, M. Emin; Kavurmacı, Havva; Avcı, Merve Ostrowski type inequalities for convex functions. Tamkang J. Math. 45 (2014), no. 4, 335–340.

402. Sarikaya, Mehmet Zeki; Yaldiz, Hatice; Erden, Samet On the weighted Ostrowski type inequalities for double integrals. Kragujevac J. Math. 38 (2014), no. 2, 303–314.

403. Latif, Muhammad Amer New Ostrowski type inequalities for co-ordinated (α, m)-convex functions. Transylv. J. Math. Mech. 6 (2014), no. 2, 139–150.

404. Dragomir, Sever S. Ostrowski type inequalities for functions whose derivatives are h-convex in absolute value. Tbilisi Math. J. 7 (2014), no. 1, 1–17.

405. İşcan, İmdat; Numan, Selim Ostrowski type inequalities for harmonically quasi-convex functions. Electron. J. Math. Anal. Appl. 2 (2014), no. 2, 189–198.

406. Cerone, Pietro; Dragomir, Sever S.; Kikianty, Eder Multiplicative Ostrowski and trapezoid inequalities. Handbook of functional equations, 57–73, Springer Optim. Appl., 95, Springer, New York, 2014.

407. Cheung, W. S.; Dragomir, Sever S. A survey on Ostrowski type inequalities for Riemann-Stieltjes integral. Handbook of functional equations, 75–104, Springer Optim. Appl., 95, Springer, New York, 2014.

408. Liu, Wenjun New bounds for the companion of Ostrowski's inequality and applications. Filomat 28 (2014), no. 1, 167–178.

409. Masjed-Jamei, Mohammad; Dragomir, Sever S. An analogue of the Ostrowski inequality and applications. Filomat 28 (2014), no. 2, 373–381.

410. Hussain, Sabir; Shehzad, Khurrem Generalized Ostrowski type inequalities on time scales with applications. Proc. Pakistan Acad. Sci. 51 (2014), no. 3, 247–257.

411. Moritoh, Shinya; Tanaka, Yumi Comparison of integral and discrete Ostrowski's inequalities in the plane. Math. Inequal. Appl. 18 (2015), no. 1, 125–132.

412. Özdemir, M. Emin; Avci Ardic, Merve Some companions of Ostrowski type inequality for functions whose second derivatives are convex and concave with applications. Arab J. Math. Sci. 21 (2015), no. 1, 53–66.

413. Awan, Khalid Mahmood; Pečarić, Josip; Penava, Mihaela Ribičić Companion inequalities to Ostrowski-Grüss type inequality and applications. Turkish J. Math. 39 (2015), no. 2, 228–234.

414. Dragomir, S. S. Exponential Pompeiu's type inequalities with applications to Ostrowski's inequality. Acta Math. Univ. Comenian. (N.S.) 84 (2015), no. 1, 39–50.

415. Dragomir, S. S. A functional generalization of Ostrowski inequality via Montgomery identity. Acta Math. Univ. Comenian. (N.S.) 84 (2015), no. 1, 63–78.

416. Akdemir, Ahmet Ocak; Tunç, Mevlüt Ostrowski type inequalities for s-logarithmically convex functions in the second sense with applications. Georgian Math. J. 22 (2015), no. 1, 1–7.

417. Qayyum, A.; Shoaib, M.; Faye, Ibrahima Some new generalized results on Ostrowski type integral inequalities with application. J. Comput. Anal. Appl. 19 (2015), no. 4, 693–712.

418. Awan, K. M.; Pečarić, J.; Vukelić, A. Harmonic polynomials and generalizations of Ostrowski-Grüss type inequality and Taylor formula. J. Math. Inequal. 9 (2015), no. 1, 297–319.

419. Anastassiou, George A. Harmonic multivariate Ostrowski and Grüss type inequalities for several functions. Demonstr. Math. 48 (2015), no. 1, 31–52.

420. Wang, Shengjun; Dou, Jingbo Ostrowski type inequalities and Hardy inequalities with boundary terms related to the generalized Greiner operator. Adv. Math. (China) 44 (2015), no. 1, 91–101.

421. Cerone, Pietro; Dragomir, Sever S.; Kikianty, Eder Ostrowski and trapezoid type inequalities related to Pompeiu's mean value theorem. J. Math. Inequal. 9 (2015), no. 3, 739–762.

422. Dragomir, Silvestru S. A survey on Ostrowski type inequalities related to Pompeiu's mean value theorem. Khayyam J. Math. 1 (2015), no. 1, 1–35.

423. Acu, Ana Maria; Gonska, Heiner Weighted Ostrowski-Grüss type inequalities. Stud. Univ. Babeş-Bolyai Math. 60 (2015), no. 2, 183–192.

424. Dragomir, S. S. Some perturbed Ostrowski type inequalities for absolutely continuous functions (I). Acta Univ. M. Belii Ser. Math. 2015, 1–16.

425. Cerone, Pietro; Dragomir, Sever S.; Kikianty, Eder Jensen-Ostrowski type inequalities and applications for f-divergence measures. Appl. Math. Comput. 266 (2015), 304–315.

426. Hussain, Sabir; Rashad, Muhammad Sami n-dimensional Ostrowski type inequalities with applications on time scales. Filomat 29 (2015), no. 7, 1623–1629.

427. Dragomir, S. S. Some perturbed Ostrowski type inequalities for absolutely continuous functions (III). Transylv. J. Math. Mech. 7 (2015), no. 1, 31–43.

428. Liu, Wenjun Ostrowski type fractional integral inequalities for MT-convex functions. Miskolc Math. Notes 16 (2015), no. 1, 249–256.

429. Noor, Muhammad Aslam; Noor, Khalida Inayat; Awan, Muhammad Uzair Fractional Ostrowski inequalities for (s, m)-Godunova-Levin functions. Facta Univ. Ser. Math. Inform. 30 (2015), no. 4, 489–499.

430. Iscan, Imdat Ostrowski type inequalities for harmonically s-convex functions. Konuralp J. Math. 3 (2015), no. 1, 63–74.

431. Anastassiou, George A. Fractional representation formulae under initial conditions and fractional Ostrowski type inequalities. Demonstr. Math. 48 (2015), no. 3, 357–378.

432. Budak, Hüseyin; Sarikaya, Mehmet Zeki A companion of Ostrowski type inequalities for functions of two variables with bounded variation. J. Adv. Math. Stud. 8 (2015), no. 2, 170–184.

433. Cerone, Pietro; Dragomir, Sever S.; Kikianty, Eder On inequalities of Jensen-Ostrowski type. J. Inequal. Appl. 2015, 2015:328, 20 pp.

434. Dragomir, S. S. Some perturbed Ostrowski type inequalities for absolutely continuous functions (II). Acta Univ. Apulensis Math. Inform. No. 43 (2015), 209–228.

435. Kiris, Mehmet Eyüp; Sarikaya, Mehmet Zeki On Ostrowski type inequalities and Čebyšev type inequalities with applications. Filomat 29 (2015), no. 8, 1695–1713.

436. Liu, Wenjun; Tuna, Adnan Diamond-α weighted Ostrowski type and Grüss type inequalities on time scales. Appl. Math. Comput. 270 (2015), 251–260.

437. Budak, Hüseyin; Sarikaya, Mehmet Zeki On generalization Ostrowski type inequalities for functions of two variables with bounded variation. Palest. J. Math. 5 (2015), no. 1, 86–97.

438. Baloch, Imran Abbas; İşcan, İmdat Some Ostrowski type inequalities for harmonically (s, m)-convex functions in second sense. Int. J. Anal. 2015, Art. ID 672675, 9 pp.

439. Dragomir, S. S. Some perturbed Ostrowski type inequalities for functions of bounded variation. Asian-Eur. J. Math. 8 (2015), no. 4, 1550069, 14 pp.

440. Dragomir, S. S. Ostrowski's type inequalities for complex functions defined on unit circle with applications for unitary operators in Hilbert spaces. Arch. Math. (Brno) 51 (2015), no. 4, 233–254.

441. Acu, Ana Maria Improvement of Grüss and Ostrowski type inequalities. Filomat 29 (2015), no. 9, 2027–2035.

442. Dragomir, S. S. Some perturbed Ostrowski type inequalities for absolutely continuous functions (I). Theory Appl. Math. Comput. Sci. 5 (2015), no. 2, 132–147.

443. Dragomir, S. S. Ostrowski type inequalities for functions whose derivatives are h-convex in absolute value. Acta Comment. Univ. Tartu. Math. 19 (2015), no. 2, 133–148.

444. Dragomir, Sever S. General Lebesgue integral inequalities of Jensen and Ostrowski type for differentiable functions whose derivatives in absolute value are h-convex and applications. Ann. Univ. Mariae Curie-Skłodowska Sect. A 69 (2015), no. 2, 17–45.

445. Qayyum, Ather; Shoaib, Muhammad; Faye, Ibrahima Companion of Ostrowski-type inequality based on 5-step quadratic kernel and applications. J. Nonlinear Sci. Appl. 9 (2016), no. 2, 537–552.

446. Tuna, Adnan; Liu, Wenjun New weighted Čebyšev-Ostrowski type integral inequalities on time scales. J. Math. Inequal. 10 (2016), no. 2, 327–356.

447. Qayyum, Ather; Kashif, Abdul Rehman; Shoaib, Muhammad; Faye, Ibrahima Derivation and applications of inequalities of Ostrowski type for n-times differentiable mappings for cumulative distribution function and some quadrature rules. J. Nonlinear Sci. Appl. 9 (2016), no. 4, 1844–1857.

448. Farid, G. New Ostrowski-type inequalities and their applications in two coordinates. Acta Math. Univ. Comenian. (N.S.) 85 (2016), no. 1, 107–112.

449. Irshad, Wajeeha; Bhatti, Muhammad Iqbal; Muddassar, Muhammad Some Ostrowski type integral inequalities for double integral on time scales. J. Comput. Anal. Appl. 20 (2016), no. 5, 914–927.

450. Erden, Samet; Sarikaya, Mehmet Zeki On exponential Pompeiu's type inequalities for double integrals with applications to Ostrowski's inequality. New Trends Math. Sci. 4 (2016), no. 1, 256–267.

451. Tuna, A.; Kutukcu, S. A new generalization of the Ostrowski inequality and Ostrowski type inequality for double integrals on time scales. J. Comput. Anal. Appl. 21 (2016), no. 6, 1024–1039.

452. Cerone, Pietro; Dragomir, Sever S.; Kikianty, Eder Ostrowski and Jensen type inequalities for higher derivatives with applications. J. Inequal. Spec. Funct. 7 (2016), no. 1, 61–77.

453. Liu, Zheng A sharp general L2 inequality of Ostrowski type. Acta Math. Acad. Paedagog. Nyházi. (N.S.) 32 (2016), no. 1, 67–78.

454. Noor, Muhammad Aslam; Noor, Khalida Inayat; Iftikhar, Sabah Fractional Ostrowski inequalities for harmonic h-preinvex functions. Facta Univ. Ser. Math. Inform. 31 (2016), no. 2, 417–445.

455. Budak, Hüseyin; Sarikaya, Mehmet Zeki A new companion of Ostrowski type inequalities for functions of two variables with bounded variation. Facta Univ. Ser. Math. Inform. 31 (2016), no. 2, 447–463.

456. Kashif, A. R.; Shoaib, M.; Latif, M. A. Improved version of perturbed Ostrowski type inequalities for n-times differentiable mappings with three-step kernel and its application. J. Nonlinear Sci. Appl. 9 (2016), no. 5, 3319–3332.

457. Erden, Samet; Budak, Hüseyin; Sarikaya, Mehmet Zeki An Ostrowski type inequality for twice differentiable mappings and applications. Math. Model. Anal. 21 (2016), no. 4, 522–532.

458. Iscan, Imdat Ostrowski type inequalities for p-convex functions. New Trends Math. Sci. 4 (2016), no. 3, 140–150.

459. Cerone, Pietro; Dragomir, Sever S.; Kikianty, Eder Erratum to: "Ostrowski and Jensen type inequalities for higher derivatives with applications" [J. Inequal. Spec. Funct. 7:1 (2016), 61–77] [MR3490720]. J. Inequal. Spec. Funct. 7 (2016), no. 2, 42–43.

460. Qayyum, Ather; Faye, Ibrahima; Shoaib, Muhammad Improvement of Ostrowski integral type inequalities with application. Filomat 30 (2016), no. 6, 1441–1456.

461. Budak, Hüseyin; Sarikaya, Mehmet Zeki New weighted Ostrowski type inequalities for mappings with first derivatives of bounded variation. Transylv. J. Math. Mech. 8 (2016), no. 1, 21–27.

462. Tunç, Mevlüt Ostrowski type inequalities for m- and (α, m)-geometrically convex functions via Riemann-Louville fractional integrals. Afr. Mat. 27 (2016), no. 5-6, 841–850.

463. Agarwal, Ravi P.; Luo, Min-Jie; Raina, R. K. On Ostrowski type inequalities. Fasc. Math. No. 56 (2016), 5–27.

464. Sarıkaya, Mehmet Zeki; Budak, Hüseyin; Tunç, Tuba; Erden, Samet; Yaldız, Hatice Perturbed companion of Ostrowski type inequality for twice differentiable functions. Facta Univ. Ser. Math. Inform. 31 (2016), no. 3, 593–607.

465. Meftah, Badreddine Ostrowski inequalities for functions whose first derivatives are logarithmically preinvex. Chin. J. Math. (N.Y.) 2016, Art. ID 5292603, 10 pp.

466. Liu, Wenjun; Gao, Xingyue; Wen, Yaqiong Approximating the finite Hilbert transform via some companions of Ostrowski's inequalities. Bull. Malays. Math. Sci. Soc. 39 (2016), no. 4, 1499–1513.

467. Dragomir, Silvestru Sever Perturbed companions of Ostrowski's inequality for absolutely continuous functions (I). An. Univ. Vest Timiş. Ser. Mat.-Inform. 54 (2016), no. 1, 119–138.

468. Kashuri, Artion; Liko, Rozana Ostrowski type fractional integral inequalities for generalized (s, m, φ)-preinvex functions. Aust. J. Math. Anal. Appl. 13 (2016), no. 1, Art. 16, 11 pp.

469. Barani, A.; Hermite–Hadamard and Ostrowski Type Inequalities on Hemispheres. Mediterr. J. Math. 13 (2016), no. 6, 4253–4263.

470. Farid, Ghulam; Straightforward Proofs of Ostrowski Inequality and Some Related Results. Int. J. Anal. 2016, Art. ID 3918483, 5 pp.

471. Liu, Zeng; More on Ostrowski Type Inequalities for some S-Convex Functions in the Second Sense. Demonstr. Math. 49 (2016), no. 4, 398–412.