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HERMITE-HADAMARD TYPE INEQUALITIES FOR k-RIEMANN LIOUVILLE FRACTIONAL INTEGRALS VIA TWO KINDS OF CONVEXITY

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ABSTRACT. In this article, a fundamental integral identity including the first order derivative of a given function via k-Riemann-Liouville fractional integral is established. This is used to obtain further Hermite-Hadamard type inequalities involving left-sided and right-sided k-Riemann-Liouville fractional integrals for m-convex and (s, m)-convex functions respectively.

Key words and phrases: Hermite-Hadamard inequality, m-convex functions, (s, m)-convex functions, k-Riemann-Liouville fractional integrals, Hölder's integral inequality, power mean inequality.

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1. INTRODUCTION

Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then f satisfies the following well-known Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

G. H. Toader defines the m-convexity in [1], an intermediate between the usual convexity and starshaped property, for details one can consult [2], [3] and [4]:

Definition 1.1. The function $f : [0, b^*] \longrightarrow \mathbb{R}$ is said to be m-convex, where $m \in [0, 1]$ and $b^* > 0$, if for every $x, y \in [0, b^*]$ and $t \in [0, 1]$, we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

Dragomir and Fitzpatrick in [5] introduced the following class of more generalized convex functions. Interesting readers can consult [6] and a recent one [12]:

Definition 1.2. The function $f : [0, b^*] \longrightarrow \mathbb{R}$ is said to be (s, m)-convex, where $(s, m) \in [0, 1]^2$ and $b^* > 0$, if for every $x, y \in [0, b^*]$ and $t \in [0, 1]$, we have

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t^s)f(y).$$

The following classes of functions: increasing, s-starshaped, starshaped, m-convex, convex and s-convex could be obtained from $(s, m) \in \{(0, 0), (s, 0), (1, 0), (1, m), (1, 1), (s, 1)\}$,.

In [7], Mubeen and Habibullah introduced the following variant of fractional integrals:

Definition 1.3. Let $f \in L[a, b]$, then k-Riemann-Lioville fractional integrals $\left({}_{RL}^{k}J_{a^+}^{\alpha}\right) f(x)$ and $\left({}_{RL}^{k}J_{b^-}^{\alpha}\right) f(x)$ of order $\frac{\alpha}{k} > 0$ with $a \geq 0$ are defined by

$$\left({}_{RL}^{k}J_{a^+}^{\alpha}\right) f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt \quad (0 \leq a < x < b)$$

and

$$\left({}_{RL}^{k}J_{b^-}^{\alpha}\right) f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt \quad (0 \leq a < x < b)$$

respectively, where k is non-negative real number and $\Gamma_k(\alpha)$ is the k-gamma function given as $\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1}e^{\frac{-t^k}{k}}dt$.

Odemir et al. in [8] and [9] used the following two important integral identities including the second order derivatives to establish many interesting Hermite-Hadamard type inequalities for m-convex and (s, m)-convex functions respectively:

Lemma 1.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be twice differentiable mapping on (a, b). If $f'' \in L[a, b]$, then the following identity holds*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt = \frac{(b-a)^2}{2}\int_0^1 t(1-t)f''(ta+(1-t)b)dt.$$

Lemma 1.2. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be twice differentiable mapping on (a,b) and $m\in(0,1]$. If $f''\in L[a,b]$, then the following identity holds*

$$\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(t)dt=\frac{(mb-a)^2}{2}\int_0^1 t(1-t)f''(ta+m(1-t)b)dt.$$

2. MAIN RESULTS

We present an important integral identity motivated by Sarikaya et al. [11] including the first order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convex functions via k-Riemann-Liouville fractional integrals.

Lemma 2.1. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be differentiable mapping on (a,b). If $f'\in L[a,b]$, then the following identity for k-fractional integrals holds*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}^{k}_{RL}J^{\alpha}_{a^+}\right)f(b)+\left({}^{k}_{RL}J^{\alpha}_{b^-}\right)f(a)\right]$$
$$=\frac{b-a}{2}\int_0^1\left[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right]f'(ta+(1-t)b)dt.$$

Proof. Considering the integral and integrating by parts

$$\frac{b-a}{2}\int_0^1\left[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right]f'(ta+(1-t)b)dt$$
$$=\frac{b-a}{2}\left[\frac{f(a)+f(b)}{(b-a)}-\frac{\alpha}{k(b-a)}\int_0^1\left[(1-t)^{\frac{\alpha}{k}-1}+t^{\frac{\alpha}{k}-1}\right]f(ta+(1-t)b)dt\right]$$

using the change of variable $ta+(1-t)b=p$, we have

$$=\frac{b-a}{2}\left[\frac{f(a)+f(b)}{b-a}-\frac{\alpha}{k(b-a)}\left(\int_b^a\left(\frac{a-p}{a-b}\right)^{\frac{\alpha}{k}-1}f(p)\frac{dp}{a-b}+\int_b^a\left(\frac{p-b}{a-b}\right)^{\frac{\alpha}{k}-1}f(p)\frac{dp}{a-b}\right)\right]$$

using the definition of k-fractional integrals, we have

$$=\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}^{k}_{RL}J^{\alpha}_{a^+}\right)f(b)+\left({}^{k}_{RL}J^{\alpha}_{b^-}\right)f(a)\right],$$

hence the proved. ∎

We have generalized the results of [10] from Riemann-Liouville fractional integrals to k-Riemann-Liouville fractional integrals, as follows:

2.1. Hermite-Hadamard-type Inequalities for m-Convex Function.

In order to prove our main results, we need the following lemma:

Lemma 2.2. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be twice differentiable mapping on (a,b). If $f''\in L[a,b]$, then the following identity for k-fractional integrals holds:*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}^{k}_{RL}J^{\alpha}_{a^+}\right)f(b)+\left({}^{k}_{RL}J^{\alpha}_{b^-}\right)f(a)\right]$$
$$=\frac{(b-a)^2}{2}\int_0^1\frac{1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}f''(ta+(1-t)b)dt.$$

Proof. Using Lemma 2.1, it suffices to verify that

$$\int_0^1 [(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}] f'(ta+(1-t)b)dt$$
$$= (b-a)\int_0^1 \frac{1-(1-t)^{\frac{\alpha}{k}+1} - t^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1} f''(ta+(1-t)b)dt$$

Note that

$$\begin{aligned}&\int_0^1 [(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}] f'(ta+(1-t)b)dt\\ &= -\int_0^1 f'(ta+(1-t)b)\, d\frac{(1-t)^{\frac{\alpha}{k}+1} + t^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}\\ &= \frac{f'(b)-f'(a)}{\frac{\alpha}{k}+1} - (b-a)\int_0^1 \frac{(1-t)^{\frac{\alpha}{k}+1} + t^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1} f''(ta+(1-t)b)dt,\end{aligned}$$

and since

$$f'(b)-f'(a) = \int_a^b f''(x)dx = (b-a)\int_0^1 f''(ta+(1-t)b)dt.$$

This completes the proof. ∎

Theorem 2.3. *Let $f : [0,b^*] \longrightarrow \mathbb{R}$ be a twice differentiable mapping with $b^* > 0$. If $|f''|^q$ is measurable and m-convex on $\left[a, \frac{b}{m}\right]$ for some fixed $q \geq 1, 0 \leq a < b$ and $m \in (0,1]$ with $\frac{b}{m} \leq b^*$, then the following inequality for k-fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}^k_{RL}J^{\alpha}_{a^+}\right) f(b) + \left({}^k_{RL}J^{\alpha}_{b^-}\right) f(a)\right]\right|$$
$$\leq \frac{\frac{\alpha}{k}(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)}\left(\frac{|f''(a)|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Proof. Suppose that $q=1$, from Lemma 2.2 we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}^k_{RL}J^{\alpha}_{a^+}\right) f(b) + \left({}^k_{RL}J^{\alpha}_{b^-}\right) f(a)\right]\right|\\ &\leq \frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-t)^{\frac{\alpha}{k}+1} - t^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}|f''(ta+(1-t)b)|dt,\end{aligned}$$

because $(1-t)^{\frac{\alpha}{k}+1} + t^{\frac{\alpha}{k}+1} \leq 1$ for any $t \in [a,b]$. Since $|f''|$ is m-convex on $\left[a, \dfrac{b}{m}\right]$, therefore, for any $t \in [0,1]$, we have

$$\left|f''(ta+(1-t)b)\right| \leq t\left|f''(a)\right| + m(1-t)\left|f''\left(\frac{b}{m}\right)\right|,$$

hence,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(b)+\left({}_{RL}^{k}J_{b^-}^{\alpha}\right)f(a)\right]\right| \\
&\leq \frac{(b-a)^2}{2(\frac{\alpha}{k}+1)}\int_0^1\left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]\left(t\left|f''(a)\right|+m(1-t)\left|f''\left(\frac{b}{m}\right)\right|\right)dt \\
&= \frac{\frac{\alpha}{k}(b-a)^2}{2(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}\left(\frac{|f''(a)|+m|f''\left(\frac{b}{m}\right)|}{2}\right)
\end{aligned}$$

which completes the proof for this case. Now, suppose that $q>1$, from Lemma 2.2 and power mean inequality for q, we get

$$\begin{aligned}
&\int_0^1\left(1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right)|f''(ta+(1-t)b)|dt \\
&\leq \left(\int_0^1\left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]|f''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

Since $(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\leq 1$ for any $t\in[a,b]$. Also, $\left|f''\right|^q$ is m-convex on $\left[a,\frac{b}{m}\right]$, therefore for any $t\in[0,1]$

$$\left|f''(ta+(1-t)b)\right|^q\leq t\left|f''(a)\right|^q+m(1-t)\left|f''\left(\frac{b}{m}\right)\right|^q.$$

Hence,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(b)+\left({}_{RL}^{k}J_{b^-}^{\alpha}\right)f(a)\right]\right| \\
&\leq \frac{(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(\int_0^1\left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]dt\right)^{1-\frac{1}{q}} \\
&\left(\int_0^1\left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]|f''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(\int_0^1\left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]dt\right)^{1-\frac{1}{q}} \\
&\left(\int_0^1\left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]\left[t|f''(a)|^q+m(1-t)\left|f''\left(\frac{b}{m}\right)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^2}{2(\frac{\alpha}{k}+1)}\left(\frac{\frac{\alpha}{k}}{\frac{\alpha}{k}+2}\right)^{1-\frac{1}{q}}\left(\frac{\frac{\alpha}{k}}{\frac{\alpha}{k}+2}\right)^{\frac{1}{q}}\left(\frac{|f''(a)|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}} \\
&= \frac{\frac{\alpha}{k}(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)}\left(\frac{|f''(a)|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}
\end{aligned}$$

which completes the proof of the theorem. ∎

Remark 2.1. With the same assumptions as in Theorem 2.3, if $|f''(x)| \leq M$ on $\left[a, \frac{b}{m}\right]$, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(b)+\left({}_{RL}^{k}J_{b^-}^{\alpha}\right)f(a)\right]\right| \leq \frac{M\frac{\alpha}{k}(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)}\left(\frac{1+m}{2}\right)^{\frac{1}{q}}.$$

Theorem 2.4. *Let $f:[0,b^*]\longrightarrow\mathbb{R}$ be twice differentiable mapping with $b^*>0$. If $|f''|^q$ is measurable and m-convex on $[a,\frac{b}{m}]$ for some fixed $q>1, 0\leq a<b$ and $m\in(0,1]$ with $\frac{b}{m}\leq b^*$ then the following inequality for k-fractional integrals holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(b)+\left({}_{RL}^{k}J_{b^-}^{\alpha}\right)f(a)\right]\right|$$
$$\leq \frac{(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{2}{p\left(\frac{\alpha}{k}+1\right)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}$$

where, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.2 and using the well-known Hölder's inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(b)+\left({}_{RL}^{k}J_{b^-}^{\alpha}\right)f(a)\right]\right|$$
$$\leq \frac{(b-a)^2}{2}\int_0^1\frac{1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}|f''(ta+(1-t)b)|dt$$
$$\leq \frac{(b-a)^2}{2(\frac{\alpha}{k}+1)}\left(\int_0^1(1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1})^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^2}{2(\frac{\alpha}{k}+1)}\left(\int_0^1(1-(1-t)^{p(\frac{\alpha}{k}+1)}-t^{p(\frac{\alpha}{k}+1)})dt\right)^{\frac{1}{p}}\left(|f''(a)|^q\int_0^1 t\,dt+m\left|f''\left(\frac{b}{m}\right)\right|^q\int_0^1(1-t)dt\right)^{\frac{1}{q}}$$
$$=\frac{(b-a)^2}{2(\frac{\alpha}{k}+1)}\left(1-\frac{2}{p(\frac{\alpha}{k}+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Here, we used

$$\left(1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right)^p\leq 1-(1-t)^{p\left(\frac{\alpha}{k}+1\right)}-t^{p\left(\frac{\alpha}{k}+1\right)}$$

for any $t\in[0,1]$, which follows from $(A-B)^p\leq A^p-B^p$ for any $A>B\geq 0$ and $p\geq 1$. This completes the proof. ∎

Corollary 2.5. *With the same assumptions as in Theorem 2.4. If $|f''(x)|\leq M$ on $\left[a,\frac{b}{m}\right]$, we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(b)+\left({}_{RL}^{k}J_{b^-}^{\alpha}\right)f(a)\right]\right|$$
$$\leq\frac{M(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{2}{p\left(\frac{\alpha}{k}+1\right)+1}\right)^{\frac{1}{p}}\left(\frac{1+m}{2}\right)^{\frac{1}{q}}.$$

Another Hermite-Hadamard type inequality for powers in terms of the second derivative is obtained as following:

Theorem 2.6. *Let $f : [0, b^*] \longrightarrow \mathbb{R}$ be twice differentiable mapping with $b^* > 0$. If $|f''|^q$ is measurable and m-convex on $\left[a, \frac{b}{m}\right]$ for some fixed $q \geq 1, 0 \leq a < b$ and $m \in (0, 1]$ with $\dfrac{b}{m} \leq b^*$, then the following inequality for k-fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(b) + \left({}_{RL}^{k}J_{b^-}^{\alpha}\right)f(a)\right]\right|$$
$$\leq \frac{(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(\frac{q\left(\frac{\alpha}{k}+1\right)-1}{q\left(\frac{\alpha}{k}+1\right)+1}\right)^{\frac{1}{q}}\left(\frac{|f''(a)|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Proof. From Lemma 2.2 and using the well-known Hölder's inequality, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[({}_{RL}^{k}J_{a^+}^{\alpha})f(b) + ({}_{RL}^{k}J_{b^-}^{\alpha})f(a)]\right|$$
$$\leq \frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-t)^{\frac{\alpha}{k}+1} - t^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}|f''(ta+(1-t)b)|\mathrm{d}t$$
$$\leq \frac{(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(\int_0^1 1dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[1-(1-t)^{\frac{\alpha}{k}+1} - t^{\frac{\alpha}{k}+1}\right]^q |f''(ta+(1-t)b)|^q\mathrm{d}t\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(|f''(a)|^q\int_0^1 [t-(1-t)^{q(\frac{\alpha}{k}+1)}t - t^{q(\frac{\alpha}{k}+1)+1}]dt\right.$$
$$\left. +m\left|f''\left(\frac{b}{m}\right)\right|^q\int_0^1 [(1-t)-(1-t)^{q\left(\frac{\alpha}{k}+1\right)+1} - t^{q\left(\frac{\alpha}{k}+1\right)}(1-t)]dt\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(\frac{q\left(\frac{\alpha}{k}+1\right)-1}{q\left(\frac{\alpha}{k}+1\right)+1}\right)^{\frac{1}{q}}\left(\frac{|f''(a)|^q + m|f''\left(\frac{b}{m}\right)|^q}{2}\right)^{\frac{1}{q}}.$$

Here, we employed

$$\left[1-(1-t)^{\frac{\alpha}{k}+1} - t^{\frac{\alpha}{k}+1}\right]^q \leq 1-(1-t)^{q\left(\frac{\alpha}{k}+1\right)} - t^{q\left(\frac{\alpha}{k}+1\right)}$$

for any $t \in [0, 1]$, which follows from $(A-B)^q \leq A^q - B^q$ for any $A > B \geq 0$ and $q \geq 1$. Hence, the proof of the theorem. ∎

Remark 2.2. From Theorem 2.3, 2.4 and 2.6 we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(b) + \left({}_{RL}^{k}J_{b^-}^{\alpha}\right)f(a)\right]\right| \leq min\{K_1, K_2, K_3\}$$

where

$$K_1 = \frac{\frac{\alpha}{k}(b-a)^2}{2(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}\left(\frac{|f''(a)|^q + m|f''\left(\frac{b}{m}\right)|^q}{2}\right)^{\frac{1}{q}},$$

$$K_2 = \frac{(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{2}{q\left(\frac{\alpha}{k}+1\right)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q + m|f''\left(\frac{b}{m}\right)|^q}{2}\right)^{\frac{1}{q}},$$

$$K_3 = \frac{(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(\frac{q\left(\frac{\alpha}{k}+1\right)-1}{q\left(\frac{\alpha}{k}+1\right)+1}\right)^{\frac{1}{q}}\left(\frac{|f''(a)|^q + m|f''\left(\frac{b}{m}\right)|^q}{2}\right)^{\frac{1}{q}}.$$

2.2. **Hermite-Hadamard-type inequalities for** (α, m)**-convex functions.**

Before we prove our main results in this section, we give the following lemma:

Lemma 2.7. *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be twice differentiable mapping on* (a, b) *with* $a < mb \leq b$*. If* $f^{''} \in L^1[a, b]$*, then the following identity for* k*-fractional integrals holds*

$$\frac{f(a) + f(mb)}{2} - \frac{\Gamma_k(\alpha + k)}{2(mb - a)^{\frac{\alpha}{k}}} \left[\left({}^k_{RL}J^{\alpha}_{a^+} \right) f(mb) + \left({}^k_{RL}J^{\alpha}_{mb^-} \right) f(a) \right]$$
$$= \frac{(mb - a)^2}{2} \int_0^1 \frac{1 - (1 - t)^{\frac{\alpha}{k}+1} - t^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k} + 1} f^{''}(ta + m(1 - t)b)\, dt.$$

Proof. The proof is similar to the proof of Lemma 2.2 on interval $[a, mb] \subset [a, b]$. ∎

Theorem 2.8. *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be twice differentiable mapping. If* $|f^{''}|^q$ *is measurable and* (s, m)*-convex on* $[a, b]$ *for some fixed* $q \geq 1$ *and* $(s, m) \in [0, 1]^2$*, then the following inequality for* k*-fractional integrals holds:*

$$\left| \frac{f(a) + f(mb)}{2} - \frac{\Gamma_k(\alpha + k)}{2(mb - a)^{\frac{\alpha}{k}}} \left[\left({}^k_{RL}J^{\alpha}_{a^+} \right) f(mb) + \left({}^k_{RL}J^{\alpha}_{mb^-} \right) f(a) \right] \right|$$
$$\leq \frac{(mb - a)^2}{2(\frac{\alpha}{k} + 1)} \left(\frac{\frac{\alpha}{k}}{\frac{\alpha}{k} + 2} \right)^{1 - \frac{1}{q}} \left[|f^{''}(a)|^q \left(\frac{1}{s + 1} - \beta_k \left(s + 1, \frac{\alpha}{k} + 2 \right) - \frac{1}{\frac{\alpha}{k} + s + 2} \right) \right.$$
$$\left. + m|f^{''}(b)|^q \left(1 - \frac{1}{s + 1} - \frac{2}{\frac{\alpha}{k} + 2} + \beta_k \left(s + 1, \frac{\alpha}{k} + 2 \right) + \frac{1}{\frac{\alpha}{k} + s + 2} \right) \right]^{\frac{1}{q}}$$

where $\beta_k(x, y), x, y \geq 0$ *is the* k*-beta function.*

Proof. Firstly, we suppose that $q = 1$. From Lemma 2.7, we have

$$\left| \frac{f(a) + f(mb)}{2} - \frac{\Gamma_k(\alpha + k)}{2(mb - a)^{\frac{\alpha}{k}}} \left[\left({}^k_{RL}J^{\alpha}_{a^+} \right) f(mb) + \left({}^k_{RL}J^{\alpha}_{mb^-} \right) f(a) \right] \right|$$
$$\leq \frac{(mb - a)^2}{2} \int_0^1 \frac{1 - (1 - t)^{\frac{\alpha}{k}+1} - t^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k} + 1} |f^{''}(ta + m(1 - t)b)|\, dt,$$

since $|f^{''}|$ is (α, m)-convex on $[a, b]$, we know that for any $t \in [0, 1]$

$$|f^{''}(ta + m(1 - t)b)| \leq t^s |f^{''}(a)| + m(1 - t^s)|f^{''}(b)|$$

therefore,

$$\left| \frac{f(a) + f(mb)}{2} - \frac{\Gamma_k(\alpha + k)}{2(mb - a)^{\frac{\alpha}{k}}} \left[\left({}^k_{RL}J^{\alpha}_{a^+} \right) f(mb) + \left({}^k_{RL}J^{\alpha}_{mb^-} \right) f(a) \right] \right|$$
$$\leq \frac{(mb - a)^2}{2} \int_0^1 \frac{1 - (1 - t)^{\frac{\alpha}{k}+1} - t^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k} + 1} \left(t^s |f^{''}(a)| + m(1 - t^s)|f^{''}(b)| \right) dt$$
$$= \frac{(mb - a)^2}{2} \left[|f^{''}(a)| \left(\frac{1}{s + 1} - \beta_k \left(s + 1, \frac{\alpha}{k} + 2 \right) - \frac{1}{\frac{\alpha}{k} + s + 2} \right) \right.$$
$$\left. + m|f^{''}(b)| \left(1 - \frac{1}{s + 1} - \frac{2}{\frac{\alpha}{k} + 2} + \beta_k \left(s + 1, \frac{\alpha}{k} + 2 \right) + \frac{1}{\frac{\alpha}{k} + s + 2} \right) \right]$$

which completes the proof for this case.
Now, suppose that $q > 1$, using Lemma 2.7 and power mean inequality for q, we obtain

$$\int_0^1 \left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]|f''(ta+m(1-t)b)|dt$$
$$\leq \left(\int_0^1 \left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]|f''(ta+m(1-t)b)|^q dt\right)^{\frac{1}{q}}.$$

Since $|f''|^q$ is (s,m)-convex on $[a,b]$, we know that for any $t \in [0,1]$

$$|f''(ta+(1-t)b)|^q \leq t^s|f''(a)|^q + m(1-t^s)|f''(b)|^q,$$

therefore,

$$\left|\frac{f(a)+f(mb)}{2} - \frac{\Gamma_k(\alpha+k)}{2(mb-a)^{\frac{\alpha}{k}}}\left[\left({}^k_{RL}J^{\alpha}_{a^+}\right)f(mb) + \left({}^k_{RL}J^{\alpha}_{mb^-}\right)f(a)\right]\right|$$
$$\leq \frac{(mb-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(\int_0^1 \left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]dt\right)^{1-\frac{1}{q}}$$

$$\left(\int_0^1 \left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]|f''(ta+m(1-t)b)|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{(mb-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(\int_0^1 \left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]dt\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1 \left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]\left(t^s|f''(a)|^q + m(1-t^s)|f''(b)|^q\right)dt\right)^{\frac{1}{q}}$$
$$= \frac{(mb-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(\frac{\frac{\alpha}{k}}{\frac{\alpha}{k}+2}\right)^{1-\frac{1}{q}}\left[|f''(a)|^q\left(\frac{1}{s+1} - \beta_k\left(s+1,\frac{\alpha}{k}+2\right) - \frac{1}{\frac{\alpha}{k}+s+2}\right)\right.$$
$$\left.+m|f''(b)|^q\left(1-\frac{1}{s+1}-\frac{2}{\frac{\alpha}{k}+2}+\beta_k\left(s+1,\frac{\alpha}{k}+2\right)+\frac{1}{\frac{\alpha}{k}+s+2}\right)\right]^{\frac{1}{q}}$$

hence the proof. ∎

Corollary 2.9. *In Theorem 2.8 if we choose $s = m = 1$, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}^k_{RL}J^{\alpha}_{a^+}\right)f(b) + \left({}^k_{RL}J^{\alpha}_{b^-}\right)f(a)\right]\right|$$
$$\leq \frac{(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(\frac{\frac{\alpha}{k}}{\frac{\alpha}{k}+2}\right)^{1-\frac{1}{q}}\left[|f''(a)|^q\left(\frac{1}{2} - \beta_k\left(2,\frac{\alpha}{k}+2\right) - \frac{1}{\frac{\alpha}{k}+3}\right)\right.$$
$$\left.+|f''(b)|^q\left(\frac{1}{2}-\frac{2}{\frac{\alpha}{k}+2}+\beta_k\left(2,\frac{\alpha}{k}+2\right)+\frac{1}{\frac{\alpha}{k}+3}\right)\right]^{\frac{1}{q}}.$$

Theorem 2.10. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be twice differentiable mapping. If $|f''|^q$ is measurable and (s,m)-convex on $[a,b]$ for some fixed $q > 1$ and $(s,m) \in [0,1]^2$, then the following inequality* *for k-fractional integrals holds:*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{\Gamma_k(\alpha+k)}{2(mb-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(mb)+\left({}_{RL}^{k}J_{mb^-}^{\alpha}\right)f(a)\right]\right|$$
$$\leq\frac{(mb-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{2}{q\left(\frac{\alpha}{k}+1\right)+1}\right)^{\frac{1}{p}}\left(|f''(a)|^q\frac{1}{s+1}+m|f''(b)|^q\frac{s}{s+1}\right)^{\frac{1}{q}}$$

where, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.7 and by using Hölder's inequality, we have,

$$\left|\frac{f(a)+f(mb)}{2}-\frac{\Gamma_k(\alpha+k)}{2(mb-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(mb)+\left({}_{RL}^{k}J_{mb^-}^{\alpha}\right)f(a)\right]\right|$$
$$\leq\frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}|f''(ta+m(1-t)b)|dt$$
$$\leq\frac{(mb-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(\int_0^1\left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(ta+m(1-t)b)|^q dt\right)^{\frac{1}{q}},$$

since $|f''|^q$ is (s,m)-convex on $[a,b]$, also we know that for any $t\in[0,1]$

$$|f''(ta+m(1-t)b)|^q\leq t^s|f''(a)|^q+m(1-t^s)|f''(b)|^q,$$

therefore,

$$\left|\frac{f(a)+f(mb)}{2}-\frac{\Gamma_k(\alpha+k)}{2(mb-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(mb)+\left({}_{RL}^{k}J_{mb^-}^{\alpha}\right)f(a)\right]\right|$$
$$\leq\frac{(mb-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(\int_0^1\left[1-(1-t)^{p\left(\frac{\alpha}{k}+1\right)}-t^{p\left(\frac{\alpha}{k}+1\right)}\right]dt\right)^{\frac{1}{p}}$$
$$\left(|f''(a)|^q\int_0^1 t^s dt+m|f''(b)|^q\int_0^1(1-t^s)dt\right)^{\frac{1}{q}}$$
$$=\frac{(mb-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{2}{p\left(\frac{\alpha}{k}+1\right)+1}\right)^{\frac{1}{p}}\left(|f''(a)|^q\frac{1}{s+1}+m|f''(b)|^q\frac{s}{s+1}\right)^{\frac{1}{q}}$$

which completes the proof. ∎

Corollary 2.11. *In Theorem 2.10 if we choose $s=m=1$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(b)+\left({}_{RL}^{k}J_{b-}^{\alpha}\right)f(b)\right]\right|$$
$$\leq\frac{(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{2}{p\left(\frac{\alpha}{k}+1\right)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 2.12. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be twice differentiable mapping. If $|f''|^q$ is measurable and (s,m)-convex on $[a,b]$ for some fixed $q>1$ and $(s,m)\in[0,1]^2$, then, the following inequality* *for k-fractional integrals holds:*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{\Gamma_k(\alpha+k)}{2(mb-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a+}^{\alpha}\right)f(mb)+\left({}_{RL}^{k}J_{mb-}^{\alpha}\right)f(a)\right]\right|$$
$$\leq \frac{(mb-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left[|f''(a)|^q\left(\frac{1}{s+1}-\beta_k\left(s+1,q\left(\frac{\alpha}{k}+1\right)+1\right)-\frac{1}{q\left(\frac{\alpha}{k}+1\right)+s+1}\right)\right.$$
$$+m|f''(b)|^q\left(1-\frac{1}{s+1}-\frac{2}{q\left(\frac{\alpha}{k}+1\right)+1}+\beta_k\left(s+1,q\left(\frac{\alpha}{k}+1\right)+1\right)\right.$$
$$\left.\left.+\frac{1}{q\left(\frac{\alpha}{k}+1\right)+s+1}\right)\right]^{\frac{1}{q}}$$

where, $\beta_k(x,y), x,y\geq 0$ is the k-beta function.

Proof. From Lemma 2.7 and by applying Hölder's inequality, we have

$$\left|\frac{f(a)+f(mb)}{2}-\frac{\Gamma_k(\alpha+k)}{2(mb-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(mb)+\left({}_{RL}^{k}J_{mb^-}^{\alpha}\right)f(a)\right]\right|$$
$$\leq \frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}+1}|f''(ta+m(1-t)b)|dt$$
$$\leq \frac{(mb-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(\int_0^1 dt\right)^{\frac{1}{p}}\left(\int_0^1\left[1-(1-t)^{\frac{\alpha}{k}+1}-t^{\frac{\alpha}{k}+1}\right]^q|f''(ta+m(1-t)b)|^q dt\right)^{\frac{1}{q}},$$

since $|f''|^q$ is (s,m)-convex on $[a,b]$, we know that for any $t\in[0,1]$

$$|f''(ta+m(1-t)b)|^q\leq t^s|f''(a)|^q+m(1-t^s)|f''(b)|^q$$

therefore,

$$\left|\frac{f(a)+f(mb)}{2}-\frac{\Gamma_k(\alpha+k)}{2(mb-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(mb)+\left({}_{RL}^{k}J_{mb^-}^{\alpha}\right)f(a)\right]\right|$$
$$\leq \frac{(mb-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left[|f''(a)|^q\int_0^1\left[t^s-(1-t)^{q\left(\frac{\alpha}{k}+1\right)}t^s-t^{q\left(\frac{\alpha}{k}+1\right)+s}\right]dt\right.$$
$$\left.+m|f''(a)|^q\int_0^1\left[(1-t^s)-(1-t)^{q\left(\frac{\alpha}{k}+1\right)}(1-t^s)-t^{q\left(\frac{\alpha}{k}+1\right)}(1-t^s)\right]dt\right)^{\frac{1}{q}}$$
$$=\frac{(mb-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left[|f''(a)|^q\left(\frac{1}{s+1}-\beta_k\left(s+1,q\left(\frac{\alpha}{k}+1\right)+1\right)-\frac{1}{q\left(\frac{\alpha}{k}+1\right)+s+1}\right)\right.$$
$$+m|f''(b)|^q\left(1-\frac{1}{s+1}-\frac{2}{q\left(\frac{\alpha}{k}+1\right)+1}+\beta_k\left(s+1,q\left(\frac{\alpha}{k}+1\right)+1\right)\right.$$
$$\left.\left.+\frac{1}{q\left(\frac{\alpha}{k}+1\right)+s+1}\right)\right]^{\frac{1}{q}}$$

which completes the proof of the theorem. ∎

Corollary 2.13. *In Theorem 2.12 if we choose* $s = m = 1$*, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(b) + \left({}_{RL}^{k}J_{b^-}^{\alpha}\right)f(a)\right]\right|$$
$$\leq \frac{(b-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left[|f''(a)|^q\left(\frac{1}{2} - \beta_k\left(2, q\left(\frac{\alpha}{k}+1\right)+1\right) - \frac{1}{q\left(\frac{\alpha}{k}+1\right)+2}\right)\right.$$
$$\left. + |f''(b)|^q\left(\frac{1}{2} - \frac{2}{q\left(\frac{\alpha}{k}+1\right)+1} + \beta_k\left(2, q\left(\frac{\alpha}{k}+1\right)+1\right) + \frac{1}{q\left(\frac{\alpha}{k}+1\right)+2}\right)\right]^{\frac{1}{q}}.$$

Remark 2.3. From Theorems 2.8, 2.10 and 2.12, we have

$$\left|\frac{f(a)+f(mb)}{2} - \frac{\Gamma_k(\alpha+k)}{2(mb-a)^{\frac{\alpha}{k}}}\left[\left({}_{RL}^{k}J_{a^+}^{\alpha}\right)f(mb) + \left({}_{RL}^{k}J_{mb^-}^{\alpha}\right)f(a)\right]\right| \leq min\{N_1, N_2, N_3\}$$

where

$$N_1 = \frac{(mb-a)^2}{2(\frac{\alpha}{k}+1)}\left(\frac{\frac{\alpha}{k}}{\frac{\alpha}{k}+2}\right)^{1-\frac{1}{q}}\left[|f''(a)|^q\left(\frac{1}{s+1} - \beta_k\left(s+1, \frac{\alpha}{k}+2\right) - \frac{1}{\frac{\alpha}{k}+s+2}\right)+\right.$$
$$\left. m|f''(b)|^q\left(1 - \frac{1}{s+1} - \frac{2}{\frac{\alpha}{k}+2} + \beta_k\left(s+1, \frac{\alpha}{k}+2\right) + \frac{1}{\frac{\alpha}{k}+s+2}\right)\right]^{\frac{1}{q}},$$

$$N_2 = \frac{(mb-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left(1 - \frac{2}{q\left(\frac{\alpha}{k}+1\right)+1}\right)^{\frac{1}{p}}\left(|f''(a)|^q\frac{1}{s+1} + m|f''(b)|^q\frac{s}{s+1}\right)^{\frac{1}{q}},$$

$$N_3 = \frac{(mb-a)^2}{2\left(\frac{\alpha}{k}+1\right)}\left[|f''(a)|^q\left(\frac{1}{s+1} - \beta_k\left(s+1, q\left(\frac{\alpha}{k}+1\right)+1\right) - \frac{1}{q\left(\frac{\alpha}{k}+1\right)+s+1}\right)+\right.$$
$$\left. m|f''(b)|^q\left(1 - \frac{1}{s+1} - \frac{2}{q\left(\frac{\alpha}{k}+1\right)+1} + \beta_k\left(s+1, q\left(\frac{\alpha}{k}+1\right)+1\right) + \frac{1}{q\left(\frac{\alpha}{k}+1\right)+s+1}\right)\right]^{\frac{1}{q}}.$$

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