**The Australian Journal of Mathematical Analysis and Applications**

AJMAA

**Volume 13**, Issue 1, Article 19, pp. 1-9, 2016

# LOWER AND UPPER BOUNDS FOR THE POINT-WISE DIRECTIONAL DERIVATIVE OF THE FENCHEL DUALITY MAP

MUSTAPHA RAÏSSOULI[12] AND MARYAM RAMEZANI[3]

*Received 31 March, 2016; accepted 3 October, 2016; published 30 November, 2016.*

[1]DEPARTMENT OF MATHEMATICS, SCIENCE FACULTY, TAIBAH UNIVERSITY, P.O.BOX 30097, ZIP CODE 41477, AL MADINAH AL MUNAWWARAH, SAUDI ARABIA.

[2]DEPARTMENT OF MATHEMATICS, SCIENCE FACULTY, MOULAY ISMAIL UNIVERSITY, MEKNES, MOROCCO.
raissouli.mustapha@gmail.com

[3]DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BOJNORD, BOJNORD, IRAN.
m.ramezani@ub.ac.ir

ABSTRACT. In this paper, we introduce the point-wise directional derivative of the Fenchel duality map and we study its properties. The best lower and upper bounds of this point-wise directional derivative are also given. We explain how our functional results contain those related to the positive bounded linear operators.

*Key words and phrases:* Fenchel duality, Point-wise directional derivative, Convex functional, Sub-differential, Positive linear operator.

2010 *Mathematics Subject Classification.* Primary 46N10; Secondary 46A20, 47N10, 39B62, 52A41.

ISSN (electronic): 1449-5910

## 1. INTRODUCTION AND BASIC NOTIONS

Let $(E, \|.\|)$ be a real or complex normed space (Banach if necessary). We denote by $E^*$ the topological dual of $E$ and by $\langle .,. \rangle$ the duality bracket between $E$ and $E^*$ i.e. $x^*(x) = \langle x, x^* \rangle$ for $x \in E$ and $x^* \in E^*$. $E^*$ is (always) a Banach space for the so-called dual norm $\|.\|_*$ defined by

$$\forall x^* \in E^* \qquad \|x^*\|_* = \sup_{x \neq 0} \frac{|\langle x, x^* \rangle|}{\|x\|} = \sup_{\|x\| \leq 1} |\langle x, x^* \rangle| = \sup_{\|x\|=1} |\langle x, x^* \rangle|.$$

To avoid any confusion, elements of $E$ will be denoted by $x, y, z$ and those of $E^*$ by $x^*, y^*, z^*$. Except explicit mention, $E^*$ is endowed with the weak$^*$ topology.

Throughout this paper, we use the notation:

$$\mathbb{R} := (-\infty, \infty),\ \widetilde{\mathbb{R}} := (-\infty, \infty] = \mathbb{R} \cup \{\infty\},\ \overline{\mathbb{R}} = [-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}.$$

A map defined from $E$ into $\overline{\mathbb{R}}$ will be called a functional and denoted by a small letter as $f, g$. We denote by $\overline{\mathbb{R}}^E$ the set of all functionals defined from $E$ into $\overline{\mathbb{R}}$. By functional map we understand a map $\Phi : \overline{\mathbb{R}}^E \longrightarrow \overline{\mathbb{R}}^F$ ($F$ being another normed space), i.e. $\Phi$ is a map whose variable is a functional. Here, we extend the structure of $\mathbb{R}$ to $\overline{\mathbb{R}}$ by setting, for all $t \in \mathbb{R}$,

$$-\infty < t < \infty,\ -\infty + t = -\infty,\ \infty + t = \infty,\ -\infty + \infty = \infty - \infty = \infty.$$

We also defined the so-called point-wise order on $\overline{\mathbb{R}}^E$ defined by, $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in E$. We say that $\Phi$ is point-wise increasing (resp. decreasing) if:

$$f \leq g \implies \Phi(f) \leq (\geq) \Phi(g)$$

and $\Phi$ is called point-wise convex (resp. concave) if:

$$\Phi\big((1-t)f + tg\big) \leq (\geq) (1-t)\Phi(f) + t\Phi(g)$$

for all real number $t \in (0, 1)$.

The following remark worth to be mentioned.

**Remark 1.1.** Throughout this paper, the involved functionals can take infinite values. According to the previous definitions, the two equalities $f = g$ and $f - g = 0$ (resp. $f \leq g$ and $f - g \leq 0$) are not always equivalent.

**N.B.** *For the sake of simplicity for the reader, we restrict ourselves in what follows to the case that $E$ is a real normed (Banach) space. The version related to the complex case can be stated in a similar manner.*

## 2. BACKGROUND MATERIAL

In this section, we recall some definitions and properties about convex analysis that will be needed throughout this paper.

The Fenchel conjugate of $f \in \overline{\mathbb{R}}^E$ is $f^* \in \overline{\mathbb{R}}^{E^*}$ defined by

$$\forall x^* \in E^* \qquad f^*(x^*) = \sup_{x \in E} \Big(\langle x, x^* \rangle - f(x)\Big). \tag{2.1}$$

The map $f \longmapsto f^*$ is then a functional map, defined from $\overline{\mathbb{R}}^E$ into $\overline{\mathbb{R}}^{E^*}$, so-called the Fenchel duality map. It is well known that such functional map is point-wise decreasing and convex. Furthermore, $f^*$ is always convex and l.s.c, even if $f$ is not. By $\Gamma_0(E)$ we denote the convex cone of all convex, lower semi-continuous (l.s.c in short) and proper functionals defined from $E$ into $\widetilde{\mathbb{R}}$ ($f$ is proper means that $f$ does not take the value $-\infty$ and is not identically equal to $\infty$). With this, $f^* \in \Gamma_0(E^*)$ whenever $f$ is proper. For each $t > 0$, we can easily see that

$(tf)^*(x^*) = tf^*(x^*/t)$ for all $x^* \in E^*$. If we define $f^{**} : E \longrightarrow \overline{\mathbb{R}}$ by $f^{**} := (f^*)^*$ then, $f = f^{**}$ if and only if $f \in \Gamma_0(E)$. We always have $f^{**} \leq f$ and $f^{***} = f^*$.

The notation $dom\ f$ refers to the domain of $f$ defined by

$$dom\ f = \{x \in E,\ \ f(x) < \infty\},$$

and $\partial f(x)$ stands for the sub-differential of $f$ at $x \in dom\ f$ defined through

$$x^* \in \partial f(x) \Longleftrightarrow \forall z \in E \qquad f(z) \geq f(x) + \langle z - x, x^* \rangle.$$

It is well known that $\partial f(x)$ is (possibly empty) closed and convex subset of $E^*$. Further, we have

$$x^* \in \partial f(x) \Longleftrightarrow f(x) + f^*(x^*) = \langle x, x^* \rangle, \tag{2.2}$$

and if $f \in \Gamma_0(E)$, $x^* \in \partial f(x)$ if and only if $x \in \partial f^*(x^*)$.

The directional derivative of $f$ in the direction $h \in E$ at $x \in dom\ f$ is defined by, [5]

$$df(x,h) := \lim_{t \downarrow 0} \frac{f(x+th) - f(x)}{t},$$

provided this limit exists in $\overline{\mathbb{R}}$. If $f$ is convex then such limit exists i.e. $df(x,h)$ always exists in $\overline{\mathbb{R}}$. If moreover, the map $h \longmapsto df(x,h)$, for fixed $x \in dom\ f$, is linear continuous then we say that $f$ is G-differentiable at $x$ and we write

$$df(x,h) = \nabla f(x)(h), \tag{2.3}$$

where $\nabla f(x)$ denotes the so-called G-gradient of $f$ at $x$. If $f$ is convex and G-differentiable at $x$ then $\partial f(x) = \{\nabla f(x)\}$.

Finally, let $f, g \in \widetilde{\mathbb{R}}^E$. The inf-convolution of $f$ and $g$ is defined through

$$\forall x \in E \qquad f \Box g(x) := \inf_{y \in E} \Big( f(y) + g(x-y) \Big). \tag{2.4}$$

It is well known that the binary law $\Box$ is commutative, associative and always satisfies $(f \Box g)^* = f^* + g^*$. Under convenient assumption, the relationship $(f+g)^* = f^* \Box g^*$ holds. For instance, such equality is satisfied provided $f, g \in \Gamma_0(E)$ and $int(dom\ \ f) \cap dom\ \ g \neq \emptyset$, where $int(dom\ \ f)$ denotes the topological interior of $dom\ \ f$. For other condition ensuring $(f+g)^* = f^* \Box g^*$, see [1] for instance.

## 3. SOME NEEDED LEMMAS

In what follows, if $E$ is a (real) Hilbert space then we identify $E^*$ with $E$ via Riesz-Frechet representation theorem. In this case, the bracket duality $\langle .,. \rangle$ is identified with the inner product of $E$. We denote by $\mathcal{B}^{+*}(E)$ the set of all self-adjoint positive invertible operators acting on $E$.

The two following lemmas, which will be needed later, may be stated.

**Lemma 3.1.** *Let $E$ be a real Hilbert space and $T \in \mathcal{B}^{+*}(E)$. Let $f$ be the real function generating by $T$, i.e.*

$$\forall x \in E \qquad f(x) = f_T(x) := \frac{1}{2}\langle Tx, x \rangle, \quad \big(f = f_T,\ \textit{in short}\big).$$

*Then the following assertions hold true:*
*(a) $f_T$ is convex if and only if $T$ is (self-adjoint) positive. If moreover $T \in \mathcal{B}^{+*}(E)$ then $(f_T)^* = f_{T^{-1}}$.*
*(b) $df(x,h) = \langle h, Tx \rangle$ for all $x, h \in E$ and so $\partial f(x) = \{Tx\}$ for every $x \in E$.*
*(c) If $g = f_S$, where $S \in \mathcal{B}^{+*}(E)$, then $f \Box g = f_{T//S}$, where*

$$T//S := \big(T^{-1} + S^{-1}\big)^{-1}$$

*is the so-called parallel sum of $T$ and $S$.*

*Proof.* It is not hard to establish it as an exercise. See also [2, 4], for instance. ∎

**Lemma 3.2.** *Let $E$ be a normed space and $p > 1$ be a real number. We set*

$$\forall x \in E \qquad f(x) = \frac{1}{p}\|x\|^p.$$

*Then the following assertions hold:*
*(a) For all $x^* \in E^*$ we have*

$$f^*(x^*) = \frac{1}{p^*}\|x^*\|_*^{p^*},$$

*where $p^*$ denotes the conjugate of $p$ defined by $1/p + 1/p^* = 1$.*
*(b) If moreover $E$ is a (real) Hilbert space then $f$ is G-differentiable at every $x \in E$ for $p \geq 2$, at each $x \neq 0$ for $p < 2$, with*

$$\nabla f(x)(h) = \|x\|^{p-2}\langle h, x\rangle.$$

*Proof.* For (a), see [3]. For (b), it is a simple exercise which we leave to the reader. ∎

## 4. POINT-WISE DIRECTIONAL DERIVATIVE OF $f \longmapsto f^*$

We start this section by stating the following definition.

**Definition 4.1.** Let $f, g \in \overline{\mathbb{R}}^E$. For $x^* \in E^*$, we set

$$[f, g]_*(x^*) := \lim_{t\downarrow 0} \frac{(f + tg)^*(x^*) - f^*(x^*)}{t},$$

provided this limit exists in $\overline{\mathbb{R}}$. In this case, $[f, g]_*$ is the point-wise directional derivative of the Fenchel duality map in the direction $g$ at $f$.

The following result asserts the existence of $[f, g]_*$ when convenient assumptions on $f$ and $g$ are added.

**Theorem 4.1.** *Let $f : E \longrightarrow \mathbb{R}$ be such that $f^*$ is proper and $g : E \longrightarrow \widetilde{\mathbb{R}}$. Then, for all $x^* \in dom\ f^*$, $[f, g]_*(x^*)$ exists in $\overline{\mathbb{R}}$, with*

$$[f, g]_*(x^*) = \inf_{t>0} \frac{(f + tg)^*(x^*) - f^*(x^*)}{t}. \tag{4.1}$$

To prove this theorem, we need the following lemma.

**Lemma 4.2.** *Let $f, g$ be as in Theorem 4.1. Then, for all $x^* \in dom\ f^*$, the map*

$$(0, \infty) \ni t \longmapsto \frac{(f + tg)^*(x^*) - f^*(x^*)}{t} \tag{4.2}$$

*is monotone increasing, i.e.*

$$t_1 \geq t_2 > 0 \Longrightarrow \frac{(f + t_1 g)^*(x^*) - f^*(x^*)}{t_1} \geq \frac{(f + t_2 g)^*(x^*) - f^*(x^*)}{t_2}.$$

*Proof.* Let $t_1 \geq t_2 > 0$. Since $dom\ f = E$ then we can write, for all $x^* \in E^*$,

$$(f + t_2 g)^*(x^*) - f^*(x^*) = \Big(\frac{t_2}{t_1}\big(f + t_1 g\big) + \big(1 - \frac{t_2}{t_1}\big)f\Big)^*(x^*) - f^*(x^*).$$

This, with the fact that the map $f \longmapsto f^*$ is point-wise convex and $0 < t_2/t_1 \leq 1$, yields

$$(f + t_2 g)^*(x^*) - f^*(x^*) \leq \frac{t_2}{t_1}\big(f + t_1 g\big)^*(x^*) + \Big(1 - \frac{t_2}{t_1}\Big)f^*(x^*) - f^*(x^*).$$

We then deduce, after simple manipulation, that

$$\frac{(f+t_2g)^*(x^*)-f^*(x^*)}{t_2} \le \frac{(f+t_1g)^*(x^*)-f^*(x^*)}{t_1},$$

provided that $x^* \in dom\ f^*$. The desired result is obtained. ∎

Since the map (4.2) is monotone increasing then

$$\inf_{t>0} \frac{(f+tg)^*(x^*)-f^*(x^*)}{t}$$

always exists in $\overline{\mathbb{R}}$, for all $x^* \in dom\ f^*$, and so

$$\lim_{t\downarrow 0} \frac{(f+tg)^*(x^*)-f^*(x^*)}{t} = \inf_{t>0} \frac{(f+tg)^*(x^*)-f^*(x^*)}{t},$$

from which Theorem 4.1 follows.

The following corollary is immediate from the equality (4.1).

**Corollary 4.3.** *Let $f, g$ be as in Theorem 4.1. Then the inequality*

$$[f,g]_*(x^*) \le (f+g)^*(x^*) - f^*(x^*) \tag{4.3}$$

*holds for all $x^* \in dom\ f^*$.*

**Theorem 4.4.** *Let $f, g : E \longrightarrow \mathbb{R}$ with $f^*$ is proper. Then for all $x^* \in dom\ f^*$ we have*

$$f^*(x^*) - (f-g)^*(x^*) \le [f,g]_*(x^*). \tag{4.4}$$

*Proof.* The following identity

$$f = \frac{1}{1+t}(f+tg) + \frac{t}{1+t}(f-g)$$

is obviously satisfied for all $t > 0$ and all $f, g$ with $dom\ g = E$. Again by virtue of the point-wise convexity of $f \longmapsto f^*$, we deduce for all $x^* \in E^*$

$$f^*(x^*) \le \frac{1}{1+t}(f+tg)^*(x^*) + \frac{t}{1+t}(f-g)^*(x^*).$$

It follows that

$$f^*(x^*) + tf^*(x^*) \le (f+tg)^*(x^*) + t(f-g)^*(x^*),$$

or equivalently, with $x^* \in dom\ f^*$,

$$tf^*(x^*) - t(f-g)^*(x^*) \le (f+tg)^*(x^*) - f^*(x^*),$$

or again

$$f^*(x^*) - (f-g)^*(x^*) \le \frac{(f+tg)^*(x^*)-f^*(x^*)}{t}.$$

We then deduce the desired inequality by letting $t \downarrow 0$ point-wisely. The proof is so complete. ∎

We end this section by stating a result summarizing the elementary properties of the binary functional map $(f, g) \longmapsto [f, g]_*$.

**Proposition 4.5.** *Let $f, g$ be as in Theorem 4.1 and $\lambda > 0$ be a real number. Then the following assertions hold:*

*(a)* $[f, \lambda g]_* = \lambda [f, g]_*$.

*(b)* $[f, g_1 + g_2]_* \le [f, g_1]_* + [f, g_2]_*$ *($g_1$ and $g_2$ are as $g$).*

*(c) The functional map $g \longmapsto [f, g]_*$, for fixed $f$, is point-wisely convex.*

*(d)* $[\lambda f, g]_* = \frac{1}{\lambda}[f, g]_*.\lambda$, *where we set* $(f.\lambda)(x) = \lambda f(x/\lambda)$.
*(e) If* $E$ *is a real Hilbert space then*

$$[f_T, f_S]_* = -f_{T^{-1}ST^{-1}},$$

*where* $T \in \mathcal{B}^{+*}(E)$ *and* $S$ *is a self-adjoint operator of* $E$.

*Proof.* The first three statements follow from the fact that the directional derivative is sub-linear in its second component. The proof of (d) and (e) is simple and therefore omitted here. ∎

## 5. IMPROVED BOUNDS FOR $[f, g]_*$

Inequalities (4.3) and (4.4) are not the best possible and we will give in this section some improvements of them. We begin by stating the following result.

**Theorem 5.1.** *Let* $f, g$ *be as in Theorem 4.1. Then the inequality*

$$[f, g]_*(x^*) \geq -g(x) \tag{5.1}$$

*holds for all* $x \in E$ *such that* $\partial f(x) \neq \emptyset$ *and* $x^* \in \partial f(x)$. *Further, inequality (5.1) refines (4.4).*

*Proof.* By (2.1) we can write, for all $x \in E$ and $x^* \in E^*$,

$$(f + tg)^*(x^*) - f^*(x^*) \geq \langle x^*, x\rangle - f(x) - tg(x) - f^*(x^*). \tag{5.2}$$

Let $x \in E$ be such that $\partial f(x) \neq \emptyset$. If we take $x^* \in \partial f(x)$ then $\langle x^*, x\rangle = f(x) + f^*(x^*)$, by (2.2). Substituting this in (5.2), with the condition $dom\ f = E$, the desired inequality follows after a simple manipulation.

We now prove that (5.1) is a refinement of (4.4). Indeed, for $x^* \in \partial f(x)$, (2.1) yields

$$(f - g)^*(x^*) \geq \langle x^*, x\rangle - f(x) + g(x) = \langle x^*, x\rangle - f(x) - f^*(x^*) + f^*(x^*) + g(x).$$

If $x^* \in \partial f(x)$ then again (2.2) implies that $\langle x^*, x\rangle - f(x) - f^*(x^*) = 0$ and so the desired result follows, so completes the proof. ∎

**Corollary 5.2.** *Let* $f, g$ *be as in Theorem 4.1. Assume that further* $f \in \Gamma_0(E)$. *Then the inequality*

$$-\inf_{x \in \partial f^*(x^*)} g(x) \leq [f, g]_*(x^*) \tag{5.3}$$

*holds for all* $x^* \in dom\ f^*$.

*Proof.* Since $f \in \Gamma_0(E)$ then the condition $x^* \in \partial f(x)$ is equivalent to $x \in \partial f^*(x^*)$. We can then say that (5.1) holds for all $x \in \partial f^*(x^*)$, whenever $x^* \in dom\ f^*$ is given. With this, (5.1) means that the real map $x \longmapsto -g(x)$ is upper bounded by $[f, g]_*(x^*)$ on the set $\partial f^*(x^*)$. It follows that

$$\sup_{x \in \partial f^*(x^*)} \big(-g(x)\big) \leq [f, g]_*(x^*),$$

from which (5.3) follows, so completing the proof. ∎

Now, a question arises from the above: Is (5.3) the best possible? That is, do exist $f, g \in \widetilde{\mathbb{R}}^E$ for which (5.3) remains an equality? The following example answers affirmatively this latter question.

**Example 5.1.** *Assume that* $E$ *is a real Hilbert space. With the notation of Lemma 3.1, let us take* $f = f_T$ *and* $g = f_S$, *where* $T \in \mathcal{B}^{+*}(E)$ *and* $S$ *is self-adjoint. It is easy to see that (detail is simple and therefore omitted here)*

$$-\inf_{x \in \partial f^*(x^*)} g(x) = -g\big(T^{-1}x^*\big) = -f_{T^{-1}ST^{-1}}(x^*),$$

*which, with Proposition 4.5,(e), implies that (5.3) is an equality.*

Now, we state a result that gives a point-wise upper bound of $[f,g]_*$.

**Theorem 5.3.** *Let $f,g$ be as in Theorem 4.1. Assume that further $f,g \in \Gamma_0(E)$. Then the inequality*

$$[f,g]_*(x^*) \leq \inf_{z^* \in E^*} \Big( g^*(z^*) + df^*(x^*, -z^*) \Big) \tag{5.4}$$

*holds for all $x^* \in dom\ f^*$. Further, (5.4) refines (4.3).*

*Proof.* By our assumption, (2.4) yields

$$(f+tg)^*(x^*) = \Big( f^* \Box t g^*(\frac{\cdot}{t}) \Big)^{**}(x^*) \leq \Big( f^* \Box t g^*(\frac{\cdot}{t}) \Big)(x^*) = \inf_{y^* \in E^*} \Big( f^*(y^*) + t g^*(\frac{x^*-y^*}{t}) \Big).$$

It follows that the inequality

$$(f+tg)^*(x^*) \leq f^*(y^*) + tg^*(\frac{x^*-y^*}{t})$$

holds for all $x^* \in dom\ f^*, y^* \in E^*$ and $t > 0$. Setting $x^* - y^* = tz^*$ we then obtain

$$(f+tg)^*(x^*) - f^*(x^*) \leq f^*(x^* - tz^*) - f^*(x^*) + tg^*(z^*).$$

Dividing by $t > 0$ and letting then $t \downarrow 0$ we then have

$$[f,g]_*(x^*) \leq g^*(z^*) + df^*(x^*, -z^*)$$

for all $x^*, z^* \in E^*$. This means that the map $z^* \longmapsto g^*(z^*) + df^*(x^*, -z^*)$, for fixed $x^* \in E^*$, is lower bounded by $[f,g]_*(x^*)$. The inequality (5.4) follows.

We now establish that (5.4) refines (4.3). In fact, since $f^*$ is convex then

$$\begin{aligned} df^*(x^*, -z^*) &= \lim_{t \downarrow 0} \frac{f^*(x^* - tz^*) - f^*(x^*)}{t} \\ &= \inf_{t>0} \frac{f^*(x^* - tz^*) - f^*(x^*)}{t} \leq f^*(x^* - z^*) - f^*(x^*). \end{aligned}$$

It follows that

$$\inf_{z^* \in E^*} \Big( g^*(z^*) + df^*(x^*, -z^*) \Big) \leq \inf_{z^* \in E^*} \Big( g^*(z^*) + f^*(x^* - z^*) \Big) - f^*(x^*).$$

Now, if we write

$$\inf_{z^* \in E^*} \Big( g^*(z^*) + f^*(x^* - z^*) \Big) = g^* \Box f^*(x^*) = f^* \Box g^*(x^*) = (f+g)^*(x^*),$$

we then deduce the desired refinement, so completes the proof. ∎

As for Example 5.1, the following one shows that inequality (5.4) is the best possible.

**Example 5.2.** *Let $E, f, g$ be as in the previous example. With Lemma 3.1, we have for all $x^*, z^* \in E$*

$$df^*(x^*, -z^*) = -\langle T^{-1}x^*, z^* \rangle \ \text{ and } \ g^*(z^*) = f_{S^{-1}}(z^*) = \frac{1}{2}\langle S^{-1}z^*, z^* \rangle.$$

*The second side of (5.4) becomes*

$$\begin{aligned} -\sup_{z^* \in E} \Big( \langle T^{-1}x^*, z^* \rangle - \frac{1}{2}\langle S^{-1}z^*, z^* \rangle \Big) &= -\big(f_{S^{-1}}\big)^*\big(T^{-1}x^*\big) = -f_S\big(T^{-1}x^*\big) \\ &= -\frac{1}{2}\langle ST^{-1}x^*, T^{-1}x^* \rangle = -\frac{1}{2}\langle T^{-1}ST^{-1}x^*, x^* \rangle = -f_{T^{-1}ST^{-1}}(x^*), \end{aligned}$$

*and so (5.4) is here an equality.*

*We can also deduce from this example, as well as from the previous one, that the functional results presented here contain those related to positive bounded linear operators.*

We now state the following corollary which summarizes the previous results.

**Corollary 5.4.** *Let $f, g$ be as in Theorem 5.3. Then the following double inequality*

$$-\inf_{x\in\partial f^*(x^*)} g(x) \le [f,g]_*(x^*) \le \inf_{z^*\in E^*}\Big(g^*(z^*) + df^*(x^*,-z^*)\Big) \tag{5.5}$$

*holds for all $x^* \in dom\ f^*$. Further (5.5) gives the best possible point-wise bounds of $[f,g]_*$.*

*Proof.* It is sufficient to combine Corollary 5.2 and Theorem 5.3, together with Example 5.1 and Example 5.2. ∎

We end this paper by stating the following corollary which, under convenient hypothesis, gives an explicit form of $[f,g]_*$.

**Corollary 5.5.** *Let $E$ be a real Hilbert space and let $f, g$ be as in Theorem 5.3. Assume that $f^*$ is G-differentiable at $x^* \in E$. Then we have*

$$[f,g]_*(x^*) = -g\big(\nabla f^*(x^*)\big). \tag{5.6}$$

*Proof.* If $f^*$ is G-differentiable at $x^*$ then

$$df^*(x^*,-z^*) = -\nabla f^*(x^*)(z^*) = -\langle \widetilde{\nabla f^*(x^*)}, z^*\rangle,$$

where $\widetilde{\nabla f^*(x^*)}$ denotes the representant of $\nabla f^*(x^*)$ guaranteed by Riesz-Frechet theorem. If we identify $\nabla f^*(x^*)$ and $\widetilde{\nabla f^*(x^*)}$ via such representation, the right side of (5.5) becomes

$$\begin{aligned}\inf_{z^*\in E^*}\Big(g^*(z^*) + df^*(x^*,-z^*)\Big) &= \inf_{z^*\in E}\Big(g^*(z^*) - \langle \nabla f^*(x^*), z^*\rangle\\ &= -\sup_{z^*\in E}\Big(\langle \nabla f^*(x^*), z^*\rangle - g^*(z^*)\Big) = -g^{**}\big(\nabla f^*(x^*)\big) = -g\big(\nabla f^*(x^*)\big),\end{aligned}$$

since $g \in \Gamma_0(E)$. Again, $f^*$ is G-differentiable at $x^*$ implies

$$\partial f^*(x^*) = \{\nabla f^*(x^*)\}$$

and so the left side of (5.5) is equal to $-g(\nabla f^*(x^*))$. The desired result follows, so completes the proof. ∎

**Remark 5.1.** Under the hypotheses of the previous corollary, (5.6) is equivalent to

$$(f+tg)^*(x^*) = f^*(x^*) - tg\big(\nabla f^*(x^*)\big) + t\,\epsilon_t(f,g)(x^*),$$

where $\epsilon_t(f,g)(x^*)$ tends to $0$ as $t \downarrow 0$. This gives an expansion approximating (point-wisely) $(f+tg)^*$ at order $1$ in $t$.

Finally, we state the following examples.

**Example 5.3.** *Let $E$ be a real Hilbert space and $T \in \mathcal{B}^{+*}(E)$. Let us take $f = f_T$ and $g = \frac{1}{p}\|.\|^p$ with $p > 1$. Following Lemma 3.1 we have $\nabla f^*(x^*) = T^{-1}x^*$ and so (5.6) gives*

$$\forall x^* \in E \qquad \Big[f_T, \frac{1}{p}\|.\|^p\Big]_*(x^*) = -\frac{1}{p}\|T^{-1}x^*\|^p.$$

**Example 5.4.** *Let $E$ be a real Hilbert space and let $f, g$ be given by*

$$\forall x \in E \qquad f(x) = \frac{1}{p}\|x\|^p, \ \ g(x) = \frac{1}{q}\|x\|^q,$$

*where $p, q > 1$. Hypotheses of Theorem 5.3 are here satisfied. According to Lemma 3.2, (5.6) yields (after a simple manipulation)*

$$\forall x^* \in E \qquad \Big[\frac{1}{p}\|.\|^p, \frac{1}{q}\|.\|^q\Big]_*(x^*) = -\frac{1}{q}\|x^*\|^{q(p^*-1)}.$$

**Example 5.5.** *With the same notation as in the previous examples, we left to the reader the task for checking that*

$$\forall x^* \in E \qquad \Big[\frac{1}{p}\|.\|^p, f_S\Big]_*(x^*) = -\|x^*\|^{2p-4} f_S(x^*),$$

*where $S$ is a self-adjoint operator of $E$.*

## REFERENCES


[1] H. Attouch and H. Brezis, Duality for the sum of Convex Functions in General Banach Spaces. In: *Aspects of Mathematics and its Applications*, J. A. Barroso, ed. North-Holland, Amsterdam, pages 125-133, 1986.

[2] J. P. Aubin, *Nonlinear Analysis and its Economic Motivations*, Masson, 1984.

[3] I. Ekeland and R. Temam, Convex Analysis and Variational Problems, *SIAM*, 1999.

[4] M. Raïssouli, On an analogue of ABA when the operator variables $A$ and $B$ are convex functionals, *Banach J. Math. Anal.*, Vol. **9**, No. 1 (2015), pp. 235-242.

[5] E. Zeidler, *Nonlinear Functional Analysis and its Applications* III, Springer-Verlag, 1984.