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BARTLE INTEGRATION IN LIE ALGEBRAS

ANDREAS BOUKAS AND PHILIP FEINSILVER

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CENTRO VITO VOLTERRA, UNIVERSITÀ DI ROMA TOR VERGATA, VIA COLUMBIA 2, 00133 ROMA, ITALY
andreasboukas@yahoo.com

DEPARTMENT OF MATHEMATICS, SOUTHERN ILLINOIS UNIVERSITY, CARBONDALE, ILLINOIS 62901, USA
pfeinsil@math.siu.edu

ABSTRACT. Using Bartle's bilinear vector integral we define stochastic integrals of bounded operator valued functions with respect to Stieltjes measures associated with the generators of the Heisenberg and Finite Difference Lie algebras. Our definition also covers the Square of White Noise and $\mathfrak{sl}(2)$ Lie algebras.

Key words and phrases: Bartle's bilinear vector integral, Heisenberg Lie algebra, $\mathfrak{sl}(2,\mathbb{R})$, Square of White Noise Lie algebra, Finite Difference Lie algebra, Fock space, Quantum random variable, Bochner's theorem.

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1. INTRODUCTION

Based on the notion of *quantum stochastic process* developed in [1], a first example of *quantum stochastic calculus* in Fock space [16] presented a noncommutative approach extending classical Itô calculus. Based on vector-valued integrators and integrands, *stochastic integrals* were defined via strong L^2-limits of matrix elements. Some generalizations of the Lebesgue integral from scalars to vectors are these:

- Bochner integral [5], [18] (Banach space valued integrand and scalar valued measure)
- Gelfand-Pettis integral [17], [14] (weak sense version of Bochner's integral)
- Bartle-Dunford-Schwartz integral [4] (scalar valued integrand and vector valued measure)
- Bartle integral [3] (vector valued integrand and measure).

In this paper, using the Bartle integral, we present a straightforward definition of integrals with respect to quantum stochastic processes, i.e., families of operators on Fock space, adapted in a certain sense, with respect to vector valued Stieltjes measures. We focus here on integrators defined in terms of the generators of the Heisenberg and Finite Difference Lie algebras in Fock space representation.

2. VECTOR VALUED MEASURES AND BARTLE'S INTEGRAL

Let $\mathscr{X}$ be a set, $\mathcal{F}$ a σ-field of subsets of $\mathscr{X}$ and Y a normed linear space. A Y*-valued measure* on $\mathscr{X}$ is a countably additive set function $m : \mathcal{F} \mapsto Y$. If $\mathscr{X} = [0, \infty)$ and $\mathcal{F}$ is the σ-field of the Borel measurable subsets of $\mathscr{X}$ then m is called a *Stieltjes measure*. The *semi-variation* of m is the extended real-valued set function $\|m\|$ defined on each $E \in \mathcal{F}$ by

$$\|m\|(E) = \sup \| \sum_{i=1}^{k} x_i \, m(E_i) \|_Y$$

where the supremum is over all finite partitions of E into disjoint sets $\{E_i\}_{i=1}^k \subset \mathcal{F}$, $k \in \mathbb{N}$, and all finite collections of elements $\{x_i\}_{i=1}^k \subset \mathrm{B}(Y)$, the space of bounded linear operators $T : Y \mapsto Y$, with $\|x_i\|_{\mathrm{B}(Y)} \leq 1$. The *variation* of m is the extended real-valued set function $|m|$ defined on each $E \in \mathcal{F}$ by

$$|m|(E) = \sup_J \sum_{i \in J} \|m(E_i)\|_Y$$

where the supremum is over all finite families $\{E_i \, ; \, i \in J\}$ of disjoint subsets of $\mathcal{F}$ such that $E = \cup_{i \in J} E_i$. If $\mathscr{X}$ is a topological space and for each $E \in \mathcal{F}$ given $\epsilon > 0$ there exist a compact set K and an open set G such that $K \subset E \subset G$ and for every $E' \in \mathcal{F}$ with $K \subset E' \subset G$, $\|m(E) - m(E')\|_Y < \epsilon$ then m is called *regular*. If $f : [0, \infty) \mapsto Y$ where Y is a normed space and $[a, b] \subset [0, \infty)$ then the *total variation* $V_a^b(f)$ of f on $[a, b]$ is defined by

$$V_a^b(f) = \sup_{\Pi} V_{\Pi}(f)$$

where

$$\Pi = \{a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b\}$$

is a partition of $[a, b]$ and

$$V_{\Pi}(f) = \sum_{i=0}^{n-1} \|f(t_{i+1}) - f(t_i)\|_Y$$

If $V_a^b(f) < \infty$ for all $[a, b] \subset [0, \infty)$ then f is said to be *with finite variation*. To each function f with finite variation there corresponds [10] a unique countably additive regular Stieltjes measure m_f with finite variation, such that for all $a, b \in \mathbb{R}$

$$\begin{aligned} m_f\left([a,b]\right) =& f(b+) - f(a-)\ ,\ m_f\left([a,b)\right) = f(b-) - f(a-) \\ m_f\left((a,b]\right) =& f(b+) - f(a+)\ ,\ m_f\left((a,b)\right) = f(b-) - f(a+) \end{aligned}$$

If Y is a Banach space and m is a Stieltjes measure then a *simple function* $f\colon \mathscr{X} \mapsto \mathrm{B}(Y)$ is a function of the form

$$f = \sum_{i=1}^{n} f_i\, \chi_{E_i}$$

where $n \in \mathbb{N}$, $E_i \in \mathcal{F}$, $\|m\|(E_i) < \infty$, $f_i \in \mathrm{B}(Y)$ for each i and $E_i \cap E_j = \emptyset$ if $i \neq j$. If $E \in \mathcal{F}$ and f is a simple function then the *integral of f over* E is defined by

$$\int_E f(s)\, m(ds) = \sum_{i=1}^{n} f_i\, m(E \cap E_i)$$

A sequence $f_n\colon \mathscr{X} \mapsto \mathrm{B}(Y)$ is said to *converge in* m*-measure* to a function $f\colon \mathscr{X} \mapsto \mathrm{B}(Y)$ if for each $\epsilon > 0$, $\|m\|\left(\{s \in \mathscr{X}\ /\ \|f_n(s) - f(s)\|_{\mathrm{B}(Y)} \geq \epsilon\}\right) \to 0$ as $n \to \infty$. A function $f\colon \mathscr{X} \mapsto \mathrm{B}(Y)$ is m*-measurable* if it is the limit in m-measure of a sequence of simple functions. Such a function f is *Bartle* m*-integrable over* $\mathscr{X}$ if there is a sequence $\{f_n\}_{n=1}^{\infty}$ of simple functions such that:

- $f_n \to f$ in m-measure
- The sequence $\{\lambda_n\}_{n=1}^{\infty}$ of integrals

$$\lambda_n(E) = \int_E f_n(s)\, m(ds)\ ,\ E \in \mathcal{F}$$

  has the property that given $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that $\|m\|(E) < \delta$ implies $\|\lambda_n(E)\|_Y < \epsilon$

If f is Bartle m-integrable over $\mathscr{X}$ then for each $E \in \mathcal{F}$ the sequence $\{\lambda_n(E)\}_{n=1}^{\infty}$ converges in $\|\cdot\|_Y$ and *the integral of* f *over* E is defined by

$$\int_E f(s)\, m(ds) = \lim_{n\to\infty} \int_E f_n(s)\, m(ds)$$

Theorem 2.1. *Let* Y *be a Banach space and let* $f\colon [0,\infty) \mapsto \mathrm{B}(Y)$ *be Bartle integrable with respect to a countably additive regular* Y*-valued measure* m *defined on the* σ*-field of Borel measurable subsets of* $[0,\infty)$*. Then, for any* $\epsilon > 0$ *there exists a step function* $g\colon [0,\infty) \mapsto \mathrm{B}(Y)$*, i.e.* $g = \sum_{i=1}^{n} a_i \chi_{I_i}$ *where* $a_i \in \mathrm{B}(Y)$ *and the* I_i*'s are pairwise disjoint bounded subintervals of* $[0,\infty)$*, such that*

$$\| \int_{[0,\infty)} f(s)\, dm(s) - \int_{[0,\infty)} g(s)\, dm(s) \|_Y < \epsilon$$

Proof. By Theorem 1 of [3] there exists a simple function $\sigma : [0,\infty) \mapsto \mathrm{B}(Y)$, say $\sigma = \sum_{j=1}^{m} b_j \chi_{B_j}$ where $b_j \in \mathrm{B}(Y)$ and the B_j's are pairwise disjoint Borel measurable subsets of $[0,\infty)$ with finite semivariation $\|m\|(B_j)$, thus $\|m(B_j)\|_Y < \infty$, such that

$$\| \int_{[0,\infty)} f(s)\, dm(s) - \int_{[0,\infty)} \sigma(s)\, dm(s) \|_Y < \frac{\epsilon}{2}$$

Now let $\delta = \frac{\epsilon}{4}\left(1 + \sum_{j=1}^{m} \|b_j\|_{\mathrm{B}(Y)}\right)$ and fix $j \in \{1, 2, ..., m\}$. By the regularity of m there exists an open set $U_j \subset [0,\infty)$ such that $B_j \subset U_j$ and $\|m(U_j) - m(B_j)\|_Y < \delta$, which implies that $\|m(U_j)\|_Y \leq \|m(B_j)\|_Y + \delta < \infty$. For each j, $U_j = \cup_{k=1}^{\infty} I_{j,k}$ where the $I_{j,k}$'s are pairwise disjoint open subintervals of $[0,\infty)$. By the countable additivity of m we have $\|\sum_{k=1}^{\infty} m(I_{j,k})\|_Y = \|m(U_j)\|_Y < \infty$. Thus, there exists k_0 such that $\|\sum_{k=k_0+1}^{\infty} m(I_{j,k})\|_Y = \|\sum_{k=0}^{\infty} m(I_{j,k}) - \sum_{k=0}^{k_0} m(I_{j,k})\|_Y < \delta$. Let $V_j = \cup_{k=1}^{k_0} I_{j,k}$. Then

$$\begin{aligned}\|m(V_j) - m(B_j)\|_Y &= \|m(U_j) - \sum_{k=k_0+1}^{\infty} m(I_{j,k}) - m(B_j)\|_Y \\ &\leq \|m(U_j) - m(B_j)\|_Y + \|\sum_{k=k_0+1}^{\infty} m(I_{j,k})\|_Y < 2\delta\end{aligned}$$

Now let $g = \sum_{j=1}^{m} b_j \chi_{V_j}$. Since each V_j is a finite union of intervals, g is a step function. Also,

$$\begin{aligned}&\|\int_{[0,\infty)} \sigma(s)\, dm(s) - \int_{[0,\infty)} g(s)\, dm(s)\|_Y = \|\sum_{j=1}^{m} b_j\, m(B_j) - \sum_{j=1}^{m} b_j\, m(V_j)\|_Y \\ &\leq \sum_{j=1}^{m} \|b_j\|_{\mathrm{B}(Y)} \|m(B_j) - m(V_j)\|_Y \leq \sum_{j=1}^{m} \|b_j\|_{\mathrm{B}(Y)}\, 2\delta \\ &\leq \left(1 + \sum_{j=1}^{m} \|b_j\|_{\mathrm{B}(Y)}\right) 2\delta \leq \frac{\epsilon}{2}\end{aligned}$$

by the definition of δ. Thus

$$\begin{aligned}&\|\int_{[0,\infty)} f(s)\, dm(s) - \int_{[0,\infty)} g(s)\, dm(s)\|_Y \\ &\leq \|\int_{[0,\infty)} f(s)\, dm(s) - \int_{[0,\infty)} \sigma(s)\, dm(s)\|_Y + \|\int_{[0,\infty)} \sigma(s)\, dm(s) - \int_{[0,\infty)} g(s)\, dm(s)\|_Y \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

□

3. BOCHNER'S THEOREM AND QUANTUM STOCHASTIC INTEGRALS

Definition 3.1. A continuous function $f : \mathbb{R} \mapsto \mathbb{C}$ is *positive definite* if

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(t-s)\phi(t)\bar{\phi}(s)\, dt\, ds \geq 0$$

for every continuous function $\phi : \mathbb{R} \mapsto \mathbb{C}$ with compact support.

Bochner's theorem [18] states that such a function can be represented as

$$f(t) = \int_{\mathbb{R}} e^{it\lambda}\, dv(\lambda)$$

where v is a non-decreasing right-continuous bounded function. If $f(0) = 1$ then such a function v defines a probability measure on $\mathbb{R}$ and Bochner's theorem says that f is the Fourier transform of a probability measure, i.e., the characteristic function of a random variable that follows the probability distribution defined by v. Moreover, the condition of positive definiteness of f is necessary and sufficient for such a representation. An example of such a positive definite function is provided by

$$f(t) = \langle \Phi, e^{itX} \Phi \rangle$$

where Φ is the normalized (i.e., $\|\Phi\| = 1$) vacuum vector of a Fock-Hilbert space $\mathscr{F}$ and X is an *observable* (self-adjoint operator on $\mathscr{F}$) also called a *quantum random variable* in which case

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(t-s)\phi(t)\bar{\phi}(s)\,dt\,ds = \| \int_{\mathbb{R}} e^{-itX}\phi(t)\,dt\ \Phi\ \|_{\mathscr{F}}^2 \geq 0$$

In this paper we consider quantum random variables X acting on the Fock space $\mathscr{F}$ associated with the Heisenberg, $\mathfrak{sl}(2,\mathbb{R})$, Square of White Noise, and Finite Difference Lie algebras.

Definition 3.2. Let $\mathcal{S}$ be a test function space and for each $f \in \mathcal{S}$ let $\psi(f)$ be the associated *exponential vector*. In particular, $\Phi = \psi(0)$ is the vacuum vector. Denote $\mathcal{E} = span\{\psi(f)\,/\,f \in \mathcal{S}\}$. If

$$\langle\cdot,\cdot\rangle : \mathcal{E}\times\mathcal{E} \mapsto [0,\infty)$$

is an inner product on $\mathcal{E}$ then the *Fock space* $\mathscr{F}$ *over* $\mathcal{S}$ is the Hilbert space completion of $(\mathcal{E}, \langle\cdot,\cdot\rangle)$.

Definition 3.3. A family $\{X(t)\,/\,t \geq 0\}$ of quantum random variables is called a *quantum stochastic process* if it is *adapted* in the following sense: for each $t \geq 0$ and $\psi(f) \in \mathcal{E}$, $X(t)\psi(f) = X_{t]}\psi(f_{t]}) \otimes \psi(f_{(t})$, where $f_{t]} = f\chi_{[0,t]}$ and $f_{(t} = f\chi_{(t,\infty)}$ [15], i.e., we have a decomposition $X(t) = X_{t]} \otimes \mathbf{1}_{(t}$ corresponding to the decomposition $\psi(f) = \psi(f_{t]}) \otimes \psi(f_{(t})$. If for each $t \geq 0$, $X_{t]}$ is a bounded linear operator then the process is said to be *bounded*.

Our goal is to define, in the sense of Bartle [3], stochastic integrals of the form

$$\int_E f(s)\,dX(s)$$

where E is a measurable subset of $[0,\infty)$ and $f\colon [0,\infty) \mapsto \mathrm{B}(\mathscr{F})$ where $\mathrm{B}(\mathscr{F})$ is the space of bounded linear operators $T\colon \mathscr{F} \mapsto \mathscr{F}$. The integrators X will in each case act on the Fock spaces $\mathscr{F}_H$ and $\mathscr{F}_{FD}$, associated with the Heisenberg [9, 11, 13, 15] and Finite Difference [6, 7, 11, 12, 13] Lie algebras respectively. The Square of White Noise [2] and $\mathfrak{sl}(2)$ integrators can be expressed as linear combinations of the Finite Difference integrators and so they are also covered. The $\mathscr{F}_H$ and $\mathscr{F}_{FD}$ Fock spaces have been essential in the development of quantum stochastic calculus.

4. MAIN EXAMPLES: THE BASIC INGREDIENTS

The basic ingredients used to build a quantum stochastic process are these:

- the test function space
- inner product of exponential vectors
- action of the integrators
- commutation relations

We present these for our main examples, $\mathscr{F}_H$ and $\mathscr{F}_{FD}$.

(1) Heisenberg Fock space $\mathscr{F}_H$: In this case $\mathcal{S} = (L^2 \cap L^\infty)([0,\infty),\mathbb{C})$ with inner product

$$\langle\psi(f),\psi(g)\rangle = e^{\int_0^\infty \bar{f}(s)g(s)\,d\mu(s)}$$

where μ is Lebesgue measure and the basic integrators are $A(f), A^\dagger(f)$ with action

$$A(f)\,\psi(g) = \left(\int_0^\infty f(t)\,\bar{g}(t)\,dt\right)\,\psi(g)\ ,\ A^\dagger(f)\,\psi(g) = \frac{\partial}{\partial\,\epsilon}|_{\epsilon=0}\,\psi(g+\epsilon\,f)$$

and commutation relations

$$[A(f), A^\dagger(g)] = \int_0^\infty \bar{f}(t)\, g(t)\, dt$$

Moreover, using the notation $X(t) = X(\chi_{[0,t]})$ we obtain [16] the matrix elements

$$\begin{aligned}
< \psi(f), A(t)\psi(g) > &= \int_0^t g(s)\ ds\ < \psi(f), \psi(g) > \\
< \psi(f), A^\dagger(t)\psi(g) > &= \int_0^t \bar{f}(s)\, ds\ < \psi(f), \psi(g) > \\
< A(t)\psi(f), A(s)\psi(g) > &= \int_0^t \bar{f}(z)\, dz \int_0^s g(w)\, dw < \psi(f), \psi(g) > \\
< A^\dagger(t)\psi(f), A^\dagger(s)\psi(g) > &= \left(\int_0^s \bar{f}(z)\, dz \int_0^t g(w)\, dw + \min\{t, s\} \right) < \psi(f), \psi(g) >
\end{aligned}$$

(2) Finite Difference Fock space $\mathscr{F}_{FD}$: In this case

$$\mathcal{S} = \{ f \colon [0, \infty) \mapsto \mathbb{R}, |f| < 1, f \text{ is a simple function} \}$$

with inner product

$$\langle \psi(f), \psi(g) \rangle = e^{-\int_0^\infty \ln(1 - f(s)g(s))\, d\mu(s)}$$

where μ is Lebesgue measure and the basic integrators are $P(f), Q(f), T(f)$ with action

$$\begin{aligned}
Q(f)\, \psi(g) &= \frac{\partial}{\partial \epsilon}|_{\epsilon=0} \left(\psi(g + \epsilon\, f) + \psi(e^{\epsilon\, f} g) \right) \\
P(f)\, \psi(g) &= \left(\int_0^\infty f(t)\, g(t)\, dt + Q(fg) \right)\ \psi(g) \\
T(f)\, \psi(g) &= \left(Q(f) + P(f) + \int_0^\infty f(t)\, g(t)\, dt \right)\ \psi(g)
\end{aligned}$$

commutation relations

$$[P(f), Q(g)] = [P(f), T(g)] = [T(f), Q(g)] = T(fg)$$

and matrix elements [8], [12], [13]

$$
\begin{aligned}
< \psi(f), P(t)\psi(g) > &= \int_0^t \frac{f(1+g)}{1-fg}(s)\,ds\ < \psi(f), \psi(g) > \\
< \psi(f), Q(t)\psi(g) > &= \int_0^t \frac{g(1+f)}{1-fg}(s)\,ds\ < \psi(f), \psi(g) > \\
< \psi(f), T(t)\psi(g) > &= \int_0^t \frac{(1+f)(1+g)}{1-fg}(s)\,ds\ < \psi(f), \psi(g) > \\
< P(t)\psi(f), P(s)\psi(g) > &= \left(\int_0^t \frac{f(1+g)}{1-fg}(z)\,dz\ \int_0^s \frac{g(1+f)}{1-fg}(w)\,dw \right.\\
&\quad \left. + \int_0^{\min\{t,s\}} \frac{fg(1+f)(1+g)}{(1-fg)^2}(z)\,dz \right) < \psi(f), \psi(g) > \\
< Q(t)\psi(f), Q(s)\psi(g) > &= \left(\int_0^s \frac{f(1+g)}{1-fg}(z)\,dz\ \int_0^t \frac{g(1+f)}{1-fg}(w)\,dw \right.\\
&\quad \left. + \int_0^{\min\{t,s\}} \frac{(1+f)(1+g)}{(1-fg)^2}(z)\,dz \right) < \psi(f), \psi(g) > \\
< T(t)\psi(f), T(s)\psi(g) > &= \left(\int_0^t \frac{(1+f)(1+g)}{1-fg}(z)\,dz\ \int_0^s \frac{(1+f)(1+g)}{1-fg}(w)\,dw \right.\\
&\quad \left. + \int_0^{\min\{t,s\}} \frac{(1+f)^2(1+g)^2}{(1-fg)^2}(z)\,dz \right) < \psi(f), \psi(g) > \ .
\end{aligned}
$$

In [2] it was shown that if $B_f^\dagger$, B_g and N_h, where f, g, h are test functions, are the generators of the Square of White Noise Lie algebra with charge c corresponding to the renormalization $\delta^2 = c\,\delta$ of the square of the Dirac delta function, and commutation relations

$$[B_g, B_f^\dagger] = 2\,c \int_0^\infty (gf)(s)\,ds + 4N_{gf}\ ,\ [N_h, B_f^\dagger] = 2B_{hf}^\dagger\ ,\ [B_g, N_h] = -2B_{gh}$$

which are a trivial central extension of the $\mathfrak{sl}(2)$ commutation relations, then the Square of White Noise operators are related to the Finite Difference operators through

$$Q(f) = \frac{1}{2}\left(B_f^\dagger + N_f\right)\ ,\ P(f) = \frac{1}{2}\left(B_f + N_f\right)\ ,\ T(f) = \int_0^\infty f(s)\,ds\,\mathbf{1} + P(f) + Q(f)$$

5. THE STIELTJES MEASURES IN THE $\mathscr{F}_H$ AND $\mathscr{F}_{FD}$ CASES

Proposition 5.1. *In the Heisenberg case, for each $f \in \mathcal{S}$ the functions*

$$F : t \in [0, +\infty) \mapsto A(t)\psi(f) \in \mathscr{F}_H$$

and

$$G : t \in [0, +\infty) \mapsto A(t)^\dagger\psi(f) \in \mathscr{F}_H$$

where $A(t) = A(\chi_{[0,t]})$ and $A(t) = A^\dagger(\chi_{[0,t]})$, are with finite variation.

Proof. Let $f \in (L^2 \cap L^\infty)([0,\infty),\mathbb{C})$, let $M = \operatorname{ess\,sup}(f)$ and let $\Pi = \{a = x_0 < x_1 < ... < x_{n-1} < x_n = b\}$ a partition of $[a,b] \subseteq [0,+\infty)$. Then

$$V_\Pi(F) = \sum_{i=0}^{n-1} \|F(t_{i+1}) - F(t_i)\|_{\mathscr{F}_H}$$

$$= \sum_{i=0}^{n-1} \left(\langle F(t_{i+1}), F(t_{i+1})\rangle - \langle F(t_{i+1}), F(t_i)\rangle - \langle F(t_i), F(t_{i+1})\rangle + \langle F(t_i), F(t_i)\rangle\right)^{1/2}$$

$$= \sum_{i=0}^{n-1} \left(\langle A(t_{i+1})\psi(f), A(t_{i+1})\psi(f)\rangle - \langle A(t_{i+1})\psi(f), A(t_i)\psi(f)\rangle\right.$$

$$\left. - \langle A(t_i)\psi(f), A(t_{i+1})\psi(f)\rangle + \langle A(t_i)\psi(f), A(t_i)\psi(f)\right)^{1/2}$$

$$= \sum_{i=0}^{n-1} \left(\int_0^{t_{i+1}} \bar{f}(s)\,ds \int_0^{t_{i+1}} f(s)\,ds - \int_0^{t_{i+1}} \bar{f}(s)\,ds \int_0^{t_i} f(s)\,ds\right.$$

$$\left. - \int_0^{t_i} \bar{f}(s)\,ds \int_0^{t_{i+1}} f(s)\,ds + \int_0^{t_i} \bar{f}(s)\,ds \int_0^{t_i} f(s)\,ds\right)^{1/2} \|\psi(f)\|$$

$$= \sum_{i=0}^{n-1} \left(\int_0^{t_{i+1}} \bar{f}(s)\,ds \int_{t_i}^{t_{i+1}} f(s)\,ds - \int_0^{t_i} \bar{f}(s)\,ds \int_{t_i}^{t_{i+1}} f(s)\,ds\right)^{1/2} \|\psi(f)\|$$

$$= \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} \bar{f}(s)\,ds \int_{t_i}^{t_{i+1}} f(s)\,ds\right)^{1/2} \|\psi(f)\| = \sum_{i=0}^{n-1} \left|\int_{t_i}^{t_{i+1}} f(s)\,ds\right| \|\psi(f)\|$$

$$\leq \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |f(s)|\,ds\ \ \|\psi(f)\| \leq M \sum_{i=0}^{n-1} (t_{i+1} - t_i)\ \|\psi(f)\| = M(b-a)\,\|\psi(f)\| < \infty$$

Thus

$$V_a^b(F) = \sup_\Pi V_\Pi(F) \leq M(b-a)\,\|\psi(f)\| < \infty$$

Similarly

$$V_\Pi(G) = \sum_{i=0}^{n-1} \|G(t_{i+1}) - G(t_i)\|_{\mathscr{F}_H}$$

$$= \sum_{i=0}^{n-1} \left(\langle G(t_{i+1}), G(t_{i+1})\rangle - \langle G(t_{i+1}), G(t_i)\rangle - \langle G(t_i), G(t_{i+1})\rangle + \langle G(t_i), G(t_i)\rangle\right)^{1/2}$$

$$= \sum_{i=0}^{n-1} \left(\langle A^\dagger(t_{i+1})\psi(f), A^\dagger(t_{i+1})\psi(f)\rangle - \langle A^\dagger(t_{i+1})\psi(f), A^\dagger(t_i)\psi(f)\rangle\right.$$

$$\left. - \langle A^\dagger(t_i)\psi(f), A^\dagger(t_{i+1})\psi(f)\rangle + \langle A^\dagger(t_i)\psi(f), A^\dagger(t_i)\psi(f)\right)^{1/2}$$

$$= \sum_{i=0}^{n-1} \left(\int_0^{t_{i+1}} \bar{f}(s)\,ds \int_0^{t_{i+1}} f(s)\,ds + t_{i+1} - \int_0^{t_i} \bar{f}(s)\,ds \int_0^{t_{i+1}} f(s)\,ds - t_i\right.$$

$$\left. - \int_0^{t_{i+1}} \bar{f}(s)\,ds \int_0^{t_i} f(s)\,ds - t_i + \int_0^{t_i} \bar{f}(s)\,ds \int_0^{t_i} f(s)\,ds + t_i\right)^{1/2} \|\psi(f)\|$$

$$= \sum_{i=0}^{n-1} \left(| \int_{t_i}^{t_{i+1}} f(s)\,ds|^2 + (t_{i+1} - t_i) \right)^{1/2} \;\; \|\psi(f)\|$$
$$\leq \sum_{i=0}^{n-1} \left(M^2 (t_{i+1} - t_i)^2 + (\sqrt{t_{i+1} - t_i})^2 \right)^{1/2} \;\; \|\psi(f)\|$$

and using $\sqrt{x^2 + y^2} \leq x + y$ for $x, y \geq 0$, we have

$$V_\Pi(G) \leq M \sum_{i=0}^{n-1} (t_{i+1} - t_i) \;\|\psi(f)\| + \sum_{i=0}^{n-1} \sqrt{t_{i+1} - t_i} \;\|\psi(f)\|$$
$$= M(b-a) \;\|\psi(f)\| + \sum_{i=0}^{n-1} \sqrt{t_{i+1} - t_i} \;\|\psi(f)\|$$

Since $V_\Pi(G)$ increases as the partition Π becomes finer and we are interested in the supremum over all partitions Π, we may assume that $\sqrt{t_{i+1} - t_i} \leq 2^{-i}$ for all $i = 0, 1, \ldots$. Thus

$$V_\Pi(G) \leq \left(M(b-a) + \sum_{i=0}^{n} \frac{1}{2^i} \right) \;\|\psi(f)\| \leq (M(b-a) + 2) \;\|\psi(f)\| < \infty$$

and so

$$V_a^b(G) = \sup_\Pi V_\Pi(G) \leq (M(b-a) + 2) \;\|\psi(f)\| < \infty$$

□

Proposition 5.2. *In the Finite Difference case, for each $f \in \mathcal{S}$ the functions*

$$F : t \in [0, +\infty) \mapsto P(t)\psi(f) \in \mathscr{F}_{FD}$$
$$G : t \in [0, +\infty) \mapsto Q(t)\psi(f) \in \mathscr{F}_{FD}$$

and

$$H : t \in [0, +\infty) \mapsto T(t)\psi(f) \in \mathscr{F}_{FD}$$

where $P(t) = P(\chi_{[0,t]})$, $Q(t) = Q(\chi_{[0,t]})$ and $T(t) = T(\chi_{[0,t]})$, are with finite variation.

Proof. Let $f = \sum_{i=1}^{k} a_k \chi_{I_k}$ be a simple function and $\Pi = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$ a partition of $[a, b] \subseteq [0, +\infty)$. Let also $\phi = \frac{f(1+f)}{1-f^2}$ Then ϕ is a also a simple function with $M = \max\{\phi(x) \,:\, x \geq 0\} < \infty$. Therefore

$$
\begin{aligned}
&V_\Pi(F) = \sum_{i=0}^{n-1} \|F(t_{i+1}) - F(t_i)\|_{\mathscr{F}_{FD}} \\
=&\sum_{i=0}^{n-1} \left(\langle F(t_{i+1}), F(t_{i+1})\rangle - \langle F(t_{i+1}), F(t_i)\rangle - \langle F(t_i), F(t_{i+1})\rangle + \langle F(t_i), F(t_i)\rangle\right)^{1/2} \\
=&\sum_{i=0}^{n-1} \left(\langle P(t_{i+1})\psi(f), P(t_{i+1})\psi(f)\rangle - \langle P(t_{i+1})\psi(f), P(t_i)\psi(f)\rangle\right. \\
&\left. - \langle P(t_i)\psi(f), P(t_{i+1})\psi(f)\rangle + \langle P(t_i)\psi(f), P(t_i)\psi(f)\rangle\right)^{1/2} \\
=&\sum_{i=0}^{n-1} \left(\int_0^{t_{i+1}} \phi(s)\,ds \int_0^{t_{i+1}} \phi(s)\,ds + \int_0^{t_{i+1}} \phi^2(s)\,ds - \int_0^{t_{i+1}} \phi(s)\,ds \int_0^{t_i} \phi(s)\,ds\right. \\
&- \int_0^{t_i} \phi^2(s)\,ds - \int_0^{t_i} \phi(s)\,ds \int_0^{t_{i+1}} \phi(s)\,ds - \int_0^{t_i} \phi^2(s)\,ds + \int_0^{t_i} \phi(s)\,ds \int_0^{t_i} \phi(s)\,ds \\
&\left. + \int_0^{t_i} \phi^2(s)\,ds\right)^{1/2} \; \|\psi(f)\| \\
=&\sum_{i=0}^{n-1} \left(\left(\int_{t_i}^{t_{i+1}} \phi(s)\,ds\right)^2 + \int_{t_i}^{t_{i+1}} \phi^2(s)\,ds\right)^{1/2} \; \|\psi(f)\| \\
\leq& M \sum_{i=0}^{n-1} \left((t_{i+1} - t_i)^2 + (t_{i+1} - t_i)\right)^{1/2} \; \|\psi(f)\|
\end{aligned}
$$

As in the proof of Proposition 5.1 we may assume that $(t_{i+1} - t_i)^2 \leq (t_{i+1} - t_i) \leq 2^{-2i}$ for all $i = 0, 1, ...$ Thus

$$
V_\Pi(F) \leq \left(\sqrt{2} M \sum_{i=0}^{n} \frac{1}{2^i}\right) \; \|\psi(f)\| \leq 2\sqrt{2} M \; \|\psi(f)\| < \infty
$$

and so

$$
V_a^b(F) = \sup_\Pi V_\Pi(F) \leq 2\sqrt{2} M \; \|\psi(f)\| < \infty
$$

Similarly letting $\sigma = \frac{1}{(1-f)^2}$, and letting $K > 1$ be such that $|\phi(x)| \leq K$ and $|\sigma(x)| \leq K$ for all $x \geq 0$ we have

$$
\begin{aligned}
&V_\Pi(G) = \sum_{i=0}^{n-1} \|G(t_{i+1}) - G(t_i)\|_{\mathscr{F}_{FD}} \\
=&\sum_{i=0}^{n-1} \left(\langle G(t_{i+1}), G(t_{i+1})\rangle - \langle G(t_{i+1}), G(t_i)\rangle - \langle G(t_i), G(t_{i+1})\rangle + \langle G(t_i), G(t_i)\rangle\right)^{1/2} \\
=&\sum_{i=0}^{n-1} \left(\langle Q(t_{i+1})\psi(f), Q(t_{i+1})\psi(f)\rangle - \langle Q(t_{i+1})\psi(f), Q(t_i)\psi(f)\rangle\right. \\
&\left. - \langle Q(t_i)\psi(f), Q(t_{i+1})\psi(f)\rangle + \langle Q(t_i)\psi(f), Q(t_i)\psi(f)\rangle\right)^{1/2} \\
=&\sum_{i=0}^{n-1} \left(\left(\int_0^{t_{i+1}} \phi(s)\, ds\right)^2 + \int_0^{t_{i+1}} \sigma(s)\, ds - \int_0^{t_i} \phi(s)\, ds \int_0^{t_{i+1}} \phi(s)\, ds - \int_0^{t_i} \sigma(s)\, ds\right. \\
&\left. - \int_0^{t_{i+1}} \phi(s)\, ds \int_0^{t_i} \phi(s)\, ds - \int_0^{t_i} \sigma(s)\, ds + \left(\int_0^{t_i} \phi(s)\right)^2 + \int_0^{t_i} \sigma(s)\, ds\right)^{1/2} \|\psi(f)\| \\
=&\sum_{i=0}^{n-1} \left(\left(\int_{t_i}^{t_{i+1}} \phi(s)\, ds\right)^2 + \int_{t_i}^{t_{i+1}} \sigma(s)\, ds\right)^{1/2} \|\psi(f)\| \\
\leq& K \sum_{i=0}^{n-1} \left((t_{i+1} - t_i)^2 + (t_{i+1} - t_i)\right)^{1/2} \|\psi(f)\| \leq 2\sqrt{2}K \ \|\psi(f)\| < \infty
\end{aligned}
$$

and so

$$
V_a^b(G) = \sup_\Pi V_\Pi(G) \leq 2\sqrt{2}K \ \|\psi(f)\| < \infty
$$

Finally, letting $\rho = \frac{1+f}{1-f}$ and $L > 1$ be such that $|\rho(x)| \leq L$ for all $x \geq 0$ we have

$$
\begin{aligned}
&V_\Pi(H) = \sum_{i=0}^{n-1} \|H(t_{i+1}) - H(t_i)\|_{\mathscr{F}_{FD}} \\
=&\sum_{i=0}^{n-1} \left(\langle H(t_{i+1}), H(t_{i+1})\rangle - \langle H(t_{i+1}), H(t_i)\rangle - \langle H(t_i), H(t_{i+1})\rangle + \langle H(t_i), H(t_i)\rangle\right)^{1/2} \\
=&\sum_{i=0}^{n-1} \left(\langle T(t_{i+1})\psi(f), T(t_{i+1})\psi(f)\rangle - \langle T(t_{i+1})\psi(f), T(t_i)\psi(f)\rangle\right. \\
&\left. - \langle T(t_i)\psi(f), T(t_{i+1})\psi(f)\rangle + \langle T(t_i)\psi(f), T(t_i)\psi(f)\rangle\right)^{1/2} \\
=&\sum_{i=0}^{n-1} \left(\left(\int_0^{t_{i+1}} \rho(s)\, ds\right)^2 + \int_0^{t_{i+1}} \rho^2(s)\, ds - \int_0^{t_{i+1}} \rho(s)\, ds \int_0^{t_i} \rho(s)\, ds - \int_0^{t_i} \rho^2(s)\, ds\right. \\
&\left. - \int_0^{t_i} \rho(s)\, ds \int_0^{t_{i+1}} \rho(s)\, ds - \int_0^{t_i} \rho^2(s)\, ds + \left(\int_0^{t_i} \rho(s)\right)^2 + \int_0^{t_i} \rho^2(s)\, ds\right)^{1/2} \|\psi(f)\| \\
=&\sum_{i=0}^{n-1} \left(\left(\int_{t_i}^{t_{i+1}} \rho(s)\, ds\right)^2 + \int_{t_i}^{t_{i+1}} \rho^2(s)\, ds\right)^{1/2} \|\psi(f) \leq 2\sqrt{2}L \ \|\psi(f)\| < \infty
\end{aligned}
$$

and so

$$V_a^b(H) = \sup_{\Pi} V_{\Pi}(H) \le 2\sqrt{2}L \, \|\psi(f)\| < \infty$$

□

Notation . For $X \in \{A, A^\dagger, P, Q, T\}$ and for a test function g we denote by $m_{X,g}$ the Stieltjes measure defined by the functions of bounded variation F, G, H as in Propositions 5.1 and 5.2.

Since they are with finite variation, it follows [10] that the measures $m_{X,g}$ are regular.

6. DEFINITION OF THE STOCHASTIC INTEGRAL

Definition 6.1. For each measurable subset E of $[0,\infty)$, for $X \in \{A, A^\dagger, P, Q, T\}$ and for $f\colon [0,\infty) \mapsto \mathrm{B}(\mathscr{F})$, where $\mathscr{F} = \mathscr{F}_H$ or $\mathscr{F} = \mathscr{F}_{FD}$, integrable with respect to the Stieltjes measure $m_{X,g}$ defined by X and the arbitrary test function g, we define the integral of f with respect to X on the exponential domain $\mathcal{E}$ as the operator

$$\int_E f(s)\, dX(s) : \mathcal{E} \mapsto \mathscr{F}$$

with

$$\int_E f(s)\, dX(s)\, \psi(g) = \int_E f(s)\, dm_{X,g}(s)$$

Proposition 6.1. *Let $X \in \{A, A^\dagger, P, Q, T\}$ and let $f\colon [0,\infty) \mapsto \mathrm{B}(\mathscr{F})$, where $\mathscr{F} = \mathscr{F}_H$ or $\mathscr{F} = \mathscr{F}_{FD}$, be integrable with respect to the Stieltjes measure $m_{X,g}$ defined by X and the arbitrary test function g and such that the map $t \mapsto f(t)$ is continuous. Then, on each interval $[a,b] \subset [0,\infty)$*

$$\int_{[a,b]} f(s)\, dX(s)\, \psi(g) = \lim_{\Pi \to 0} \sum_{\lambda=0}^{n} f(s_\lambda)\left(\phi(s_{\lambda+1}) - \phi(s_\lambda)\right)$$

where $\Pi = \{s_0 = a < s_1 < \ldots < s_n = b\}$ is any partition of $[a,b]$, $\phi \in \{F, G, H\}$ where F, G, H are as in Propositions 5.1 and 5.2 and the limit is in the $\mathscr{F}$ norm sense.

Proof. For each $n \ge 1$ define $f_n : [0,\infty) \mapsto \mathrm{B}(\mathscr{F})$ by

$$f_n(s) = \begin{cases} \sum_{\lambda=0}^{n} f(s_\lambda)\chi_{[s_\lambda, s_{\lambda+1})} & \text{if } s \in [a,b] \\ f(s) & \text{otherwise} \end{cases}$$

Then

$$\int_{[a,b]} f_n(s)\, dm_{X,g}(s) = \sum_{\lambda=0}^{n} f(s_\lambda)\, m_{X,g}\left([s_\lambda, s_{\lambda+1})\right)$$

We will show that f_n converges to f uniformly on $[0,\infty)$. Since f_n agrees with f outside $[a,b]$, it suffices to show that f_n converges to f uniformly on $[a,b]$. Let $\epsilon > 0$ be given. Since f is uniformly continuous on $[a,b]$ there exists $\delta > 0$ such that $\|f(s) - f(t)\|_{\mathrm{B}(\mathscr{F})} < \epsilon$ for all $s, t \in [a,b]$ with $|s-t| < \delta$. Let $n_0 \ge 1$ be such that for all $\lambda \in \{0, 1, 2, \ldots, n_0\}$, $s_{\lambda+1} - s_\lambda < \delta$. Such an n_0 exists since the mesh of the partition tends to zero. Then, for all $n \ge n_0$ (corresponding to finer partitions) we also have that $s_{\lambda+1} - s_\lambda < \delta$ for all $\lambda \in \{0, 1, 2, \ldots, n\}$, and, by the definition of f_n, for $s \in [a,b]$

$$\|f(s) - f_n(s)\|_{\mathrm{B}(\mathscr{F})} \le \max_{0 \le \lambda \le n} \|f(s) - f(s_\lambda)\|_{\mathrm{B}(\mathscr{F})} < \epsilon$$

since if $s \in [s_\lambda, s_{\lambda+1})$ then $s - s_\lambda < s_{\lambda+1} - s_\lambda < \delta$. Thus ([3], Theorem 8)

$$\begin{aligned}
&\int_{[a,b]} f(s)\, dm_{X,g}(s) \\
&= \lim_{n\to\infty} \int_{[a,b]} f_n(s)\, dm_{X,g}(s) \\
&= \lim_{n\to\infty} \sum_{\lambda=0}^{n} f(s_\lambda)\, m_{X,g}\left([s_\lambda, s_{\lambda+1})\right) \\
&= \lim_{n\to\infty} \sum_{\lambda=0}^{n} f(s_\lambda)\, \left(\phi(s_{\lambda+1}) - \phi(s_\lambda)\right) \\
&= \lim_{\Pi\to 0} \sum_{\lambda=0}^{n} f(s_\lambda)\left(\phi(s_{\lambda+1}) - \phi(s_\lambda)\right)
\end{aligned}$$

□

7. COMPUTATION OF MATRIX ELEMENTS

In the case of adapted processes, matrix elements

$$\langle \int_0^t f(s)\, dX(s)\, \psi(g), \psi(h)\rangle$$

and

$$\langle \int_0^t f_1(s)\, dX(s)\, \psi(g), \int_0^t f_2(s)\, dX_2(s)\, \psi(h)\rangle$$

are computed with the use of Theorem 6.1 as follows and result in the familiar formulas of quantum stochastic calculus [6, 12, 16]:

$$\begin{aligned}
&\langle \int_0^t f(s)\, dX(s)\, \psi(g), \psi(h)\rangle \\
=&\langle \lim_{\Pi\to 0} \sum_{\lambda=0}^{n} f(s_\lambda)\left(\phi(s_{\lambda+1}) - \phi(s_\lambda)\right), \psi(h)\rangle \\
=& \lim_{\Pi\to 0} \sum_{\lambda=0}^{n} \langle f(s_\lambda)\left(\phi(s_{\lambda+1}) - \phi(s_\lambda)\right), \psi(h)\rangle
\end{aligned}$$

and assuming decompositions $f(t) = f_{t]} \otimes \mathbf{1}_{(t}$, $X(t) = X_{t]} \otimes \mathbf{1}_{(t}$, $\psi(g) = \psi(g_{t]}) \otimes \psi(g_{(t})$, $\psi(h) = \psi(h_{t]}) \otimes \psi(h_{(t})$, as in Definition 3.3, we have

$$\begin{aligned}
&\langle \int_0^t f(s)\, dX(s)\, \psi(g), \psi(h)\rangle = \lim_{\Pi\to 0} \sum_{\lambda=0}^{n} \langle f(s_\lambda)\, \left(X(s_{\lambda+1}) - X(s_\lambda)\right) \psi(g), \psi(h)\rangle \\
&= \lim_{\Pi\to 0} \sum_{\lambda=0}^{n} \langle f_{s_\lambda]}\, \psi(g_{s_\lambda]}), \psi(h_{s_\lambda]})\rangle \langle X_{(s_\lambda, s_{\lambda+1}]}\, \psi(g_{(s_\lambda, s_{\lambda+1}]}), \psi(h_{(s_\lambda, s_{\lambda+1}]})\rangle \langle \psi(g_{(s_{\lambda+1}}), \psi(h_{(s_{\lambda+1}})\rangle \\
&= \lim_{\Pi\to 0} \sum_{\lambda=0}^{n} \langle f_{s_\lambda]}\, \psi(g_{s_\lambda]}), \psi(h_{s_\lambda]})\rangle \int_{s_\lambda}^{s_{\lambda+1}} w_X(s)\, ds\, \langle \psi(g_{(s_\lambda, s_{\lambda+1}]}), \psi(h_{(s_\lambda, s_{\lambda+1}]})\rangle \langle \psi(g_{(s_{\lambda+1}}), \psi(h_{(s_{\lambda+1}})\rangle
\end{aligned}$$

$$= \lim_{\Pi \to 0} \sum_{\lambda=0}^{n} \int_{s_\lambda}^{s_{\lambda+1}} w_X(s)\, ds\, \langle f_{s_\lambda]}\, \psi(g_{s_\lambda]}), \psi(h_{s_\lambda]}) \rangle \langle \psi(g_{(s_\lambda, s_{\lambda+1}]}), \psi(h_{(s_\lambda, s_{\lambda+1}]}) \rangle \langle \psi(g_{(s_{\lambda+1}}), \psi(h_{(s_{\lambda+1}}) \rangle$$
$$= \lim_{\Pi \to 0} \sum_{\lambda=0}^{n} \int_{s_\lambda}^{s_{\lambda+1}} w_X(s)\, ds\, \langle f(s_\lambda)\, \psi(g), \psi(h) \rangle$$

where w_X is a simple function. In fact we may assume [8] that w_X is a step function and that the partition Π is so small that each interval $[s_\lambda, s_\lambda + 1]$ is contained in one of the disjoint intervals defining w_X. Let $w_X(s_\lambda)$ be the value of w_X in that interval. Then

$$\langle \int_0^t f(s)\, dX(s)\, \psi(g), \psi(h) \rangle$$
$$= \lim_{\Pi \to 0} \sum_{\lambda=0}^{n} w_X(s_\lambda)\, (s_{\lambda+1} - s_\lambda)\ \langle f(s_\lambda)\, \psi(g), \psi(h) \rangle$$
$$= \int_0^t w_X(s) \langle f(s)\, \psi(g), \psi(h) \rangle\, ds$$

The computation of

$$\langle \int_0^t f_1(s)\, dX(s)\, \psi(g), \int_0^t f_2(s)\, dX_2(s)\, \psi(h) \rangle$$

is similar.

REFERENCES

[1] L. ACCARDI, A. FRIGERIO and J.T. LEWIS, Quantum Stochastic processes, *Publications of the R.I.M.S.*, Kyoto, 18 (1982), pp. 97–133.

[2] L. ACCARDI and M. SKEIDE, On the relation of the square of white noise and the finite difference algebra, Infinite dimensional analysis, quantum probability and related topics, **Vol. 3**, no. 1, (2000), pp. 185–189.

[3] R. G. BARTLE, A general bilinear vector integral, *Studia Math.*, **15**, (1956), pp. 337–352.

[4] R.G. BARTLE, N. DUNFORD and J.T. SCHWARTZ, Weak compactness and vector measures, *Canad. J. Math.*, **7**, (1955), pp. 289–305.

[5] S. BOCHNER, Integration von Functionen, deren Wert die Elemente eines Vektorraumes sind, **Fund. Math.**, 20, (1933), pp. 262–276.

[6] A. BOUKAS, An Example of a Quantum Exponential Process, *Mh. Math.*, **112**, (1991), pp. 209–215.

[7] A. BOUKAS, Stochastic Calculus on the Finite - Difference Fock space, *Quantum Probability and Related Topics*, **vol. VI.**, World Scientific, (1992), pp. 205–218.

[8] A. BOUKAS, *Quantum Stochastic Analysis: a non-Brownian case*, PhD thesis, Southern Illinois University, USA, 1988.

[9] A. M. CHEBOTAREV, *Lectures on Quantum Probability*, Aportaciones Matematicas 14, Sociedad Matematica Mexicana, 2000.

[10] N. DINCULEANU, *Vector Measures*, International Series of Monographs in Pure and Applied Mathematics, **Vol. 95**, Pergamon Press, 1967.

[11] P. J. FEINSILVER, Discrete analogues of the Heisenberg-Weyl algebra, *Monatshefte fur Mathematik*, **104**, (1987).

[12] P. J. FEINSILVER, Lie algebras and recurrence relations I, *Acta Applicandae Mathematicae*, **13**, (1988), pp. 291–333.

[13] P. J. FEINSILVER and R. SCHOTT, *Algebraic Structures and Operator Calculus. Volumes I and III*, Kluwer, 1993.

[14] I. M. GEL'FAND, Sur un lemme de la theorie des espaces lineaires, *Zap. Naukovodosl. Inst. Mat. Mekh. Kharkov. Mat. Tov.*, **13**(1), (1936), pp. 35–40

[15] R.L. HUDSON, Quantum stochastic calculus in Fock space: a review, *Fundamental Aspects of Quantum Theory*, Plenum (1986).

[16] R. L. HUDSON, K. R. PARTHASARATHY, Quantum Ito's formula and stochastic evolutions, *Commun. Math. Phys.*, **93**, (1984), pp. 301–323.

[17] B. J. PETTIS, On integration in vector spaces, *Trans. Amer. Math. Soc.*, **44**, (1938), pp. 277–304.

[18] K. YOSIDA, *Functional Analysis*, Springer-Verlag, 6th ed., 1980.