The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 13, Issue 1, Article 6, pp. 1-5, 2016

ON OPERATORS FOR WHICH $T^2 \geq -T^{*2}$

MESSAOUD GUESBA[1] AND MOSTEFA NADIR[2]

Received 11 November, 2015; accepted 9 February, 2016; published 20 June, 2016.

[1]DEPARTMENT OF MATHEMATICS, UNIVERSITY OF EL OUED 39000, ALGERIA

[2]DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MSILA 28000, ALGERIA
guesbamessaoud2@gmail.com[1]
mostefanadir@yahoo.fr[2]

ABSTRACT. In this paper we introduce the new class of operators for which $T^2 \geq -T^{*2}$ acting on a complex Hilbert space H. We give some basic properties of these operators. we study the relation between the class and some other well known classes of operators acting on H.

Key words and phrases: Positive operators, Hermitian operators, Cartesian decomposition, Sum of operators.

2000 *Mathematics Subject Classification.* 47A05, 47A55, 47B15.

ISSN (electronic): 1449-5910

The authors are grateful to the referee for his comments and suggestions.

1. INTRODUCTION

Throughout this paper, $B(H)$ denotes to the algebra of all bounded linear operators acting on a complex Hilbert space H. If $T \in B(H)$ then T^* is its adjoint and $T = A + iB$ is its cartesian decomposition. Many classes of operator in $B(H)$ are defined according to the relation between T and T^*, for example T is normal if $TT^* = T^*T$; self- adjoint if $T^* = T$; skew-adjoint if $T^* = -T$; positive if $T^* = T$ and $\langle Tx, x\rangle \geq 0$ for all $x \in H$, and skew-normal if $T^2 = T^{*^2}$; quasinormal if $TT^*T = T^*T^2$; projection if $T^2 = T = T^*$. For an operator $T \in B(H)$ if $\|Tx\| = \|x\|$for all $x \in H$ (or equivalently $T^*T = I$), then T is called an isometry; T is called unitary if $TT^* = T^*T = I$. An operator T on H is called hyponormal if $TT^* \leq T^*T$.

In this paper we consider operator in $B(H)$ for which $T^2 \geq -T^{*^2}$. In section two we study some of the basic properties of operators. In section three we study sum of two operators and the direct sum and the tensor product. In section four we study the relation between the class and some other previously studied classes of operators in $B(H)$.

2. SOME BASIC PROPERTIES

We start this section by a characterization of operators for which $T^2 \geq -T^{*^2}$.

Proposition 2.1. *If $T = A + iB$, then $T^2 \geq -T^{*^2}$ if and only if $A^2 \geq B^2$.*

Proof. By direct calculations we have

$$\begin{aligned} T^2 &= (A + iB)^2 \\ &= A^2 - B^2 + i(AB + BA) \end{aligned}$$

and

$$T^{*^2} = A^2 - B^2 - i(AB + BA).$$

We obtain,

$$T^2 + T^{*^2} = 2\left(A^2 - B^2\right) \geq 0,$$

which implies that $A^2 \geq B^2$. ∎

Proposition 2.2. *If $T \in B(H)$ such that $T^2 \geq 0$ then $T^2 \geq -T^{*^2}$.*

Proposition 2.3. *Let $T, S \in B(H)$ are unitarily equivalent and if $T^2 \geq -T^{*^2}$, then so S.*

Proof. By assumption, there is a unitary operator $U \in B(H)$ such that $S = U^{-1}TU$, which implies that

$$\begin{aligned} S^* &= U^*T^*\left(U^{-1}\right)^* \\ &= U^*T^*\left(U^*\right)^{-1}. \end{aligned}$$

Thus we have

$$\begin{aligned} S^2 &= \left(U^{-1}TU\right)\left(U^{-1}TU\right) \\ &= U^{-1}T^2U \end{aligned}$$

and

$$-S^{*^2} = -U^*T^{*2}(U^*)^{-1}.$$

Since U is unitary $(U^{-1} = U^*)$ and using the fact that $T^2 \geq -T^{*2}$ we conclude that

$$U^{-1}T^2U \geq -U^*T^{*2}(U^*)^{-1}.$$

Thus $S^2 \geq -S^{*2}$. ∎

3. DIRECT SUM AND TENSOR PRODUCT OF TWO OPERATORS

In this section, we study the sum of two operators, and the direct sum and the tensor product.

Proposition 3.1. *If* $T^2 \geq -T^{*^2}$ *and* $S^2 \geq -S^{*^2}$ *such that* $TS = -ST$. *Then*

$$(T+S)^2 \geq -(T+S)^{*^2}.$$

Proof. Since $TS = -ST$ then $TS + ST = 0$ which implies that

$$(T+S)^2 = T^2 + S^2,$$

and

$$(T+S)^{*^2} = T^{*^2} + S^{*^2}.$$

$T^2 \geq -T^{*^2}$and $S^2 \geq -S^{*^2}$ implies

$$T^2 + S^2 \geq -\left(T^{*^2} + S^{*^2}\right).$$

Since $(T+S)^2 = T^2 + S^2$ and $-\left(T^{*^2} + S^{*^2}\right) = -(T+S)^{*^2}$, we have

$$(T+S)^2 \geq -(T+S)^{*^2}.$$

∎

Proposition 3.2. *If* $T^2 \geq -T^{*^2}$ *and* $S^2 \geq -S^{*^2}$, *then*

$$(T \oplus S)^2 \geq -(T \oplus S)^{*^2}$$

and

$$(T \otimes S)^2 \geq -(T \otimes S)^{*^2}.$$

Proof. Let $x = x_1 \oplus x_2$ be an element of $H \oplus H$, then

$$\begin{aligned}
(T \oplus S)^2 x &= (T \oplus S)^2 (x_1 \oplus x_2) \\
&= \left(T^2 \oplus S^2\right)(x_1 \oplus x_2) \\
&= T^2 x_1 \oplus S^2 x_2 \\
&\geq \left(-T^{*^2}\right) x_1 \oplus \left(-S^{*^2}\right) x_2 \\
&= -\left(T^{*^2} \oplus S^{*^2}\right)(x_1 \oplus x_2) \\
&= -(T \oplus S)^{*^2} x.
\end{aligned}$$

Thus $(T \oplus S)^2 \geq -(T \oplus S)^{*^2}$.

Also,

$$\begin{aligned}(T\otimes S)^2 x &= (T\otimes S)^2 (x_1\otimes x_2)\\ &= \left(T^2\otimes S^2\right)(x_1\otimes x_2)\\ &= T^2x_1\otimes S^2x_2\\ &\geq \left(-T^{*^2}\right)x_1\otimes\left(-S^{*^2}\right)x_2\\ &= -\left(T^{*^2}\otimes S^{*^2}\right)(x_1\otimes x_2)\\ &= -(T\otimes S)^{*^2}x.\end{aligned}$$

Thus $(T\otimes S)^2 \geq -(T\otimes S)^{*^2}$, which completes the proof. ∎

4. RELATION BETWEEN THE CLASS OPERATORS FOR WHICH $T^2 \geq -T^{*^2}$ AND SOME OTHER CLASSES OF OPERATORS IN $B(H)$

Proposition 4.1. *If $T\in B(H)$ is hermitian operator, such that $T^2 \geq -T^{*^2}$. Then T is skew-normal.*

Proof. Since $T^2 \geq -T^{*^2}$and T is hermitian, the last inequality implies that $T^2\geq 0$. Thus T^2 is positive, then $T^2 = (T^2)^* = (T^*)^2$. Thus T is skew-normal. ∎

Proposition 4.2. *If $T\in B(H)$ is skew-adjoint such that $T^2\geq -T^{*^2}$. Then $T=0$.*

Proof. Let $T\in B(H)$ is skew-adjoint and let $T=A+iB$ be its cartesian decomposition. Since T is skew-adjoint $(T=-T^*)$ which implies that $A=0$, thus $A^2=0$. Since $T^2\geq -T^{*^2}$then be Proposition 2.1, $B^2\leq 0$ but $B^2\geq 0$ which implies $B^2=0$.

Since B is hermitian that $B=0$. Thus $T=0$. ∎

Proposition 4.3. *If $T^2\geq -T^{*^2}$and T is idempotent, then $T+T^*$ is positive.*

Proof. Since $T^2 \geq -T^{*^2}$, since T is idempotent $(T^2=T)$ which implies $T^{*^2} = T^*$, then $T^2\geq -T^{*^2}$implies $T\geq -T^*$. Hence $T+T^*\geq 0$. Thus $T+T^*$ is positive. ∎

Corollary 4.4. *If $T^2\geq -T^{*^2}$and T is similar to an idempotent. Then $T+T^*$ is positive.*

Proof. Since any operator similar to an idempotent is idempotent, then T is idempotent. The result now follows immediately from Proposition 4.3.

In [4] the author introduced the class of subprojection operators in $B(H)$: $T\in B(H)$ is called a subprojection if $T^2=T^*$. The class of all subprojections is denoted by $S(H)$. ∎

Proposition 4.5. *If $T^2\geq -T^{*^2}$and $T\in S(H)$, then $T+T^*$ is positive.*

Proof. Since $T \in S(H)$, then $T^2 = T^*$ which implies that $T^* \geq -T$. Thus $T + T^* \geq 0$, hence $T + T^*$ is positive. ∎

Proposition 4.6. *If $T^2 \geq -{T^*}^2$ and T is orthogonal projection. Then T is positive.*

Proof. Since $T^2 \geq -{T^*}^2$ and T is orthogonal projection $(T^2 = T = T^*)$ which implies $T \geq -T$, then $T \geq 0$. Thus T is positive. ∎

REFERENCES

[1] J. B. CONWAY, *A Course in Functional Analysis*, (1985) New York, Spring-Verlag.

[2] I. GOHBERG, S. GOLDBERG, M. A. KAASHOEK, *Basic of Classes of Linear Operators*, Birkhäuser Verlag, Basel, 2003.

[3] A. A. S, JIBRIL, On $\mu-$Operators, *International Mathematical Forum*, Vol. **8**, (2013), no 25, pp. 1215-1224.

[4] A. A. S, JIBRIL, On subprojection sperators, *International Mathematical Journal*, **4**(3), (2003), pp. 229-238.

[5] E. KAMEI, Operators with skew commutative parts, *Math. Japonica*, **25** (4), pp. 431-432.