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ON THE CONSTANT IN A TRANSFERENCE INEQUALITY FOR THE VECTOR-VALUED FOURIER TRANSFORM

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ABSTRACT. The standard proof of the equivalence of Fourier type on $\mathbb{R}^d$ and on the torus $\mathbb{T}^d$ is usually stated in terms of an implicit constant, defined as the minimum of a sum of powers of sinc functions. In this note we compute this minimum explicitly.

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1. INTRODUCTION

The motivation of this paper comes from a well-known transference result for the vector-valued Fourier transform. Let X be a complex Banach space. The *Fourier transform* of a function $f \in L^1(\mathbb{R}^d; X)$ is defined by

$$\mathscr{F}_{\mathbb{R}^d} f(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)\,dx, \quad \xi \in \mathbb{R}^d.$$

Likewise, the *Fourier transform* of a function $f \in L^1(\mathbb{T}^d; X)$ is defined by

$$\mathscr{F}_{\mathbb{T}^d} f(k) := \int_{\mathbb{T}^d} e^{-2\pi i k\cdot t} f(t)\,dt, \quad k \in \mathbb{Z}^d.$$

Proposition 1.1. *Let X be a complex Banach space, fix $d \geq 1$ and $p \in (1,2]$, and let $\frac{1}{p} + \frac{1}{q} = 1$. The following assertions are equivalent:*

(i) *$\mathscr{F}_{\mathbb{R}^d}$ extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ into $L^q(\mathbb{R}^d; X)$;*
(ii) *$\mathscr{F}_{\mathbb{T}^d}$ extends to a bounded operator from $L^p(\mathbb{T}^d; X)$ into $\ell^q(\mathbb{Z}^d; X)$.*

In this situation, denoting the norms of these extensions by $\varphi_{p,X}(\mathbb{R}^d)$ and $\varphi_{p,X}(\mathbb{T}^d)$, we have

$$\varphi_{p,X}(\mathbb{R}^d) \leq \varphi_{p,X}(\mathbb{T}^d) \leq C_q^{-d/q} \varphi_{p,X}(\mathbb{R}^d),$$

where C_q is the global minimum of the periodic function

$$x \mapsto \sum_{m\in\mathbb{Z}} \Big|\frac{\sin(\pi(x+m))}{\pi(x+m)}\Big|^q, \quad x \in \mathbb{R}.$$

This function, as well as several others considered below, have removable singularities. It is understood that we will always be working with their unique continuous extensions.

A complex Banach space X which has the equivalent properties (i) and (ii) is said to have *Fourier type* p; this notion has been introduced in [5]. Proposition 1.1 goes back to [4]; in its stated form the result can be found in [2, 3]. Related results may be found in [1]. These references do not comment on the location of the global minimum. A quick computer plot (see Figure 1) suggests that the minimum is taken in the points $\frac{1}{2} + \mathbb{Z}$. To actually *prove* this turns out to be surprisingly difficult. This is the modest objective of the present note:

Proposition 1.2. *For every real number $r \geq 1$, the function $f_r : [0,1] \to \mathbb{R}$ defined by*

$$f_r(x) := \sum_{m\in\mathbb{Z}} \Big|\frac{\sin(\pi(x+m))}{\pi(x+m)}\Big|^{2r}, \quad x \in [0,1],$$

has a global minimum at $x = \frac{1}{2}$.

Our proof has developed essentially by trial and error. We believe it is perfectly possible that a truly pedestrian proof can be given, but we failed to find one despite many hours of efforts. As a consequence of Proposition 1.2 we obtain the explicit estimate

$$\varphi_{p,X}(\mathbb{R}^d) \leq \varphi_{p,X}(\mathbb{T}^d) \leq \frac{\pi^d}{\big(2(2^q-1)\zeta(q)\big)^{d/q}} \cdot \varphi_{p,X}(\mathbb{R}^d),$$

noting that

$$\sum_{m\in\mathbb{Z}} \frac{1}{|\frac{1}{2}+m|^q} = 2(2^q-1)\zeta(q).$$

For even integers $q = 2n$, the constant on the right-hand side may be evaluated explicitly in terms of the Bernoulli numbers. To further estimate this constant, recall that for any $x \in \ell^2(\mathbb{Z})$ the function $q \mapsto ||x||_q := (\sum_{m\in\mathbb{Z}} |x_m|^q)^{1/q}$ is decreasing on $[2,\infty)$ and $\lim_{q\to\infty} ||x||_q =$

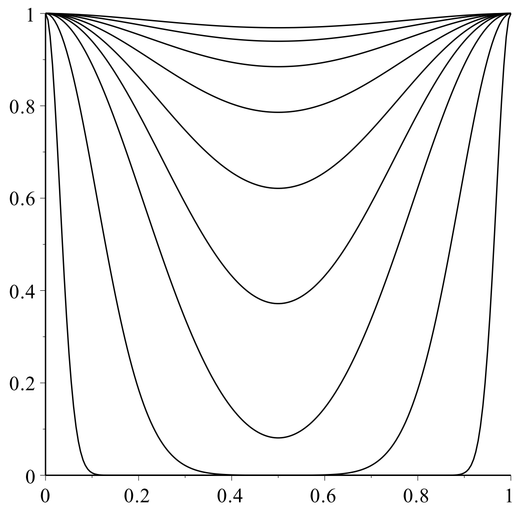

Figure 1: A plot of f_r, where $r = 1.02^k$ for $k = 1, 2, 4, \ldots, 256$.

$\sup_{i\in\mathbb{Z}} |x_i|$. Taking $x_m := |\frac{1}{2} + m|^{-1}$ we find $(\sum_{m\in\mathbb{Z}} |\frac{1}{2} + m|^{-q})^{1/q} \geq 2$ for every $q \geq 2$, and hence in particular

$$\varphi_{p,X}(\mathbb{R}^d) \leq \varphi_{p,X}(\mathbb{T}^d) \leq (\tfrac{1}{2}\pi)^d \varphi_{p,X}(\mathbb{R}^d).$$

2. THE MAIN RESULT

The proof of the proposition is based on the following lemmas. The main idea is contained in the first lemma.

Lemma 2.1. *Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing convex function, and let $x_1, \ldots, x_n \in \mathbb{R}_+$ and $y_1, \ldots, y_n \in \mathbb{R}_+$ be such that*

(i) $x_1 + \cdots + x_n \geq y_1 + \cdots + y_n$;
(ii) *there exists $t \in \mathbb{R}_+$ such that*
- $x_i \leq y_i$ *if* $y_i < t$;
- $x_i \geq y_i$ *if* $y_i \geq t$.

Then $g(x_1) + \cdots + g(x_n) \geq g(y_1) + \cdots + g(y_n)$.

Proof. We will prove the lemma by induction on n. The case $n = 1$ is clear: $x_1 \geq y_1$ implies that $g(x_1) \geq g(y_1)$ since g is non-decreasing. Suppose now that the lemma has been proved for $n = 1, \ldots, m - 1$.

If $x_i = y_i$ for some index $1 \leq i \leq m$, then we may remove x_i and y_i and apply the induction hypothesis.

If $x_i \geq y_i$ for every index $1 \leq i \leq m$, then again the result is immediate since g is non-decreasing. Therefore, we may assume that $x_i < y_i$ for some index $1 \leq i \leq m$. Then, by the first condition in the lemma, there is also an index j for which $x_j > y_j$. By the second condition in the lemma we then have $x_i < y_i < t \leq y_j < x_j$.

Let $\epsilon := \min(y_i - x_i, x_j - y_j)$ and define $x'_i := x_i + \epsilon$, $x'_j := x_j - \epsilon$, and $x'_k := x_k$ for all other indices. Then $x'_1, \ldots, x'_m, y_1, \ldots, y_m$ satisfy the conditions in the lemma (with the same t) and $x'_i = y_i$ or $x'_j = y_j$. Hence, by the induction hypothesis, we have

$$g(x'_1) + \cdots + g(x'_m) \geq g(y_1) + \cdots + g(y_m). \tag{2.1}$$

Since $x_i \le x'_i \le x'_j \le x_j$, we can write $x'_i = \lambda x_i + (1-\lambda)x_j$ for some $\lambda \in [0,1]$. Since $x'_j = x_i + x_j - x'_i$, we have $x'_j = (1-\lambda)x_i + \lambda x_j$. By the convexity of g it follows that

$$g(x'_i) + g(x'_j) \le (\lambda g(x_i) + (1-\lambda)g(x_j)) + ((1-\lambda)g(x_i) + \lambda g(x_j)) = g(x_i) + g(x_j). \tag{2.2}$$

Combining inequalities (2.1) and (2.2) we obtain the lemma for $n = m$, thus completing the induction step. ∎

In order to apply this lemma we need a number of technical facts. The first (cf. [2, (6.14)]) is elementary and is left as an exercise.

Lemma 2.2. $f_1(x) = 1$ *for all* $x \in [0,1]$.

Let $h : \mathbb{R} \to \mathbb{R}$ be defined by

$$h(x) := \operatorname{sinc}^2(\pi x) = \Big(\frac{\sin(\pi x)}{\pi x}\Big)^2, \quad x \in \mathbb{R}.$$

Lemma 2.3. *Let* $r \ge 1$. *The following assertions hold on the interval* $[0,1]$*:*

(i) *the function* $h(x) + h(x-1)$ *has a global minimum at* $x = \frac{1}{2}$*;*

(ii) *for all* $m = 1,2,3,\dots$, $h(x+m) + h(x-(m+1))$ *has a global maximum at* $x = \frac{1}{2}$*;*

(iii) *the function*

$$h(x) + h(x-1) - (h(x)^r + h(x-1)^r)^{1/r}$$

has a global maximum at $x = \frac{1}{2}$*;*

(iv) *for all* $m = 1,2,3,\dots$ *and* $r \ge 1$,

$$(h(x+m) + h(x-(m+1)))^r - h(x+m)^r - h(x-(m+1))^r$$

has a global maximum at $x = \frac{1}{2}$.

Assuming the lemmas for the moment, let us first show how the proposition can be deduced from them.

Proof of Proposition 1.2. Fix $r \ge 1$ and set, for $x \in [0,1]$,

$$s_m(x) := h(x+m) + h(x-(m+1)) \qquad (m = 0,1,2,\dots)$$

and

$$\widetilde{s}_0(x) := ((h(x))^r + (h(x-1))^r)^{1/r}.$$

In view of part (iv) of Lemma 2.3 it suffices to prove that

$$\widetilde{s}_0^{\,r} + s_1^r + s_2^r + \cdots$$

has a global minimum at $x = \frac{1}{2}$.

Fix an arbitrary $x \in [0,1]$ and set

$$x_m := s_m(x), \quad y_m := s_m(\tfrac{1}{2}) \qquad (m = 0,1,2,\dots)$$

and

$$\widetilde{x}_0 := ((h(x))^r + h(x-1)^r)^{1/r}, \quad \widetilde{y}_0 := ((h(\tfrac{1}{2}))^r + h(-\tfrac{1}{2})^r)^{1/r}.$$

In view of parts (i) and (ii) of Lemma 2.3 we have

$$x_0 \ge y_0, \quad x_i \le y_i \qquad (i = 1,2,\dots) \tag{2.3}$$

Lemma 2.2 implies

$$x_0 + x_1 + x_2 + \cdots = y_0 + y_1 + y_2 + \cdots \tag{2.4}$$

By (2.3) and (2.4),

$$x_0 + x_1 + \cdots + x_n \ge y_0 + y_1 + \cdots + y_n \qquad (n = 0,1,2,\dots) \tag{2.5}$$

Part (iii) of Lemma 2.3 implies

$$\widetilde{x}_0 - x_0 \geq \widetilde{y}_0 - y_0. \tag{2.6}$$

By (2.5) and (2.6),

$$\widetilde{x}_0 + x_1 + \cdots + x_n \geq \widetilde{y}_0 + y_1 + \cdots + y_n \qquad (n = 0, 1, 2, \ldots) \tag{2.7}$$

Finally, by (2.3) and (2.6),

$$\widetilde{x}_0 \geq \widetilde{y}_0. \tag{2.8}$$

A simple calculation shows that $\widetilde{y}_0 > \frac{4}{\pi^2}$ and $y_i < \frac{4}{\pi^2}$ for $i = 1, 2, \ldots$. Taking $t = \frac{4}{\pi^2}$ in Lemma 2.1 and $g(x) := x^r$ now implies, by virtue of (2.3), (2.7), and (2.8), that

$$\widetilde{x}_0^r + x_1^r + \cdots + x_n^r \geq \widetilde{y}_0^r + y_1^r + \cdots + y_n^r$$

holds for every n. Taking limits for $n \to \infty$ completes the proof. ∎

3. PROOF OF LEMMA 2.3

This section is devoted to the proof of Lemma 2.3, which is based on the following observations:

Lemma 3.1. *On the interval* $[0, 1]$*:*

(i) $\dfrac{\cos(\frac{1}{2}\pi x)}{1 - x^2}$ *takes a global maximum at* $x = 0$*;*

(ii) $\dfrac{(x^2 + 1)\cos^2(\frac{1}{2}\pi x)}{(1 - x^2)^2}$ *takes a global minimum at* $x = 0$.

Proof. We start by showing that

$$\sqrt{2}\sin(\tfrac{1}{4}\pi x) \geq x \quad \text{for all } x \in [0, 1]. \tag{3.1}$$

To this end, consider $f(x) := \sqrt{2}\sin(\frac{1}{4}\pi x) - x$. Observe that $f'(x) = \frac{\pi\sqrt{2}}{4}\cos(\frac{1}{4}\pi x) - 1$ is decreasing on $[0, 1]$, hence f is concave. Since $f(0) = f(1) = 0$ this implies that $f(x) \geq 0$ for $x \in [0, 1]$, which proves the claim.

(i): The value at $x = 0$ of the given function equals 1, so it suffices to show that $\cos(\frac{1}{2}\pi x) \leq 1 - x^2$ for all $x \in [0, 1]$. This follows from the double-angle formula for cosine and (3.1):

$$\cos(\tfrac{1}{2}\pi x) = 1 - 2\sin^2(\tfrac{1}{4}\pi x) \leq 1 - x^2.$$

(ii): The given function has value 1 at $x = 0$, hence it suffices to show that for all $x \in [0, 1]$,

$$(x^2 + 1)\cos^2(\tfrac{1}{2}\pi x) \geq (1 - x^2)^2.$$

On the interval $[\frac{1}{2}, 1]$ we substitute $x = 1 - y$. We then must prove that for $y \in [0, \frac{1}{2}]$,

$$(2 - 2y + y^2)\sin^2(\tfrac{1}{2}\pi y) \geq (2y - y^2)^2.$$

Since $2y \in [0, 1]$, we can use (3.1) to obtain $\sqrt{2}\sin(\frac{1}{4}\pi \cdot 2y) \geq 2y$, and hence $\sin^2(\frac{1}{2}\pi y) \geq 2y^2$. This implies that

$$(2 - 2y + y^2)\sin^2(\tfrac{1}{2}\pi y) \geq (2 - 2y + y^2)(2y^2) = (y^2 + (2 - y)^2)y^2 \geq (2 - y)^2 y^2 = (2y - y^2)^2,$$

which concludes the proof on the interval $[\frac{1}{2}, 1]$.

For $x \in [0, \frac{1}{2}]$ we have

$$\begin{aligned}(x^2+1)\cos^2(\tfrac{1}{2}\pi x) &\geq (x^2+1)\Big(1-\frac{\pi^2}{8}x^2\Big)^2 \\ &= (x^2+1)(1-\frac{\pi^2}{4}x^2+\frac{\pi^4}{64}x^4) \\ &\geq 1+(1-\frac{\pi^2}{4})x^2+(\frac{\pi^4}{64}-\frac{\pi^2}{4})x^4 \\ &= 1+(1-\frac{\pi^2}{4})x^2+(\frac{\pi^4}{64}-\frac{\pi^2}{4}-1)x^4+x^4 \\ &\geq 1+\Big[(1-\frac{\pi^2}{4})+\frac{1}{4}(\frac{\pi^4}{64}-\frac{\pi^2}{4}-1)\Big]x^2+x^4 \\ &\geq 1-2x^2+x^4 \\ &= (1-x^2)^2,\end{aligned}$$

noting that $\frac{\pi^4}{64}-\frac{\pi^2}{4}-1<0$ and $(1-\frac{\pi^2}{4})+\frac{1}{4}(\frac{\pi^4}{64}-\frac{\pi^2}{4}-1)\approx -1.9537471\cdots > -2$ ∎

Proof of Lemma 2.3. (i): We have

$$h(x)+h(x-1)=\frac{\sin^2(\pi x)}{\pi^2x^2}+\frac{\sin^2(\pi x)}{\pi^2(x-1)^2}=\frac{(2x^2-2x+1)\sin^2(\pi x)}{\pi^2x^2(x-1)^2}=: g(x).$$

We must show that

$$f(x) := g(x+\frac{1}{2}) = \frac{8}{\pi^2}\frac{4x^2+1}{(4x^2-1)^2}\cos^2(\pi x)$$

has a global minimum in $x=0$ on the interval $[-\frac{1}{2},\frac{1}{2}]$. But this follows from Lemma 3.1 and the fact that f is even.

(ii): For $m = 1, 2, 3, \dots$ we have

$$h(x+m)+h(x-(m+1))=\frac{[2x^2-2x+(m+1)^2+m^2]\sin^2(\pi x)}{\pi^2[(x+m)^2(x-(m+1))^2]}=: g_m(x).$$

We must show that

$$f_m(x) := g_m(x+\frac{1}{2}) = \frac{8}{\pi^2}\frac{4x^2+4m^2+4m+1}{[(4x^2-(2m+1)^2]^2}\cos^2(\pi x)$$

has a global maximum in $x=0$ on the interval $[-\frac{1}{2},\frac{1}{2}]$. For this, it suffices to check that the functions

$$\frac{4x^2+1}{(4x^2-M^2)^2}\cos^2(\pi x) \quad\text{and}\quad \frac{1}{(4x^2-M^2)^2}\cos^2(\pi x)$$

are decreasing on $[0,\frac{1}{2}]$ for each $M \geq 3$, or equivalently, that

$$\frac{\sqrt{x^2+1}}{M^2-x^2}\cos(\tfrac{1}{2}\pi x) \quad\text{and}\quad \frac{1}{M^2-x^2}\cos(\tfrac{1}{2}\pi x)$$

are decreasing on $[0,1]$ for each $M \geq 3$. It suffices to prove this for the first function, since this will immediately imply the result for the second function.

Straightforward algebra shows that the derivative of the function

$$\psi_M(x) := \frac{\sqrt{x^2+1}}{M^2-x^2}\cos(\tfrac{1}{2}\pi x)$$

has a zero at x if and only if

$$2x(x^2+2+M^2)\cos(\tfrac{1}{2}\pi x) = \pi(M^2-x^4+(M^2-1)x^2)\sin(\tfrac{1}{2}\pi x).$$

But,

$$2x(M^2+2+x^2)\cos(\tfrac{1}{2}\pi x) \le 2x(M^2+2+x^2)$$

and, since $0 \le x \le 1$,

$$\pi(M^2-x^4+(M^2-1)x^2)x \le \pi(M^2-x^4+(M^2-1)x^2)\sin(\tfrac{1}{2}\pi x),$$

while also, using that $M \ge 3$ and $0 \le x \le 1$,

$$2(M^2+2+x^2) \le 2(M^2+2+(M^2-1)x^2) < \pi(M^2-1+(M^2-1)x^2) \le \pi(M^2-x^4+(M^2-1)x^2)$$

since $2(M^2+2) < \pi(M^2-1)$ for $M \ge 3$. It follows that the derivative of ψ_M has no zeros on $(0,1]$, and then from

$$\psi_M(0) = \frac{1}{M^2} > 0 = \psi_M(1)$$

it follows that ψ_M is decreasing on $[0,1]$.

(iii): Proceeding as in (i), we have

$$\begin{aligned} &h(x)+h(x-1)-((h(x))^r+(h(x-1))^r)^{1/r} \\ &\quad = \frac{1}{\pi^2}\Big[\frac{1}{x^2}+\frac{1}{(1-x)^2}-\Big(\frac{1}{x^{2r}}+\frac{1}{(1-x)^{2r}}\Big)^{1/r}\Big]\sin^2(\pi x) =: g(x). \end{aligned}$$

We must show that

$$f(x) := g(\tfrac{1}{2}+x) = \frac{1}{\pi^2}\Big[(\tfrac{1}{2}-x)^2+(\tfrac{1}{2}+x)^2-((\tfrac{1}{2}-x)^{2r}+(\tfrac{1}{2}+x)^{2r})^{1/r}\Big]\frac{\cos^2(\pi x)}{(\frac{1}{4}-x^2)^2}$$

has a global maximum in $x=0$ on the interval $[-\frac{1}{2},\frac{1}{2}]$. The function f is even, and by Lemma 3.1, $\cos^2(\pi x)/(\frac{1}{4}-x^2)^2$ takes its maximum at $x=0$. It thus remains to show that on the interval $[0,\frac{1}{2}]$ the function

$$\phi_r(x) := (\tfrac{1}{2}-x)^2+(\tfrac{1}{2}+x)^2-((\tfrac{1}{2}-x)^{2r}+(\tfrac{1}{2}+x)^{2r})^{1/r}$$

is decreasing on $[0,\frac{1}{2}]$. The derivative of this function equals

$$\phi_r'(x) = 4x-2\big((\tfrac{1}{2}-x)^{2r}+(\tfrac{1}{2}+x)^{2r}\big)^{1/r-1}\big((\tfrac{1}{2}+x)^{2r-1}-(\tfrac{1}{2}-x)^{2r-1}\big).$$

To show that $\phi_r'(x) \le 0$ we must show that

$$\big((\tfrac{1}{2}+x)^{2r}+(\tfrac{1}{2}-x)^{2r}\big)^{1/r-1}\big((\tfrac{1}{2}+x)^{2r-1}-(\tfrac{1}{2}-x)^{2r-1}\big) \ge 2x$$

for $x \in [0,\frac{1}{2}]$, or, after substituting $a=\frac{1}{2}+x$ and $b=\frac{1}{2}-x$, that

$$a^{2r-1}-b^{2r-1} \ge (a-b)\big(a^{2r}+b^{2r}\big)^{1-1/r}$$

for all $a \in [\frac{1}{2},1]$. In view of

$$\begin{aligned} \big(a^{2r}+b^{2r}\big)^{1-1/r} &= \Big[\big(a^{2r}+b^{2r}\big)^{1/(2r)}\Big]^{2r-2} \\ &\le \Big[\big(a^{2r-1}+b^{2r-1}\big)^{1/(2r-1)}\Big]^{2r-2} = \big(a^{2r-1}+b^{2r-1}\big)^{1-1/(2r-1)}, \end{aligned}$$

with $p := 2r-1$ it suffices to show that

$$a^p-b^p \ge (a-b)(a^p+b^p)^{1-1/p}$$

for all $a \ge b \ge 0$. We can further simplify this upon dividing both sides by b^p. In the new variable $x=a/b$ we then have to prove that

$$x^p-1 \ge (x-1)(x^p+1)^{1-1/p}$$

for all $x \ge 1$.

Using that $(1+y)^\alpha \le 1+\alpha y$ for $y\ge 0$ and $0\le\alpha\le 1$, we have

$$(x-1)(x^p+1)^{1-1/p} = (x^p - x^{p-1})(1+x^{-p})^{1-1/p} \le (x^p - x^{p-1})[1+(1-\frac{1}{p})\,x^{-p}].$$

Therefore it remains to prove that for $x\ge 1$ and $p\ge 1$ we have

$$x^p - 1 \ge (x^p - x^{p-1})[1+(1-\frac{1}{p})\,x^{-p}],$$

or, multiplying both sides with x, that

$$x^{p+1} - x \ge x^{p+1} - x^p + (1-\frac{1}{p})\,(x-1).$$

that is, we must show that

$$f_p(x) := x^p \ge x + (1-\frac{1}{p})(x-1) =: g_p(x).$$

Now

$$f_p'(x) = px^{p-1}, \quad g_p'(x) = 2-\frac{1}{p}.$$

It follows that $f_p'(x) \ge g_p'(x) \ge 0$ for $x\ge 1$, since $p \ge 2-\frac{1}{p}$ (multiply both sides by p). Together with $f_p(1) = g_p(1)$ it follows that $f_p(x)\ge g_p(x)$ for $x\ge 1$ and $p\ge 1$. This concludes the proof of (iii).

(iv): Fix $m\ge 1$. For $x\in[-\frac12,\frac12]$ we have

$$\begin{aligned}
&(h(x+\tfrac12+m)+h(x+\tfrac12-(m+1)))^r - h(x+\tfrac12+m)^r - h(x+\tfrac12-(m+1))^r \\
&= \Bigg[\Big(\frac{1}{(x+(m+\frac12))^2}+\frac{1}{(x-(m+\frac12))^2}\Big)^r - \Big(\frac{1}{(x+(m+\frac12))^2}\Big)^r - \Big(\frac{1}{(x-(m+\frac12))^2}\Big)^r\Bigg] \\
&\qquad\qquad \times \pi^{-2r}\big(\cos^2(\pi x)\big)^r.
\end{aligned}$$

We must show that this function has a global maximum on $[-\frac12,\frac12]$ at $x=0$. Since by Lemma 3.1 $\cos(\pi x)/(1-4x^2)$ has a global maximum at $x=0$, it suffices to prove that

$$\big[((a+x)^{-2}+(a-x)^{-2})^r - (a+x)^{-2r} - (a-x)^{-2r}\big]\cdot(1-4x^2)^{2r}$$

has a global maximum at $x=0$, where we have written $a := m+\frac12 \ge \frac32$. Since the function $x\mapsto x^r$ is convex, we have $\frac12(a+x)^{-2r}+\frac12(a-x)^{-2r} \ge (\frac12(a+x)^{-2}+\frac12(a-x)^{-2})^r$ and hence

$$2^{1-r}((a+x)^{-2}+(a-x)^{-2})^r - (a+x)^{-2r} - (a-x)^{-2r} \le 0$$

with equality for $x=0$. Therefore, it suffices to show that

$$(1-2^{1-r})((a+x)^{-2}+(a-x)^{-2})^r(1-4x^2)^{2r}$$

has a global maximum at $x=0$. It is enough to show that the function $g(x) := ((a+x)^{-2} + (a-x)^{-2})(1-4x^2)^2$ is decreasing on $[0,\frac12]$.

Computing the derivative of g we find

$$\begin{aligned}
g'(x) &= -16x(1-4x^2)((a+x)^{-2}+(a-x)^{-2}) + (1-4x^2)^2(-2(a+x)^{-3}+2(a-x)^{-3}) \\
&= (1-4x^2)(a+x)^{-3}(a-x)^{-3}k(x),
\end{aligned}$$

where

$$\begin{aligned} k(x) &= -16x(a^2-x^2)((a+x)^2+(a-x)^2)+(1-4x^2)(2(a+x)^3-2(a-x)^3) \\ &= -16x\cdot 2(a^4-x^4)+(1-4x^2)\cdot 4x\cdot(3a^2+x^2) \\ &= 4x[-8(a^4-x^4)+(1-4x^2)(3a^2+x^2)] \\ &= 4x[4x^4+(1-12a^2)x^2+(3a^2-8a^4)]. \end{aligned}$$

Since $a > \sqrt{\frac{3}{8}}$, the function $p(y) := 4y^2+(1-12a^2)y+(3a^2-8a^4)$ has a positive and a negative root. The sum of the two roots equals $\frac{12a^2-1}{4}$ and therefore the positive root is larger than $3a^2-\frac{1}{4} \geq \frac{26}{4}$. It follows that p is negative on $[0,\frac{1}{4}]$ and hence that $g'(x) = (1-4x^2)(a+x)^{-3}(a-x)^{-3}\cdot 4x\cdot p(x^2) \leq 0$ on $[0,\frac{1}{2}]$, which finishes the proof. ∎

Added in proof. After this paper had been accepted for publication, Tom Koornwinder sent us the following interesting proof for the case that the parameter r in Proposition 1.2 is integral. With his kind permission we reproduce it here.

We consider $f_r(x)$ on $(0,1)$. In terms of the Hurwitz zeta-function $\zeta(s,q)$ (see [6, Eq. 25.11.1]) we have

$$f_r(x) = \pi^{-2r}\sin^{2r}(\pi x)(\zeta(2r,x)+\zeta(2r,1-x)), \quad r=1,2,\ldots$$

In terms of the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$ (see [6, Eq. 25.11.12]) this can be rewritten as

$$f_r(x) = \frac{\pi^{-2r}\sin^{2r}(\pi x)}{(2r-1)!}\left(\frac{d}{dx}\right)^{2r-1}(\psi(x)-\psi(1-x)).$$

Applying the reflection formula $\psi(1-z)-\psi(z) = \pi\cot(\pi z)$ (see [6, Eq. 5.5.4]) we obtain

$$f_r(x) = \frac{-\pi^{1-2r}\sin^{2r}(\pi x)}{(2r-1)!}\left(\frac{d}{dx}\right)^{2r-1}\cot(\pi x).$$

Substitution of $t = \pi x$ simplifies this expression to

$$\frac{(2r-1)!}{\sin^{2r} t}\cdot f_r(t/\pi) = -\left(\frac{d}{dt}\right)^{2r-1}\cot t.$$

Since $(d/dt)\cot t = -1/\sin^2 t$, we have $f_1(t/\pi) = 1$. Also, we obtain the following recursion relation:

$$\frac{(2r+1)!}{\sin^{2r+2} t}\cdot f_{r+1}(t/\pi) = -\left(\frac{d}{dt}\right)^{2r+1}\cot t \;=\; (2r-1)!\left(\frac{d}{dt}\right)^2\frac{f_r(t/\pi)}{\sin^{2r} t}. \tag{3.2}$$

A small computation shows that

$$\begin{aligned} \left(\frac{d}{dt}\right)^2\frac{f_r(t/\pi)}{\sin^{2r} t} &= \left(\frac{d}{dt}\right)\left[(\sin^{-2r} t)\left(\frac{d}{dt}\right)f_r(t/\pi) - 2r(\cos t)(\sin^{-2r-1} t)f_r(t/\pi)\right] \\ &= (\sin^{-2r} t)\left(\frac{d}{dt}\right)^2 f_r(t/\pi) - 4r(\cos t)(\sin^{-2r-1} t)\left(\frac{d}{dt}\right)f_r(t/\pi) \\ &\quad + \left(2r(2r+1)\cos^2 t\sin^{-2r-2} t + 2r\sin^{-2r} t\right)\cdot f_r(t/\pi). \end{aligned}$$

Hence, (3.2) implies that

$$\begin{aligned}\frac{(2r+1)!}{(2r-1)!}f_{r+1}(t/\pi) &= (\sin^2 t)\left(\frac{d}{dt}\right)^2 f_r(t/\pi) - 4r(\cos t)(\sin t)\left(\frac{d}{dt}\right) f_r(t/\pi)\\ &\quad + (2r(2r+1)\cos^2 t + 2r\sin^2 t)\cdot f_r(t/\pi). \end{aligned}\tag{3.3}$$

Set $y := \cos^2 t$ and $D := d/dy$. So $d/dt = -2(\sin t\cos t)D$ and

$$\begin{aligned}\left(\frac{d}{dt}\right)^2 &= \frac{d}{dt}(-2\sin t\cos t)D\\ &= -2(\cos^2 t - \sin^2 t)D - (2\sin t\cos t)\left(\frac{d}{dt}\right)D\\ &= -2(\cos^2 t - \sin^2 t)D + (4\sin^2 t\cos^2 t)D^2\\ &= (-4y+2)D + 4y(1-y)D^2.\end{aligned}$$

Equation (3.3) can therefore be rewritten as

$$\begin{aligned}\frac{(2r+1)!}{(2r-1)!}f_{r+1}(t/\pi) &= \big[4y(1-y)^2D^2 + ((8r-4)y+2)(1-y)D + 2r(2ry+1)\big] f_r(t/\pi)\\ &= \big[4y(r-yD)^2 + 8(r-yD)yD + 2yD + 2r + 4yD^2 + 2D\big] f_r(t/\pi).\end{aligned}\tag{3.4}$$

Observe that $(r-yD)y^k = (r-k)y^k$. Hence, if $p = p(y)$ is a polynomial of degree $n < r$ with nonnegative coefficients, then the same holds for $(r-yD)p$. The recursion (3.4) and the fact that $f_1(t/\pi) = 1$ now imply that $f_r(t/\pi)$ is a polynomial in y of degree $r-1$ with nonnegative coefficients. The first few are given explicitly by

$$\begin{aligned}f_1(t/\pi) &= 1\\ f_2(t/\pi) &= \frac{1}{3} + \frac{2}{3}\cos^2 t\\ f_3(t/\pi) &= \frac{2}{15} + \frac{11}{15}\cos^2 t + \frac{2}{15}\cos^4 t\\ f_4(t/\pi) &= \frac{17}{315} + \frac{4}{7}\cos^2 t + \frac{38}{105}\cos^4 t + \frac{4}{315}\cos^6 t\\ f_5(t/\pi) &= \frac{62}{2835} + \frac{1072}{2835}\cos^2 t + \frac{484}{945}\cos^4 t + \frac{247}{2835}\cos^6 t + \frac{2}{2835}\cos^8 t\end{aligned}$$

For integers r, Proposition 1.2 is an immediate consequence.

For half-integers $r = n + \frac{1}{2}$ one could observe that the identity

$$\psi^{(2n)}(x) = -(2n)!\sum_{m=0}^{\infty}\frac{1}{(x+m)^{2n+1}}$$

allows one to express the inequality of Proposition 1.2 in terms of the polygamma functions $\psi^{(2n)}$. We have not been able, however, to use this fact to give a simpler proof in that case.

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